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Stability, chaos and multiple attractors: A single agent makes a difference*

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Abstract

This paper provides an example in which a slight behavioral heterogeneity may fundamentally change the qualitative properties of a nonlinear cobweb market with a quadratic cost function and an isoelastic demand function. We consider two types of producers; adaptive and naive. In a market of naive agents a single adaptive agent stabilizes the otherwise exploding market. In a market of adaptive agents a single naive agent may destabilize the market; without him there exists at most one periodic attractor in the market but with him there may appear many coexisting periodic attractors of arbitrarily large periods.

JEL classification: D21; E32; C61

Key words: 2-D Nonlinear cobweb model; Behavioral heterogeneity; Chaos; Homoclinic bifurcations; Coexisting periodic attractors

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1 Introduction

A producer can choose his own way from the many available techniques to adjust production capacity and many different types of behavior coexist in reality. However, monotypic behavior dominates in economic theories. A representative agent typifies preferences and technologies as well as rational behavior of the whole society of agents.

One possible argument in favor of simplifying a model by assuming a 'representative rational agent' apparatus is that all the different behavior has already died out and only the representative agent survives [Lucas (1986)]. Evolutionary economics shows, however, that survival probabilities depend on the environment of agents and the selection mechanism [Axelrod (1984)]. Different types of behavior can survive simultaneously.

Another possible defense of assuming a representative agent is that a majority of the agents behaves in the same way and their behavior determines the dynamics of the market. In the stock market, however, a small group of risky traders could disturb the behavior of stock prices. The type of the market may determine whether the behavior of a majority determines the market outcome or whether the outcome depends on a minority of agents. The 'representative rational agent' is a theoretical apparatus that works with certainty only when all agents behave in the same way.

There are different behavioral techniques available and a dynamical process of switching to successful technologies seems to be plausible. However, even a unique superior technology does not necessarily extinguish all different types of technologies. At least one producer may sometimes behave differently. If the market is still in a phase of transition, this producer still uses the 'old' technology because he is a late adopter. If we are in a steady state, this producer tries a 'new' technology to improve profits. Thus heterogeneity, or diversity, of agents' behavior is a natural feature in our daily life, but not in traditional economics. Only recently, dynamical economics has considered heterogeneous agents [Gallegati/Kirman (1999), Delli-Gatti/Gallegati/Kirman (2000), Den-Haan (2001) and Kirman/Zimmermann (2001)]. The literature separates three different kinds of heterogeneity; personal characteristics like preferences or income, the way expectations are formed, and behavioral rules that agents use due to their bounded rationality. Some important results are already available.

In a growth model, agents with heterogenous preferences for income are examined by Cardak (1999); in a dynamic economy with progressive tax system heterogeneity in the rate of impatience is studied by Sarte (1997) and in an overlapping generation model heterogeneity in income and talent is analyzed by Chiu (1998). Heterogeneous general preferences in a perfect-foresight equilibrium of a finance-constrained economy allows Hopf cycles to be entirely consistent with a wide range of elasticities of substitution [Barinci (2001)].

Brock/Hommes (1997), Gaunersdorfer (2000) and Goeree/Hommes (2000) study dynamical models where agents update their expectations according to an observed measure such as net profits. Bomfin (2001) shows that if some agents solve their inference problems based on simple forecasting rules of thumb, there is a significant effect on the aggregate properties of the economy. In a cobweb model, where two different forecasting procedures are considered, either one destabilizes the price dynamics if it is uniformly adopted by all firms; or the price equilibrium becomes locally stable if firms are heterogeneous, and the two rules are suitably mixed within the population [Franke/Nesemann (1999)].

Day/Huang (1990), Chiarella (1992), Lux (1995) and Lux/Marchesi (2000) study how heterogeneous behavior of traders generates complex motion of financial markets. Cooper (1998) considers heterogeneity of agents in a standard stochastic growth model by assuming that agents react with different probabilities to current values of relevant state variables.

In the present paper we would like to investigate whether a slight behavioral heterogeneity could be a factor that generates complex dynamics of the market. We consider a nonlinear cobweb market with a quadratic cost function and an isoelastic demand function. Two types of producers' behavior are assumed; one is 'adaptive' and the other is 'naive'. An adaptive producer adjusts his output toward the profit-maximizing quantity as a target, while a naive producer produces the profit-maximizing quantity instantaneously.

We show that a single adaptive agent may change the complexity of the market behavior. If there is no adaptive agent and demand is inelastic enough for the market to explode, a single adaptive agent can stabilize the market in the sense that it would not explode, but only by causing chaos. On the other hand, when there are exclusively adaptive agents, there exists at most one periodic attractor for the market. If a single naive agent appears, then there may appear many (and even infinitely many) coexisting periodic attractors of arbitrarily large period.

2 Model

In this section, we derive a two-dimensional nonlinear cobweb model including naive and adaptive agents from a general, n-dimensional model including n-types of adaptive agents.

