Chapter 5: Exponential and Logarithmic Functions


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Chapter 5. Exponential and Logarithmic Functions

Exponential functions find applications in economics in relation to growth and economic dynamics. In these fields, quite often the choice variable is time and economists are trying to determine the best timing for certain economic activities to take place. An exponential function is one in which the independent variable appears in the exponent. Very often that exponent is time. A simple exponential function of time \( t \) takes the form

\[
y = f(t) = b^t \quad \text{where } b > 1
\]

We rule out negative numbers for \( b \) because we do not want to take the square root of a negative number. If \( b = 1 \), then \( y \) will also be 1 or a constant function and, finally, if \( b \) is a positive fraction, we get

\[
y = \left( \frac{1}{3} \right)^t = 3^{-t}
\]

where the base can be rewritten into one with a base greater than 1.

![Figure 1](image)

The exponential function in Figure 1 is continuous and smooth everywhere, so it is differentiable at any point in the domain. While \( t \) can take negative values and the domain contains both negative and positive numbers, the dependent variable \( y \) is always positive, so the range of the function includes only positive values. In its general form, the exponential function can be expressed as

\[
y = ab^ct
\]

where the parameters \( a \) and \( c \) are compressing or extending agents. Note that when \( a \) takes negative values, \( y \) is no longer positive and can be plotted below the horizontal axis. For example, in the case of \( y = -b^t \), that is, when \( a = -1 \), the graph will be symmetrical to that of \( y = b^t \) but the function will take only negative values (see Figure 2).

**The Exponential Number \( e \)**

In mathematics, some bases are more convenient to use than others. For example, the exponential number \( e = 2.71828 \), also known as Napier’s number, is used as the base of the natural exponential function. Examples of it are

\[
y = e^t \quad y = Ae^{rt}
\]
Figure 2

Why is $e$ a preferred base? It turns out that for the simple exponential function $y = e^t$, the first derivative with respect to the unknown variable $t$ is equal to the function itself. In other words,

$$(e^t)' = \frac{de^t}{dt} = e^t$$

How can we use this knowledge to find the derivative of a more complicated natural exponential function such as $y = Ae^{rt}$, which finds wide application in economics? One way to proceed is to rewrite the function in the form

$$y = Ae^{rt} = A(e^t)^r$$

Now, by the chain rule, we can find the derivative of $y$ with respect to $t$, which is

$$y' = Ar(e^t)^{r-1}e^t = Ae^{rt}$$

Interestingly enough, the number $e$ also has an economic meaning. From mathematics it is known that the number $e$ may also be defined as the limit of the function $f(m) = \left(1 + \frac{1}{m}\right)^m$ as $m \to \infty$. That is,

$$e = \lim_{m \to \infty} f(m) = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

The number $e$ finds application in the process of interest compounding. For example, let’s assume we have an initial principal of $1$, and let’s assume for simplicity that we get an interest of 100% per year. This means that at the end of the year we will have a total of

$$V(1) = 1(1 + 100\%) = 1(1 + 1) = 2$$

If interest is compounded semiannually, or twice for the entire period, we then have 50% in the middle of the year and another 50% at the end of it. We obtain

$$V(2) = 1(1 + 50\%)(1 + 50\%) = 1 \left(1 + \frac{1}{2}\right)^2$$

With three compoundings per year, we have, respectively,

$$V(3) = 1 \left(1 + \frac{1}{3}\right)^3$$

and in the general case of $m$ compoundings per year, we have

$$V(m) = 1 \left(1 + \frac{1}{m}\right)^m$$

$^1$ It is proven that the number $e$ can be expressed as an infinite series, i.e., an expression involving an infinite number of additive terms, such that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = 2.7182819$. 

\[1\]
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\[ V(m) = \left(1 + \frac{1}{m}\right)^m \]

Assuming that interest is compounded continuously, that is, \( m \to \infty \), we obtain

\[ \lim_{m \to \infty} V(m) = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e \]

or the number \( e \) in dollar terms becomes the value of an initial principal of $1, if an interest of 100% is compounded continuously. A more general discrete formula for interest compounding is the following:

\[ V(m) = A \left(1 + \frac{r}{m}\right)^{mt} \]

where the initial principal or asset is \( A \), the interest rate is \( r \), the number of compoundings per year again is \( m \), and the interest is compounded \( t \) number of years. We use some mathematical transformations to turn this discrete formula into a continuous one. Setting \( \frac{m}{r} = w \), we get

\[ V(m) = A \left(1 + \frac{r}{m}\right)^{mt} = A \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^t = A \left[\left(1 + \frac{1}{w}\right)^w\right]^t \]

As the number of compoundings increases infinitely such that \( m \to \infty \), so does \( w \), and we have \( w \to \infty \). Thus, the parenthesized expression equals the number \( e \) by definition, so we obtain in the continuous form

\[ V = \lim_{m \to \infty} V(m) = Ae^{rt} \]

We notice that, as opposed to the discrete formula, the continuous formula for interest compounding does not contain the number of compoundings \( m \). Thus, our two familiar functions \( y = e^t \) and \( y = Ae^{rt} \) appropriate an economic meaning that relates to continuous interest compounding.

With discrete compounding, we find the future value of a stream of flows using annual interest payments. This is where the knowledge of geometric series comes at hand. Suppose that a firm has annual revenue of $100 million and the interest rate is 5% per year. How could the firm find the future value of its revenue after three years computed on an annual (discrete) basis?

The firm will receive exactly $100 million in the first year. In the second year the expected revenue will be \( 100 \left(1 + \frac{5}{100}\right) = 105 \), while in the third year it will be \( 100 \left(1 + \frac{5}{100}\right)^2 = 110.25 \). Hence, the total revenue expected by the firm is

\[ 100 + 100 \left(1 + \frac{5}{100}\right) + 100 \left(1 + \frac{5}{100}\right)^2 = 100 + 105 + 110.25 = 315.25 \]

Such computations are easy to do with the help of a calculator, if we have a small number of terms. But imagine the firm wants to find its total revenue after 50 years! Then we will have to add many more terms, and our work will become truly tedious. To facilitate our task, we resort to finite and infinite geometric series.

Let us have \( n \) numbers \( A, Ak, Ak^2, ..., Ak^{n-1} \) such that each term is obtained by multiplying the previous one by the constant \( k \). Then we can express their sum as

\[ s_n = A + Ak + Ak^2 + ... + A^{n-2} + Ak^{n-1} \]

This sum represents a finite geometric series with quotient \( k \). In our specific example, the respective values of the parameters are \( A = 100 \), \( k = 1.05 \) and \( n = 3 \). We can find a short-cut formula for
determining the sum in the geometric series. We use a trick whereby we multiply both sides of the equation by the constant $k$. This gives

$$ks_n = Ak + Ak^2 + Ak^3 + ... + Ak^{n-1} + Ak^n$$

Subtracting one equation from the other gives

$$s_n - ks_n = A - Ak^n$$

since all other terms cancel. If $k = 1$, then clearly $s_n = nA$. However, when $k \neq 1$, then

$$s_n = A(1 - k^n) = A(k^n - 1)$$

where the denominator is nonzero. Hence, we conclude that for a finite geometric series

$$s_n = A + Ak + Ak^2 + ... + Ak^{n-2} + Ak^{n-1} = \frac{A(k^n - 1)}{k - 1}$$

for $k \neq 1$

To check our computations with our specific example, we substitute

$$s_3 = \frac{A(k^3 - 1)}{k - 1} = \frac{100(1.05^3 - 1)}{1.05 - 1} = 315.25$$

which is exactly the result obtained previously, this time using many fewer computations.

What if the firm will receive this revenue infinitely? Then the number of years of compounding tends to infinity, or we have $n \to \infty$. Hence, in the formula we already obtained, we get an infinite sum, the value of which will depend on the term $k^n$. Thus, if $|k| > 1$, the number $k^n$ does not tend to any limit and is divergent. If $-1 < k < 1$, the number $k^n$ tends to 0. In this last case, the sum $s_n$ of the first $n$ terms tends to the limit $\frac{A}{1 - k}$ as $n \to \infty$, that is,

$$s_n = A + Ak + Ak^2 + ... + Ak^{n-2} + Ak^{n-1} = \frac{A}{1 - k}$$

for $|k| < 1$ and

$$\sum_{n=1}^\infty Ak^{n-1} = \frac{A}{1 - k}$$

for $|k| < 1$

Thus, an infinite geometric series is a special case of a finite one. If $|k| \geq 1$, we say that the infinite geometric series diverges and has no (finite) sum. When $|k| < 1$, the infinite geometric series converges to $\frac{A}{1 - k}$.

Various economic functions experience the process of exponential growth. Examples are population, wealth, or capital. The rate at which these grow is given by the instantaneous rate of growth. For the function $Ae^{rt}$ the instantaneous rate of growth is $r$, where by rate of growth of $V$ we mean the first derivative $V'$ expressed in percentage terms of the value of $V$ itself or the marginal function over the total function. Thus,

$$\text{rate of growth of } V = \frac{V'}{V} = \frac{\text{marginal}}{\text{total}}$$

In the case of the function $Ae^{rt}$ the instantaneous rate of growth is

$$\frac{V'}{V} = rAe^{rt} = r$$

Closely related to the instantaneous rate of growth is discounting and negative growth. While through compounding we are looking for the future value $V$ from an initial principal or present value $A$, the
process of discounting is the opposite. It requires that we find the present value \( A \) from a known future value \( V \). Using the continuous formula, we obtain

\[
A = \frac{V}{e^{rt}} = Ve^{-rt}
\]

It is easy to see that the instantaneous rate of growth of the present value \( A \) is negative, or \(-r\). This is why it is called a rate of decay. While interest compounding represents positive growth, discounting illustrates the opposite process, that of negative growth.

**Logarithmic Functions**

The logarithmic function is an inverse to the exponential function such that for the function \( y = b^t \) the logarithmic function obtainable is

\[
t = \log_b y
\]

The function graphs as in Figure 3. This time, the independent variable is \( y \) and the dependent variable is \( t \). Alternatively, \( \log_b y \) is the power to which \( b \) must be raised in order to obtain the value \( y \). Thus, we can write

\[
b^{\log_b y} = y
\]

This formula is also very important for base conversion. For example, if we want to convert the base \( b \) into the number \( e \), which you recall is convenient to use in economic and mathematical analysis, it is enough to write

\[
b = e^{\ln b}
\]

The method of base conversion is very important in solving economic problems of optimal timing, where it is much easier to work with \( e \) as the base when it comes to finding first-order and second-order derivatives of objective functions. Note that when the base is 10, we speak of common \( \log \). When the base is \( e \), we speak of a natural \( \log \).

**Figure 3**

**Rules of Logarithms**

1. **Log of a product**
   \[
   \ln(uv) = \ln u + \ln v \quad u, v > 0
   \]

2. **Log of a quotient**
   \[
   \ln\left(\frac{u}{v}\right) = \ln u - \ln v \quad u, v > 0
   \]
3. Log of a power
\[ \ln u^a = a \ln u \quad u > 0 \]

4. Conversion of log base
\[ \log_b u = \frac{\log_a u}{\log_a b} \quad u > 0 \]
which translates into
\[ \frac{\log_b u}{\log_a u} = \log_a u \]

5. Inversion of log base
\[ \log_b a = \frac{1}{\log_a b} \]
Knowing the rules of logarithms helps us solve exponential equations where the unknown is in the exponent of a particular function as given by
\[ ab^x = c \quad a, b, c > 0 \]
Taking the log of both sides of the equation, we obtain
\[ \ln a + x \ln b = \ln c \]
\[ x = \frac{\ln c - \ln a}{\ln b} \]
\[ x = \frac{\ln c}{\ln b} \]
\[ x = \log_b \frac{c}{a} \]

Derivatives of Exponential and Logarithmic Functions

We already know that the derivative of the function \( e^t \) with respect to \( t \) is the function itself, that is,
\[ \frac{de^t}{dt} = e^t \]

It is also known that the derivative of the simple logarithmic function \( \ln t \) with respect to \( t \) is
\[ \frac{d\ln t}{dt} = \frac{1}{t} \]

In the more general form, the rules can be applied to cases where instead of the variable \( t \) we have some function of it \( f(t) \). Thus, we get the following derivatives:
\[ \frac{de^{f(t)}}{dt} = f'(t)e^{f(t)} \]
\[ \frac{d\ln f(t)}{dt} = \frac{f'(t)}{f(t)} \]
We can easily check that the first two derivatives are special cases of the last two when \( f(t) = t \). A more interesting case is when the base is a number different from \( e \), for example, some \( b \). The derivative of the function \( b^t \) with respect to \( t \) is
\[ \frac{db^t}{dt} = \frac{de^{\ln b}}{dt} = e^{\ln b} \ln b = b^t \ln b \]
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\[
\frac{d \log_b t}{dt} = \frac{d}{dt} \left( \frac{\ln t}{\ln b} \right) = \frac{1}{t \ln b}
\]

In the case of a function of \( t \) such as \( f(t) \), we get

\[
\frac{db^{f(t)}}{dt} = \frac{de^{f(t) \ln b}}{dt} = f'(t)e^{f(t) \ln b} \ln b = f'(t)b^{f(t) \ln b}
\]

\[
\frac{d \log_b f(t)}{dt} = \frac{d}{dt} \left( \frac{\ln f(t)}{\ln b} \right) = \frac{f'(t)}{f(t) \ln b}
\]

Optimal Timing

An example of an optimal timing problem is timber cutting, where for a given planted forest the value of timber is an increasing function of time given by

\[ V = 3^t \]

If the discount rate is \( r \), what is the optimal time to cut the timber for sale? We use the continuous formula to find the present value of timber, where

\[ A(t) = Ve^{-rt} = 3^t e^{-rt} = e^{\ln 3} e^{-rt} = e^{\ln 3 - rt} \]

\[ A'(t) = \frac{dA}{dt} = e^{\ln 3 - rt} \left( \frac{\ln 3}{2\sqrt{t}} - r \right) = 0 \]

The first derivative \( A'(t) \) can be zero only when \( r = \frac{\ln 3}{2\sqrt{t}} \) or for \( t^* = \frac{\ln^2 3}{4r^2} \), which gives the optimal number of years after which the timber should be cut. Given an interest rate \( r = 10\% \), the optimal number of years is

\[ t^* = \left( \frac{1.098}{0.2} \right)^2 \approx 30 \text{ years} \]

To prove that present value is, indeed, maximized, we resort to the second-order condition:

\[ A''(t) = A'(t) \left( \frac{\ln 3}{2\sqrt{t}} - r \right) + e^{\ln 3 - rt} \left( -\frac{\ln 3}{4t\sqrt{t}} \right) = e^{\ln 3 - rt} \left( -\frac{\ln 3}{4t\sqrt{t}} \right) < 0 \]

The first term in the second derivative is 0 because the first derivative is \( A'(t) = 0 \), while the second term has a negative sign. Thus, we have proved that the second derivative is negative and \( t^* = 30 \) years is a maximizing solution.

