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## **Logarithmic Quasi-Homothetic Preferences**

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# Logarithmic Quasi-Homothetic Preferences

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## Abstract

We study a class of symmetric, quasi-homothetic preferences that result in demands logarithmic in own prices when these have a negligible impact on aggregate price indices (as in monopolistic competition models). Thus marginal revenues are computationally friendly, and decreasing whenever demands are elastic. Preferences can be represented either by an additive negative exponential direct utility function, or by an expenditure function which depends on two price indices. A parameter accounts for any number of commodities. Another one affects the relevant demand elasticities, which are not constant. Commodities can be either substitutes or complements according to the size of consumption.

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**Keywords:** Negative exponential utility function, logarithmic demand, quasi-homotheticity, price indices.

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## 1. Introduction

Dixit-Stiglitz (1977: sections I and III) popularized the use of constant-elasticity-of-substitution (CES) homothetic preferences to model the Chamberlinian “large group” monopolistic competition. However, they also showed that more general additive forms (allowing more general commodity substitutability) highlight different results: see e.g. Dixit-Stiglitz (1977: section II) and Krugman (1979). In this paper we consider a class of non-homothetic preferences that explicitly exhibit variable demand elasticities. Preferences are symmetric, and no commodity plays a special role independently from prices and income. In fact, the expenditure function (the indirect utility function) only depends on a price index and a price dispersion index. Preferences are indeed additive, which is both useful, because implies a parsimonious parameterization, and restrictive, because rules out inferior goods and net complements. However, it turns out that commodities can be either gross substitutes or complements, according to the size of consumption (this can be “controlled” using a single parameter). In addition, while the expenditure shares depend on the income, as it is economically reasonable, the Engel curves are linear, which is formally convenient (i.e., preferences are quasi homothetic: see e.g. Deaton-Muellbauer, 1980: section 5.4). Finally, once a large group of commodities is considered (a parameter accounts for any number of them), both uncompensated (Marshallian) and compensated (Hicksian) demands just depend on the logarithm of the own price, which is computationally friendly (marginal revenues are decreasing whenever demands are elastic).

## 2. The negative exponential utility function

Suppose that consumer preferences over a large number  $N$  of commodities can be represented by the following “negative exponential”<sup>1</sup> utility function:

$$U(\mathbf{x}) = -\frac{1}{\alpha} \sum_{h=1}^N e^{-\alpha x_h}, \quad (1)$$

where  $x_h \geq 0$  indicates the consumption of commodity  $h = 1, N$  and  $\alpha$  is a positive parameter. Clearly, preferences are additive: see e.g. Deaton-Muellbauer (1980: section 5.3). Note that if only commodities indexed from 1 to  $n$  are actually consumed, this is easily accounted for by introducing the negative-integer parameter  $k = n - N$ :

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<sup>1</sup> As is well known, the so-called “constant absolute risk aversion” utility function  $U(x) = -\alpha^{-1}e^{-\alpha x}$  is used in macroeconomics because it implies a variable “instantaneous” elasticity of substitution: see for example Blanchard-Fisher (1988: p. 44).

$$U(\mathbf{x}) = -\frac{1}{\alpha} \left( k - \sum_{h=1}^n e^{-\alpha x_h} \right). \quad (1')$$

Obviously,  $U(\cdot)$  embodies a “taste for variety”, since it is strictly concave and increasing with respect to  $n$ . Note that it is well defined over the non negative orthant of the relevant Euclidean space. Thus, according to standard results, it implies regular and well-defined demand functions for (strictly) positive prices and income.

By using the condition:

$$MRS_{ij}(\mathbf{x}) = e^{-\alpha(x_i - x_j)} = \frac{p_i}{p_j}, \quad (2)$$

where  $p_i$  is the price of commodity  $i$ , and  $MRS_{ij}$  stands for marginal rate of substitution between commodities  $i$  and  $j$  ( $i, j = 1, n$ ), it is easily computed that the expenditure function dual to (1') is given by:

$$E(\mathbf{p}, u) = -\frac{1}{\alpha} P(\mathbf{p}) (H(\mathbf{p}) + \ln(k - \alpha u)), \quad (3)$$

where  $u$  is the utility “level”.  $P(\mathbf{p}) = \sum_h p_h$  is a linearly homogeneous price index, and  $H(\mathbf{p}) = \sum_h \hat{p}_h \ln \hat{p}_h$ , where  $\hat{p}_i = p_i/P$ , is the so-called “entropy index” of price dispersion (heterogeneity): see e.g. Stuart-Ord (1994: p. 115). Note that  $H \in (-\infty, 0)$  for positive prices, achieves its minimum  $-\ln n$  if  $p_i = p$  for all  $i = 1, n$ , and it is homogeneous of degree zero.

