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Working Paper
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ICIR Working Paper Series, No. 33/19

Provided in Cooperation with:
International Center for Insurance Regulation (ICIR), Goethe University Frankfurt

Suggested Citation: Paulusch, Joachim; Schlütter, Sebastian (2019) : Making the square-root formula compatible with capital allocation, ICIR Working Paper Series, No. 33/19, Goethe University Frankfurt, International Center for Insurance Regulation (ICIR), Frankfurt a. M.

This Version is available at:
http://hdl.handle.net/10419/196909

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ICIR Working Paper Series No. 33/2019

Edited by Helmut Gründl and Manfred Wandt

Making the square-root formula compatible with capital allocation*

Joachim Paulusch†  Sebastian Schlütter‡

This version: 30th April 2019

Abstract

Modern regulatory capital standards, such as the Solvency II standard formula, employ a correlation based approach for risk aggregation. The so-called “square-root formula” uses correlation parameters between, for example, market risk, non-life insurance risk and default risk to determine the company’s aggregate capital requirement. To support business steering, companies will allocate the required capital back to business segments and risk drivers. We demonstrate that capital allocations based on the square-root formula can substantially differ from those based on the true risk distribution if correlations are viewed as Pearson or tail correlations. Thereby, an EVA-maximizing insurer receives misleading steering signals which can induce mispricing of risk and a default probability substantially above the desired level. To make the square-root formula feasible for business steering, we derive partial-derivative-implied correlations which reflect how marginal exposure changes impact the aggregate capital requirement. We show that those partial-derivative-implied correlations imply capital allocations in line with the true risk distribution.

JEL classification: G22, G28, G32.

Keywords: Solvency II, Tail correlation, Risk aggregation, Capital allocation.

*This paper represents the authors’ personal opinions, not necessarily those of their employers. We thank Bruce Auchinleck, Gregor Fels and Helmut Gründl for worthwhile discussions.
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1 Introduction

Risk diversification is at the core of an insurer’s business model. Regulatory capital requirements should therefore appropriately account for risk dependencies and diversification between different risk categories or lines of business. Several modern regulatory capital standards, such as the Solvency II standard formula, employ a correlation-based approach for risk aggregation, which determines the aggregate capital requirement of the company in two steps.\(^1\) Firstly, the stand-alone capital requirements for \(n\) univariate risks are determined, resulting in a capital requirement vector \(x \in \mathbb{R}^n\). Secondly, an \(n \times n\)-matrix \(R\) of correlation parameters is used to determine the aggregate capital requirement, \(\text{CapR}\), by the square-root formula:

\[
\text{CapR} = \sqrt{x^T R x}
\]  

(1)

In this context, a major challenge lies in the calibration of \(R\). Only under the restrictive assumption that risks follow an elliptical distribution, such as the multivariate Gaussian or Student distribution, does the matrix \(R\) reflect the Pearson correlation matrix. In general, calibrating \(R\) with Pearson correlations can cause that the square-root formula to substantially understate or overstate the aggregate capital requirement.\(^2\)

Several studies highlight that an improper calibration of the risk aggregation approach can have detrimental consequences for the companies’ risk taking behavior. Fischer and Schlüetter (2015) simulate an insurer whose capital requirement is determined by the standard formula. If the standard formula does not accurately reflect the correlation between

\(^{1}\)BIS (2010) provide an overview of the risk aggregation approaches used in regulatory capital standards. More recently, the International Capital Standard (ICS) makes use of a correlation-based risk aggregation approach.

the insurer’s asset and liability risks, the insurer’s optimal combination of equity capital and stock risk can be a corner solution. Hence, small changes in the standard formula’s calibration may have dramatic consequences for the insurer’s preferred investment strategy. Braun et al. (2017) study a life insurer’s optimal asset allocation if capital requirements are determined either by the Solvency II standard formula or by an partial internal model. The authors find that the standard formula does not accurately account for diversification effects, substantially restricts the set of feasible asset allocations and may induce an inefficient portfolio selection. Given that an insurer evaluates its portfolio with the return on risk-adjusted capital (RORAC) in combination with the capital required by the standard formula, Braun et al. (2018) find that less-diversified portfolios with high asset risk will be optimal. Chen et al. (2019) provide empirical evidence for moral hazard induced by deficiencies in the risk aggregation approach of the risk-based capital (RBC) system in the United States. The RBC system employs the square-root formula under the assumption that the major risk categories are uncorrelated (i.e., the matrix $R$ in Eq. 1 is the identity matrix). Chen et al. highlight that questionable marginal capital requirements resulting from this set-up have increased the insurers’ risk-taking in terms of their fixed-income securities portfolio.

The aim of this paper is to find an approach to calibrate the square-root formula such that it provides appropriate risk taking and risk management incentives. Given that insurance companies face substantial costs of holding capital, the regulatory capital requirement influences various of their risk management decisions, such as their investment, insurance pricing or reinsurance strategy. In this sense, an important advantage of square-root formula is that it defines a differentiable and homogeneous risk measure, meaning that

---

the capital requirement defined by (1) can be allocated to risk drivers or lines of businesses by the Euler (synonymous: gradient) principle. The Euler capital allocation reflects how marginal exposure changes impact the aggregate capital requirement. Hence, the Euler allocation provides economically reasonable steering signals when optimizing the company’s risk/return profile, e.g. based on the performance measure such as Economic Value Added (EVA) or RORAC.$^5$

Our analysis starts with some examples demonstrating that the Euler capital allocations strongly hinge on the calibration of the matrix $R$. When dealing with non-elliptical distributions, a popular approach is to calibrate $R$ based on tail correlations such as Value-at-Risk (VaR) implied correlations. The aim of VaR-implied correlations is to minimize the distance between the result of (1) and the aggregate capital requirement which results from the true distribution of the risks. For $n = 2$ risks, the correlation between the risks can be set such that (1) matches the aggregate capital requirement according to the true risk distribution.$^6$ given that $R$ is symmetric and its diagonal elements are 1, there is only one free parameter in $R$ and hence, the VaR-implied matrix $R$ is unique. This way to determine $R$ is also recommended by the European insurance supervisory authority (EIOPA 2014, p. 9). For $n \geq 3$, Mittnik (2014) suggests to identify the entries of $R$ jointly such that (1) optimally reflects the capital requirement of a set of $m \geq n$ portfolios.

Our examples point out that neither Pearson correlations nor VaR-implied correlations calibrate $R$ such that the Euler capital allocations of (1) reflect those of the true risk distribution. Most strikingly, for $n = 2$ risks, the Euler capital allocations of (1) in con-

$^5$The maximization of EVA and/or RORAC in connection with capital allocation has been studied by Tasche (2008), Buch et al. (2011) and Diers (2011).

$^6$Cf. Campbell et al. (2002).
neation with the unique VaR-implied matrix can substantially deviate from those based on the true risk distribution. Practically speaking, this means that insurance companies that follow EIOPA’s guidance for the calibration of $R$ and base their risk management decisions on the standard formula, will misstate how their true risk profile reacts to marginal exposure changes. Apart from the first-order derivatives of the aggregate capital requirement with respect to exposure changes, the VaR-implied matrix implies that the second-order derivatives of (1) do not match those of the true risk distribution. As shown by Buch et al. (2011), those derivatives are relevant for the company to choose the magnitude of exposure changes when maximizing RORAC.