Redefining Rate of Growth

We already saw that the rate of growth of a function \( f(t) \) over time \( t \) is the marginal over the total function, or

\[ r = \frac{f'(t)}{f(t)} \]

But from the rules of derivatives of logarithmic functions, we also obtained that the ratio \( \frac{f'(t)}{f(t)} \) is the derivative of the natural log of \( f(t) \) with respect to time \( t \). Thus, we obtain for the instantaneous rate of growth \( r \)
We can use our knowledge of logarithmic functions to find the rate of growth of a combination of functions. Let us assume that the function \( y \) is a product of two other functions of time \( u \) and \( v \), respectively, such that

\[
y = uv
\]

where \( u = f(t) \) and \( v = g(t) \)

We also know the rates of growth of the three functions \( r_y \), \( r_u \), and \( r_v \), respectively. The rules of logarithms allow us to find a relationship for the rates of growth of a combination of functions.

For example, if we take the natural log of both sides of the equation for the \( y \) function, we obtain

\[
\ln y = \ln u + \ln v
\]

Differentiating both sides of this equation with respect to time \( t \) yields a relationship for the three rates of growth:

\[
\frac{d \ln y}{dt} = \frac{d \ln u}{dt} + \frac{d \ln v}{dt}, \text{ or }
\]

\[
r_y = r_u + r_v
\]

where by definition each rate of growth is the derivative of the natural log of the given function with respect to time. Similarly, if a function \( y \) is a quotient of two other functions \( u \) and \( v \) such that

\[
y = \frac{u}{v}
\]

we can easily see that

\[
r_y = r_u - r_v
\]

A more complex case is when the function \( y \) is a sum or a difference of the functions \( u \) and \( v \). Let us first consider the first case when it is a sum of the two functions:

\[
y = u + v
\]

Using derivatives,

\[
r_y = \frac{y'}{y} = \frac{(u + v)'}{u + v} = \frac{u'}{u + v} + \frac{v'}{u + v} = \frac{u}{u + v}r_u + \frac{v}{u + v}r_v
\]

or the rate of growth of a sum is a weighted average of the rates of growth of the components.

Similarly, when we take the difference

\[
y = u - v
\]

\[
r_y = \frac{y'}{y} = \frac{(u - v)'}{u - v} = \frac{u'}{u - v} - \frac{v'}{u - v} = \frac{u}{u - v}r_u - \frac{v}{u - v}r_v
\]

As an example, let us take the total exports of a country to be \( X \) consisting of the exports of agricultural goods \( A \) and those of industrial goods \( I \). If the exports of agricultural goods are known to grow at the rate \( \alpha \) while those of industrial goods at the rate \( \beta \), then what is the growth rate of total exports? Substituting in the formula for a combination of functions, we get that the rate of growth is equal to the share of agricultural goods in total exports times its growth rate plus the share of industrial goods in total exports times its growth rate, or

\[
r_X = \frac{A}{X}\alpha + \frac{I}{X}\beta
\]
Point Elasticity Revisited

Our knowledge of logarithms allows us to revisit the concept of point elasticity, where we have a function \( y = f(x) \). By definition, the point elasticity of \( y \) with respect to \( x \) is the ratio of the marginal over the average function:

\[
E_{xy} = \frac{dy}{dx} \cdot \frac{y}{x} = \frac{dy}{dx} \cdot \frac{x}{y}
\]

where \( \frac{dy}{dx} \) gives the marginal over the total function. We know that this result can be expressed in log terms in the form

\[
\frac{dy}{dx} = \frac{d}{dx} \ln y
\]

Substituting in the point elasticity formula, we get

\[
E_{xy} = \frac{dy}{dx} \cdot \frac{\ln y}{x} = \frac{d}{x} \ln y
\]

Again, by rules of logarithmic functions, the derivative of a natural log of a function \( x \) with respect to \( x \) gives

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

where we substitute for \( x \) in the point elasticity, and we obtain

\[
E_{xy} = \frac{dy}{dx} \cdot \frac{\ln y}{x} = \frac{d}{x} \ln y = \frac{d}{x} \ln x
\]

Note that whereas the rate of growth is the ratio of the marginal and the total function, the point elasticity is the marginal over the average function.

A short way of finding the point elasticity of demand for the function \( q = \frac{a}{p^2} \) is

\[
\ln q = \ln a - 2 \ln p
\]

\[
E_d = \frac{d \ln q}{d \ln p} = -2
\]

To verify this result we can use differentiation, where

\[
E_d = \frac{dq}{dp} = -\frac{2ap^2}{p^3 a} = -2
\]

which shows a relatively elastic demand. An interesting chain rule exists for point elasticities where we have a composite function such that \( y = g(w) \) and \( w = h(x) \). Then we have a chain rule of point elasticities:

\[
E_{yx} = E_{yw} E_{wx}
\]

To prove this chain rule, we simply investigate the right side of the equation

\[
E_{yw} E_{wx} = \frac{dy}{dw} \cdot \frac{dw}{dx} = \frac{dy}{dx} = E_{yx}
\]

One application of the log form of point elasticities is with a log linear regression. Applied economists use and analyze data to determine some pattern or the influence of one or more independent variables on another (dependent) variable. For instance, when they try to estimate a firm’s demand function \( q \), they take into account factors such as the own price of the good \( p_x \), the price of a competitive
Problem: (substitute) product \( p_y \), the amount of advertising of the firm \( A \), the income of the consumers \( I \), etc. The relationship between the quantity demanded and all these factors may not be linear but, rather, log linear of the type
\[
q = ap_r^{\alpha_1} p^2 r^{\alpha_2} A^{\alpha_3} I^{\alpha_4}
\]
When taking the log of both sides of this equation, we obtain a multiple log linear regression equation of the type
\[
\ln q = \ln a + \alpha_1 \ln p_x + \alpha_2 \ln p_y + \alpha_3 \ln A + \alpha_4 \ln I
\]
But we already know that the derivative of \( \ln q \) with respect to \( \ln p_x \) is nothing but the own-price elasticity of demand for the firm, that is,
\[
E_{d_x} = \frac{d \ln q}{d \ln p_x} = \alpha_1
\]
Similarly,
\[
E_{d_y} = \frac{d \ln q}{d \ln p_y} = \alpha_2
\]
is the cross-price elasticity of demand,
\[
E_A = \frac{d \ln q}{d \ln A} = \alpha_3
\]
is the own-advertising elasticity of the firm and
\[
E_I = \frac{d \ln q}{d \ln I} = \alpha_4
\]
is the income elasticity of demand.

Knowing economic theory, we can further try to predict the signs of the respective elasticities and, depending on whether their absolute values are smaller or bigger than 1, we can establish whether demand is inelastic or elastic with respect to those variables.

Problems

1. If you put $200 into a savings account at an interest rate of 10%, how much money will you have in four years?

Solution:
Using the formula for future value \( V = A(1 + r)^t \), we obtain
\[
V = 200(1 + 0.1)^4 = 200(1.4641) = $292.82
\]

2. Sally deposits $100 in a bank account that pays 10% annual interest, compounded semiannually. What will be the value of her account after 6 months, after 1 year, after 18 months, after 2 years, and after \( n \) times of compounding?

Solution:
Using the general compound-interest formula,
\[
V(m) = A \left(1 + \frac{r}{m}\right)^{mt}
\]
where \( r = 0.10 \) and \( m = 2 \), or we have 2 compounds per year, since compounding is said to be semiannual. We can easily find \( V \) for any period of time:
\[
V(6 \text{ months}) = 100 \left(1 + \frac{0.1}{2}\right)^{2(0.5)} = 100(1.05) = $105
\]
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\[ V(1 \text{ year}) = 100 \left(1 + \frac{0.1}{2}\right)^{2(1)} = 100(1.05)^2 = 100(1.1025) = $110.25 \]

\[ V(1.5 \text{ years}) = 100 \left(1 + \frac{0.1}{2}\right)^{2(1.5)} = 100(1.05)^3 = $115.76 \]

\[ V(2 \text{ years}) = 100 \left(1 + \frac{0.1}{2}\right)^{2(2)} = 100(1.05)^4 = $121.55 \]

If we have \( n \) number of compoundings within one year, then the formula is:

\[ V = 100 \left(1 + \frac{0.1}{n}\right)^{n(t)} \]

For \( t \) years, we have for the value of Sally’s account:

\[ V = 100 \left(1 + \frac{0.1}{n}\right)^{nt} \]

3. Andy deposits $500 in a bank that pays 9% annual interest compounded continuously. What is the function that describes the value of his account at any time \( t \)? What will be the value of his account after one year? After 18 months?

Solution:

The value of Andy’s account at time \( t \) is

\[ V(t) = 500e^{0.09t} \]

For the specific periods, the value is found similarly, using the continuous formula:

\[ V(1) = 547.09 \]

\[ V(1.5) = 572.27 \]

4. Sales of a new product tend to decrease over time. Sales levels are described by the function

\[ S(t) = 1,500 + 750e^{-0.2t} \]

where \( t \) is measured in years. Show the decay of sales by evaluating the sales level for \( t = 0, 1, 4 \), and 10 years.

Solution:

Simply substituting the respective value of time in the sales function,

\[ S(0) = 1,500 + 750 \approx 2,250 \]

\[ S(1) = 1,500 + 750e^{-0.2} \approx 2,114 \]

\[ S(4) = 1,500 + 750e^{-0.8} \approx 1,837 \]

\[ S(10) = 1,500 + 750e^{-2} \approx 1,601 \]

We see that the sales to the firm get smaller every single year, as the product matures.

5. Eddie has deposited $4,000 in a bank account that pays 8% annual interest compounded continuously. He hopes that in 4 years, when he graduates from college, the value of the account will be $10,000 and he can buy a new car. When will Eddie’s account be worth $10,000?

Solution:

The future value of Eddie’s account is
\( F(t) = 4000e^{0.08t} \)
\( 10,000 = 4000e^{0.08t} \)
\( e^{0.08t} = 2.5 \)

Taking the natural logarithm of both sides yields
\[ 0.08t = \ln 2.5 \]
\[ t = \frac{\ln 2.5}{0.08} = \frac{0.916}{0.08} \approx 11.45 \text{ years} \text{ is the approximate time when Eddie’s account will reach 10,000.} \]

6. How long will it take money to double in value if invested in a bank that pays 12% compounded continuously?

Solution:
If we assume the value of the deposit to be \( A \), we should have
\[ 2A = Ae^{0.12t} \]
\[ 2 = e^{0.12t} \]
\[ \ln 2 = 0.12t \]
\[ t = \frac{\ln 2}{0.12} \approx 5.8 \text{ years} \text{ is the approximate time when the value of the deposit will double.} \]

7. Suppose you deposited $100 in a bank that compounds interest continuously, but you forgot to ask the annual interest rate. All you know is that your deposit of $100 today will be worth $150 in 5 years. What is the annual rate of interest that this bank pays?

Solution:
Using the future-value formula, we have:
\[ 150 = 100e^{5r} \]
\[ 1.5 = e^{5r} \]
\[ \ln 1.5 = 5r \]
\[ r = \frac{\ln 1.5}{5} \approx 0.08 \]
Therefore, the interest rate that will increase our deposit of $100 to $150 in 5 years is 8%.

8. If the interest rate is 10% and cash flows are $1,000 at the end of year one, $2,000 at the end of year two, and $3,000 at the end of year three, then what is the present value of these cash flows?

Solution:
Using the familiar formula for discounting cash flows,
\[ A = \frac{R_1}{1 + r} + \frac{R_2}{(1 + r)^2} + \frac{R_3}{(1 + r)^3} \]
\[ A = \frac{1,000}{1 + 0.1} + \frac{2,000}{(1 + 0.1)^2} + \frac{3,000}{(1 + 0.1)^3} = 909.09 + 1,652.89 + 2,253.9 \approx 4,815.92 \]
At the end of year three, we will have approximately $4,816.
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9. If the annual interest rate is 7% and cash flows are $2,000 at the end of year one, $2,500 at the end of year two, and $3,000 at the end of year three, then what is the present value of these cash flows? Instead of this stream of revenues, would you rather consider receiving a lump sum of $6,000 now?

Solution:
Using the well-known formula for discounting a stream of revenues,
\[ A = \frac{2,000}{1 + 0.07} + \frac{2,500}{(1 + 0.07)^2} + \frac{3,000}{(1 + 0.07)^3} = 1,869.16 + 2,183.6 + 2,448.9 \approx 6,501.65. \]

is the approximate value of a stream of revenues receivable within three years. Since the present value of the stream is greater than receiving the lump sum now, it is preferable to take the stream within three years.

10. If the interest rate is 5% and cash flows are $500 at the end of year one, $1,000 at the end of year two, and $2,000 at the end of year three, what is the present value of these cash flows? Instead of this stream of revenues, would you rather consider receiving a lump sum of $3,200 now?

Solution:
With the help of the discounting formula for cash flows, we obtain
\[ A = \frac{500}{1 + 0.05} + \frac{1,000}{(1 + 0.05)^2} + \frac{2,000}{(1 + 0.05)^3} = 476.2 + 907.03 + 1,727.68 \approx 3,110.9. \]

The present value of the stream of revenues is lower than the lump sum of $3,200 receivable immediately, so it is better to accept this sum rather than the stream of cash flows within three years.

11. What is the value of a principal of $1,000 at 6% interest for a period of 3 years if compounded annually, semiannually, and continuously?

Solution:
If compounded annually, the value of the principle would grow to
\[ V = A(1 + r)^t = 1,000(1 + 0.06)^3 \approx 1,191.02. \]

Semiannually, the amount will grow according to the formula
\[ V = A \left(1 + \frac{r}{m}\right)^{mt} = 1,000 \left(1 + \frac{0.06}{2}\right)^{(3)} \approx 1,194.05 \]
and continuously,
\[ V = Ae^{rt} = 1,000e^{0.06(3)} \approx 1,197.21. \]

12. The value of a newly purchased computer is \( V_o \). The computer depreciates at a rate of 30% each year over a period of 6 years. Express the value function of the computer \( V(t) \) at time \( t \). What will its value be in the sixth year?

Solution:
A year from the purchase, the value of the computer will be
\[ V_o - 0.3V_o = (0.7)V_o \]
In the second year, the value computer will fall to
\[ 0.7V_o - 0.3(0.7)V_o = (0.7)^2V_o. \]
Note that \((0.7)^1 > (0.7)^2\) since \(0.7 > 0.49\). Thus, \(t\) years from the purchase the value will be 
\[ V(t) = (0.7)^t V_o \]

In the sixth year, the value of the computer would depreciate to 
\[ V(6) = (0.7)^6 V_o \approx 0.118 V_o \]
or, after 6 years, the computer would have lost nearly 90% of its value and would only be worth
11.7% of its initial value.

13. The population of the European Union after the last enlargement in 2007 is approximately 300 million people. It is increasing at an annual rate of 1.5%. Express the function of the European population at time \(t\). What is the doubling time for it?

**Solution:**
To express the time function of European population we start from the initial period:
\[ P_o = 300 \]
\[ P(1) = 300(1 + 0.015) \]
\[ P(2) = 300(1 + 0.015)^2 \]

Thus, \(t\) years from the enlargement, European population will grow according to the function 
\[ P(t) = 300(1.015)^t \]

In order for the population to double,
\[ 600 = 300(1.015)^t \]
\[ 2 = 1.015^t \]
\[ \ln 2 = t \ln 1.015 \]
\[ t \approx 46.56 \]
or, it will take more than 46 years for the population of the European Union to double.