By Shephard’s Lemma, the compensated demand for commodity  $i$  is given by:

$$h_i(\mathbf{p}, u) = \frac{\partial E(\mathbf{p}, u)}{\partial p_i} = \frac{1}{\alpha} (\ln P(\mathbf{p}) - \ln p_i - \ln(k - \alpha u)), \quad (4)$$

which implies that the elements of the Slutsky matrix are given by:

$$\frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{1}{\alpha P(\mathbf{p})}, \quad \frac{\partial h_i(\mathbf{p}, u)}{\partial p_i} = \frac{1}{\alpha} \left( \frac{1}{P(\mathbf{p})} - \frac{1}{p_i} \right). \quad (5)$$

Equations (3)-(5) summarize the main feature of preferences underlying (1). In particular, note from (5) that net substitutability takes a very simple structure (no special relationship exists between any couple of compensated demands *independently* from prices).

In addition, the expenditure function (3) has the Gorman *generalized* polar form, which implies that the Engel curves are linear: see e.g. Deaton-Muellbauer (1980: pp. 130-1 and 144-6). To see quasi homotheticity, solve (3) with respect to  $u$  to get the following indirect utility function:

$$V(\mathbf{p}, y) = \frac{k - e^{-\frac{\alpha y}{P(\mathbf{p})} + H(\mathbf{p})}}{\alpha}, \quad (6)$$

where  $y$  is the nominal expenditure and  $y/P$  is a real expenditure (quantity) index. By Hotelling's Lemma, the uncompensated demand for commodity  $i$  is given by:

$$x_i(\mathbf{p}, y) = -\frac{\partial V(\mathbf{p}, y)}{\partial p_i} / \frac{\partial V(\mathbf{p}, y)}{\partial y} = \frac{y}{P(\mathbf{p})} - \frac{1}{\alpha} (\ln p_i - \ln P(\mathbf{p}) - H(\mathbf{p})). \quad (7)$$

Note that (7) is linear with respect to  $y$  (all goods are normal) and that, if  $p_i = p$  for all  $i = 1, n$ , then  $x_i = y/(np)$ . On the contrary, in general the Marshallian demand function (7) will depend on  $\ln p_i$  and on the vector price  $\mathbf{p}$  both through  $P$  and  $H$ .

However, one can exploit the assumption of additive preferences (see e.g. Deaton-Muellbauer, 1980: pp. 138-42) to show that, if  $n$  is large enough, the marginal utility of outlay  $\lambda = \partial V / \partial y$  is approximately constant with respect to single prices. In particular, by manipulating the first-order condition  $e^{-\alpha x_i} = \lambda p_i$ , one get:

$$x_i(\mathbf{p}, y) = -\frac{1}{\alpha} (\ln p_i + \ln \lambda(\mathbf{p}, y)). \quad (8)$$

It can be shown that  $\partial \ln \lambda / \partial \ln y = -\alpha y / P$  and thus  $\varepsilon_i = \partial \ln x_i / \partial \ln y = y / (\alpha x_i P)$ . Note that the parameter  $\alpha$  is the reciprocal of the income elasticity of demand in any symmetric equilibrium with  $p_i = p$  for all  $i = 1, n$ . Since:

$$\theta_i(\mathbf{p}, y) = \frac{\partial \ln \lambda(\mathbf{p}, y)}{\partial \ln p_i} = -\frac{p_i}{P(\mathbf{p})} (\alpha x_i(\mathbf{p}, y) - 1), \quad (9)$$

it follows that, if  $p_i$  and  $x_i$  are not disproportionate,  $\theta_i$  is of the same order of magnitude as  $1/n$ . Thus (8) can be used as a Marshallian demand which is logarithmic in its own price, with:

$$\varepsilon_{ii}(\mathbf{p}, y) = \frac{\partial \ln x_i(\mathbf{p}, y)}{\partial \ln p_i} = -\frac{\theta_i(\mathbf{p}, y) + 1}{\alpha x_i(\mathbf{p}, y)} \approx -\frac{1}{\alpha x_i(\mathbf{p}, y)}, \quad (10)$$

$$\varepsilon_{ij}(\mathbf{p}, y) = \frac{\partial \ln x_i(\mathbf{p}, y)}{\partial \ln p_j} = -\frac{\theta_j(\mathbf{p}, y)}{\alpha x_i(\mathbf{p}, y)} \approx 0. \quad (11)$$

Of course, one might also *assume* that the influence of single prices on price indices  $P$  and  $H$  is negligible, leading to the same results (this might be appropriate in macroeconomic contexts or in monopolistic competition settings). In fact, one might start from assuming that the expenditure function (the indirect utility function) only depends on price indices, as exploited by Datta-Dixon (2000) and (2001) in the case of homothetic preferences.