To make the square-root formula an appropriate basis for business steering, we propose to take a different view on the matrix $R$. For elliptical distributions, the entries of $R$ globally coincide with the second-order partial derivative of the squared aggregate capital requirement with respect to changes in the capital requirement of the univariate risks. For general distributions, the second-order partial derivatives uniquely define the matrix $R$. We show that the square-root formula in connection with this partial-derivative-implied matrix has useful properties for business steering: it provides the aggregate capital requirement, Euler capital allocations and second-order partial derivatives of the aggregate capital requirement in line with the results from the true risk distribution. Like an ordinary correlation matrix, $R$ is symmetric, but its diagonal elements are not necessarily one.

In an impact assessment, we consider an insurance company that determines the EVA-optimal portfolio in terms of the volumes of its lines-of-business; the capital requirement is defined as the 99.5% Value-at-Risk over one year. In a benchmark case, the capital requirement is determined based on the true risk distribution. Afterwards, it is deter-
mined by the square-root formula in connection with either the VaR-implied matrix or the partial-derivative-implied matrix. In this context, the VaR-implied matrix implies that the EVA-optimal portfolio is substantially led astray from the optimal portfolio in the benchmark case. This may result in a true one-year default probability substantially above the prescribed level of 0.5% (in an extreme case, the EVA-optimal portfolio attains a default probability of 14.9%). The square-root formula in connection with the partial-derivative-implied matrix results in the same EVA-optimal portfolio as in the benchmark case as long as the matrix is calibrated based on a portfolio that is close enough to the optimal one. If the calibration portfolio differs from the optimal one, the partial-derivative-implied matrix still leads to an EVA-optimal portfolio whose true default probability is close to the prescribed level of 0.5%.

2 The square-root formula and capital allocation

Suppose an insurance company is confronted with $n$ risks. The random variable $X_i$, $i \in \{1, \ldots, n\}$, models the loss in the insurer’s equity capital due to risk $i$ over one year. In line with Solvency II regulation, the stand-alone capital requirement for risk $i$ is defined based on the Value-at-Risk at confidence level $\alpha$ of the unexpected loss, $X_i - \mathbb{E}(X_i)$:

$$x_i = \text{CapR}_\alpha(X_i) = \text{VaR}_\alpha(X_i - \mathbb{E}(X_i)) = \text{VaR}_\alpha(X_i) - \mathbb{E}(X_i)$$ (2)

The vector $x = (x_i)_{i=1,...,n} \in \mathbb{R}^n$ contains all $n$ stand-alone capital requirements. The aggregate change in the company’s equity capital is

$$X = \sum_{i=1}^{n} X_i,$$ (3)
and the aggregate capital requirement is given by

$$\text{CapR}_\alpha(X) = \text{VaR}_\alpha(X - \mathbb{E}(X))$$  \hspace{1cm} (4)$$

To derive risk-adjusted performance measures and support decision making, it is reasonable to reallocate the capital requirement in (4) to the $n$ risks. To this end, Tasche (2008) suggests to consider the function

$$u = (u_1, \ldots, u_n)^T \mapsto f_X(u) = \text{CapR}_\alpha\left(\sum_{i=1}^{n} u_i \cdot X_i\right)$$  \hspace{1cm} (5)$$

The vector $(u_1, \ldots, u_n)^T$ can be viewed as the company’s exposures to risks $1, \ldots, n$. Without loss of generality, we may assume that the $X_i$ are scaled such that the coordinates $u = 1_n = (1, \ldots, 1)^T$ reflect the company’s actual portfolio with $f_X(1_n) = \text{CapR}_\alpha(X)$.

Tasche (2008) suggests to compute the contributory capital requirement of risk $k$ as the partial derivative of the aggregate capital requirement with respect to the exposure to risk $k$: \footnote{Similarly, Myers and Read (2001, pp. 557-559) propose to allocate capital to lines of insurance based on the marginal change in a line’s size. In contrast to Tasche, who assumes a homogeneous risk measure, Myers and Read assume that the marginal default values (the sensitivity of the present value of the insurer’s option to default when marginally changing a line’s size) are equal across the lines-of-insurance.}

$$\frac{\partial}{\partial u_k} f_X(1_n) = \left. \frac{d}{dh} \text{CapR}_\alpha(X + h \cdot X_k) \right|_{h=0}$$  \hspace{1cm} (6)$$

Tasche (2008) shows that it is reasonable to calculate the RORAC of risk $k$ based on the contributory capital requirement as defined in (6): if the RORAC of risk $k$ increases the RORAC of the entire company, a small increase in the exposure to risk $k$ enhances the RORAC of the company.
Given that the function in (5) is homogeneous of degree 1, Euler’s theorem implies that

\[ \sum_{k=1}^{n} \frac{\partial}{\partial u_k} f_X (1_n) = \text{CapR}_\alpha (X) \]  

(7)

Hence, the contributory capital requirements add up to the company’s aggregate capital requirement. Since (4) and (6) are based on the complete multivariate risk distribution, we will denote them later on as the “true” aggregate and “true” contributory capital requirement.

The capital requirement defined by the square-root formula is a homogeneous function in \( x \):

\[ x = (x_1, \ldots, x_n)^T \mapsto \sqrt{\sum_{i,j=1}^{n} \varrho_{ij} x_i x_j} = \sqrt{x^T R x}, \quad R = (\varrho_{ij})_{i,j=1}^{n}, \]  

(8)

where \( x_i \) is interpreted as the stand-alone capital requirement of risk \( i \), analogously to Eq. (2). Using the Hadamard product \( u \circ x = (u_1 x_1, \ldots, u_n x_n)^T \in \mathbb{R}^n \), the analogy to function \( f_X \) of equation (5) in terms of the square-root formula is

\[ u = (u_1, \ldots, u_n)^T \mapsto g_x (u) = \sqrt{\sum_{i,j=1}^{n} \varrho_{ij} u_i x_i u_j x_j} = \sqrt{(u \circ x)^T R (u \circ x)}, \]  

(9)

where the subscript \( x \) indicates the dependence on the actual portfolio \( x \) and must not be confused with a derivative. Again, \( g_x \) is a homogeneous function in \( u \). Hence, the contributory capital requirement of risk \( k \) according to the square-root formula is given by

\[ \frac{\partial}{\partial u_k} g_x (u) = \frac{\sum_{i=1}^{n} \varrho_{ki} u_i x_i}{\sqrt{(u \circ x)^T R (u \circ x)}} \cdot x_k, \]  

(10)
with
\[ \sum_{k=1}^{n} \frac{\partial}{\partial u_k} g_x(1_n) = \sqrt{x^T R x} \] (11)

Analogously to the contributory capital requirement in (6), \( \frac{\partial}{\partial u_k} g_x(1_n) \) measures the change in the aggregate capital requirement when the exposure to risk \( k \) is marginally increased. In matrix notation and at \( u = 1_n \), the vector of contributory capital requirements is determined as
\[ \frac{(Rx) \circ x}{\sqrt{x^T R x}} \] (12)