14. The population of Eastern Europe is decreasing every year at a rate of 2.5% annually. Find out in how many years the population will be reduced in half.

**Solution:**
Starting from the initial population \(P_o\),
\[ P(1) = P_o (1 - 0.025) \]
\[ P(2) = P_o (1 - 0.025)^2 \]

Thus, \(t\) years from now East European population will decline by the formula 
\[ P(t) = 0.975^t P_o \]

In order for the population to decrease in half, we should solve the equation 
\[ \frac{P}{2} = 0.975^t P_o \]
\[ \frac{1}{2} = 0.975^t \]
\[ \ln 1/2 = t \ln 0.975 \]
\[ t \approx 27.38 \]
or, it will take approximately 27 years for Eastern European population to go down by 50%.
15. The price $P(x)$ of a product depends on the number of units $x$ produced. For the price function $P(x) = \log\left(10 + \frac{x}{4}\right)$, how many units will be produced at a price of $3$?

**Solution:**

Directly substituting for price in the formula, we can solve for $x$:

$$3 = \log\left(10 + \frac{x}{4}\right)$$

$$10^3 = 10 + \frac{x}{4}$$

$$1,000 = 10 + \frac{x}{4}$$

$$\frac{x}{4} = 990$$

$$x = 3,960$$ units will be produced at a price of $3$ given the particular price function.

16. Suppose the value of a business investment is given by the function $I(t) = 100,000 e^{0.5t}$, $(t > 0)$. Determine the relative rate of growth of this investment after 10 years.

**Solution:**

Using the formula for the rate of growth of an investment,

$$r = \frac{I'(t)}{I(t)} = \frac{d \ln I(t)}{dt}$$

Taking the log of both sides of the investment function and differentiating with respect to time,

$$\ln I(t) = \ln 100,000 \cdot e^{0.5t} = \ln 100,000 + 0.5t$$

$$\frac{d \ln I(t)}{dt} = (0.5) \frac{1}{2\sqrt{t}} = \frac{1}{4\sqrt{t}}$$

Since $t = 10$ years, the rate of growth is $r = \frac{1}{4\sqrt{10}} = 0.079$, or approximately 8%.

17. Show that if the value of an investment increases according to the function $y(t) = Pe^{rt}$, then the relative rate of increase is constant. What is this relative rate of increase?

**Solution:**

Taking the log of the investment function and differentiating with respect to time,

$$\ln y = \ln P + rt$$

$$\frac{d \ln y}{dt} = r$$ is a constant rate of increase.

18. Use logarithms to find the point elasticity for the demand function $q = ap^{-n}$.

**Solution:**

Taking the log of the demand function and expressing $\frac{d \ln q}{d \ln p}$ as the elasticity,

$$\ln q = \ln a - n \ln p$$
where we should have \( n > 0 \) for ordinary goods.

19. Using logarithms, find the point elasticity for the demand function \( p = \frac{b}{\sqrt{q}} \), where \( b > 0 \).

**Solution:**

We first need to find the direct demand function:

\[
\sqrt{q} = \frac{b}{p}
\]

\[
q = \frac{b^2}{p^2}
\]

Taking the log of both sides of this direct demand function helps us express point elasticity:

\[
\ln q = 2 \ln b - 2 \ln p
\]

\[
E_d = \frac{d \ln q}{d \ln p} = -2 \quad \text{shows demand for this good is elastic.}
\]

20. Calculate the price elasticity of demand if the demand function is \( p = 100 - 30 \ln(q + 2) \) for \( q = 10 \).

**Solution:**

To use the formula for price elasticity \( E_d = \frac{dp}{dq} \), we need to find \( \frac{dq}{dp} = \frac{1}{dp/dq} \). Differentiating price with respect to output results in

\[
\frac{dp}{dq} = -\frac{30}{q + 2}
\]

Substituting in the elasticity formula,

\[
E_d = \frac{\left[100 - 30 \ln(q + 2)\right] (q + 2)}{30q}
\]

For \( q = 10 \)

\[
E_d = -\frac{(100 - 30 \ln 12) 12}{300} \approx -1.02
\]

Demand is nearly unit elastic.

21. A firm has total revenue \( TR(q) = 3 \ln(q^2 + 4q) \) and total cost \( TC(q) = 2q \). Find the optimal output level and profit for the firm. Use a second-order condition.

**Solution:**

We express profit by subtracting total cost from total revenue and we differentiate it with respect to \( q \):

\[
\pi(q) = 3 \ln(q^2 + 4q) - 2q
\]

\[
\pi'(q) = \frac{3(2q + 4)}{(q^2 + 4q)} - 2 = 0 \quad \text{at the point of profit maximization.}
\]
\[
\frac{3(q + 2)}{(q^2 + 4q)} = 1
\]

\[
3q + 6 = q^2 + 4q
\]

\[
q^2 + q - 6 = 0
\]

which gives

\[
q_1 = -3 \quad \text{and} \quad q_2 = 2 \quad \text{acceptable}
\]

For the second-order condition, we have

\[
\pi''(q) = (6) \frac{(q^2 + 4q) - (q + 2)(2q + 4)}{(q^2 + 4q)^2} = (6) \frac{q^2 + 4q - 2q^2 - 8q - 8}{(q^2 + 4q)^2} = (6) \frac{(-q^2 - 4q - 8)}{(q^2 + 4q)^2} =
\]

\[
= (-6) \frac{(q^2 + 4q + 8)}{(q^2 + 4q)^2} < 0 \quad \text{for any} \quad q. \quad \text{It turns out} \quad q = 2 \quad \text{is optimal output. The maximum profit is:}
\]

\[
\pi(2) = 3\ln(4 + 8) - 4 = 3\ln12 - 4 \approx 3.45
\]

**22.** The demand function for a good is given by \( q = 4,600e^{-0.04p} \), where fixed costs of the firm equal $9.37 while unit variable costs are $20. Find the optimal price that maximizes profit to this firm. Prove that the profit is indeed maximum at this point of production. Find the maximum profit.

**Solution:**

We express total cost as the sum of fixed and variable costs and we subtract it from revenue:

\[
\pi(p) = pq - cq - TFC = q(p - c) - TFC = 4,600e^{-0.04p}(p - 20) - 9.37
\]

\[
\pi'(p) = (-0.04)4,600e^{-0.04p}(p - 20) + 4,600e^{-0.04p} = 0
\]

\[
\pi'(p) = 4,600e^{-0.04p}(-0.04p + 0.8 + 1) = 0
\]

\[
0.04p = 1.8
\]

\[
p^* = 45 \quad \text{gives the optimal price the firm should charge to maximize its profit.}
\]

\[
\pi''(p) = (-0.04)4,600e^{-0.04p}(-0.04p + 1.8) - 0.04(4,600)e^{-0.04p} =
\]

\[
= -0.04\pi'(p) - 184e^{-0.04p} =
\]

\[
= 0 - 184e^{-0.04p} < 0
\]

Therefore, profit is maximized. The maximum profit is

\[
\pi(45) = 4,600e^{-0.04(45)}(45 - 20) - 9.37 = 4,600e^{-1.8}(25) - 9.37 = 19,009.37 - 9.37 = 19,000
\]

**23.** The demand function for a good is given by \( q = ae^{-bp} \) where the parameters \( a, b \) are positive. It is also known that unit variable costs equal \( c \), while fixed costs of the firm are given by \( d \). Find the optimal price that maximizes profit to this firm. Prove that the profit is indeed maximum at this point of production. Express the maximum profit.

**Solution:**

Expressing profit as the difference between total revenue and total cost and then maximizing it

\[
\pi = pq - cq - d
\]

\[
\pi(p) = pae^{-bp} - cae^{-bp} - d
\]

\[
\pi'(p) = ae^{-bp} - bpace^{-bp} + abce^{-bp} = 0
\]

\[
\pi'(p) = a(1 - bp)e^{-bp} + abce^{-bp} = 0
\]
\[ \pi'(p) = ae^{-bp}(1-bp+bc) = 0 \]
\[ 1-bp+bc = 0 \]
\[ p^* = \frac{1+bc}{b} \]
is optimal price. To prove maximum profit, we resort to the second-order condition:
\[ \pi^*(p) = -b\left[ ae^{-bp}(1-bp+bc)\right] - abe^{-bp} = -b\pi'(p) - abe^{-bp} = -abe^{-bp} < 0 \]
Therefore, we have a maximum for the profit at the critical value of \( p \). This maximum profit is
\[ \pi(p^*) = \left(\frac{1+bc}{b}\right) ae^{-b(1+bc)} - cae^{-b(1+bc)} - d = \left(\frac{1+bc}{b}\right) ae^{-1-bc} - cae^{-1-bc} - d = ae^{-1-bc} \left(\frac{1+bc}{b} - d\right) \]

24. A firm’s inverse demand function is \( p = 6.4e^{-0.025q} \), while the unit cost of production is \( 2.8e^{-0.025q} \). Find the maximum profit to the firm using a second-order condition. What is the profit-maximizing price?

Solution:
Subtracting total cost from total revenue and then maximizing the obtained profit function gives optimal output:
\[ \pi(q) = (p-c)q = (6.4e^{-0.025q} - 2.8e^{-0.025q})q = 3.6e^{-0.025q} \]
\[ \pi'(q) = 3.6e^{-0.025q} + 3.6e^{-0.025q}(-0.025)q = 3.6e^{-0.025q}(1-0.025q) = 0 \quad \text{or} \quad 1-0.025q = 0 \]
\[ q = 40 \]
\[ p = 6.4e^{-0.025(40)} = 6.4e^{-1} \approx 2.35 \]
\[ \pi(40) = 3.6e^{-0.025(40)} \approx 52.97 \]
For the second-order condition, we obtain maximum, since
\[ \pi^*(q) = 3.6e^{-0.025q}(-0.025)(1-0.025q) + 3.6e^{-0.025q}(-0.025) = 0 - 0.09e^{-0.025q} < 0 \]

25. Maximize the total revenue of a firm that has the demand function \( p = 8.75e^{-0.04q} \). What are the revenue maximizing output and price? What is that maximum revenue?

Solution:
To maximize total revenue, we directly differentiate it with respect to output:
\[ TR(q) = pq = 8.75e^{-0.04q}q \]
\[ TR'(q) = 8.75e^{-0.04q} - 0.04q = 8.75e^{-0.04q}(1-0.04q) = 0 \quad \text{or} \quad 1-0.04q = 0 \]
\[ q = 25 \]
Substituting output \( q \) in the demand function allows finding revenue-maximizing price:
\[ p = 8.75e^{-0.04(25)} = 8.75e^{-1} \approx 3.22 \]
26. A firm’s short-run production function is \( q = kL^a e^{-bL} \), where \( L \) gives the number of workers hired by the firm. It is also known that \( k > 0 \), \( a > 1 \), and \( 0 < b < 1 \). Find the number of workers that maximizes the product per worker \( AP_L \). Prove that at that point of production \( AP_L \) is indeed maximized. How is optimal number of workers related to the parameters \( a \) and \( b \)? How many workers should be hired to maximize total product?

**Solution:**

Dividing total product by the number of workers \( L \) gives the average-product function:

\[
AP_L = kL^{a-1} e^{-bL} = k e^{(a-1)\ln L - bL}
\]

\[
AP_L' = k \left( \frac{a-1}{L} - b \right) e^{(a-1)\ln L - bL} = 0
\]

\[
\frac{a-1}{L} = b
\]

\( L^* = \frac{a-1}{b} \) is the number of workers that gives highest productivity per worker. It is evident that the optimal number of workers \( L^* \) is positively related to \( a \) and negatively to \( b \).

\[
AP_L'' = AP_L' \left( \frac{a-1}{L} - b \right) - ke^{(a-1)\ln L - bL} \left( \frac{a-1}{L^2} \right) = -AP_L \left( \frac{a-1}{L^2} \right) < 0 \text{ ensures maximum } AP_L.
\]

\( q = kL^a e^{-bL} = k e^{a\ln L - bL} \)

\[
q'(L) = k \left( \frac{a}{L} - b \right) e^{a\ln L - bL} = 0
\]

\[
\frac{a}{L} - b = 0
\]

\( L^* = \frac{a}{b} \) maximizes total product.

27. A firm’s short-run production function is \( q = L e^{-0.5\ln L} \), where \( L \) gives the number of workers hired by the firm. The price at which the product sells is $20, while the average worker’s hourly wage is $1. Find the number of workers that maximizes profit to this firm. Prove that at that point of production the profit is indeed maximized.

**Solution:**

We can express profit as the difference between total revenue and total cost:

\[
q = L e^{-0.5\ln L} = L(e^{\ln L})^{-0.5} = LL^{-0.5} = L^{0.5}
\]

\( \pi(L) = pq - wL = 20L^{0.5} - L \)

\( \pi'(L) = 10L^{-0.5} - 1 = 0 \)

\[
L^{-0.5} = \frac{1}{10}
\]

\[
\frac{1}{\sqrt{L}} = \frac{1}{10}
\]
100 workers maximize profit to the firm. To prove maximum profit, 
\[ \pi^*(L) = -5L^{-1.5} < 0 \]
Hence, 100 workers maximize the profit to this firm.

28. A firm’s short-run production function is \( q = Le^{-a\ln L} \), where \( L \) gives the number of workers hired by the firm. It is also known that \( 0 < a < 1 \). The price at which the product sells is \( p \), while the average worker’s wage is \( w \). Find the number of workers that maximizes profit to this firm. Prove that at that point of production profit is indeed maximized. How is the optimal number of workers related to wage and the price of the finished product?

Solution:

We first transform the production function and then express profit so that to find \( L^* \),
\[
q = Le^{-a\ln L} = L(e^{\ln L})^{-a} = LL^{-a} = L^{1-a} \quad \text{or alternatively,} \quad q = Le^{-a\ln L} = e^{\ln L}e^{-a\ln L} = e^{\ln L-a\ln L} = e^{(1-a)\ln L}
\]
\[\pi(L) = pl^{-a} - wL\]
\[\pi'(L) = p(1-a)L^{-a} - w = 0\]
\[L^{-a} = \frac{w}{p(1-a)}\]
\[L^a = \frac{w}{p(1-a)}\]
\[L^* = \left[ \frac{p(1-a)}{w} \right]^{1/a}\]
It is evident that the optimal amount of labor is negatively related to wage and positively to the price of the finished product. Hence, the firm is likely to hire more workers, the lower their average wage and the higher the market price of its product. To prove maximum profit, 
\[\pi^*(L) = -ap(1-a)L^{-a-1} < 0 \quad \text{for} \quad 0 < a < 1\]

29. The production function is given by \( q = 2KL - \sqrt{K} - \sqrt{L} \), where \( K = 5t + 5 \) and \( L = 6e^{0.1t} \) when \( t \) is time. Find the total change in output with time. Estimate this change at moment four, that is, \( t = 4 \).