2.1 Definition of adaptive and naive behavior

Before presenting the model, we start by defining precisely the notion of 'naive' and 'adaptive' behavior. Let us consider the following profit-maximizing problem: to decide in period t the production $\tilde{x}_{t+1}$ for period $t+1$ subject to a quadratic cost function $ax^2/2,\ a > 0$ and naive price expectations, which state that his price expectation for the next period is equal to the current price p_t. The resulting quantity is

$$\tilde{x}_{t+1} = \frac{p_t}{a}.$$

If a supplier produces this quantity instantaneously, i.e.

$$x_{t+1} = \tilde{x}_{t+1}, \tag{1}$$

we call him a naive supplier.

On the other hand, as considered in Onozaki/Sieg/Yokoo (2000), an adaptive supplier adjusts his last period's production x_t in the direction of $\tilde{x}_{t+1}$. Thus his adjustment behavior is described by the following formula:

$$x_{t+1} = x_t + \alpha\left(\tilde{x}_{t+1} - x_t\right), \tag{2}$$

where $\alpha \in [0,1]$ is the speed of adjustment. If $\alpha = 1$, then Eq. (2) is identical to Eq. (1), which means that naive behavior is a special case of adaptive behavior where $\alpha = 1$.

2.2 General model

Let us consider a general model where there are N (a positive integer) types of adaptive suppliers[1]. All groups of suppliers share the same cost function considered above. Production $x_{i,t+1}$ in period $t+1$ of the i-th type of suppliers is determined by

$$x_{i,t+1} = (1-\alpha_i)x_{i,t} + \frac{\alpha_i p_t}{a}, \quad i = 1, 2, \cdots, N$$

where $\alpha_i \in [0,1]$ is a speed of adjustment of the i-th type of suppliers. Therefore, the aggregate supply per capita X_t at period t is given by

$$X_t = \sum_{i=1}^{N} n_i x_{i,t} \quad \text{with} \quad n_i \in [0,1] \quad \text{and} \quad \sum_{i=1}^{N} n_i = 1$$

where n_i is the relative size of the i-th group of suppliers.

We assume an inverse demand function which is isoelastic with a price elasticity of $1/\sigma$:

$$p_t = \frac{b}{Y_t^{\sigma}}, \quad b > 0, \quad \sigma > 0.$$

[1]Another possible introduction of heterogeneity is to assume that cardinality of types is a continuum represented as the unit interval $[0,1]$.

Equating the aggregate supply and demand, $X_t = Y_t$, gives an N -dimensional discrete-time dynamical system:

$$x_{i,t+1} = (1-\alpha_i)x_{i,t} + \frac{b\alpha_i}{a\left(\sum_{i=1}^{N} n_i x_{i,t}\right)^{\sigma}}, \quad i = 1, 2, \cdots, N.$$

Applying a variable transformation,

$$z_{i,t} := \left(\frac{a}{b}\right)^{\frac{1}{1+\sigma}} x_{i,t},$$

we obtain the final form:

$$z_{i,t+1} = (1-\alpha_i)z_{i,t} + \frac{\alpha_i}{\left(\sum_{i=1}^{N} n_i z_{i,t}\right)^{\sigma}}, \quad i = 1, 2, \cdots, N. \tag{3}$$

2.3 Reduced models

Using the general model (3), we can derive a 'standard', homogeneous cobweb model and an adaptive, homogeneous cobweb model [Onozaki/Sieg/Yokoo (2000)]. If we assume $N = 1$ and $\alpha = 1$, then (3) reduces to a one-dimensional, discrete-time dynamical equation

$$z_{t+1} = \frac{1}{z_t^{\sigma}}, \tag{4}$$

which preserves the properties of the standard cobweb model. The behavior of (4) depends on price elasticity; if price elasticity is greater than one ($\sigma < 1$), price trajectories converge to a stable fixed point $z^* = 1$. If price elasticity is equal to one ($\sigma = 1$), price trajectories exhibit 2-period cycles. However, if price elasticity is less than one ($\sigma > 1$), price trajectories oscillate and explode to infinity.

If we assume $N = 1$ and $\alpha \in [0, 1)$, then (3) reduces to a one-dimensional, discrete-time dynamical equation

$$z_{t+1} = (1-\alpha)z_t + \frac{\alpha}{z_t^{\sigma}}, \tag{5}$$

the behavior of which is studied by Onozaki/Sieg/Yokoo (2000) and Onozaki/Sawada (2001) to show that if $\sigma < (2-\alpha)/\alpha$, price trajectories converge to a unique stable fixed point $z^* = 1$. The fixed point undergoes a period doubling bifurcation at $\sigma = (2-\alpha)/\alpha$. If $\sigma > (2-\alpha)/\alpha$, price trajectories exhibit periodical cycles or chaos. Because $(2-\alpha)/\alpha > 1$, the last inequality implies that $\sigma > 1$. We can state that adaptive behavior prevents the unstable market from going to infinity, only by causing periodical cycles or chaotic behaviors. Adaptive behavior stabilizes the market in this sense.

However, the assumption that all agents behave homogeneously is unrealistic. To get a better picture of a cobweb market, we concentrate on a rather simple type of heterogeneity. We consider a model that is a little more general than (4) and (5) by including two categories of behavior; adaptive and naive. Reducing the N-dimensional model to a two-dimensional model makes not only analytical treatment but also the graphical depiction much easier and makes it possible to show the difference between one-dimensional and two-dimensional model.