Finally, the second-order partial derivatives of the square-root formula \( g_x \) with respect to exposure changes are obtained as
\[ \frac{\partial^2}{\partial u_k \partial u_l} g_x(u) = -\sum_{i=1}^{n} \varrho_{ki} u_i x_i x_k \cdot \sum_{j=1}^{n} \varrho_{lj} u_j x_j x_l \left( u \circ x \right)_{1.5} + \rho_{k,l} \frac{x_k x_l}{\sqrt{(u \circ x)^T R (u \circ x)}}, \]
with
\[ \frac{\partial^2}{\partial u_k \partial u_l} g_x(1_n) = \frac{\rho_{k,l} x_k x_l - \frac{\partial g_x}{\partial u_k} \frac{\partial g_x}{\partial u_l}}{\sqrt{x^T R x}} \] (13)

3 Pitfalls of the square-root formula

3.1 Set-up

The subsequent numerical examples highlight pitfalls of the square-root formula in connection with a Pearson or VaR-implied matrix \( R \) when risks are not elliptically distributed. We assume that \( X_1, \ldots, X_n \) are independent and identically distributed (iid) random variables which follow the Gamma distribution with shape parameter \( k \) and scale parameter \( \vartheta \). Hence, \( X \) is Gamma distributed with shape parameter \( n \cdot k \) and scale param-
eter $\vartheta$. The setting offers the advantage that the examples are manageable without Monte Carlo simulation, yet insightful. We let the shape parameter $k$ vary in $\{0.2, 0.5, 1, 2, 5\}$. For simplicity, we set the scale parameter $\vartheta = k^{-1}$ such that $\mathbb{E}(X_i) = k \cdot \vartheta = 1$. The two smaller shape parameter values may reflect an insurer’s aggregate non-life insurance risk, which exhibits a relatively heavy tail.\(^8\) The larger values of the shape parameter reflect risks with lighter tails, such as the market risk resulting from a well-diversified portfolio.\(^9\)

Throughout section 3, the capital requirement corresponds to the Value-at-Risk of the undertaking subject to a confidence level of 99.5%.

### 3.2 Bias of the aggregate capital requirement

Pfeifer and Strassburger (2008) have pointed out that the square-root formula in connection with a Pearson correlation matrix may substantially understate or overstate the aggregate capital requirement if risks are independently Beta distributed. This section demonstrates that VaR-implied correlations may even cause a larger misstatement than Pearson correlations. In line with Pfeifer and Strassburger (2008), we measure the bias based on the relative error as

\[
\text{Relative Error} = \frac{\sqrt{x^T R_x \text{CapR}_\alpha(X)}}{\text{CapR}_\alpha(X)} - 1
\]

\(^8\)Bernard et al. (2018, p. 847) assume the distribution $200 \cdot \text{LogNormal}(0, 1)$ for aggregate Non-Life insurance risks. This implies a coefficient of variation of 131.1% and a ratio between the capital requirements at a 99.5% confidence level and a 90% confidence level of 5.884. The Gamma distribution implies a coefficient of variation of 141.6% for $k = 0.5$ and a ratio between the 99.5% and 90% capital requirements of 6.309 for $k = 0.2$. A complete overview of the characteristics of the assumed Gamma distribution parameterizations is provided in Table 1.

\(^9\)Bernard et al. (2018, p. 847) assume a normal distribution for the aggregate market risk, which implies a ratio between the 99.5% and 90% capital requirements of 2.01. This value is achieved by the Gamma distribution when setting the shape parameter $k$ to infinity.
Our first approach to parameterize $R$ is the Pearson correlation matrix, which is the $n \times n$ identity matrix due to the independence of the $X_i$. Secondly, we follow the guidance of EIOPA (2014) and set the entries of $R = (\varrho_{i,j})_{i,j=1}^n$ in pairs to VaR-implied correlation parameters (cf. Campbell et al., 2002, p. 89):

$$
\varrho_{i,j} = \frac{\text{CapR}_\alpha(X_i + X_j)^2 - \text{CapR}_\alpha(X_i)^2 - \text{CapR}_\alpha(X_j)^2}{2 \cdot \text{CapR}_\alpha(X_i) \cdot \text{CapR}_\alpha(X_j)}
$$

(15)

if $i \neq j$ and otherwise $\varrho_{i,i} = 1$. For example, for the shape parameter $k = 0.2$ this results in

$$
\text{CapR}_{99.5\%}(X_1) = \text{VaR}_{99.5\%}(X_1) - \mathbb{E}(X_1) = 13.773 - 1 = 12.773
$$

$$
\text{CapR}_{99.5\%}(X_1 + X_2) = \text{VaR}_{99.5\%}(X_1 + X_2) - \mathbb{E}(X_1 + X_2) = 18.018 - 2 = 16.018
$$

$$
\varrho_{1,2} = \frac{16.018^2 - 12.773^2 - 12.773^2}{2 \cdot 12.773 \cdot 12.773} = -0.214
$$

For $n \geq 3$, all entries of $R$ aside from the diagonal are equal to $\varrho_{1,2}$ due to the symmetry in the example.

Table 1 depicts the characteristics of the employed parameterizations (in terms of the coefficient of variation and the ratio between the capital requirements of a single risk at the 99.5% and 90% level) as well as the relative error. If $R$ is the Pearson correlation matrix, the square-root formula overstates the aggregate capital requirement by up to 37.1% in case of $n = 6$ heavy-tailed risks. If $R$ includes the pairwise VaR-implied correlations, the square-root formula determines the aggregate capital requirement in line with the true risk distribution as long as there are only $n = 2$ risks. If $n \geq 3$, the square-root formula may substantially understate the aggregate capital requirement (for instance, by $-55.9\%$ if $k = 2$ and $n = 6$). Moreover, $x^T Rx$ can be negative meaning that $\sqrt{x^T Rx}$ is not defined
in real numbers. For instance, taking up on the example that \( k = 0.2 \) in connection with \( n = 6 \) risks,

\[
x^T R x = \ n \cdot \text{CapR}_{99.5\%}(X_1)^2 + n \cdot (n - 1) \cdot \varrho_{1,2} \cdot \text{CapR}_{99.5\%}(X_1)^2
\]

\[
= 6 \cdot 12.773^2 + 30 \cdot (-0.214) \cdot 12.773^2 = -67.181
\]

Table 1: Relative error of the square-root formula when \( R \) is the Pearson correlation matrix or determined by pairwise VaR-implied correlation coefficients. “n/a” means that \( \sqrt{x^T R x} \) is not defined because \( x^T R x < 0 \).

\[
\begin{array}{ccccccc}
\text{k} & \text{Coef. of Var.} & \text{CapR}_{99.5\%}/\text{CapR}_{90\%} & \text{Relative error} \\
 & & & \text{Pearson correlation} & \text{Pairwise VaR-implied cor.} \\
 & & & n = 2 & n = 4 & n = 6 & n = 2 & n = 4 & n = 6 \\
0.2 & 223.6\% & 6.309 & 12.8\% & 27.8\% & 37.1\% & 0.0\% & -23.5\% & n/a \\
0.5 & 141.4\% & 4.034 & 13.2\% & 26.7\% & 34.3\% & 0.0\% & -25.9\% & n/a \\
1 & 100.0\% & 3.3 & 11.9\% & 23.2\% & 29.2\% & 0.0\% & -22.7\% & n/a \\
2 & 70.7\% & 2.874 & 10.1\% & 18.9\% & 23.4\% & 0.0\% & -17.9\% & -55.9\% \\
5 & 44.7\% & 2.537 & 7.4\% & 13.5\% & 16.4\% & 0.0\% & -12.1\% & -32.7\%
\end{array}
\]