Solution:

We can express total change by totally differentiating output with time through two channels, capital and labor, since both are functions of time:
\[
\frac{dq}{dt} = \frac{\partial q}{\partial K} \frac{dK}{dt} + \frac{\partial q}{\partial L} \frac{dL}{dt}
\]
\[
\frac{dq}{dt} = q'(K) \frac{dK}{dt} + q'(L) \frac{dL}{dt}
\]
\[
q'(K) = \frac{\partial q}{\partial K} = MP_K = 2L - \frac{1}{2\sqrt{K}} \quad \quad \quad \quad \quad q'(L) = \frac{\partial q}{\partial L} = MP_L = 2K - \frac{1}{2\sqrt{L}}
\]
\[
\frac{dq}{dt} = 5\left(2L - \frac{1}{2\sqrt{K}}\right) + 0.6e^{0.1t}\left(2K - \frac{1}{2\sqrt{L}}\right)
\]
For \( t = 4 \) we have the following values for capital and labor:

\[
K = 5(4) + 5 = 25 \quad L = 6e^{0.1(4)} \approx 9
\]

Substituting \( t = 4 \) in the total derivative,

\[
\frac{dq}{dt} = 5\left(18 - \frac{1}{10}\right) + 0.6e^{0.1(4)}\left(50 - \frac{1}{6}\right) = 89.5 + 0.9(49.83) \approx 134.1
\]

30. A production function is defined implicitly so that \( q^{1+\ln q} = AK^\alpha L^\beta \), where \( q = q(K, L) \) is the level of output. Express the marginal products of capital and labor. What is the \( MRTS \) for this production function? Assume all parameters are positive.

**Solution:**

Taking the log of both sides of this implicit production function,

\[
(1 + \ln q) \ln q = \ln A + \alpha \ln K + \beta \ln L
\]

\[
F(q, K, L) = \ln q + c(\ln q)^2 - \ln A - \alpha \ln K - \beta \ln L = 0
\]

We know that by the implicit-function rule,

\[
MP_k = \frac{\partial q}{\partial K} = -\frac{F_k}{F_q} \quad \text{and} \quad MP_L = \frac{\partial q}{\partial L} = -\frac{F_L}{F_q}
\]

\[
F_q = \frac{1}{q} + 2c(\ln q) = 1 + 2c \ln q
\]

Hence, substituting for the two marginal products,

\[
MP_k = -\frac{F_k}{F_q} = \frac{\alpha q}{K(1 + 2c \ln q)} \\
MP_L = -\frac{F_L}{F_q} = \frac{\beta q}{L(1 + 2c \ln q)}
\]

To find the \( MRTS \), again using the implicit-function rule, we apply

\[
\frac{\partial K}{\partial L} = \frac{F_L}{F_K} = -\frac{\beta K}{\alpha L}
\]

31. Find \( MU \), if \( TU = x^2 \ln(x^2 + 1) \).

**Solution:**

Directly differentiating total utility with respect to the quantity \( x \) of the good gives marginal utility:

\[
MU = 2x \ln(x^2 + 1) + \frac{2x^3}{x^2 + 1}
\]

32. Using logarithms, find the marginal utility \( MU \) for a consumer consuming \( x \) amount of a given commodity, where his total utility function is \( TU = y = \frac{x^3}{(x^2 + 1)(3x - 4)} \).

**Solution:**

We use the formula \( \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)} \), where for \( f'(t) \) we have \( f'(t) = f(t) \frac{d}{dt} \ln f(t) \). Since \( y = f(t) \) in this case we should have
$y' = y \frac{d \ln y}{dx}$

Taking the log of the function \( y = \frac{x^3}{(x^2 + 1)(3x - 4)} \) leads to

\[
\ln y = 3 \ln x - \ln(x^2 + 1) - \ln(3x - 4)
\]

\[
d \ln y = \frac{3 - 2x}{x^2 + 1} - \frac{3}{3x - 4}
\]

The derivative \( y' \) that gives marginal utility \( MU \) then is

\[
MU = y' = \left( \frac{3 - 2x}{x^2 + 1} - \frac{3}{3x - 4} \right) \frac{x^3}{(x^2 + 1)(3x - 4)}
\]

33. Use properties of logarithms to find \( MU \), if \( TU = \log \frac{\sqrt{x^2 + 2x + 1}}{\sqrt{x}} \).

**Solution:**

Notice that this is a common logarithm. We can transform it into a natural logarithm:

\[
TU = \ln \frac{\sqrt{x^2 + 2x + 1}}{\sqrt{x}} = \ln \frac{(x + 1)^2}{\sqrt{x}} = \ln (x + 1) - \frac{1}{2} \ln x
\]

and through direct differentiation of total utility we obtain marginal utility:

\[
MU = \frac{1}{\ln 10} \left( \frac{1}{x + 1} - \frac{1}{2x} \right) = \frac{1}{\ln 10} \frac{2x - x - 1}{2x(x + 1)} = \frac{x - 1}{2x(x + 1) \ln 10}
\]

34. Find the maximum total utility of a consumer consuming some positive amount of a good \( x \) where it is known that \( TU(x) = \frac{\ln x^2}{x} \).

**Solution:**

We can transform total utility into \( TU(x) = \frac{\ln x^2}{x} = \frac{2 \ln x}{x} \) and differentiate with respect to \( x \):

\[
MU = \frac{2x - 2 \ln x}{x^2} = \frac{2(1 - \ln x)}{x^2} = 0 \text{ or }
\]

\[
1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e
\]

Thus, \( TU_{\text{max}} = TU(e) = \frac{2 \ln e}{e} = \frac{2}{e} \)

35. A speculator in precious stones has purchased a ruby that is increasing in value according to the function \( V(t) = V_0 e^{ct} \), where \( t \) is time measured in years. If the discount rate is 8% per year, when should the speculator sell the ruby to maximize profits?

**Solution:**

By the process of discounting, we can find the present value of the ruby, where \( A(t) = Ve^{-\rho t} \), so
Chapter 5. Exponential and Logarithmic Functions

\[ A(t) = V_0 e^{\sqrt{t}} e^{-rt} = V_0 e^{\sqrt{t} - rt} = V_0 e^{\sqrt{t} - 0.08t} \]

\[ A'(t) = V_0 e^{\sqrt{t} - 0.08t} \left( \frac{1}{2\sqrt{t}} - 0.08 \right) = 0 \]

The optimal time to sell the ruby is

\[ \frac{1}{2\sqrt{t}} = 0.08 \]

\[ t = \frac{1}{0.16} \approx 6.25 \text{ years} \]

\[ A'(t) = A' \left( \frac{1}{2\sqrt{t}} - 0.08 \right) + V_0 e^{\sqrt{t} - 0.08t} \left( -\frac{1}{4t^{\frac{3}{2}}} \right) = 0 + V_0 e^{\sqrt{t} - 0.08t} \left( -\frac{1}{4t^{\frac{3}{2}}} \right) < 0 \], which proves that present value is indeed maximized.

36. An investment is valued approximately by the function \( f(t) = 50,000e^{0.2t^2} \). What will be the relative change in the value of this investment, that is, its rate of growth, in 10 years?

**Solution:**

Using the formula for the rate of growth, we divide the derivative by the total function:

\[ f'(t) \]

\[ f(t) \]

\[ \frac{f'(t)}{f(t)} = \frac{0.2}{2t} \frac{50,000e^{0.2t^2}}{50,000e^{0.2t^2}} = \frac{0.1}{t} = \frac{0.1}{\sqrt{10}} = 0.0316 \text{ or approximately } 3.16\% \]

37. A coin and stamp dealer calculates that the value \( V(t) \) in dollars that a collection will appreciate after \( t \) years is given by the formula \( V(t) = 1,000e^{\sqrt{t}} \). If the annual discount rate is 8%, when should the collection be sold to maximize the return?

**Solution:**

Expressing the present value of the collection using the discounting formula \( A(t) = Ve^{-rt} \),

\[ A(t) = 1000e^{\sqrt{t} - 0.08t} = 1000e^{\frac{1}{2\sqrt{t}} - 0.08t} \]

\[ A'(t) = 1000e^{-\frac{1}{4t^{\frac{3}{2}}} - 0.08} = 0 \]

\[ \sqrt{t} = \frac{1}{0.32} \]

\[ t^* \approx 9.8 \text{ years} \] maximize the return from the collection. By the second-order condition,

\[ A''(t) = A'(t) \left( \frac{1}{4\sqrt{t}} - 0.08 \right) + 1000e^{-\left( -\frac{1}{8t\sqrt{t}} \right)} = 1000e^{-\left( -\frac{1}{8t\sqrt{t}} \right)} < 0 \text{ maximum} \]

38. A large wood products company in the state of Washington has just planted hybrid trees and has determined that the value \( V(t) \) of this timber (in millions of dollars) is increasing over time according to the exponential function

\[ V(t) = 16^{\log(8t+16)} \]
where \( t \) is time measured in years starting from the date the trees were planted. Assuming that their value is discounted at an interest rate \( r \) of 6\% per year, when should the company cut the timber for maximum profit? Use a second-order condition to prove that it is maximized.

Solution:

With the help of the discounting formula \( A(t) = Ve^{-rt} \), we find the present value:

\[
A(t) = 16^{lg(8t+16)} e^{-0.06t} = e^{2ln4lg(8t+16)-0.06t}
\]

\[
A'(t) = e^{-\left[\frac{2ln4(8)}{ln10(8t+16)}-0.06\right]} = e^{-\left(\frac{2lg4}{t+2} - 0.06\right)} = 0
\]

The parenthesized expression must be 0 for the first-order condition to be followed.

\[
\frac{2lg4}{t+2}-0.06 = 0
\]

\[
\frac{lg4}{t+2} = 0.03
\]

\( t^\ast \approx 18 \) years is the best time to cut the timber. The second-order condition requires that

\[
A''(t) = A'(t)\left(\frac{2ln4}{t+2} - 0.06\right) + e^{-\left[-\frac{2ln4}{(t+2)^2}\right]} < 0
\]

As \( A'(t) = 0 \), the optimal number of years \( t^\ast \) is indeed a maximum.

39. A Swedish timber producing company knows that the value \( V(t) \) of its timber stock (in millions of euros) is increasing over time according to the exponential function

\[
V(t) = 18^{lg(5t-10)}
\]

where \( t \) is time measured in years starting from the planting of the trees. Assuming that their value is discounted at an interest rate \( r \) of 5\% per year, when should the company cut the timber for maximum profit? Use a second-order condition to prove that value is maximized.

Solution:

Through the process of discounting, we find present value \( A \) from future value \( V \):

\[
A(t) = 18^{lg(5t-10)} e^{-0.05t} = e^{ln18lg(5t-10)-0.05t}
\]

\[
A'(t) = e^{-\left[\frac{5ln18}{ln10(5t-10)}-0.05\right]} = e^{-\left(\frac{lg18}{t-2} - 0.05\right)} = 0
\]

The parenthesized expression must be 0 for the first-order condition to be satisfied.

\[
\frac{lg18}{t-2} - 0.05 = 0
\]

\( t^\ast \approx 27 \) years is the optimal time for the company to cut the timber. According to the second-order condition,

\[
A''(t) = A'(t)\left(\frac{lg18}{t-2} - 0.05\right) + e^{-\left[-\frac{lg18}{(t-2)^2}\right]} < 0
\]

As \( A'(t) = 0 \), the optimal number of years \( t^\ast \) indeed indicates a maximum.

40. A speculator in precious stones has purchased a ruby that is increasing in value according to the function

\[
V(t) = 9^{log_{57}(\sqrt{57}+25)}
\]
where \( t \) is time measured in years starting from the date the ruby was bought. If its value is discounted at an interest rate \( r \) of 15% per year, when should the speculator sell the ruby to maximize his profit? Use a second-order condition to prove that profit is maximized.

**Solution:**

We first transform the function \( V \) for convenience:

\[
V(t) = 9 \log_{10} \sqrt{15t+25} = 2 \log_{10} (15t+25) = 3 \log_{10} (15t+25) = 15t + 25
\]

Thus, the present value of the ruby is the discounted future value, or

\[
A(t) = (15t + 25)e^{-0.15t} = e^{\ln(15t+25) - 0.15t}
\]

\[
A'(t) = e^{-\left( \frac{15}{15t + 25} - 0.15 \right)} = 0
\]

It follows that the parenthesized expression equals 0.

\[
\frac{1}{15t + 25} = 0.01
\]

\[
100 = 15t + 25
\]

\[
t^* = 7.5\text{ years}
\]

The second-order condition for optimization is

\[
A''(t) = A'(t^*) \left( \frac{15}{15t + 25} - 0.15 \right) + e^{-\left( \frac{15^2}{(15t + 25)^2} \right)} < 0 \text{ implying a maximum for the value.}
\]

41. A speculator in precious stones has purchased a ruby whose value grows according to the function

\[
V(t) = 27 \log_{10} \sqrt{20t+30}
\]

where \( t \) is time measured in years since the purchase of the ruby. Knowing that its value is discounted at an interest rate \( r \) of 10% per year, when should the speculator sell the ruby to maximize his profit? Show that the profit is maximized using a second-order condition.

**Solution:**

Transforming future value \( V \) and expressing present value with the help of the formula \( A(t) = Ve^{-rt} \),

\[
V(t) = 9 \log_{10} \sqrt{20t+30} = 3 \log_{10} (20t+30) = (20t + 30)^{\frac{3}{4}}
\]

\[
A(t) = (20t + 30)^{\frac{3}{4}} e^{-0.1t} = e^{\frac{3}{4} \ln(20t+30) - 0.1t}
\]

\[
A'(t) = e^{-\left( \frac{3(20)}{4(20t + 30)} - 0.1 \right)} = 0
\]

It follows that at the optimum the parenthesized expression equals 0.

\[
\frac{3}{4t + 6} = 0.1
\]

\[30 = 4t + 6\]

\[4t = 24\]

\[t^* = 6\text{ years is the best time to sell the ruby.}\]

The second-order condition for optimization is
42. A speculator in precious stones has purchased a ruby that is increasing in value according to the function

\[ V(t) = 8^{\log_2 \sqrt{b+2t}} \]

where \( t \) is time measured in years starting from the date the ruby was bought. Assuming that its value is discounted at an interest rate \( r \) of 10% per year, when should the speculator sell the ruby to maximize his profit? Use a second-order condition to prove that the profit is maximized.

**Solution:**

Through transformation, we obtain for future value:

\[ V(t) = 2^{3 \log_2 (6 + 2t)} = (6 + 2t)^{\frac{3}{2}} \]

The present value of the ruby is thus

\[ A(t) = (6 + 2t)^{\frac{3}{2}} e^{-0.1t} = e^{\ln(6 + 2t) - 0.1t} \]

We know that at the optimum the parenthesized expression should equal 0.

\[ \frac{3}{6 + 2t} = 0.1 \]

\[ 30 = 6 + 2t \]

\[ 2t = 24 \]

\[ t^* = 12 \text{ years} \]

The second-order condition for optimization is

\[ A''(t) = A'(t^*) \left( \frac{3}{6 + 2t} - 0.1 \right) + e^{-0.1t} \left[ - \frac{6}{(6 + 2t)^2} \right] < 0 \text{ maximum.} \]

43. A large orange producer in Greece has determined that the value \( V(t) \) of the produce (in millions of dollars) is increasing over time according to the exponential function

\[ V(t) = a^\ln \sqrt{bt+c} \]

where \( t \) is time for oranges to ripe. Assuming that the discount rate is \( r \), what is the optimal time \( t \) for this producer to pick oranges such that the present value of the harvest is maximized? Use a second-order condition to prove that this time \( t \) is indeed optimal. How is \( t \) related to the three coefficients \( a, b, c \) and the interest rate \( r \)?