It is easy to derive a two-type suppliers model from the general expression (3). Let us suppose $N = 2$, denote an adaptive supplier by $i = 1$ and a naive supplier by $i = 2$. The relative size of adaptive suppliers' group n_1 is replaced by m, so that the relative size of naive suppliers' group is $1 - m$. Since a naive supplier produces the profit-maximizing amount immediately, his adjustment speed α_2 is obviously unity. Letting $z_{1,t} = u_t$ and $z_{2,t} = v_t$ gives

$$u_{t+1} = (1-\alpha)u_t + \frac{\alpha}{[mu_t + (1-m)v_t]^{\sigma}}, \tag{6}$$

$$v_{t+1} = \frac{1}{[mu_t + (1-m)v_t]^{\sigma}}, \tag{7}$$

where $\alpha \in (0, 1)$ expresses the adjustment speed of the adaptive supplier.

3 Analysis of the model

The main purpose of this section is to show that heterogeneity in agents' production adjustment behavior can give rise to qualitatively different and more complicated dynamic features than those of the homogeneous production case.

Eliminating u's from Eqs. (6) and (7), we obtain the following second order difference equation:

$$v_{t+2}^{-\frac{1}{\sigma}} = (1-m)v_{t+1} - (1-m)(1-\alpha)v_t + (1-\alpha)v_{t+1}^{-\frac{1}{\sigma}} + \alpha m v_{t+1}. \tag{8}$$

Letting

$$x_t = v_t^{-\frac{1}{\sigma}} \quad \text{and} \quad x_{t+1} = y_t,$$

Eq. (8) can then be transformed into the two-dimensional dynamical system $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$, where

$$F(x, y) = (y, f(y) + (1-m)\, h(x, y)) \tag{9}$$

with

$$\begin{aligned} f(y) &= (1-\alpha)y + \alpha y^{-\sigma} \quad \text{and} \\ h(x, y) &= (1-\alpha)\left[y^{-\sigma} - x^{-\sigma}\right]. \end{aligned}$$

In order to indicate the dependence of F and f on the parameter σ and m, we sometimes write these as $F_{\sigma,m}$ and f_{σ}. In this section, the parameter $\alpha \in (0, 1)$ is assumed to be always arbitrarily fixed in $(0, 1)$.

3.1 Some implications of homoclinic bifurcation

It has been widely known in economic literature that complex dynamics can arise via homoclinic bifurcations; see, e.g., Palis/Takens (1991) for a detailed mathematical treatment of this subject. We will show that the economic system given by (9) has a hyperbolic fixed saddle whose stable and unstable manifolds have homoclinic tangencies that unfold generically. As a result, the system is shown to exhibit complex dynamics such as strange attractors, infinitely many periodic attractors, creation of horseshoes, and cascades of period-doubling sequences.

Note first that the map $F_{\sigma,m}$ in (9) has a unique fixed point $p = (1, 1)$, which is independent of parameters. For a suitable choice of parameters, p is a dissipative hyperbolic saddle. Let us denote by $W^s_{\sigma,m}(p)$ the stable manifold of the fixed point p for the map $F_{\sigma,m}$. Similarly, $W^u_{\sigma,m}(p)$ denotes the unstable manifold of p for $F_{\sigma,m}$.

LEMMA HT (HOMOCLINIC TANGENCY LEMMA): *There exists $\varepsilon \in (0, 1)$ such that for any $m \in (\varepsilon, 1)$ and for some $\hat{\sigma} = \hat{\sigma}(m)$, the map $F_{\hat{\sigma},m}$ has the following properties:*

(i) *the fixed point p is a dissipative hyperbolic saddle, i.e., the Jacobian matrix $DF_{\hat{\sigma},m}(p)$ has two real eigenvalues λ_1 and λ_2 such that $|\lambda_1| > 1$, $0 < |\lambda_2| < 1$ and $|\lambda_1\lambda_2| < 1$;*

(ii) *the stable manifold $W^s_{\hat{\sigma},m}(p)$ and the unstable manifold $W^u_{\hat{\sigma},m}(p)$ have a quadratic homoclinic tangency that unfolds generically with respect to σ.*

PROOF: See Appendix 6.1.
The relation between stable and unstable manifolds as a function of σ is depicted in Fig. 1.

*** Fig. 1(a)-(c) about here ***

The dynamical complexities stated in the following proposition are due to homoclinic bifurcations. For complex dynamics due to homoclinic bifurcations in an overlapping generations model, see e.g. de Vilder (1996).