Finally, note that (8) implies that the revenue function  $R(p_i) = p_i x_i$  is strictly concave with respect to  $p_i$ , and that there exists a finite *choke-off* price  $\bar{p}_i$  such that  $x_i = 0$  if  $p_i \geq \bar{p}_i$  (as it is also evident in (7)). This is sufficient to ensure that the problem of optimally setting  $p_i$  in order to maximize profit is well behaved. In fact, marginal revenue is decreasing with respect to  $x_i$  whenever demand is elastic (marginal revenue is positive only if  $\alpha x_i < 1$ ). Also note that, from (9) and (11), commodities  $i$  and  $j$  are gross complements if and only if  $\alpha x_j > 1$ .<sup>2</sup> To compare logarithmic quasi-homothetic preferences with CES (homothetic) preferences (see e.g. Dixit-Stiglitz, 1977, section I), notice that in the former case the (partial) Allen-Uzawa elasticities of substitution (which can be derived from (3)) are given by  $\sigma_{ij} = y/(\alpha x_j P)$ . Thus, similarly to the others, elasticities of substitution are not constant and not necessarily equal. Note that, as is well known, the condition of gross substitutability between commodities  $i$  and  $j$  is  $\sigma_{ij} > \varepsilon_j$ . Also notice that  $\sigma$  reduces to  $y/(\alpha x)$  in any symmetric equilibrium with  $p_i = p$  (and  $x_i = x$ ) for all  $i = 1, n$  (see Dixit-Stiglitz, 1977: p. 304).

### 3. The case of two commodities

To fully grasp the implication of (1), consider the case of only two goods. The Marshallian demand of commodity  $i$  (7) reduces to ( $i \neq j$ ,  $i, j = 1, 2$ ):

$$x_i(p_1, p_2, y) = \frac{\alpha y - p_j \ln \frac{p_i}{p_j}}{\alpha(p_1 + p_2)}. \quad (12)$$

$x_i(\cdot)$  is always decreasing and strictly concave with respect to  $p_i$  when  $x_i > 0$ . Notice that the *choke-off* price  $\bar{p}_i(p_j, y, \alpha)$  is given by:

$$\bar{p}_i(p_j, y, \alpha) = p_j e^{(\alpha y)/p_j}. \quad (13)$$

There might also exist a price  $\underline{p}_i$  such that (12) reduces  $x_i = y/p_i$  for all  $p_i \leq \underline{p}_i$  (commodity  $j$  is not bought at all). If it exists,  $\underline{p}_i(p_j, y, \alpha)$  is implicitly defined by the condition:

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<sup>2</sup> Notice that this condition is not symmetric due to non-homotheticity.

$$\ln p_j = \frac{\alpha y}{\underline{p}_i} + \ln \underline{p}_i, \quad (14)$$

which can be satisfied only if  $\alpha y e \leq p_j$  (note that if  $\underline{p}_i(p_j, y, \alpha)$  is well defined, it must be the case that  $\alpha y / p_j < 1$ ). This case is illustrated in Figure 1.

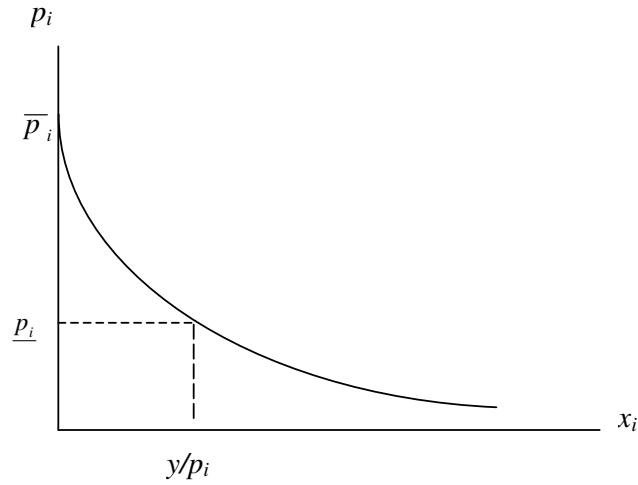


Figure 1: the direct demand curve

$x_i(\cdot)$  is increasing with respect to  $p_j$  (commodities are gross substitutes) only if  $\alpha x_j > 1$ , and in such a case the cross demand is also convex (note that this can be controlled by using the parameter  $\alpha$ ). On the contrary, cross demand is concave if:

$$\frac{p_j}{p_i} + 1 \geq \ln \frac{p_i}{p_j}. \quad (15)$$

Define  $\hat{p}_i(p_j, y, \alpha)$  by the condition  $x_i(\hat{p}_i, p_j, y) = 1/\alpha$ : two cases are possible, according to the existence of  $\underline{p}_i(p_j, y, \alpha)$ . They are illustrated in Figure 2.