Mittnik (2014) proposes to determine the entries of \( R \) jointly for a set of calibration portfolios which include portions of the risks that are to be aggregated. To this end, one identifies those correlation parameters \( \varrho_{i,j} \) that minimize the expression

\[
\sum_{k=1}^{l} \left[ \text{CapR}_\alpha \left( \sum_{i=1}^{n} w_i^{(k)} X_i \right)^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \varrho_{ij} \text{CapR}_\alpha \left( w_i^{(k)} X_i \right) \text{CapR}_\alpha \left( w_j^{(k)} X_j \right) \right]^2
\]

(16)

where \( w_i^{(k)} \) is the weight of risk \( i \) in portfolio \( k \) with \( \sum_{i=1}^{n} w_i^{(k)} = 1 \) for all portfolios \( k \in \{1, \ldots, l\} \). For a given set of portfolios and assuming that \( \varrho_{ij} = \varrho_{ji} \) for all \( i, j \in \{1, \ldots, n\} \) and \( \varrho_{ii} = 1 \) for all \( i \in \{1, \ldots, n\} \), Mittnik (2014) derives closed-form solutions for the set of correlation parameters \( \varrho_{i,j} \) minimizing (16). There are various possible sets of calibration portfolios. Mittnik (2014) distinguishes between an “exact” identification if the number of portfolios coincides with the number of correlation parameters to be determined, \( l = n(n - 1)/2 \), and an “overidentified” identification if the number of portfolios exceeds that
of the correlation parameters, \( l > n(n - 1)/2 \). With an exact identification, the target function in (16) attains zero and hence, the square-root formula determines the capital requirement accurately for all \( n(n - 1)/2 \) portfolios. As the next Lemma states, the set of portfolios can be chosen such that the square-root formula determines the aggregate capital requirement of a company in line with the true risk distribution.

**Lemma 1.** Let \( x_i = \text{CapR}_\alpha(X_i) \neq 0 \) for all \( i \in \{1, ..., n\} \). Then there exists a matrix \( R \) such that \( \sqrt{x^\top R x} = \text{CapR}_\alpha(X) \).

One issue about this result is that \( R \) is in general not unique if \( n \geq 3 \). More importantly, as the next subsection demonstrates, the VaR-implied matrices provide misleading signals for business steering.

### 3.3 Bias of contributory capital requirements

The subsequent examples demonstrate that the contributory capital requirements according to the square-root formula can substantially deviate from those calculated based on the true distribution.

The first example deals with \( n = 2 \) risks with \( X_1 \sim \Gamma(k = 0.5, \vartheta = 0.5) \) and \( X_2 \sim \Gamma(k = 2, \vartheta = 2) \).\(^{10}\) The stand-alone capital requirements are

\[
x = \begin{pmatrix} 6.879 \\ 2.715 \end{pmatrix}
\]

\(^{10}\)If both risks were identically parameterized, the square-root formula as well as the true distribution imply that all risks receive the same contributory capital requirements. In this case, trivially, the square-root formula determines the contributory capital requirements in line with the true risk distribution if the aggregate capital requirement is stated correctly.
To determine the true contributory capital requirements in (6), we need to calculate the percentile of \(X + h \cdot X_i\) for small values of \(h\). The random variable \(X + h \cdot X_i\) is not Gamma distributed, but, based on Moschopoulos (1985), its distribution function can be expressed analytically. Using this representation for the distribution function, we have obtained percentiles of \(X + h \cdot X_i\) numerically by the Newton method. According to (6), we have calculated the true contributory capital requirements of risk 1 as 6.6523 and that of risk 2 as 0.4042, meaning that \(\text{CapR}_{99.5\%}(X) = 6.6523 + 0.4042 = 7.0565\). The VaR-implied correlation parameter is determined as

\[
\varrho_{1,2} = \frac{\text{CapR}_\alpha(X)^2 - \text{CapR}_\alpha(X_1)^2 - \text{CapR}_\alpha(X_2)^2}{2 \cdot \text{CapR}_\alpha(X_1) \cdot \text{CapR}_\alpha(X_2)} = \frac{7.0565^2 - 6.879^2 - 2.715^2}{2 \cdot 6.879 \cdot 2.715} = -0.1313
\]

The contributory capital requirements according to the square-root formula are determined by (12) as

\[
\begin{align*}
(Rx) \odot x &= \frac{1}{7.0565} \begin{pmatrix} 1 & -0.1313 \\ -0.1313 & 1 \end{pmatrix} \begin{pmatrix} 6.879 \\ 2.715 \end{pmatrix} \odot \begin{pmatrix} 6.879 \\ 2.715 \end{pmatrix} = \begin{pmatrix} 6.3593 \\ 0.6972 \end{pmatrix} \\
\end{align*}
\]

Hence, the contributory capital requirement of risk 1 is understated by \(6.3593/6.6523 - 1 = -4.4\%\) and the one of risk 2 is overstated by \(0.6972/0.4042 - 1 = 72.5\%\).

The second example demonstrates that the choice of calibration portfolios to determine the VaR-implied matrix \(R\) substantially impacts the contributory capital requirements resulting from the square-root formula. We consider \(n = 5\) risks with \(X_i \sim \Gamma(k = 0.5, \vartheta = 0.5)\) for \(i \in \{1, 2, 3\}\) and \(X_i \sim \Gamma(k = 2, \vartheta = 2)\) for \(i \in \{4, 5\}\). According to (6), the true contributory capital requirements of risks 1, 2 and 3 are each 3.1725 and those of risks
4 and 5 are each 0.2963. Summing up, this coincides with $\text{CapR}_{99.5\%}(X) = 10.1101$. To calibrate $R$ by minimizing (16), Mittnik (2014) uses equally-weighted portfolios of up to $n$ risks; Table 2 sketches all possible equally-weighted portfolios for $n = 5$. There are $\binom{5}{2} = 10$ equally-weighted portfolios with 2 risks, $\binom{5}{3} = 10$ portfolios with 3 risks, $\binom{5}{4} = 5$ portfolios with 4 risks and one portfolio with 5 risks.

Table 2: Possible equally-weighted portfolios to calibrate $R$ in case of $n = 5$ risks.