**Solution:**

Through the process of discounting, present value can be found as

\[ A(t) = a^\ln \sqrt{bt+c} e^{-rt} = e^{\frac{1}{2} \ln(a) - rt} \]

\[ A'(t) = e^{-rt} \left[ \frac{b \ln a}{2(bt + c)} - r \right] = 0 \]

By first-order condition, the parenthesized expression must equal 0. It follows that
\[
\frac{b \ln a}{2(bt + c)} = r
\]
\[
t^* = \frac{\ln a}{2r} \cdot \frac{c}{b} \quad \text{where } \ln a > 0.
\]
We could easily check that \( t^* \) is positively related to \( a, b \) and negatively related to \( r \) and \( c \). A second-order condition requires finding the second derivative:
\[
A''(t) = A'(t) \left[ \frac{b \ln a}{2(bt + c)} - r \right] + e^{-r} \left[ -\frac{b^2 \ln a}{2(bt + c)^2} \right] < 0
\]
The first term is 0 due to \( A'(t^*) = 0 \). Since \( \ln a > 0 \), we get that \( A''(t^*) < 0 \), which proves maximum.

44. A large banana producer in Ecuador has determined that the value \( V(t) \) of the produce (in millions of dollars) is increasing over time according to the exponential function
\[
V(t) = a^{\ln \sqrt{t - bc}} a, b, c > 0 \quad \text{and} \quad a > e
\]
where \( t \) is time for bananas to ripen. Assuming that the discount rate is \( r \), what is the optimal time \( t \) for this producer to pick bananas such that the present value of the harvest is maximized? Use a second-order condition to prove that this time \( t \) is indeed optimal. How is \( t \) related to the three coefficients \( a, b, c \) and the interest rate \( r \)?

Solution:

Obtaining present value from future value with the help of the formula \( A(t) = V e^{-rt} \) gives:
\[
A(t) = a^{\ln \sqrt{t - bc}} e^{-rt} = e^{\frac{1}{2} \ln a \ln(t - bc) - rt}
\]
\[
A'(t) = e^{-r} \left[ \frac{\ln a}{2(t - bc)} - r \right] = 0
\]
By first-order condition, the parenthesized expression must equal 0. Therefore,
\[
\frac{\ln a}{2(t - bc)} = r
\]
\[
t^* = \frac{\ln a}{2r} + bc \quad \text{where } \ln a > 0.
\]
We could easily check that \( t^* \) is positively related to \( a, b, c \) and negatively related to \( r \). A second-order condition requires finding the second derivative:
\[
A''(t) = A'(t) \left[ \frac{\ln a}{2(t + bc)} - r \right] + e^{-r} \left[ -\frac{\ln a}{2(t + bc)^2} \right] < 0
\]
The first term is 0 due to \( A'(t^*) = 0 \). Since \( \ln a > 0 \), we get that \( A''(t^*) < 0 \), which proves maximum.

45. A large olive producer in Spain knows that the value \( V(t) \) of the produce changes with time according to the exponential function
\[
V(t) = a^{\log_8 \sqrt{t + d}} \quad a > b > 1 \quad \text{and} \quad c, d > 0
\]
where \( t \) is the time for olives to ripen. If the interest rate is \( r \), what is the optimal time \( t \) for this producer to pick olives in order to maximize the present value of the harvest? Use a second-order condition to prove that this time \( t \) is, indeed, optimal. How is \( t \) related to the coefficients \( a, b, c, d \) and the interest rate \( r \)?
Solution:

By the process of discounting, we express present value \( A \) from the given function \( V \):

\[
A(t) = a^{\log_b \sqrt{ct+d}} e^{-rt} = e^{\frac{1}{2}\ln a \log_b (ct+d) - rt} = e^{\frac{1}{2}\ln b \ln(ct+d) - rt} = e^{\frac{1}{2}\log_b a \ln (ct+d) - rt}
\]

\[
A'(t) = e^{-r} \left[ \frac{c \log_b a}{2(c(t + d))} - r \right] = 0
\]

By the first-order condition, the parenthesized expression must equal 0. It follows that

\[
\frac{c \log_b a}{2(c(t + d))} = r
\]

\[
t^* = \frac{\log_b a}{2r} - \frac{d}{c} \quad \text{where} \quad \log_b a > 0.
\]

We could easily check that \( t^* \) is positively related to \( a, c \) and negatively related to \( r, d \) and \( b \). The second-order condition requires finding the second derivative:

\[
A''(t) = A'(t) \left[ \frac{c \log_b a}{2(c(t + d))} - r \right] + e^{-r} \left[ -\frac{c^2 \log_b a}{2(c(t + d))^2} \right] < 0
\]

The first term is 0 due to \( A'(t^*) = 0 \). Since \( \log_b a > 0 \), we get that \( A''(t^*) < 0 \), which proves a maximum.

46. A large olive producer in Spain has determined that the value \( V(t) \) of the produce is increasing over time according to the exponential function

\[
V(t) = a^{\log_b \sqrt{ct+d}} \quad a > b > 1 \text{ and } c, d > 0
\]

where \( t \) is time for olives to ripen. Assuming that the interest rate is \( r \), what is the optimal time \( t \) for this producer to pick olives such that the present value of the harvest is maximized? Use a second-order condition to prove that this time \( t \) is, indeed, optimal. How is \( t \) related to the coefficients \( a, b, c, d \) and the interest rate \( r \)?

Solution:

Using \( A(t) = V e^{-rt} \), we express present value from future value:

\[
A(t) = a^{\log_b \sqrt{ct+d}} e^{-rt} = e^{\frac{1}{2}\ln a \log_b (ct+d) - rt} = e^{\frac{1}{2}\ln b \ln(ct+d) - rt} = e^{\frac{1}{2}\log_b a \ln (ct+d) - rt}
\]

\[
A'(t) = e^{-r} \left[ \frac{\log_b a}{2(t-cd)} - r \right] = 0
\]

By the first-order condition, the derivative must equal zero so the parenthesized expression is also 0.

\[
\frac{\log_b a}{2(t-cd)} = r
\]

\[
t^* = \frac{\log_b a}{2r} + cd \quad \text{where} \quad \log_b a > 0.
\]

We could easily check that \( t^* \) is positively related to \( a, c, d \) and negatively related to \( r \) and \( b \). The second-order condition requires finding the second derivative:

\[
A''(t) = A'(t) \left[ \frac{\log_b a}{2(t-cd)} - r \right] + e^{-r} \left[ -\frac{\log_b a}{2(t-cd)^2} \right] < 0
\]

The first term is 0 due to \( A'(t^*) = 0 \). Since \( \log_b a > 0 \), we get that \( A''(t^*) < 0 \), which proves a maximum.
47. The famous Prado museum in Madrid owns a painting by El Greco. Experts in the field have evaluated that its value \( V(t) \) is increasing over time according to the exponential function

\[
V(t) = 2^{\sqrt{2t+5}}
\]

where \( t \) is time. Assuming that the annual interest rate is constant at 8%, what is the optimal time \( t \) for Prado to sell El Greco’s painting? Use a second-order condition to prove that this time \( t \) is indeed optimal.

**Solution:**

From the future value, we obtain the discounted present value of the painting:

\[
A(t) = 2^{\sqrt{2t+5}} e^{-rt} = e^{\sqrt{2t+5} \ln 2} e^{-rt} = e^{\sqrt{2t+5} \ln 2 - rt}
\]

Differentiating the present value of the painting with respect to time,

\[
A'(t) = e^{\sqrt{2t+5} \ln 2 - rt} \left( \frac{2 \ln 2}{2\sqrt{2t+5}} - r \right) = e^{\sqrt{2t+5} \ln 2 - rt} \left( \frac{\ln 2}{\sqrt{2t+5}} - r \right) = 0
\]

\[
0.08 = \frac{\ln 2}{\sqrt{2t+5}}
\]

\[
2t + 5 = \left( \frac{\ln 2}{0.08} \right)^2
\]

\[
2t + 5 = 75
\]

\[
t \approx 35 \text{ years in which the Prado museum should sell El Greco’s painting.}
\]

\[
A''(t) = A'(t) \left( \frac{\ln 2}{\sqrt{2t+5}} - r \right) + e^{\sqrt{2t+5} \ln 2 - rt} \left( - \frac{2 \ln 2}{2\sqrt{(2t+5)^3}} \right) = 0 - e^{\sqrt{2t+5} \ln 2 - rt} \frac{\ln 2}{\sqrt{(2t+5)^3}} < 0 , \text{ which implies a maximum for the present value of the painting.}
\]

48. The famous Thyssen museum in Madrid owns a painting by Picasso. Experts in the field have evaluated that its value \( V(t) \) is increasing over time according to the exponential function

\[
V(t) = 2^{\sqrt{2t+8}}
\]

where \( t \) is time. Assuming that the annual interest rate is constant at 10%, what is the optimal time \( t \) for Thyssen to sell Picasso’s painting? Use a second-order condition to prove that this time \( t \) is optimal.

**Solution:**

Transforming future value into present value and differentiating it with respect to time yields

\[
A(t) = 2^{\sqrt{2t+8}} e^{-rt} = e^{\sqrt{2t+8} \ln 2} e^{-rt} = e^{\sqrt{2t+8} \ln 2 - rt}
\]

\[
A'(t) = e^{\sqrt{2t+8} \ln 2 - rt} \left( \frac{2 \ln 2}{2\sqrt{2t+8}} - r \right) = e^{\sqrt{2t+8} \ln 2 - rt} \left( \frac{\ln 2}{\sqrt{2t+8}} - r \right) = 0
\]

\[
0.1 = \frac{\ln 2}{\sqrt{2t+8}}
\]

\[
2t + 8 = \left( \frac{\ln 2}{0.1} \right)^2
\]

\[
2t + 8 = 48
\]

\[
t \approx 20 \text{ years is the best time to sell the Picasso.}
\]
A"(t) = A'(t) \left( \frac{\ln 2}{\sqrt{2t + 8}} - r \right) + e^{\ln 2 + \ln 2 - \eta} \left( -\frac{2 \ln 2}{2\sqrt{(2t + 8)^3}} \right) = 0 - e^{\ln 2 + \ln 2 - \eta} \frac{\ln 2}{\sqrt{(2t + 8)^3}} < 0\), which implies a maximum for the present value of the painting.

49. The utility function of an individual consuming three commodities is given by

\[ U(x, y, z) = \left( \frac{x - y}{x + y} \right)^z \]

where the quantities of the three goods are \( x \), \( y \), and \( z \), respectively. Find the total differential \( dU \) of the utility function. Express the three partial point elasticities \( \varepsilon_{Ux}, \varepsilon_{Uy}, \) and \( \varepsilon_{Uz} \).

**Solution:**

Expressing the total differential of utility with the help of marginal utilities,

\[ dU = U_x dx + U_y dy + U_z dz \]

\[ U_x = z \left( \frac{x - y}{x + y} \right)^{z-1} \frac{x + y - x + y}{(x + y)^2} = \left( \frac{x - y}{x + y} \right)^{z-1} \frac{2yz}{(x + y)^2} \]

\[ U_y = z \left( \frac{x - y}{x + y} \right)^{z-1} \frac{(-x - y - x + y)}{(x + y)^2} = -\left( \frac{x - y}{x + y} \right)^{z-1} \frac{2xz}{(x + y)^2} \]

To find \( U_z \) we need to take the natural logarithm of both sides of the utility function:

\[ \ln U = z \ln \left( \frac{x - y}{x + y} \right) \]

We know that \( \frac{\partial \ln U}{\partial z} = \frac{U_z}{U} \), so

\[ U_z = U \frac{\partial \ln U}{\partial z} = \left( \frac{x - y}{x + y} \right)^{z-1} \ln \left( \frac{x - y}{x + y} \right) \]

For the three point elasticities, we substitute in the well-known formula

\[ \varepsilon_{Ux} = \frac{U_x}{U} = \left( \frac{x - y}{x + y} \right)^{z-1} \frac{2xyz}{(x + y)^2} \left( \frac{x + y}{x - y} \right) = \frac{2xyz}{x^2 - y^2} \]

\[ \varepsilon_{Uy} = \frac{U_y}{U} = -\left( \frac{x - y}{x + y} \right)^{z-1} \frac{2xyz}{(x + y)^2} \left( \frac{x + y}{x - y} \right) = -\frac{2xyz}{x^2 - y^2} \]

\[ \varepsilon_{Uz} = \left( \frac{x - y}{x + y} \right)^z \ln \left( \frac{x - y}{x + y} \right) \left( \frac{x + y}{x - y} \right)^z = z \ln \left( \frac{x - y}{x + y} \right) \]

50. The utility function of an individual consuming three commodities is given by

\[ U(x, y, z) = \left( \frac{x + y}{x - y} \right)^z \]

where the quantities of the three goods are \( x \), \( y \), and \( z \), respectively. Find the total differential \( dU \) of the utility function. Express the three partial point elasticities \( \varepsilon_{Ux}, \varepsilon_{Uy}, \) and \( \varepsilon_{Uz} \).
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51. The utility function of an individual consuming three commodities is given by

$$U(x, y, z) = \left(\frac{x-y}{xy}\right)^{z}$$

where the quantities of the three goods are $x$, $y$, and $z$, respectively. Find the total differential $dU$ of the utility function. Express the three partial point elasticities $\varepsilon_{Ux}$, $\varepsilon_{Uy}$, and $\varepsilon_{Uz}$.

Solution:

The total differential of the utility function can be expressed as

$$dU = U_x dx + U_y dy + U_z dz .$$

$$U_x = z \left(\frac{x-y}{xy}\right)^{z-1} \left(\frac{-y}{x^2 y^2} \right) = \frac{z}{x^2} \left(\frac{x-y}{xy}\right)^{z-1}$$

$$U_y = z \left(\frac{x-y}{xy}\right)^{z-1} \left(\frac{-x}{x^2 y^2} \right) = -\frac{z}{y^2} \left(\frac{x-y}{xy}\right)^{z-1}$$

Taking the natural log of both sides of the expression gives

$$\ln U = z \ln \left(\frac{x-y}{xy}\right)$$

Differentiating with respect to $z$ yields
\[
\frac{\partial \ln U}{\partial z} = \frac{U_z}{U} = \ln \frac{x-y}{xy} \quad U_z = \ln \frac{x-y}{xy} \left( \frac{x-y}{xy} \right)^z
\]

This immediately gives \( E_{Uz} \) since \( E_{Uz} = \frac{U_z}{U / z} \), or \( E_{Uz} = z \ln \frac{x-y}{xy} \).

\[
E_{Ux} = \frac{z}{x^2} \left( \frac{x-y}{xy} \right)^{z-1} \left( \frac{xy}{x-y} \right)^z x = \frac{yz}{x-y}
\]

\[
E_{Uy} = -\frac{z}{y^2} \left( \frac{x-y}{xy} \right)^{z-1} \left( \frac{xy}{x-y} \right)^z y = -\frac{xz}{x-y}
\]

52. The utility function of an individual consuming three commodities is given by

\[
U(x, y, z) = \left( \frac{xy}{x-y} \right)^z
\]

where the quantities of the three goods are \( x, y, \) and \( z \), respectively. Find the total differential \( dU \) of the utility function. Express the three partial point elasticities \( e_{Ux} \), \( e_{Uy} \), and \( e_{Uz} \).