PROPOSITION 1 (COMPLEX DYNAMICS): *Take $m \in (\varepsilon, 1)$ and $\hat{\sigma}$ as in Lemma HT. Let $\delta > 0$ be a sufficiently small number and let $I = (\hat{\sigma}-\delta, \hat{\sigma}+\delta)$. Then the following holds:*

(i) *There exists an interval $H \subset I$ such that for each $\sigma \in H$, $F_{\sigma,m}$ has a horseshoe. That is, there exists an $F_{\sigma,m}$-invariant set Λ_σ on which $F_{\sigma,m}$ has infinitely many saddle-type periodic orbits of arbitrarily large period;*

(ii) *There exists a set of σ-values $E \subset I$ with positive Lebesgue measure such that for each σ, $F_{\sigma,m}$ has a strange attractor;*

(iii) *There exists a sequence $\{\sigma_n\} \subset I$ with $\sigma_n \to \hat{\sigma}$ as $n \to \infty$ such that for each σ_n, $F_{\sigma_n,m}$ undergoes a period-doubling bifurcation;*

(iv) *For each $k \geq 1$, there exists an interval $J_k \subset I$ such that for each $\sigma \in J_k$, $F_{\sigma,m}$ has at least k coexisting periodic attractors. Furthermore, there exist infinitely many subintervals $I_n \subset I$ and a dense subset $N_n \subset I_n$ such that for each $\sigma \in N_n$, $F_{\sigma,m}$ has infinitely many periodic attractors of arbitrarily large period.*

PROOF: See, e.g., Palis/Takens (1993, Chapter 2) for (i), Mora/Viana (1993) for (ii), Yorke/Alligood (1983) for (iii), and Robinson (1983) for (iv).

3.2 A single heterogeneous agent makes a difference

Once a single agent (to be more precise, a sufficiently small fraction of agents) of a different type is put into a homogeneous group, what will happen in the market? We will show that such a single heterogeneous agent may drastically change the qualitative dynamic feature of a market.

First, we show that if there is no adaptive agent and demand is inelastic enough for the market to explode, a single adaptive agent can stabilize the market in the sense that it would not explode.

PROPOSITION 2 (A SINGLE ADAPTIVE AGENT MAKES A DIFFERENCE): *For $m = 0$ and $\sigma > 1$, the trajectory of F for any initial condition $(u_0, v_0) \in \mathbb{R}^2_{++}$ explodes unless $v_0 = 1$. On the other hand, for $m \in (0, 1]$, every trajectory starting from $\mathbb{R}^2_{++}$ is trapped into a compact region in $\mathbb{R}^2_{++}$.*

PROOF: See Appendix 6.2.

Conversely, if a single naive agent appears in a market where there exist exclusively adaptive agents, then there may appear many (and even infinitely many) coexisting periodic attractors in the market. Multiplicity of attractors cannot occur in a market solely occupied by adaptive agents.

PROPOSITION 3 (A SINGLE NAIVE AGENT MAKES A DIFFERENCE): *For $m = 1$, there exists at most one periodic attractor for the map $F_{\sigma,1}$. On the other hand, for any $m < 1$ sufficiently close to 1 and for any $k \geq 1$, there* *exists an interval J_k of σ-values such that for each $\sigma \in J_k$, $F_{\sigma,m}$ has at least k coexisting periodic attractors. Furthermore, for $m < 1$ sufficiently close to 1, there exist intervals $\{I_i\}_{i=1}^{\infty}$ of σ-values and dense subsets $\{N_i \subset I_i\}$ such that for each $\sigma \in N_i$, $F_{\sigma,m}$ exhibits infinitely many coexisting periodic attractors of arbitrarily large period.*

PROOF: See Appendix 6.3.

4 Numerical simulations

In this section we present some results of numerical simulations of Eq. (6) and (7) and discuss their implications.

First, we show the graph of a strange attractor for the parameter constellation $(\alpha, \sigma, m) = (0.2, 8.1, 0.8)$ in Fig. 2, which exhibits a fractal structure.

*** Fig. 2 about here ***

A one-parameter bifurcation diagram with respect to m is depicted in Fig. 3 and a part of it is enlarged into Fig. 4. From PROPOSITION 2 we can state that if there appears one adaptive supplier in an otherwise unstable cobweb market ($m = 0$ and $\sigma > 1$) then the market will not explode but will begin to behave chaotically. Furthermore, from Fig. 3 it is observed that as the relative size m of adaptive suppliers increases, the amplitude of price trajectory gets smaller. In these senses, adaptive behavior stabilizes a cobweb market.

*** Fig. 3-4 about here ***

The model has three key parameters, α, σ and m, so that we can draw two-parameter bifurcation diagrams with one of them fixed. These diagrams are depicted in Fig. 5-7, where each color corresponds to period's number of cycles as shown in the table in Fig.5. The red area exhibits pairs of parameter values for which every trajectory converges to a unique stable fixed point. The orange area consists of pairs of parameter values for which every trajectory converges to a period-2 cycle. The mustard-colored area corresponds to a period-3 cycle, the yellow area corresponds to a period-4 cycle, the emeraldine area corresponds to a period-6 cycle, and the light-blue area corresponds to a period-8 cycle, etc. The black area corresponds to a cycle of period more than 16. For the set of parameters which belongs to the white area, our model exhibits observable chaos in the sense of a positive Lyapunov exponent.