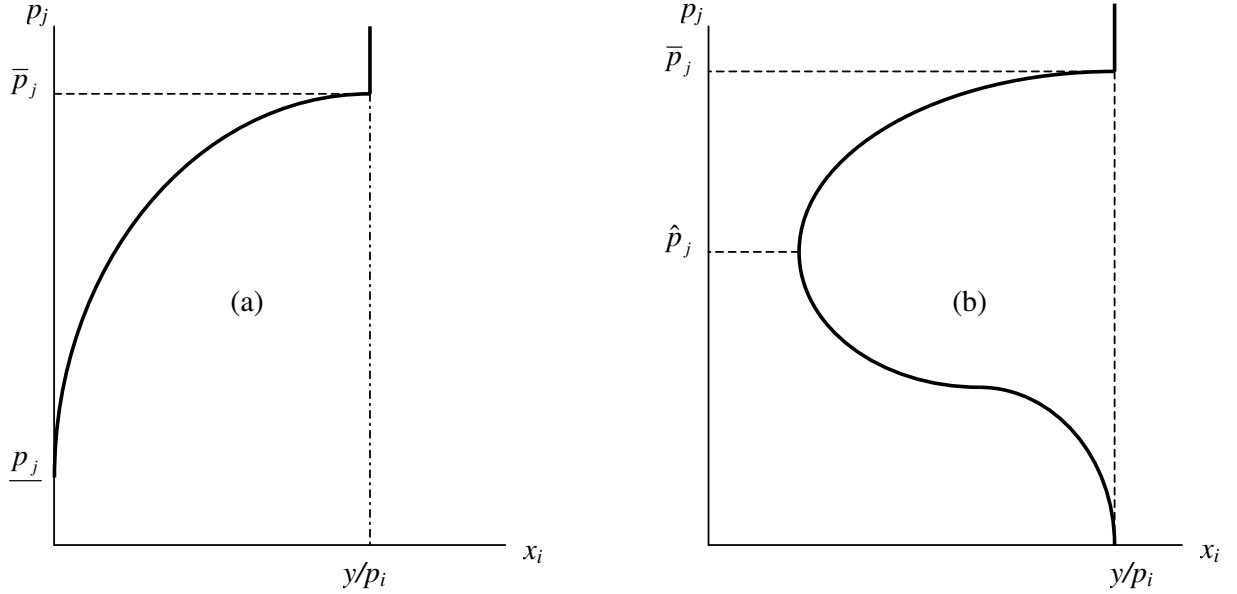


Figure 2: the cross demand curve  
 case (a):  $\underline{p}_j$  exists; case (b):  $\underline{p}_j$  does not exist

#### 4. Conclusion

In this paper we have studied the case of symmetric, logarithmic quasi-homothetic preferences. They can account for any number of goods and generate demand curves that are more general than the commonly used ones, which come from CES preferences. In particular, they have demand elasticities (both with respect to income and prices) which are not constant, and exhibit finite *choke-off* prices. Nevertheless, by exploiting the properties of additive preferences, we can argue that when the number of good is so large that the influence over the marginal value of income of single prices is negligible, the demands are just logarithmic with respect to the own price, and the revenue functions are well defined, computationally friendly and strictly concave.

Our results are related to those presented in Datta-Dixon (2000) and (2001), where homothetic preferences are developed which generate demands that depend on aggregate price indices and are simple functions of the own price. In our case, the relevant dual<sup>3</sup> functions depend on a straightforward price index and on a well-known index of price dispersion. Moreover, they also account for some non homothetic behaviour. Logarithmic quasi-homothetic preferences should then be useful in all the applications suggested by Datta-Dixon (2001), namely the analysis of pass-through with price-setting behaviour, and of (asymmetric) Cournot oligopoly. In addition, they can

<sup>3</sup> Differently from the case of Datta-Dixon (2001: pp. 161-2), we are able to present also the primal utility function representation for the class of logarithmic preferences.

be used in international trade models *à la* Krugman (1979) (see Bertolotti, 1998), and should also prove suitable for multi-sector growth and monopolistic competition settings.

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