Solution:

The total differential of the utility function can generally be written as \( dU = U_x dx + U_y dy + U_z dz \).

\[
U_x = z \left( \frac{xy}{x-y} \right)^{z-1} \left[ \frac{y(x-y) - xy}{(x-y)^2} \right] = -\frac{zy^2}{(x-y)^2} \left( \frac{xy}{x-y} \right)^{z-1}
\]

\[
U_y = z \left( \frac{xy}{x-y} \right)^{z-1} \left[ \frac{x(x-y) + xy}{(x-y)^2} \right] = \frac{zx^2}{(x-y)^2} \left( \frac{xy}{x-y} \right)^{z-1}
\]

Taking the natural log of both sides of the expression gives

\[
\ln U = z \ln \frac{xy}{x-y}
\]

Differentiating with respect to \( z \) yields

\[
\frac{\partial \ln U}{\partial z} = \frac{U_z}{U} = \ln \frac{xy}{x-y} \quad U_z = \ln \frac{xy}{x-y} \left( \frac{xy}{x-y} \right)^z
\]

This immediately gives \( E_{Uz} \) since \( E_{Uz} = \frac{U_z}{U / z} \), or \( E_{Uz} = z \ln \frac{xy}{x-y} \).

\[
E_{Ux} = -\frac{zy^2}{(x-y)^2} \left( \frac{xy}{x-y} \right)^{z-1} \left( \frac{x-y}{xy} \right)^z x = -\frac{yz}{x-y}
\]

\[
E_{Uy} = \frac{zx^2}{(x-y)^2} \left( \frac{xy}{x-y} \right)^{z-1} \left( \frac{x-y}{xy} \right)^z y = \frac{xz}{x-y}
\]

53. A consumer has to choose between present and future consumption. His present consumption is \( c_1 \), and his future one is \( c_2 \). His current income is \( m_1 \), and he expects to earn income \( m_2 \) in the future. His utility function is given by

\[
U = \ln c_1 + \ln \frac{c_2}{1+r}
\]
where his utility of consuming in the second period is discounted by the interest rate \( r \). How much should the consumer consume at the optimum?

**Solution:**

In the second period, the consumer’s consumption will be

\[ c_2 = m_2 + (1 + r)(m_1 - c_1) \]

or the sum of his income earned in the second period and the compounded savings from the first period. Thus, if currently the consumer saves so that \( m_1 - c_1 > 0 \), his future consumption is greater than his future income and he has positive net savings from the first period compounded by the interest rate \( r \). If, however, his current income is smaller than his consumption, that is, \( m_1 - c_1 < 0 \), then he will be a net borrower now and, hence, his future income will be reduced by the amount of the loan compounded by the interest rate. Substituting in the utility function,

\[ U = \ln c_1 + \frac{\ln [m_2 + (1 + r)(m_1 - c_1)]}{1 + r} \]

and differentiating with respect to \( c_1 \),

\[ \frac{dU}{dc_1} = \frac{1}{c_1} - \frac{(1 + r)}{(1 + r)[m_2 + (1 + r)(m_1 - c_1)]} = 0 \]

\[ \frac{1}{m_2 + (1 + r)(m_1 - c_1)} = \frac{1}{c_1} \]

\[ c_1 = m_2 + (1 + r)m_1 - (1 + r)c_1 \]

\[ c_1(2 + r) = m_2 + (1 + r)m_1 \]

\[ c_1 = \frac{m_2 + (2 + r - 1)m_1}{2 + r} \]

\[ c_1^* = m_1 + \frac{m_2 - m_1}{2 + r} \]

For the optimal amount of present consumption, we get that it is the sum of current income \( m_1 \) and a discretionary amount, depending on the change in income. Thus, if the future income is greater than the current and \( m_2 > m_1 \), the consumer would be a net borrower in the present moment. He will consume more than he earns presently and will substitute future consumption with present. But if he expects a lower income in the future so that \( m_2 < m_1 \), then he will choose to be a net lender in the present moment. Thus, he sacrifices present consumption for future. Expressing consumption in the second period,

\[ c_2^* = m_2 + (1 + r)\left( m_1 - m_1 - \frac{m_2 - m_1}{2 + r} \right) = m_2 - (1 + r)\left( \frac{m_2 - m_1}{2 + r} \right) = \]

\[ = \frac{m_2 + (1 + r)m_1}{2 + r} = \frac{m_1 + m_2 - m_1}{2 + r} \]

We also find that the consumption in the two periods would be the same and that the difference in his lender or borrower status would depend solely on the differential between his income levels in the two periods. Thus, for example, when those two incomes are the same (that is, the consumer earns the same in the present and in the future), in each period he will consume exactly up to the amount of his income. Hence, if the consumer does not expect an intertemporal change in his income, he would neither save, nor borrow.
54. A consumer has to choose between present and future consumption. His present consumption is \( c_1 \), and his future one is \( c_2 \). His current income is \( m_1 \), and he expects to earn income \( m_2 \) in the future. His utility function is the product of his current consumption and his future one discounted by the interest rate \( r \).

\[
U = c_1 \left( \frac{c_2}{1 + r} \right)
\]

How much should the consumer consume at the optimum?

**Solution:**

In the second period, the consumer’s consumption will be

\[ c_2 = m_2 + (1 + r)(m_1 - c_1) \]

or the sum of his future income and the compounded savings from the first period. Thus, if currently the consumer saves and \( m_1 - c_1 > 0 \), his future consumption is greater than his future income and he has positive net savings from the first period compounded by the interest rate \( r \). If, however, his income presently is smaller than his consumption, that is, \( m_1 - c_1 < 0 \), then he is a net borrower presently and, hence, his future income will be reduced by the amount of the loan compounded by the interest rate. Substituting in the utility function,

\[
U = c_1 \left( \frac{m_2 + (1 + r)(m_1 - c_1)}{1 + r} \right)
\]

and differentiating with respect to \( c_1 \) using the product rule,

\[
\frac{dU}{dc_1} = m_2 + (1 + r)(m_1 - c_1) - c_1 = 0
\]

\[
c_1 = \frac{m_2}{2(1 + r)} + \frac{m_1}{2}
\]

\[
c_1^* = m_1 + \frac{m_2 - (1 + r)m_1}{2(1 + r)}
\]

The optimal amount of present consumption is the sum of current income \( m_1 \) and a discretionary amount depending on the change in income. Thus, if the future income is sufficiently above the compounded present income, i.e., \( m_2 > (1 + r)m_1 \), the consumer would be a net borrower in the present moment. He will consume more than he earns presently and will substitute future consumption with present. But if he expects a lower income in the future so that \( m_2 < (1 + r)m_1 \), then he will choose to be a net lender in the present moment. Thus, he sacrifices present consumption for future. From the expression for present consumption, we also see that it is adversely related to the interest rate. When the interest rate is high, the opportunity costs of present consumption are high and the consumer would be enticed to save rather than to consume at present. But if the interest rate is rather low, then the consumer would prefer to consume at present. For future consumption,

\[
c_2^* = m_2 + (1 + r)\left[ m_1 - \frac{m_1}{2} - \frac{m_2}{2(1 + r)} \right] = m_2 + \frac{(1 + r)m_1 - m_2}{2} = \frac{m_2 + (1 + r)m_1}{2}
\]

We see that future consumption is positively related to the interest rate. That is, if the interest rate is high, the consumer would save in the first moment to consume in the future. But if the interest rate is low, his incentive to save is rather low. This way, he would be left with little consumption in the second period.

55. Consider the aggregate production function

\[
F(K,L,t) = AK^\alpha L^\beta e^{\lambda t}
\]

\( A = \text{const.} \quad \alpha, \beta > 0 \quad \text{and} \quad \alpha + \beta = 1 \)
where aggregate output grows exponentially and present output depends on the level of knowledge $A$, physical capital $K$, and the size of the labor force $L$. Physical capital $K$ and labor $L$ are both functions of time $t$, and both grow steadily at the rate $\lambda$. Express the total change in aggregate output $F(K, L, t)$ with time. What is the rate of growth of the aggregate production function?

**Solution:**

The total change in aggregate output can be found by totally differentiating it with respect to time:

$$
\frac{dF}{dt} = \frac{\partial F}{\partial K} \frac{dK}{dt} + \frac{\partial F}{\partial L} \frac{dL}{dt} + \frac{\partial F}{\partial t}
$$

$$
\frac{dF}{dt} = \alpha AK^{-1}L^\beta e^{\lambda t} \frac{dK}{dt} + \beta AK^\alpha L^{-1} e^{\lambda t} \frac{dL}{dt} + \lambda AK^\alpha L^\beta e^{\lambda t}
$$

$$
\frac{dF}{dt} = AK^\alpha L^\beta e^{\lambda t} \left( \frac{\alpha dK}{Kdt} + \beta dL \frac{dL}{dt} + \lambda \right)
$$

$$
\frac{dF}{dt} / F = \alpha \frac{dK}{dt} + \beta \frac{dL}{dt} + \lambda
$$

$r_F = \alpha r_K + \beta r_L + \lambda = \alpha \lambda + \beta \lambda + \lambda = \lambda (1 + \alpha + \beta)$

Alternatively, taking the log of both sides of the aggregate production function in order to find the rate of growth,

$$
\ln F = \ln A + \alpha \ln K + \beta \ln L + \lambda t
$$

Given that $A$ is a constant, this translates into

$$
\frac{d\ln F}{dt} = \alpha \frac{d\ln K}{dt} + \beta \frac{d\ln L}{dt} + \lambda \quad \text{or the result we obtained previously},
$$

$r_F = \alpha r_K + \beta r_L + \lambda = \alpha \lambda + \beta \lambda + \lambda = \lambda (1 + \alpha + \beta)$

56. Consider the aggregate production function

$$
F(K, L, t) = H(K, L)e^{\lambda t} \quad \lambda = \text{const.} \quad \lambda > 0
$$

where aggregate output grows exponentially and present output $H(K, L)$ depends on physical capital $K$ and the size of the labor force $L$. Physical capital $K$ and labor $L$ are both functions of time $t$ and both grow steadily at the rate $\lambda$. Express the total change in aggregate output $F(K, L, t)$ with time.

What is the rate of growth of the aggregate production function?

**Solution:**

Time affects aggregate output through three channels: directly and through capital and labor. Thus,

$$
\frac{dF}{dt} = \frac{\partial F}{\partial H} \frac{dH}{dt} + \frac{\partial F}{\partial K} \frac{dK}{dt} + \frac{\partial F}{\partial L} \frac{dL}{dt} + \frac{\partial F}{\partial t}
$$

$$
\frac{dF}{dt} = \frac{\partial H}{\partial K} \frac{dK}{dt} e^{\lambda t} + \frac{\partial H}{\partial L} \frac{dL}{dt} e^{\lambda t} + \lambda He^{\lambda t}
$$

$$
\frac{dF}{dt} / F = \frac{\partial H e^{\lambda t}}{\partial H e^{\lambda t}} \frac{dK}{dt} / K + \frac{\partial H e^{\lambda t}}{\partial H e^{\lambda t}} \frac{dL}{dt} / L + \lambda
$$

$r_F = E_{HK} r_K + E_{HL} r_L + \lambda = E_{HK} \lambda + E_{HL} \lambda + \lambda = \lambda (1 + E_{HK} + E_{HL})$

Alternatively, taking the log of both sides of the production function in order to find the rate of growth,

$$
\ln F = \ln H + \lambda t$
57. The aggregate production function of a country is given by the equation
\[ Y = A L^\alpha K^\beta b^t \]
where \( A \) is a positive parameter, \( t \) is time, and \( 0 < \alpha, \beta < 1 \). Both capital \( K \) and labor \( L \) change with time. Express the total change in aggregate output \( Y \) with time. What is its growth rate?

**Solution:**
Time affects aggregate output through three channels: directly, through labor, and through capital. Expressing this as a total derivative,
\[ \frac{dY}{dt} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial L} \frac{dL}{dt} + \frac{\partial Y}{\partial K} \frac{dK}{dt} \]
\[ \frac{\partial Y}{\partial L} = \alpha A L^{\alpha-1} K^\beta b^t \]
\[ \frac{\partial Y}{\partial K} = \beta A L^\alpha K^{\beta-1} b^t \]
Using the rule for finding the derivative of an exponential function,
\[ \frac{dY}{dt} = \ln b A L^\alpha K^\beta b^t \]
Thus, the total derivative becomes
\[ \frac{dY}{dt} = \ln b A L^\alpha K^\beta b^t + \alpha A L^{\alpha-1} K^\beta b^t \left( \frac{dL}{dt} \right) + \beta A L^\alpha K^{\beta-1} b^t \left( \frac{dK}{dt} \right) \]
To find the rate of growth of aggregate output, we can divide both sides by \( Y \) or use logarithms:
\[ \frac{dY}{dt} \frac{1}{Y} = \frac{A L^\alpha K^\beta b^t}{A L^\alpha K^\beta b^t} \left( \ln b + \alpha \frac{dL}{dt} + \beta \frac{dK}{dt} \right) = \ln b + \alpha \dot{L} + \beta \dot{K} \]
where \( \dot{L} \) and \( \dot{K} \) are the growth rates of labor and capital, respectively. Taking the log of both sides,
\[ \ln Y = \ln A + \alpha \ln L + \beta \ln K + \ln b \]
and differentiating further,
\[ \dot{Y} = \frac{d \ln Y}{dt} = \frac{\alpha d \ln L}{dt} + \frac{\beta d \ln K}{dt} + \ln b = \alpha \dot{L} + \beta \dot{K} + \ln b \]
which is the result obtained previously.

58. The aggregate production function of Romania is estimated to be
\[ Y = 2.5 L^{0.64} K^{0.32} (1.08)^t \]
where \( Y \) is Romanian net aggregate output, \( K \) is capital stock, \( L \) is labor, and \( t \) is time. Both capital \( K \) and labor \( L \) change with time. Express the total change in Romanian aggregate output \( Y \) with time and its growth rate. The growth of which factor of production is more important for the general growth of the economy?

**Solution:**
Expressing the total derivative generally,
For the total derivative, we get
\[
\frac{dY}{dt} = 2.5L^{0.64}K^{0.32}(1.08)\ln1.08 + 1.6L^{-0.36}K^{0.32}(1.08)\left(\frac{dL}{dt}\right) + 0.8L^{0.64}K^{-0.68}(1.08)\left(\frac{dK}{dt}\right)
\]

Dividing both sides by \(Y\) to obtain the rate of growth,
\[
\frac{dY}{Y} = \frac{2.5L^{0.64}K^{0.32}(1.08)\ln1.08 + 1.6L^{-0.36}K^{0.32}(1.08)\left(\ln1.08 + \frac{0.64dL}{Ldt} + \frac{0.32dK}{Kdt}\right)}{2.5L^{0.64}K^{0.32}(1.08)\ln1.08 + 0.64L + 0.32K}
\]

where \(\dot{L}\) and \(\dot{K}\) are the growth rates of labor and capital, respectively. We see that in the Romanian economy much of the growth in aggregate output comes from the increase in the labor force and less so from capital.