*** Fig. 5-7 about here ***

Among the factors which determine whether the trajectories are chaotic, the most important parameters are speed of adjustment and price elasticity of demand. The faster the speed of adjustment and the less elastic the demand, the more likely the price fluctuates chaotically. The relative size m of adaptive agents, compared to these two parameters, influences the situation only slightly. The relatively larger the size of adaptive agents, the less likely it is that the price fluctuates chaotically. This result confirms the importance of adaptive behavior and price elasticity of demand already stressed by Onozaki/Sieg/Yokoo (2000).

The theoretical fact that introducing naive agents may change market behavior is also observed in numerical simulations. Fig. 8 depicts a two-parameter bifurcation diagram for the reduced model (5) where there are exclusively adaptive agents, while Fig. 5 depicts the case where the relative size of naive agents is 20 percent. Comparing Fig. 5 to 8 shows that there is a crucial difference between one-dimensional model and two-dimensional model in the process of how bifurcations occur; in Fig. 5 the red area of convergence is smaller and partly replaced by the orange area of alternating prices. More importantly, as a result of introducing naive agents, the extreme bottom-left chaotic area (the first window of period-doubling bifurcations) in Fig. 5 is larger, and within the chaotic area there appear many 'fishhooks'[2].

*** Fig. 8 about here ***

One part of Fig. 5 including some fishhooks is enlarged and shown in Fig. 9. The geometrical shapes in Fig. 9 are also observed in the first windows of Fig. 6 and 7. Striking features of the fishhooks are that (1) their shapes

[2] The word 'fishhook' was first used in Fraser/Kapral (1982).

are similar in the same diagram, (2) they basically consist of a combination of regions exhibiting period-doubling bifurcations, and (3) they accumulate in increasing order of period as σ increases or m decreases.

*** Fig. 9 about here ***

In order to clarify the structure of a fishhook, it is useful to calculate bifurcation curves, which are the loci in the parameter space where the system has a periodic orbit exhibiting the relevant bifurcation. Bifurcation curves related to a period-5 orbit, projected on the (α, σ) -plane, are depicted in Fig. 10 where thick curves correspond to saddle-node (or fold or tangent) bifurcations and thin curves correspond to period-doubling bifurcations.

*** Fig. 10 about here ***

It is noted that the two thick curves in Fig. 10 touch each other at their tips and form a cusp-like shape. To understand the formation of the cusp, let us employ the concept of *periodic point surface* coined by Sannami (1994). Let n be a positive integer and $G_n : \mathbb{R}^5_+ \to \mathbb{R}^2$ be a map defined by $G_n(x, y, \alpha, \sigma, m) = F^n(x, y) - (x, y)$. As $G_n = (0, 0)$ denotes a periodic point of period-n, a point on $K_n = G_n^{-1}(0, 0)$ indicates the coordinate of a periodic point and the corresponding parameters. K_n is called the periodic point surface in $\mathbb{R}^5_+$. The periodic point surface K_5 near the cusp point in Fig. 10 is conceptually represented in Fig. 11. Thick bifurcation curves in Fig. 10 are the projections of the folds of the periodic point surface K_5 on the (α, σ)-plane.

*** Fig. 11 about here ***

Fig. 12 and 13 depict basins of attraction for different constellations of parameters, which again exhibit a fractal structure. There coexist initial points such that every trajectory starting from these converges to either period-12 or -30 cycles in Fig. 12 and either period-10 or -18 cycles in Fig. 13. Colors in these figures follow the order shown in the table in Fig. 5, but colors for period-n ($n > 16$) cycles are assigned to the same color as those for period-$(n - 16)$.

*** Fig. 12 and 13 about here ***

5 Conclusion

We have investigated the dynamics of a nonlinear, two-dimensional cobweb model which contains two types of heterogeneous agents—adaptive and naive suppliers. Even a single heterogeneous agent may change the qualitative behavior of the market. If there are exclusively naive agents and demand is inelastic enough for the market to explode, a single adaptive agent can stabilize the market in the sense that it would not explode, but only by causing chaos. On the other hand, when there are exclusively adaptive agents, there exists at most one periodic attractor for the market. If a single naive agent appears in such a market, then there may appear many (and even infinitely many) coexisting periodic attractors of arbitrarily large period.

Onozaki/Sieg/Yokoo (2000) hypothesis states that in a market with exclusively adaptive agents, low price elasticities and fast adjustment may cause the market to behave chaotically. In this paper, we extend this hypothesis so as to hold for a market with heterogenous agents. More importantly, however, market behavior is not determined by the behavior of a majority of agents but even a single heterogenous agent may have a profound impact on the qualitative behavior of the market. Therefore, the assumption that the theoretical concept of homogenous agents is an appropriate approximation of the reality of heterogenous agents, which is common in traditional economic theory, seems questionable. Heterogeneity, or diversity, of agents may be the mother of rich dynamics and therefore possibly the source of stability, oscillation and chaos.