<table>
<thead>
<tr>
<th>Portfolio no., $k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>10</th>
<th>11</th>
<th>...</th>
<th>20</th>
<th>21</th>
<th>...</th>
<th>25</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>0</td>
<td>1/4</td>
<td>...</td>
<td>0</td>
<td>1/5</td>
</tr>
<tr>
<td>$w_2$</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>0</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>1/3</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>...</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>1/3</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
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<tr>
<td>$w_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>1/3</td>
<td>0</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
</tbody>
</table>

According to Lemma 1, portfolio no. 26 and nine further portfolios are needed to ensure that the square-root formula determines the aggregate capital requirement exactly. There are $\binom{25}{9} = 2,042,975$ sets of portfolios that achieve this target. Calibrating $R$ based on all these sets would require extensive calculation time. For simplicity, we have performed the calibration for 1,000 randomly chosen sets of portfolios. Based on each of these sets, we have also determined the contributory capital requirements resulting from the square-root formula in connection with $R$ (cf. line (10)). Figure 1 depicts for each set the contributory capital requirements of risks $X_1$ and $X_4$. All calibrations lead to the aggregate capital requirement of 10.1101, but in many cases the calculated contributory capital requirements give the insurance company an odd view on the interplay of the risks, since they deviate strongly from the results implied by the true risk distribution. For example, several sets of portfolios imply that the contributory capital requirement of $X_4$ is negative, indicating that this risk provides a natural hedge of the aggregate risk and that an extension of the exposure to $X_4$ reduces the aggregate risk. In fact, however, the true aggregate capital requirement increases by such an extension. Moreover, in most
cases the calculated contributory capital requirements of risks 1, 2 and 3 are not identical; the same applies to risks 4 and 5. Hence, the insurance company would rank these risks differently, even though they are in fact identical.

Figure 1: Contributory capital requirement of risks $X_1$ and $X_4$ according to square-root formula in connection with matrix $R$ based on 1,000 randomly chosen sets of portfolios as well as according to true risk distribution.

4 Partial-derivative-implied correlations

4.1 Approach

To improve the suitability of the square-root formula in conjunction with the Euler capital allocation principle, we propose an alternative view on the meaning of the matrix $R$. If the random vector $(X_1, \ldots, X_n)$ follows an elliptical distribution and $\text{CapR}_\alpha$ is proportional to the standard deviation, we have

$$f_X(u) = \sqrt{\sum_{i,j=1}^{n} \varrho_{ij}u_iu_jx_i},$$

(17)
where $x_i$ denote the stand-alone capital requirements of $X_i$ (cf. (2)) and $\varrho_{ij}$ the respective Pearson correlation coefficients. This implies

$$\frac{\partial}{\partial u_k} f^2_X(u) = 2x_k \sum_{j=1}^n \varrho_{kj} u_j x_j,$$

(18)

and

$$\varrho_{k\ell} = \frac{1}{2x_k x_\ell} \frac{\partial^2}{\partial u_k \partial u_\ell} f^2_X(u)$$

(19)

Hence, the entries of $R$ coincide with the second-order partial derivative of the squared capital requirement with respect to exposures to the respective risks, divided by twice the stand-alone capital requirements of these risks. For elliptical distributions, Eq. (19) holds globally for any exposure vector $u$.

If the risks are not elliptically distributed, Eq. (19) does not hold in general. However, for portfolios $u \circ x$ that do not differ too much from the actual portfolio $u = 1_n$, Eq. (19) still provides reasonable guidance for the calibration of $R$. At $u = 1_n$, Eq. (19) can be rewritten as

$$\varrho_{k\ell} = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_\ell} f^2_X(1_n)$$

(20)

Eq. (20) provides a unique definition of a matrix $R = (\varrho_{k\ell})_{k,l=1}^n$, which we will subsequently call the partial-derivative-implied matrix. Theorem 1 shows that this matrix is a suitable basis of the square-root formula.

**Theorem 1.** Let $U \subseteq \mathbb{R}^n$ be open, $x \in U$, and $f : U \rightarrow \mathbb{R}$ be a homogeneous function that is twice continuously differentiable. Let $f(x) > 0$. Then, the matrix $R = R(x) = (\varrho_{k\ell})_{k,l=1}^n$ defined by

$$\varrho_{k\ell} = \varrho_{k\ell}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_\ell} f^2(x)$$

(21)
is symmetric. Writing \( f_x(u) = f(u \circ x) \) and

\[
g_x(u) = \sqrt{(u \circ x)^T R(x) (u \circ x)},
\]

(22)

the following holds:

\[
g_x(1_n) = f_x(1_n),
\]

(23)

\[
\frac{\partial}{\partial u_i} g_x(1_n) = \frac{\partial}{\partial u_i} f_x(1_n),
\]

(24)

\[
\frac{\partial^2}{\partial u_k \partial u_i} g_x(1_n) = \frac{\partial^2}{\partial u_k \partial u_i} f_x(1_n).
\]

(25)

According to Theorem 1, the square-root formula in connection with the partial-derivative-based matrix \( R \) has useful properties. For the actual portfolio, the square-root formula provides the aggregate capital requirement, the contributory capital requirements of all risks (cf. Eq. (12)) as well as all second-order derivatives of the aggregate capital requirement (cf. Eq. (13)) in accordance with the true multivariate risk distribution.

As demonstrated in section 3.3, it can be impossible to find a correlation matrix \( R \) such that the square-root formula matches the true risk distribution in terms of the aggregate capital requirement and the contributory capital requirements. To achieve this matching, the diagonal entries of the partial-derivative-based matrix can deviate from one. If only \( n = 2 \) risks are to be aggregated, the number of free parameters in \( R \) increases from 1 to 3. For a larger number of risks, the number of additional free parameters is moderate. For \( n = 5 \), for example, the partial-derivative based matrix has 15 free parameters whereas a correlation matrix has 10.
4.2 Example

We continue the example from section 3.3 with \( n = 2 \) risks, i.e. \( X_1 \sim \Gamma(k = 0.5, \vartheta = 0.5) \) and \( X_2 \sim \Gamma(k = 2, \vartheta = 2) \). According to line 31, we obtain the matrix \( R \) as

\[
R = \begin{pmatrix}
1.0244 & -0.0824 \\
-0.0824 & 0.5958
\end{pmatrix}
\] (26)

Incorporating this matrix into the square-root formula implies that the contributory capital requirements are (cf. Eq. (12))

\[
\frac{(Rx) \circ x}{\sqrt{x^T R x}} = \frac{1}{7.0565} \begin{pmatrix}
1.0244 & -0.0824 \\
-0.0824 & 0.5958
\end{pmatrix} \begin{pmatrix}
6.879 \\
2.715
\end{pmatrix} \circ \begin{pmatrix}
6.879 \\
2.715
\end{pmatrix} = \begin{pmatrix}
6.6523 \\
0.4042
\end{pmatrix}
\]

Moreover, the second-order derivatives of the capital requirement according to the square-root formula are obtained by Eq. (13) as

\[
\frac{\partial^2}{\partial u_1 \partial u_1} g_x(u) = \frac{1.0244 \cdot 6.879^2 - 6.6523^2}{7.0565} = 0.5993
\]

\[
\frac{\partial^2}{\partial u_1 \partial u_2} g_x(u) = \frac{(-0.0824) \cdot 6.879 \cdot 2.715 - 6.6523 \cdot 0.4042}{7.0565} = -0.5993
\]

\[
\frac{\partial^2}{\partial u_2 \partial u_2} g_x(u) = \frac{0.5958 \cdot 2.715^2 - 0.4042^2}{7.0565} = 0.5993
\]

As stated by Theorem 1, the just calculated contributory capital requirements and second-order derivatives based on the square-root formula coincide with the results when all calculations are based on the true risk distribution.
5 Implications for business steering

5.1 Set-up

The feasibility of the matrix $R$ for capital allocation is highly relevant for business steering. This section illustrates that a misspecification of $R$ has detrimental consequences for policyholders in terms of insurance premiums as well as the default risk level that the insurer actually attains.