59. Consider a Solow output model given by the equation
\[
Y(t) = K^\alpha (AL)^{1-\alpha}
\]

where \(K\) is physical capital, \(L\) denotes the number of workers, and \(A\) is “knowledge” or “effectiveness of labor.” Knowledge \(A\) and labor \(L\) enter multiplicatively and therefore \(AL\) is referred to as effective labor. Physical capital \(K\), knowledge \(A\), and labor \(L\) are all functions of time \(t\) such that \(\dot{K}, \dot{A}\), and \(\dot{L}\) are their rates of growth, respectively. Express the total change in output \(Y\) with time. What is the partial point elasticity \(\varepsilon_{YK}\) of output \(Y\) with respect to capital \(K\)? If \(\dot{Y}\) is the rate of growth of output, prove that \(\dot{Y} = \varepsilon_{YK} \dot{K} + \varepsilon_{YA} \dot{A} + \varepsilon_{YL} \dot{L}\).

Solution:
Since capital, knowledge, and labor are all functions of time, it affects output through three channels:
\[
\frac{dY}{dt} = \frac{\partial Y}{\partial K} \frac{dK}{dt} + \frac{\partial Y}{\partial A} \frac{dA}{dt} + \frac{\partial Y}{\partial L} \frac{dL}{dt}
\]

Substituting these values into the total derivative,
\[
\frac{dY}{dt} = \alpha K^{\alpha-1} (AL)^{1-\alpha} \frac{dK}{dt} + (1-\alpha)K^{\alpha} A^{1-\alpha} L^{1-\alpha} \frac{dL}{dt} + (1-\alpha)K^{\alpha} A^{1-\alpha} L^{1-\alpha} \frac{dA}{dt}
\]

or \(\dot{Y} = \varepsilon_{YK} \dot{K} + \varepsilon_{YA} \dot{A} + \varepsilon_{YL} \dot{L}\)

\(\varepsilon_{YK} = \frac{\partial Y/\partial K}{Y/K} = \frac{\alpha AK^{\alpha-1}(AL)^{1-\alpha} K}{AK^{\alpha}(AL)^{1-\alpha}} = \alpha\)
Alternatively, taking the log of both sides of the original national income function, we get
\[ \ln Y = \alpha \ln K + (1 - \alpha) \ln A + (1 - \alpha) \ln L \]

Through partial differentiation, we obtain the partial point elasticities
\[
\varepsilon_{yK} = \frac{\partial \ln Y}{\partial \ln K} = \alpha \quad \varepsilon_{yA} = \frac{\partial \ln Y}{\partial \ln A} = 1 - \alpha \quad \varepsilon_{yL} = \frac{\partial \ln Y}{\partial \ln L} = 1 - \alpha
\]

Then,
\[
\dot{Y} = \frac{d}{dt} \ln Y = \frac{d}{dt} \left[ \alpha \ln K + (1 - \alpha) \ln A + (1 - \alpha) \ln L \right] = \alpha \frac{d}{dt} \ln K + (1 - \alpha) \frac{d}{dt} \ln A + (1 - \alpha) \frac{d}{dt} \ln L =
\]
\[
= \varepsilon_{yK} \dot{K} + \varepsilon_{yA} \dot{A} + \varepsilon_{yL} \dot{L}
\]

60. Consider a Lucas output model given by the equation
\[ Y(t) = A K^\alpha H^\beta L^{1-\alpha-\beta} \quad A = \text{const.} \quad \alpha, \beta > 0 \quad \text{and} \quad \alpha + \beta < 1 \]
where \( A \) is the level of knowledge, \( K \) is physical capital, \( H \) is the stock of human capital, while \( L \) denotes the number of workers. Physical capital \( K \), human capital \( H \), and labor \( L \) are all functions of time \( t \) such that \( \dot{K} \), \( \dot{H} \), and \( \dot{L} \) are their rates of growth, respectively. Express the total change in output \( Y \) with time. What is the partial point elasticity \( \varepsilon_{yK} \) of output \( Y \) with respect to capital \( K \)? If \( \dot{Y} \) is the rate of growth of output, prove that \( \dot{Y} = \varepsilon_{yK} \dot{K} + \varepsilon_{yH} \dot{H} + \varepsilon_{yL} \dot{L} \).

**Solution:**

There are three channels through which time affects aggregate output: physical capital, human capital, and labor. Thus, we have:
\[
\frac{dY}{dt} = \frac{\partial Y}{\partial K} \frac{dK}{dt} + \frac{\partial Y}{\partial H} \frac{dH}{dt} + \frac{\partial Y}{\partial L} \frac{dL}{dt}
\]
\[
\frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1} H^\beta L^{1-\alpha-\beta}
\]
\[
\frac{\partial Y}{\partial H} = \beta AK^\alpha H^{\beta-1} L^{1-\alpha-\beta}
\]
\[
\frac{\partial Y}{\partial L} = (1 - \alpha - \beta) AK^\alpha H^\beta L^{(1-\alpha-\beta)-1}
\]

Substituting these partial derivatives into the total derivative,
\[
\frac{dY}{dt} = \alpha AK^\alpha H^\beta L^{1-\alpha-\beta} \frac{dK}{dt} + \beta AK^\alpha H^\beta L^{1-\alpha-\beta} \frac{dH}{dt} + (1 - \alpha - \beta) AK^\alpha H^\beta L^{1-\alpha-\beta} \frac{dL}{dt} =
\]
\[
= Y \left[ \alpha \dot{K} + \beta \dot{H} + (1 - \alpha - \beta) \dot{L} \right]
\]
or
\[
\dot{Y} = \alpha \dot{K} + \beta \dot{H} + (1 - \alpha - \beta) \dot{L}
\]

\[
\varepsilon_{yK} = \frac{\partial Y/\partial K}{Y/K} = \frac{\alpha AK^{\alpha-1} H^\beta L^{1-\alpha-\beta} K}{AK^{\alpha-1} H^\beta L^{1-\alpha-\beta}} = \alpha
\]

It is easy to see that
\[
\dot{Y} = \varepsilon_{yK} \dot{K} + \varepsilon_{yH} \dot{H} + \varepsilon_{yL} \dot{L}
\]

Alternatively, taking the log of both sides of the original national income function, we get
\[ \ln Y = \ln A + \alpha \ln K + \beta \ln H + (1 - \alpha - \beta) \ln L \]

Through partial differentiation, we obtain the partial point elasticities
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\[
\varepsilon_{YK} = \frac{\partial \ln Y}{\partial \ln K} = \alpha \\
\varepsilon_{YH} = \frac{\partial \ln Y}{\partial \ln H} = \beta \\
\varepsilon_{YL} = \frac{\partial \ln Y}{\partial \ln L} = 1 - \alpha - \beta
\]

Then,

\[
\dot{Y} = \frac{\partial \ln Y}{\partial t} = \frac{d}{dt} \left[ \ln A + \alpha \ln K + \beta \ln H + (1 - \alpha - \beta) \ln L \right] = \alpha \frac{d \ln K}{dt} + \beta \frac{d \ln H}{dt} + (1 - \alpha - \beta) \frac{d \ln L}{dt} = \\
= \varepsilon_{YK} \dot{K} + \varepsilon_{YH} \dot{H} + \varepsilon_{YL} \dot{L}
\]

61. Consider a Cobb-Douglas production function given by the equation

\[ Q = \phi L^\alpha K^\beta \]

where \( \phi \) denotes the level of technology, \( K \) is physical capital, and \( L \) denotes the number of workers. Find the partial point elasticities \( E_{\phi Q}, E_{QL} \), and \( E_{QK} \). Given that technology, labor, and capital are all functions of time such that \( \dot{\phi}, \dot{L} \), and \( \dot{K} \) are their rates of growth, respectively, prove the condition \( \dot{Q} = E_{\phi Q} \dot{\phi} + E_{QL} \dot{L} + E_{QK} \dot{K} \). If the specific forms of the functions are \( \phi = 6t \), \( K = \frac{5t}{L} \), and \( L = \frac{4}{t} \), express the total change in output \( Q \) with time \( t \). 

**Solution:**

We can directly take the log of the two sides of the production function

\[ \ln Q = \ln \phi + \alpha \ln L + \beta \ln K \]

where each partial point elasticity is

\[ E_{\phi Q} = \frac{\partial \ln Q}{\partial \ln \phi} = 1 \quad E_{QL} = \frac{\partial \ln Q}{\partial \ln L} = \alpha \quad E_{QK} = \frac{\partial \ln Q}{\partial \ln K} = \beta \]

From the first equation, we obtain

\[ \frac{d \ln Q}{dt} = \frac{d \ln \phi}{dt} + \alpha \frac{d \ln L}{dt} + \beta \frac{d \ln K}{dt} \]

which is the same as

\[ \dot{Q} = E_{\phi Q} \dot{\phi} + E_{QL} \dot{L} + E_{QK} \dot{K} \]

In the general form, the total derivative of output \( Q \) with respect to time is

\[
\frac{dQ}{dt} = \frac{\partial Q}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial Q}{\partial L} \frac{dL}{dt} + \frac{\partial Q}{\partial K} \left( \frac{\partial K}{\partial t} + \frac{\partial K}{\partial L} \frac{dL}{dt} \right) \\
\frac{\partial Q}{\partial \phi} = L^\alpha K^\beta \\
\frac{\partial Q}{\partial L} = \alpha \phi L^{a-1} K^\beta \\
\frac{\partial Q}{\partial K} = \beta \phi L^a K^{\beta-1} \\
\frac{d\phi}{dt} = 6 \\
\frac{dL}{dt} = -\frac{4}{t^2} \\
\frac{dK}{dt} = \frac{5}{L} \\
\frac{dL}{dt} = -\frac{5t}{L^2}
\]

62. For the production function in the previous problem, express the total change in output \( Q \) with time \( t \), if the specific forms of the functions are \( \phi = 5t \), \( K = \frac{2t}{L} \), and \( L = \frac{4}{t} \).

**Solution:**

In the general form, the total derivative of output \( Q \) with respect to time is
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\[ \frac{dQ}{dt} = \frac{dQ}{d\varphi} \frac{d\varphi}{dt} + \frac{dQ}{dL} \frac{dL}{dt} + \frac{dQ}{dt} \left( \frac{\partial K}{\partial t} + \frac{\partial K}{\partial L} \frac{dL}{dt} \right) \]

\[ \frac{\partial Q}{\partial \varphi} = \alpha \varphi L^{\alpha - 1} K^\beta \quad \frac{\partial Q}{\partial L} = \beta \varphi L^{\alpha - 1} K^\beta - 1 \quad \frac{d\varphi}{dt} = -\frac{4}{t^2} \quad \frac{\partial K}{\partial t} = \frac{2}{L} \quad \frac{\partial K}{\partial L} = -\frac{2t}{L^2} \]

63. A profit-maximizing firm pays for labor the nominal wage. The price it charges for its own product is \( p \). The average price level in the country is \( \bar{p} \). If the rate of growth of the average price level is \( \bar{p}' \), what is the growth rate of the firm’s own price? If the productivity of labor and the real wage do not change substantially with time as is often the case in economic reality, by how much should the firm raise its price to stay at the optimum?

**Solution:**

Since for profit maximization

\[ VMP_L = pMP_L = w_n \]

and we know that the real wage is the ratio of the nominal wage and the average price level, or

\[ w_r = \frac{w_n}{\bar{p}} \]

Hence,

\[ w_r' = w_n' + \bar{p}' \]

That is, the rate of growth of the nominal wage is the sum of the growth rates of real wage and the price level. Thus, for the value of the marginal product of labor, we have

\[ \dot{p} + MP_L = w_n \quad \text{or} \quad \dot{p} + MP_L = w_r + \bar{p} \]

At the optimum the rate of growth of the product price is equal to the growth rates of real wage and the average price level less that of the marginal product of labor. Since the productivity of labor and the real wage rarely change with time, their growth rates are zero. Then we get

\[ \dot{p} = \bar{p} \]

or the firm should keep up with the level of inflation when setting its price. For example, if the inflation rate is 6%, the firm should increase the price of its product exactly by 6%.

64. If population grows according to the function \( H = H_0 e^{kt} \) and consumption by the function \( C = C_0 e^{at} \), find the rates of growth of population, consumption, and per capita consumption by using natural log.

**Solution:**

We need to find

\[ r_H = \frac{H'}{H} \quad \text{rate of growth of population} \]
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\( r_c = \frac{C'}{C} \)  
rate of growth of consumption

\[ H' = \left[ H_o (2)^{kt} \right]' = H_o k 2^{kt} \ln 2 \]

\( r_H = \frac{H_o k 2^{kt} \ln 2}{H_o 2^{kt}} = k \ln 2 \)  \hspace{1cm} (1)

\( r_c = \frac{C_o a e^{at}}{C_o e^{at}} = a \)  \hspace{1cm} (2)

With the help of logarithms, we know that

\[ r = \frac{V_1}{V} = \frac{d \ln V}{dt} \]

Taking the natural logarithm of both sides of \( H = H_o (2)^{kt} \), we obtain

\( \ln H = \ln H_o + kt \ln 2 \)

\[ \frac{d \ln H}{dt} = k \ln 2 \] same as (1)

\( C = C_o e^{at} \)

\( \ln C = \ln C_o + at \)

\[ \frac{d \ln C}{dt} = a \] same as (2)

Since per capita consumption can be expressed as \( \frac{C}{H} \), its rate of growth \( \frac{r_c}{H} \) can be obtained through

the formula \( r_c = r_C - r_H \). Therefore, \( \frac{r_c}{H} = a - k \ln 2 \).

65. The money supply in the economy is defined by the equation \( M = C + D \), where \( C \) is the cash on deposit in banks and \( D \) is the total of all time- and demand deposits. If \( c \) and \( d \) are the respective rates of growth of \( C \) and \( D \), find the rate of growth of money supply \( M \).

Solution:

Directly using the well-known formula for the growth rate of a function representing the sum of two functions,

\[ m = \frac{C}{C + D} + \frac{D}{C + D} \frac{d}{dt} \]

66. Total savings in an economy represent the sum of private and government savings where government savings can be represented as the difference between tax revenues and government spending such that \( S_g = T - G \). If \( t \) is the rate of growth of taxes and \( g \) is the rate of growth of government spending, what is the rate of growth of government savings?

Solution:

Using the formula for rate of growth with the help of a derived function,

\[ s = \frac{S_g'}{S_g} = \frac{(T - G)'}{T - G} = \frac{T'}{T} - \frac{G'}{T - G} = \frac{T}{S_g} - \frac{G}{S_g} \]
67. Total savings in an economy represent the sum of private and government savings where private savings can be represented as the difference between disposable income and household consumption such that $S_p = Y_d - C$. If $y$ is the rate of growth of disposable income and $c$ is the rate of growth of household consumption, what is the rate of growth of private savings?