6 Appendix

6.1 Proof of Lemma HT

We first consider the case where $m = 1$. In this case, the map $F_{\sigma,m}$ given in (9) reduces to the following singular (thus noninvertible) map:

$$F_{\sigma,1}(x, y) = (y, f_\sigma(y)).$$

The map $F_{\sigma,1}$ is clearly equivalent to the one-dimensional map f_σ in the sense that f_σ on $\mathbb{R}$ is topologically conjugate to $F_{\sigma,1}$ on $Im(F_{\sigma,1})$ through the conjugacy $\varphi(x) = (x, f_\sigma(x))$. The dynamics of f_σ was studied in Onozaki/Sieg/Yokoo (2000) where f_σ was shown to be strictly convex and unimodal with its global minimum at

$$\theta = \left(\frac{\alpha\sigma}{1-\alpha}\right)^{\frac{1}{1+\sigma}}.$$

That is, $f'_\sigma(\theta) = 0$ and $f''_\sigma(x) > 0$ for every $x > 0$.

We can see that if σ is large enough and m is close enough to unity (depending on σ), then a unique fixed point $p = (1,1)$ of $F_{\sigma,m}$ is a hyperbolic and dissipative saddle. More precisely, we state

LEMMA 1 (LEMMA HT (i)): *Given $\alpha \in (0,1)$, if $\sigma > (2-\alpha)/\alpha$ then there is a $\delta = \delta(\sigma) \in (0,1)$ such that for $m \in (\delta, 1)$ the two eigenvalues λ_1 and λ_2 of the Jacobian matrix $DF_{\sigma,m}$ evaluated at $p = (1,1)$ satisfy (i) $0 < |\lambda_1 \lambda_2| = (1-m)\sigma < 1$ and (ii) $|\lambda_1| > 1$ and $0 < |\lambda_2| < 1$.*

PROOF OF LEMMA 1: Since

$$DF_{\sigma,m}(p) = \begin{pmatrix} 0 & 1 \\ (1-m)(1-\alpha)\sigma & 1-\alpha-\alpha\sigma-(1-m)(1-\alpha)\sigma \end{pmatrix},$$

for $m = 1$ we have $\lambda_1 = 1-\alpha-\alpha\sigma$ and $\lambda_2 = 0$. Thus, by the continuity of eigenvalues with respect to m, if $\sigma > (2-\alpha)/\alpha$ then (i) and (ii) hold for m sufficiently close to 1. □

We will denote the globally stable (unstable) manifold of the fixed point $p = (1,1)$ of the map $F_{\sigma,m}$ by $W^s_{\sigma,m}(p)$ ($W^u_{\sigma,m}(p)$, respectively). We will abuse this notation for the singular case when $m = 1$.

LEMMA 2: *For any $\alpha \in (0,1)$ and for any sufficiently large σ, the unstable manifold $W^u_{\sigma,1}(p)$ contains an arc $\gamma^u_{\sigma,1} = \{(x,y) : x \in [f_\sigma(\theta), f^2_\sigma(\theta)], y = f_\sigma(x)\}$.*

PROOF OF LEMMA 2: Note that from Lemma 3 in Onozaki/Sieg/Yokoo (2000), we know that $f_\sigma(\theta) < 1 < \theta < f^2_\sigma(\theta)$ for sufficiently large σ. The assertion of Lemma 2 follows from the fact that $f_\sigma([f_\sigma(\theta), \theta]) \supset [\theta, f^2_\sigma(\theta)]$ and from Lemma 2 in Onozaki/Sieg/Yokoo (2000). □

LEMMA 3: *For $\alpha \in (0,1)$ and $m = 1$, the stable manifold $W^s_{\sigma,1}(p)$ contains a horizontal line segment (depending on σ)*

$\gamma^s_{\sigma,1} = \{(x,y) : x \in [f_\sigma(\theta), f^2_\sigma(\theta)],\ y = \hat{q}(\sigma)$ with $f^n(\hat{q}) = 1$ for some $n > 1\}$

with the property that

(P1) *$\gamma^s_{\sigma_1,1}$ and $\gamma^u_{\sigma_1,1}$ have no intersection;*

(P2) *$\gamma^s_{\sigma_2,1}$ and $\gamma^u_{\sigma_2,1}$ have two transverse intersections for some σ_1 and σ_2 with $\sigma_1 < \sigma_2$.*

PROOF OF LEMMA 3: From the proof of Proposition 2 in Onozaki/Sieg/Yokoo (2000), we know that there are σ_1 and σ_2 such that $f^2_{\sigma_1}(\theta(\sigma_1)) < \tilde{q}(\sigma_1)$ and

$f^2_{\sigma_2}(\theta(\sigma_2)) > \tilde{q}(\sigma_2)$, where $\tilde{q}$ is an eventual fixed point with $\tilde{q} > \theta$. This implies that $f_{\sigma_1}(\theta(\sigma_1)) > \hat{q}(\sigma_1) = f^{-1}_{\sigma_1}(\tilde{q}(\sigma_1)) \cap (0, \theta(\sigma_1))$ and $f_{\sigma_2}(\theta(\sigma_2)) < \hat{q}(\sigma_2) = f^{-1}_{\sigma_2}(\tilde{q}(\sigma_2)) \cap (0, \theta(\sigma_2))$. Since $f(\theta) \leq f(x)$ for $x > 0$, (P1) follows. Since $f(x) \to +\infty$ as $x \to 0$ and $x \to +\infty$ and since f is strictly decreasing on $(0, \theta)$ and strictly increasing on (θ, ∞), $\gamma^s_{\sigma_2,1}$ and $\gamma^u_{\sigma_2,1}$ have two (and only two) intersections by continuity. They are transverse because $f'(x) \neq 0$ for $x \neq \theta$, which proves (P2). □

The above situation is called *inevitable tangency* coined by Takens (1992). See Fig. 14.