Assume that an insurer operates with $n$ lines of business (lob’s). The scalars $u_i$ represent the volume of lob $i$ in terms of the number of insurance contracts. Suppose that the $u_i$ are scaled, for example, in 100,000 contracts such that we may disregard the integer restriction. The total claims costs of lob $i$ are modelled by $u_i \cdot X_i$. Moreover, we assume that the diversification within each lob does not vary in $u_i$ such that the total claims costs of lob $i$ are modelled by $u_i \cdot X_i$. The connection between the volume $u_i$ and the premium $p_i$ of lob $i$ is determined by an isoelastic demand function,\(^\text{11}\)

$$u_i(p_i) = n_i \cdot p_i^{\epsilon_i},\quad (27)$$

where $n_i > 0$ calibrates demand to market size and $\epsilon_i < -1$ is the price elasticity of demand which is constant in $p_i$. We consider a representative insurer whose objective is to maximize its economic value added (EVA).\(^\text{12}\) In our model, the insurer’s EVA is the

\(^{11}\)To simplify the notation, $p_i$ is also scaled. If $u_i$ are specified per 100,000 contracts, $p_i$ is the premium income per 100,000 contracts.

\(^{12}\)In the context of an insurer’s asset management, Braun et al. (2018) use the return on risk-adjusted capital (RORAC) as the insurer’s objective. In the context of evaluating the performance of an insurer’s lob’s, the EVA is advantageous, since the contributory capital of a lob can become negative (or zero) implying that the RORAC is not meaningful (or not defined); cf. Diers (2011).
expected profit, deducted by the cost of capital, which are modelled by a hurdle rate \( r_h \) times the insurer’s capital requirement:

\[
EVA(u) = \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot \text{CapR}_\alpha \left( \sum_{i=1}^{n} u_i \cdot X_i \right),
\]

where \( p_i(u_i) \) is the inverse of the demand function in Eq. (27).

The EVA-maximizing strategy can be identified by the Newton method. To this end, let \( \nabla_uEVA \) denote the gradient of the EVA with respect to \( u \) and \( H_uEVA \) the respective Hessian matrix. Starting from the vector \( u^{(k)} \in \mathbb{R}^n \), the next iteration is to choose the volumes

\[
u^{(k+1)} = u^{(k)} - [H_uEVA]^{-1}\nabla_uEVA
\]

(28)

The optimal strategy is characterized by \( \nabla_uEVA = 0 \), which can equivalently be expressed in terms of the optimal premiums

\[
p_i = \frac{1}{1+1/\epsilon_i} \cdot \left( \mathbb{E}[X_i] + r_h \cdot \frac{\partial}{\partial u_i} \text{CapR}_\alpha \left( \sum_{i=1}^{n} u_i \cdot X_i \right) \right) \quad \forall i \in \{1, \ldots, n\}
\]

(29)

Hence, the optimal premium for lob \( i \) reflects the expected claims costs within this lob, \( \mathbb{E}[X_i] \), plus the additional cost of capital caused by a marginal extension of this lob, multiplied with a loading factor which is the larger, the closer the price elasticity of demand is at -1.

### 5.2 Example with 2 risks

We will now use some numerical examples to study how strongly insurance premiums and the insurer’s default risk can be distorted if the square-root formula is calibrated
with VaR-implied or partial-derivative-implied correlation parameters. We start with
the example with \( n = 2 \) risks from section 3.3, i.e. \( X_1 \sim \Gamma(k = 0.5, \vartheta = 0.5) \) and
\( X_2 \sim \Gamma(k = 2, \vartheta = 2) \). We set \( \epsilon_i = -9 \) for both lob’s \( i \),\(^{13} \) and \( r_h = 5\% \).

As a benchmark case, we determine the Value-at-Risk based on the true risk distribution
and derive the corresponding EVA-maximizing strategy. If the demand function param-
ters are \( n_1 = 38.2587 \) and \( n_2 = 3.4564 \), the EVA-maximizing strategy is characterized by
\( u = (1, 1)^T = I_2 \). Hence, the true contributory capital requirements are those calculated
in section 3.3 and the optimal premiums are determined by Eq. (29) as

\[
\begin{align*}
    p_1 &= \frac{1}{1 + 1/-9} \cdot (1 + 5\% \cdot 6.6523) = 1.499 \\
    p_2 &= \frac{1}{1 + 1/-9} \cdot (1 + 5\% \cdot 0.4042) = 1.148
\end{align*}
\]

The larger contributory capital requirement of lob 1 with heavy-tailed risks makes in-
surance contracts substantially more expensive than those of lob 2. Figure 2 depicts the
contour lines of the EVA-function. Point A, i.e. \( u = (1, 1)^T \), reflects the optimal strategy
that has just been calculated. In addition, Table 3 shows all relevant calculation results.

Suppose now that the capital requirement in the EVA is determined by the square-
root formula in connection with the partial-derivative-implied matrix, which is calibrated
based on the exposures \( u = (1, 1)^T \). Given that the contributory capital requirements
coincide with the true ones according to Theorem 1, the strategy \( u = (1, 1)^T \) and \( p =
(1.499, 1.148)^T \) is again EVA-optimal.

\(^{13}\)In terms of the empirical results of Yow and Sherris (2008, p. 318), this may reflect the price elasticity
in motor insurance. According to Yow and Sherris (2008), customers are more price sensitive in other
lines of business, such as liability insurance or fire insurance.
Next, suppose that the square-root formula is used in connection with the VaR-implied matrix, which, again, is calibrated for \( u = (1, 1)^T \). Even at \( u = (1, 1)^T \), the first and second-order derivatives of \( \text{CapR}_\alpha \left( \sum_{i=1}^n u_i \cdot X_i \right) \) with respect to \( u_i \) deviate from those calculated based on the true risk distribution (cf. section 3.3). Since these derivatives enter the gradient \( \nabla_u \text{EVA} \) and the Hessian matrix \( H_u \text{EVA} \) in Eq. (28), the optimal strategy is led astray to \( u = (1.0616, 0.9286)^T \) and \( p = (1.489, 1.1572)^T \). Recall that the VaR-implied matrix has been calibrated such that the square-root formula determines the aggregate capital requirement in accordance with the true risk distribution only for \( u = (1, 1)^T \). Given that the insurer is incentivized to offer more insurance contracts with heavy-tailed risks, the square-root formula understates the aggregate risk. If the insurer adjusts its equity capital in accordance with the capital required by the standard formula, the true insolvency probability increases to 0.509%.