Solution:
Expressing rate of growth as the ratio of the derived and the total savings function,

$$\frac{dS_p}{S_p} = \frac{(Y_d - C)'}{(Y_d - C)} = \frac{Y_d'Y_d - C'C}{(Y_d - C)Y_d} - \frac{Y_dY_d'}{Y_d - C} = \frac{Y_d - C}{Y_d - C}$$

68. Bulgaria exports white cheese and grapes where current revenues from cheese in millions of euros are $c_o = 10$ and those from grapes are $g_o = 8$. Assume that earnings from cheese $c$ grow by 11.4 percent and those from grapes $g$ by 15 percent. What is the rate of growth of export revenues?

Solution:
Since Bulgaria’s total exports comprise white cheese and grapes, total earnings from exports must be $E = c + g$

Using the formula for a composite function $y = u + v$,

$$r_y = \frac{u + v}{u + v}r_u + \frac{v}{u + v}r_v$$

$$r_E = \frac{c + g}{c + g}r_c + \frac{g}{c + g}r_g = \frac{10}{10 + 8} (0.114) + \frac{8}{10 + 8} (0.15) = 0.13$$

Bulgaria’s export revenues grow by 13%.

69. Bulgaria’s population declines every year according to the function $P(t) = 7,600,000e^{-\frac{3t}{2}}$. Find the rate of growth of Bulgarian population at $t = 8$.

Solution:
Taking the log of both sides of the population function, we obtain

$$\ln P(t) = \ln 7,600,000 - \frac{3t}{2}$$

Differentiating the left side with respect to $t$ gives the rate of growth

$$\frac{d}{dt} \ln P(t) = -\frac{3}{2} = -\frac{1}{6t^{3/2}}$$

$$r_p = \frac{d}{dt} \ln P(t) = -\frac{1}{6t^{3/2}} = -\frac{1}{6(8)^{3/2}} = -\frac{1}{6(4)^{3/2}} = -\frac{1}{24} \approx -0.042$$

The rate of growth of Bulgarian population is negative at approximately -4.2%.

70. The sales of product $A$ of a firm are described by the function $S_A(t) = 100e^{3.2t}$, while those of product $B$ by $S_B(t) = 250e^{0.8t}$. What is the rate of growth of the entire sales of the firm at $t = 4$? Compare it to the individual rates of growth of the two products.
Solution:

For the entire sales of the product, we have

\[ S(t) = S_A(t) + S_B(t) \]

and their rate of growth is

\[ r_s = \frac{S'(t)}{S(t)} = \frac{S'_A(t) + S'_B(t)}{S_A(t) + S_B(t)} \]

\[ S(t) = 100e^{1.2\sqrt{t}} + 250e^{0.8\sqrt{t}} \]

\[ S'(t) = 100e^{1.2\sqrt{t}} \left( 1.2 \frac{1}{2t^{1/2}} \right) + 250e^{0.8\sqrt{t}} \left( 0.8 \frac{1}{2t^{1/2}} \right) = \frac{100}{t^{1/2}} \left( 0.6e^{1.2\sqrt{t}} + e^{0.8\sqrt{t}} \right) \]

\[ r_s = \frac{S'(t)}{S(t)} = \frac{0.6e^{1.2\sqrt{t}} + e^{0.8\sqrt{t}}}{100 \left( e^{1.2\sqrt{t}} + 2.5e^{0.8\sqrt{t}} \right) t^{1/2}} = \frac{0.6e^{1.2\sqrt{t}} + e^{0.8\sqrt{t}}}{e^{1.2\sqrt{t}} + 2.5e^{0.8\sqrt{t}}} t^{1/2} \]

At \( t = 4 \)

\[ r_s = \frac{0.6e^{1.2\sqrt{4}} + e^{0.8\sqrt{4}}}{e^{1.2\sqrt{4}} + 2.5e^{0.8\sqrt{4}}} 4^{1/2} = \frac{0.6e^{2.4} + e^{1.6}}{2 \left( e^{2.4} + 2.5e^{1.6} \right)} = \frac{0.6(11.02) + 4.95}{2(11.02 + 12.38)} = \frac{11.562}{46.8} = 0.247 \]

The firm’s sales grow at approximately 24.7%. For product \( A \), the rate of growth can be expressed as

\[ \ln S_A = \ln 100 + 1.2\sqrt{t} \]

\[ r_{s_A} = \frac{d}{dt} \ln S_A = \frac{0.6}{\sqrt{t}} = \frac{0.6}{2} = 0.3 \]

For product \( B \), the rate of growth can be found as

\[ \ln S_B = \ln 250 + 0.8\sqrt{t} \]

\[ r_{s_B} = \frac{d}{dt} \ln S_B = \frac{0.4}{\sqrt{t}} = \frac{0.4}{2} = 0.2 \]

The sales of the first product grow at a rate of 30%, while those of the second grow at 20%. The growth of all sales is 24.7%.

71. The minority population in a country changes according to the function \( M(t) = 52,000e^{0.21\sqrt{t}} \), while the rest of the population is described by \( R(t) = 260,000e^{-0.24\sqrt{t}} \). Find the rate of growth of the entire population at moment \( t = 8 \). Compare it to the rates of growth of minorities and the majority at that moment. How would the structure of the population change from the present to that moment?

Solution:

The entire population comprises the minority and the majority such that

\[ P(t) = M(t) + R(t) \]

and

\[ r_p = \frac{P'(t)}{P(t)} = \frac{M'(t) + R'(t)}{M(t) + R(t)} \]

\[ P(t) = 52,000e^{0.21\sqrt{t}} + 260,000e^{-0.24\sqrt{t}} \]

\[ P'(t) = 52,000e^{0.21\sqrt{t}} \left( 0.21 \frac{1}{3t^{1/3}} \right) + 260,000e^{-0.24\sqrt{t}} \left( -0.24 \right) \frac{1}{3t^{1/3}} = \]

\[ = \frac{52,000e^{0.21\sqrt{t}}}{3t^{1/3}} - \frac{624,000e^{-0.24\sqrt{t}}}{3t^{1/3}} \]

At \( t = 8 \)

\[ r_p = \frac{52,000e^{0.21\sqrt{8}}}{3(8)^{1/3}} - \frac{624,000e^{-0.24\sqrt{8}}}{3(8)^{1/3}} = \]

\[ = \frac{52,000e^{0.21\sqrt{8}}}{24} - \frac{624,000e^{-0.24\sqrt{8}}}{24} = \frac{52,000}{24} e^{0.21\sqrt{8}} - \frac{624,000}{24} e^{-0.24\sqrt{8}} = \]

\[ = \frac{2166.67}{24} e^{0.21\sqrt{8}} - \frac{26,200}{24} e^{-0.24\sqrt{8}} = \frac{90.27}{24} e^{0.21\sqrt{8}} - \frac{1083.33}{24} e^{-0.24\sqrt{8}} = \]

\[ = 3.765 e^{0.21\sqrt{8}} - 45.27 e^{-0.24\sqrt{8}} \]

The rate of growth of the entire population is approximately 3.765%. The rate of growth of the minority population is \( 52,000e^{0.21\sqrt{8}} / 52,000 = e^{0.21\sqrt{8}} = 1.477 \), and the rate of growth of the majority population is \( 260,000e^{-0.24\sqrt{8}} / 260,000 = e^{-0.24\sqrt{8}} = 0.692 \). The structure of the population will change from a majority of 52% minority and 48% majority to a majority of 47% minority and 53% majority at that moment.
At $t = 8$

\[
0.07e^{0.21t} - 0.4e^{-0.24t}/\left(e^{0.21t} + 5e^{-0.24t}\right)^{2/3} = 0.07e^{0.04t} - 0.4e^{-0.048}/\left(e^{0.04t} + 5e^{-0.048}\right)^{2/3}
\]

\[
= 0.07(1.52) - 0.4(0.62) = 0.106 - 0.248 = 0.142 \
\]

Total population is declining at approximately 0.76%. For minorities the rate of growth is:

\[
\ln M = \ln 52,000 + 0.21t
\]

\[
r_M = \frac{d\ln M}{dt} = \frac{0.07}{t^{2/3}} = \frac{0.07}{8^{2/3}} = \frac{0.07}{4} = 0.0175
\]

For the majority the rate of growth is:

\[
\ln R = \ln 260,000 - 0.24t
\]

\[
r_R = \frac{d\ln R}{dt} = \frac{0.08}{t^{2/3}} = \frac{0.08}{8^{2/3}} = \frac{0.08}{4} = -0.02
\]

Minorities grow at a rate of 1.75%, while the rest of the population declines at 2%. The overall growth is negative at 0.76%.

At $t = 0$

\[
M(0) = 52,000e^{0.21(0)} = 52,000 \
R(t) = 260,000e^{-0.24(0)} = 260,000
\]

At $t = 8$

\[
M(8) = 52,000e^{0.21(8)} = 79,142 \
R(8) = 260,000e^{-0.24(8)} = 160,883
\]

Presently minorities are a fifth of the rest of the population. After time $t = 8$ minorities will be nearly half of the size of the majority, or one-third of the total population.

72. Aggregate consumption in Brazil is represented by the function $C = C_o e^{6t^2 + \sqrt{t}\ln 8}$, whereas the population function can be expressed by $H = H_o e^{\sqrt{t}/\ln 8}$, $t$ representing the time factor. Find the rate of growth of per capita consumption.

Solution:

Using the formula for rate of growth of a composite function, we find the rate of growth of per capita consumption:

\[
r_C = \frac{r_C}{r_H}
\]

\[
r_C = \frac{C'}{C} = \frac{C_o e^{6t^2 + \sqrt{t}\ln 8} \left(12t + \frac{\ln 8}{2\sqrt{t}}\right)}{C_o e^{6t^2 + \sqrt{t}\ln 8}} = 12t + \frac{\ln 8}{2\sqrt{t}}
\]
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\[ H = H_o \cdot 8^{\frac{3t}{\ln 8}} = H_o e^{\ln 8^{\frac{3t}{\ln 8}}} \]

\[ H' = H_o e^{\ln 8^{\frac{3t}{\ln 8}} \left( \frac{\ln 8}{2\sqrt{t}} + 6t \right)} \]

\[ r_H = \frac{H_o e^{\ln 8^{\frac{3t}{\ln 8}} \left( \frac{\ln 8}{2\sqrt{t}} + 6t \right)}}{H_o e^{\ln 8^{\frac{3t}{\ln 8}}}} = \frac{\ln 8}{2\sqrt{t}} + 6t \]

Substituting for the rate of growth of per capita consumption,

\[ r_c = \frac{12t + \ln \left( \frac{\ln 8}{2\sqrt{t}} - 6t - \ln \left( \frac{\ln 8}{2\sqrt{t}} \right) \right) = 6t}{2\sqrt{t}} \]

73. Aggregate consumption in Congo is represented by the function \( C = C_o e^{4t^2 + \frac{\ln 5}{2\sqrt{t}}} \), whereas the population function can be expressed by \( H = H_o 5^{\frac{2t^2}{\ln 5}} \), \( t \) representing the time factor. Find the rate of growth of per capita consumption.

**Solution:**

The rate of growth of per capita consumption is the difference between the rate of growth of aggregate consumption and the rate of growth of population:

\[ \frac{r_c}{\pi} = r_c - r_H \]

\[ r_c = \frac{C'}{C} = \frac{C_o e^{4t^2 + \frac{\ln 5}{2\sqrt{t}} \left( 8t + \frac{\ln 5}{2\sqrt{t}} \right)}}{C_o e^{4t^2 + \frac{\ln 5}{2\sqrt{t}}}} = 8t + \frac{\ln 5}{2\sqrt{t}} \]

\[ H = H_o 5^{\frac{2t^2}{\ln 5}} = H_o e^{\ln 5^{\frac{2t^2}{\ln 5}}} \]

\[ H' = H_o e^{\ln 5^{\frac{2t^2}{\ln 5}} \left( \frac{\ln 5}{2\sqrt{t}} + 4t \right)} \]

\[ r_H = \frac{H_o e^{\ln 5^{\frac{2t^2}{\ln 5}} \left( \frac{\ln 5}{2\sqrt{t}} + 4t \right)}}{H_o e^{\ln 5^{\frac{2t^2}{\ln 5}}}} = \frac{\ln 5}{2\sqrt{t}} + 4t \]

Substituting the specific values for the rate of growth of per capita consumption,

\[ \frac{r_c}{\pi} = 8t + \frac{\ln 5}{2\sqrt{t}} - 4t - \frac{\ln 5}{2\sqrt{t}} = 4t \]

74. The total exports of Zaire given by the function \( X = 100e^{(6t^{\frac{1}{\sqrt{t}}}) \ln 20} \) include the exports of goods and the exports of services. The derived function of services with respect to the time factor \( t \) is

\[ S' = \frac{(60)20^{t^{\frac{1}{\sqrt{t}}}} \ln 20}{\sqrt{t}} \].

What is the rate of growth of goods, if the rate of growth of services is known to be \( r_s = \frac{3\ln 20}{\sqrt{t}} \)?
Solution:

We know that $r_s = \frac{S'}{S}$ so $S = \frac{S'}{r_s}$. Substituting the specific values,

$S = \frac{(60)20^{6\sqrt{t} - 10}}{3\sqrt{t} \ln 20} = (20)20^{6\sqrt{t} - 10}$

$X = 100e^{(6\sqrt{t} - 10) \ln 20} = (100)20^{6\sqrt{t} - 10}$

We obtain exports of goods:

$G = X - S = (80)20^{6\sqrt{t} - 10} = 80e^{(6\sqrt{t} - 10) \ln 20}$

$G' = (80) \frac{3 \ln 20 e^{(6\sqrt{t} - 10) \ln 20}}{\sqrt{t}}$  

$r_G = \frac{G'}{G} = \frac{240 \ln 20 e^{(6\sqrt{t} - 10) \ln 20}}{80 \sqrt{t} e^{(6\sqrt{t} - 10) \ln 20}} = \frac{3 \ln 20}{\sqrt{t}}$

75. The total exports of Venezuela given by the function $X = 70e^{(6\sqrt{t} - 20) \ln 10}$ include the exports of goods and the exports of services. The derived function of services with respect to the time factor $t$ is

$S' = \frac{(90)10^{6\sqrt{t} - 20} \ln 10}{\sqrt{t}}$. What is the rate of growth of goods, if the rate of growth of services is known to be $r_s = \frac{3 \ln 10}{\sqrt{t}}$?

Solution:

We know that $r_s = \frac{S'}{S}$ so $S = \frac{S'}{r_s}$. Substituting the given values yields,

$S = \frac{(90)10^{6\sqrt{t} - 20}}{3 \sqrt{t} \ln 10} = (30)10^{6\sqrt{t} - 20}$

$X = 70e^{(6\sqrt{t} - 20) \ln 10} = (70)10^{6\sqrt{t} - 20}$

We obtain exports of goods

$G = X - S = (40)10^{6\sqrt{t} - 20} = 40e^{(6\sqrt{t} - 10) \ln 10}$

$G' = (40) \frac{3 \ln 10 e^{(6\sqrt{t} - 10) \ln 10}}{\sqrt{t}}$

$r_G = \frac{G'}{G} = \frac{120 \ln 10 e^{(6\sqrt{t} - 10) \ln 10}}{40 \sqrt{t} e^{(6\sqrt{t} - 10) \ln 10}} = \frac{3 \ln 10}{\sqrt{t}}$