*** Fig. 14(a)-(b) a bout here ***

Now we will perturb the singular map $F_{\sigma,1}$ into non-singular maps by making m slightly smaller.

LEMMA 4 (LEMMA HT (ii)): *Let σ_1 and σ_2 be as in Lemma 3. Then there exists $\varepsilon \in (\delta, 1)$ such that for every $m \in (\varepsilon, 1)$, the map $F_{\sigma,m}$ has arcs $\hat{\gamma}^s{}_{\sigma,m} \subset W^s_{\sigma,m}(p)$ and $\hat{\gamma}^u{}_{\sigma,m} \subset W^u_{\sigma,m}(p)$ satisfying the following:*

(i) $\hat{\gamma}^s{}_{\sigma_1,m} \cap \hat{\gamma}^u{}_{\sigma_1,m} = \phi$;

(ii) $\hat{\gamma}^s{}_{\sigma_2,m}$ *and* $\hat{\gamma}^u{}_{\sigma_2,m}$ *have two transverse intersections, and*

(iii) *For some $\sigma^* \in (\sigma_1, \sigma_2)$, $\hat{\gamma}^s{}_{\sigma^*,m}$ and $\hat{\gamma}^u{}_{\sigma^*,m}$ have a quadratic homoclinic tangency that unfolds generically with respect to σ.*

PROOF OF LEMMA 4: Since the fixed point p of the singular map $F_{\sigma,1}$ is hyperbolic (eigenvalue of zero is allowed), the non-singular map $F_{\sigma,m}$ for m sufficiently close to 1 has an arc $\hat{\gamma}^u{}_{\sigma,m} \subset W^u_{\sigma,m}(p)$ which is C^r-close ($r \geq 0$) to $\gamma^u_{\sigma,1}$ (obtained in Lemma 2) by continuous dependence of the unstable manifold on $F_{\sigma,m}$ in the C^r topology. Furthermore, it is easily seen that the arc $\gamma^s_{\sigma,1}$ consists of regular points, i.e., $Im(DF^n_{\sigma,1}(x)) + T_{F^n_{\sigma,1}(x)}(\gamma^s_{\sigma,1}) = \mathbb{R}^2$ for every $x \in \gamma^s_{\sigma,1}$ and for n such that $F^n_{\sigma,1}(x) = p$. Thus by Proposition 1 in Appendix 4 in Palis and Takens (1993, p.182), the non-singular map $F_{\sigma,m}$ for m sufficiently close to 1 has an arc $\hat{\gamma}^s{}_{\sigma,m} \subset W^s_{\sigma,m}(p)$ which is C^r-close ($r \geq 0$) to $\gamma^s_{\sigma,1}$. By stability of transversality and by Lemma 3, the inevitable tangency conditions, (i) and (ii), follow. Evidently, the existence of σ^* for which $\hat{\gamma}^s{}_{\sigma^*,m}$ and $\hat{\gamma}^u{}_{\sigma^*,m}$ have a homoclinic tangency follows from the inevitable tangency. By Takens' weakened generic conditions [see Takens

(1992)] for analytic diffeomorphisms, we immediately have the quadraticity and the generic unfolding of the homoclinic tangency because the ratio $-\log(\lambda_2(\sigma))/\log(\lambda_1(\sigma))$ of eigenvalues of $DF_{\sigma,m}(p)$ is clearly non-constant with respect to σ. $\square$

6.2 Proof of Proposition 2

(i): Since $v_t = v_0^{(-\sigma)^t}$, we have $\limsup_{t\to+\infty} v_t = +\infty$ for $v_0 > 0$ and $v_0 \neq 1$. Furthermore, if $v_0 = 1$, then $u_{t+1} = (1-\alpha)u_t + \alpha$ and thus $u_t \to 1$ as $t \to +\infty$ for any u_0.

(ii): Suppose first that the (positive) sequence $\{v_t\}$ is eventually bounded in the sense that there exist an integer K (depending on v_0 and u_0) and $\bar{v} < +\infty$ such that $0 < v_t \leq \bar{v}$ for any $t > K$. Noting that $u_{t+1} = (1-\alpha)u_t + \alpha v_{t+1} \leq (1-\alpha)u_t + \alpha\bar{v}$ for $t > K$, we obtain

$$\begin{aligned} u_t &= (1-\alpha)^{t-K}u_K + \alpha\left[\sum_{i=0}^{t-K-1}(1-\alpha)^i v_{t-i}\right] \\ &\leq (1-\alpha)^{t-K}u_K + \alpha\bar{v}\left[\sum_{i=0}^{t-K-1}(1-\alpha)^i\right] \\ &= (1-\alpha)^{t-K}u_K + \bar{v}\left[1-(1-\alpha)^{t-K}\right] \\ &\longrightarrow \bar{v} \quad \text{as} \quad t \to +\infty. \end{aligned}$$