So far, we have made the strong assumption that the calibration portfolio for \( R \) coincides with the EVA-maximizing portfolio in the benchmark scenario (i.e. when the Value-at-Risk is calculated based on the true risk distribution); in practical considerations, these two portfolios might deviate. We will now demonstrate that those deviations hardly impact the EVA-maximizing strategy if partial-derivative-implied correlations are used, but they intensify the distortions caused by VaR-implied correlations. Suppose that the regulator calibrates \( R \) based on the portfolio \( \tilde{u} = (1.1, 0.9)^T \) which includes more heavy-tailed and less light-tailed risks (cf. point B in Figure 2). If partial-derivative-implied correlations are used, the EVA based on the square-root formula is optimal for \( u = (0.998, 1.003)^T \), which is close to the optimum based on the true risk distribution. If VaR-implied correlations are used, an insurer using the square-root formula opts for \( u = (1.057, 0.912)^T \). The arrows in Figure 2 depict these distortions graphically.
When using VaR-implied correlations, a poor choice of the calibration portfolio can imply that the insurer’s true default probability is clearly above 0.5%. Suppose that the regulator calibrates $R$ based on less heavy-tailed and more light-tailed risks (points D and D’ in Figure 2). The square-root formula together with VaR-implied correlations induces a strategy with a larger portion of heavy-tailed risks. If the calibration portfolio is in point D, i.e. $\tilde{u} = (0.9, 1.1)^T$, the insurer chooses the portfolio $u = (1.067, 0.947)^T$ which results in a true default probability of 0.524%. If the calibration portfolio is in point D’, the insurer chooses $u = (1.073, 0.965)^T$ and the true default probability of 0.54%; The square-root formula in connection with partial-derivative-implied correlations distorts the EVA-optimal strategy less strongly and captures the resulting aggregate risk more accurately. Hence, the use of partial-derivative-implied correlations ends up in a true default probability of 0.5% for point D and of 0.496% for point D’, respectively.

Figure 3 shows for a wide range of calibration portfolios that VaR-implied correlations distort the true default probability much more than partial-derivative-implied correlations. The capital requirement in the EVA is calculated based on the square-root formula in connection with a matrix $R$. The coordinates in Figure 3 depict the calibration portfolio $\tilde{u} \in [0.7, 1.3] \times [0.7, 1.3]$. The colors in the graph on the left-hand side of Figure 3 depict the true default probabilities if $R$ includes VaR-implied correlations. Here, only a narrow area of calibration portfolios ensures a true default probability of 0.5%. According to the graph on the right-hand side, the partial-derivative-implied matrix ensures that the EVA-optimal strategy has a default probability close to 0.5% for a wide area of calibration portfolios.
Table 3: EVA-optimal strategies based on square-root formula with partial-derivative-implied (pd-impl.) and VaR-implied matrices $R$, calibrated at five coordinates A, B, C, D, D’ (cf. Fig. 2). $X_1 \sim \Gamma(k = 0.5, \vartheta = 0.5)$ (heavy-tailed) and $X_2 \sim \Gamma(k = 2, \vartheta = 2)$ (light-tailed).

<table>
<thead>
<tr>
<th>Type of R</th>
<th>$R$ calibrated at</th>
<th>Optimal volumes</th>
<th>Optimal premiums</th>
<th>True Default Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1$</td>
<td>$u_2$</td>
<td>$u_1$</td>
<td>$u_2$</td>
</tr>
<tr>
<td>A</td>
<td>pd-impl.</td>
<td>1.0</td>
<td>1.0</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>VaR-impl.</td>
<td>1.0</td>
<td>1.0</td>
<td>1.062</td>
</tr>
<tr>
<td>B</td>
<td>pd-impl.</td>
<td>1.1</td>
<td>0.9</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>VaR-impl.</td>
<td>1.1</td>
<td>0.9</td>
<td>1.057</td>
</tr>
<tr>
<td>C</td>
<td>pd-impl.</td>
<td>0.9</td>
<td>0.9</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>VaR-impl.</td>
<td>0.9</td>
<td>0.9</td>
<td>1.062</td>
</tr>
<tr>
<td>D</td>
<td>pd-impl.</td>
<td>0.9</td>
<td>1.1</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>VaR-impl.</td>
<td>0.9</td>
<td>1.1</td>
<td>1.067</td>
</tr>
<tr>
<td>D’</td>
<td>pd-impl.</td>
<td>0.8</td>
<td>1.2</td>
<td>0.975</td>
</tr>
<tr>
<td></td>
<td>VaR-impl.</td>
<td>0.8</td>
<td>1.2</td>
<td>1.073</td>
</tr>
</tbody>
</table>

Figure 2: Contour lines of the function $EVA(u)$. Points A, B, C, D and D’ reflect the portfolios based on which $R$ is calibrated. The arrows show how the portfolio is adjusted if $R$ is calibrated based on volumes $\tilde{u} = (1.1, 0.9)^T$ with partial-derivative-implied correlations (upper arrow) or VaR-implied correlations (lower arrow).
5.3 Example with 5 risks

We now study the EVA-optimal strategy based on the example from section 3.3 with \( n = 5 \) risks, i.e., \( X_1, X_2 \) and \( X_3 \) are heavy-tailed and \( X_4 \) and \( X_5 \) are light-tailed. Setting the size parameters of the demand function to \( n_1 = n_2 = n_3 = 10.862 \) and \( n_4 = n_5 = 3.295 \) and calculating the capital requirement based on the true risk distribution, \( u = 1_5 \) is EVA-optimal. These exposures correspond to insurance premiums \( p_1 = p_2 = p_3 = 1.303 \) and \( p_4 = p_5 = 1.142 \).

At \( u = 1_5 \), we calibrate the partial-derivative-implied matrix as

\[
R = \begin{pmatrix}
1.1398 & -0.2240 & -0.2240 & -0.0178 & -0.0178 \\
-0.2240 & 1.1398 & -0.2240 & -0.0178 & -0.0178 \\
-0.2240 & -0.2240 & 1.1398 & -0.0178 & -0.0178 \\
-0.0178 & -0.0178 & -0.0178 & 0.5321 & 0.0093 \\
-0.0178 & -0.0178 & -0.0178 & 0.0093 & 0.5321 \\
\end{pmatrix}
\]
The square-root formula in connection with this matrix leads to the EVA-optimal strategy $u = 1_5$, matching the optimal strategy based on the true risk distribution.

Recall from section 3.3 that the VaR-implied matrix is not unique for $n = 5$, but depends on the set of calibration portfolios. For each of the 1,000 sets of portfolios considered in section 3.3, we have determined the EVA-optimal strategy based on the square-root formula in connection with the VaR-implied matrix calibrated at $u = 1_5$. In each of the 1,000 instances, the EVA-optimal strategy results in a true default probability above 0.5%. For 10% of the sets of calibration portfolios, the true default probability is above 0.75%; in one case it is even 14.9%. Here, the EVA-optimal exposures to the heavy-tailed risks rank between 0.587 and 0.716 and those to the light-tailed risks are 2.359 and 1.988. The square-root formula understates the Value-at-Risk for these exposures which leads to a substantially higher default probability than the originally desired one.14

6 Conclusion

This article highlights that Pearson as well as VaR-implied correlations do not provide a sound basis to calibrate the matrix $R$ of the square-root formula if steering signals received from capital allocations matter. As an alternative, we propose to calibrate $R$ based on second-order partial derivatives of the (squared) capital requirement with respect to exposure changes. The proposed partial-derivative-implied matrix $R$ makes the square-root formula compatible with the Euler capital allocation principle. Those capital allocations support to evaluate marginal changes of the current portfolio. Under the prerequisite that $R$ is calibrated based on a portfolio which is not too far away from

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14 As a robustness check, we have also performed the VaR-implied calibration of $R$ based on the two-step approach proposed by Mittnik (2014, p. 71 f.), which ensures that $R$ is positive semidefinite. However, this alternative specification had only a minor impact for the EVA-optimal strategies.
the optimal portfolio, they help to find the optimal portfolio. In terms of Solvency II regulation, the evaluation of marginal portfolio changes around current exposures seems to be within the intended purpose of the standard formula. If an insurer takes substantial portfolio changes into consideration, Solvency II would require to evaluate the risk of the potential new portfolio with appropriate methods in a so-called Own Risk and Solvency Assessment (ORSA). As outlined by Gründl and Schmeiser (2007), capital allocation is not helpful in this context.

Our examples highlighting the pitfalls of Pearson and VaR-implied correlation matrices make use of the complete multivariate risk distribution. Hence, the pitfalls do not result from sampling or estimation issues, but from the concept of the square-root formula in connection with those matrices. When thinking about an estimator for $R$ based on a finite sample, it has been unclear up to now how to assess the bias of such an estimator: since the VaR-implied matrix $R$ is not unique for $n > 3$ risks, its entries are not uniquely defined. The suggested partial-derivative-implied matrix $R$ is uniquely defined based on the complete multivariate risk distribution, and it makes the square-root formula consistent with the true aggregate capital requirement of the current portfolio as well as with the first two partial derivatives. An important question for future research is to derive an appropriate estimator of $R$. In this context, the thresholds derived by Bernard et al. (2018) can be helpful to ensure that the square-root formula in connection with $R$ provides a suitable aggregate capital requirement.
Appendix

Proof of Lemma 1. For each \( s \in \{1, \ldots, n(n-1)/2-1\} \), choose the indices \( 1 \leq k_s < \ell_s \leq n \) and define a portfolio \( w^{(s)} \in \mathbb{R}^n \) with \( w^{(s)}_{k_s} = w^{(s)}_{\ell_s} = 1 \) and \( w^{(s)}_i = 0 \) for all other entries \( i \). The indices shall be chosen such that the \( n(n-1)/2 - 1 \) portfolios are pairwise different. The last portfolio’s entries are set to \( w_k^{(n(n-1)/2)} = 1 \) for all \( k = 1, \ldots, n \). For \( s \in \{1, \ldots, n(n-1)/2\} \), we set up the equation

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \varrho_{ij} \text{CapR}_\alpha(w^{(s)}_i X_i) \text{CapR}_\alpha(w^{(s)}_j X_j) = \text{CapR}_\alpha\left(\sum_{i=1}^{n} w^{(s)}_i X_i \right)^2,
\]

which is linear in the variables \( \varrho_{ij} \). The first \( n(n-1)/2 - 1 \) equations simplify to

\[
2\varrho_{k_s \ell_s} \text{CapR}_\alpha(X_{k_s}) \text{CapR}_\alpha(X_{\ell_s}) = \text{CapR}_\alpha\left(X_{k_s} + X_{\ell_s} \right)^2
\]

and can be uniquely solved for \( \varrho_{k_s \ell_s} \). The last equation simplifies to

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \varrho_{ij} \text{CapR}_\alpha(X_i) \text{CapR}_\alpha(X_j) = \text{CapR}_\alpha\left(\sum_{i=1}^{n} X_i \right)^2
\]

and it can be solved for the last unknown parameter \( \varrho_{k \ell} \). Taking the square-root on both sides of Eq. (30) implies \( \sqrt{x^T R x} = \text{CapR}_\alpha(X) \).

\( \square \)

Proof of Theorem 1. We write \( \partial_{\ell} = \partial/\partial x_{\ell} \) and \( \partial_{\ell} = \partial/\partial u_{\ell} \), applying to functions of the variable \( x \), or \( u \), respectively. Note that

\[
\varrho_{k \ell}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_{\ell}} f^2(x) = \frac{\partial}{\partial x_{\ell}} \left\{ f(x) \frac{\partial f}{\partial x_k} (x) \right\} = \partial_{\ell} \{ f(x) \partial_k f(x) \}
\]

(31)
Schwarz’s Theorem on the symmetry of second-order derivatives shows the symmetry of $R$, and the product rule implies

$$
\partial_\ell \left\{ f(x) \sum_k x_k \partial_k f(x) \right\} = \sum_k x_k \partial_\ell \left\{ f(x) \partial_k f(x) \right\} + f(x) \partial_\ell f(x). \tag{32}
$$

We use this and Euler’s Theorem on homogeneous functions (cf. Tasche, 2008, p. 4), namely

$$
\sum_\ell x_\ell \partial_\ell f(x) = f(x), \tag{33}
$$

and derive

$$
\sum_{k\ell} x_k x_\ell \ell_k = \sum_{k\ell} x_k x_\ell \partial_\ell \left\{ f(x) \partial_k f(x) \right\} = \sum_\ell x_\ell \left[ \sum_k x_k \partial_\ell \left\{ f(x) \partial_k f(x) \right\} \right]
= \sum_\ell x_\ell \left[ \partial_\ell \left\{ f(x) \sum_k x_k \partial_k f(x) \right\} - f(x) \partial_\ell f(x) \right] \tag{34}
= \sum_\ell x_\ell \left[ \partial_\ell f^2(x) - f(x) \partial_\ell f(x) \right] = \sum_\ell f(x) x_\ell \partial_\ell f(x) = f^2(x).
$$

The assumption $f(x) > 0$ now implies (23). To prove equation (24), we note that

$$
f_x(1_n) = f(x) \text{ and } \partial_\ell f_x(1_n) = x_\ell \partial_\ell f(x). \tag{35}
$$

We use this and equations (10), (23), (32) and (33) to show

$$
\partial_\ell g_x(1_n) = \frac{x_\ell}{g_x(1_n)} \sum_{k=1}^n \ell_k x_k = \frac{x_\ell}{f_x(1_n)} \sum_k x_k \partial_\ell \left\{ f(x) \partial_k f(x) \right\}
= \frac{x_\ell}{f_x(1_n)} \left[ \partial_\ell \left\{ f(x) \sum_k x_k \partial_k f(x) \right\} - f(x) \partial_\ell f(x) \right] \tag{36}
= \frac{x_\ell}{f_x(1_n)} \left[ \partial_\ell f^2(x) - f(x) \partial_\ell f(x) \right] = \frac{x_\ell f(x) \partial_\ell f(x)}{f_x(1_n)} = \partial_\ell f_x(1_n).
$$
This is equation (24). To prove Equation (25), we check that

$$
\partial_{k\ell} f_x^2(1_n) = x_k x_\ell \partial_{k\ell} f^2(x)
$$

and derive

$$
\partial_{k\ell} g_x^2(1_n) = 2 x_k x_\ell g_{k\ell} = 2 x_k x_\ell \partial_{\ell} \{ f(x) \partial_k f(x) \} = x_k x_\ell \partial_{k\ell} f^2(x) = \partial_{k\ell} f_x^2(1_n).
$$

This implies

$$
\partial_{k\ell} g_x(1_n) \partial_k g_x(1_n) + g_x(1_n) \partial_{k\ell} g_x(1_n) = \partial_{k\ell} f_x(1_n) \partial_k f_x(1_n) + f_x(1_n) \partial_{k\ell} f_x(1_n).
$$

By (23) and (24), we deduce Equation (25) and finish the proof.
References


European Insurance and Occupational Pensions Authority (EIOPA), 2014. The underlying assumptions in the standard for-