Thus it suffices to show that $\{v_t\}$ is eventually bounded. Suppose not. Then, for any $\varepsilon' > 0$, there exists an integer K such that $v_{K+3} > \varepsilon'$ or

$$0 < mu_{K+2} + (1-m)v_{K+2} < (\varepsilon')^{-1/\sigma} \equiv \varepsilon,$$

which implies

$$u_{K+2} < \frac{\varepsilon}{m} \quad \text{and} \quad v_{K+2} < \frac{\varepsilon}{1-m}.$$

Thus we get

$$(1-\alpha)u_{K+1} + \frac{\alpha}{[mu_{K+1} + (1-m)v_{K+1}]^\sigma} < \frac{\varepsilon}{m} \quad \text{and} \tag{10}$$

$$\frac{1}{[mu_{K+1} + (1-m)v_{K+1}]^\sigma} < \frac{\varepsilon}{1-m}. \tag{11}$$

From (10) we obtain

$$u_{K+1} < \frac{\varepsilon}{m(1-\alpha)}. \tag{12}$$

Rearranging (11) gives

$$\left[\frac{1-m}{\varepsilon}\right]^{\frac{1}{\sigma}} < mu_{K+1} + (1-m)v_{K+1}. \tag{13}$$

Combining (12) and (13) we obtain

$$v_{K+1} > \frac{1}{1-m}\left[\frac{1-m}{\varepsilon}\right]^{\frac{1}{\sigma}} - \frac{\varepsilon}{(1-\alpha)(1-m)} \equiv \Delta(\varepsilon). \tag{14}$$

From (12) and (14), it follows that

$$\begin{aligned}\frac{\varepsilon}{m(1-\alpha)} &> u_{K+1} = (1-\alpha)u_K + \frac{\alpha}{[mu_K + (1-m)v_K]^{\sigma}} \\ &> (1-\alpha)u_K + \alpha v_{K+1} \\ &> (1-\alpha)u_K + \alpha\Delta(\varepsilon) \\ &> \alpha\Delta(\varepsilon).\end{aligned}$$

Hence we obtain

$$\varepsilon > m\alpha(1-\alpha)\Delta(\varepsilon).$$

Since $\Delta(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, we get a contradiction. This completes the proof. □

6.3 Proof of Proposition 3

(i): Note first that the Schwarzian derivative of f_σ is given by[3]

$$Sf_\sigma(x) = -\frac{\alpha\sigma(1+\sigma)\left[\alpha\sigma(\sigma-1) + 2(1-\alpha)(2+\sigma)x^{1+\sigma}\right]}{2\left[(\alpha-1)x^{2+\sigma} + \alpha\sigma x\right]^2}.$$

We see that

$$Sf_\sigma(x) < 0 \qquad \text{for} \quad \sigma \geq 1 \quad \text{and} \quad x > 0.$$

Therefore, by Singer's theorem [Singer (1978)], f has at most one periodic attractor for $\sigma \geq 1$, and so does $F_{\sigma,1}$.

Now consider the case of $\sigma \in (0,1)$. Since we have a unique fixed point at $x = 1$, it is sufficient to show that f has no periodic attractor of period greater than or equal to 2. Since Sarkovskii's ordering says that for any continuous map on an interval, the existence of a periodic point of period greater than 2 implies that of period-2, all we need to do is to show that f

[3]See Onozaki/Sieg/Yokoo (2000), p.108.

cannot have a periodic point of period-2 for $\sigma \in (0,1)$. For this we define a strictly decreasing function l by

$$l(x) = \frac{1}{x^\sigma} \qquad \text{for} \quad \sigma \in (0,1).$$

Since

$$f(x) - l(x) = (1-\alpha)\left[\frac{x^{1+\sigma}-1}{x^\sigma}\right],$$

we have

$$f(1) = l(1) = 1, \quad f(x) < l(x) \quad \text{for} \quad x \in (0,1), \quad \text{and} \quad f(x) > l(x) \quad \text{for} \quad x > 1.$$

Since $\sigma \in (0,1)$, we can easily see that

$$x < l^2(x) \quad \text{for} \quad x \in (0,1) \quad \text{and} \quad l^2(x) < x \quad \text{for} \quad x > 1.$$

Suppose that the map f has a (least) periodic point p of period-2, that is, $f^2(p) = p$ and $f(p) \neq p$. Suppose $p > 1$. (For $p \in (0,1)$, the same argument applies.) Since $f(p) > l(p)$ and l is strictly decreasing, we have

$$l^2(p) > l(f(p)). \tag{15}$$

In order for p to be a periodic point of period-2, it is necessary that $f(p) < 1$ holds. To see this, note that it would otherwise imply that either $f(p) > p > 1$ or $p > f(p) > 1$. Let $g(x) = f(x) - x$. For the case of $f(p) > p > 1$, we obtain $g(p) = f(p) - p > 0$ and $g(f(p)) = f^2(p) - f(p) = p - f(p) < 0$, which implies that there is a fixed point $q \in [p, f(p)]$ of f. This contradicts the uniqueness of a fixed point of f. Similarly, for $p > f(p) > 1$, we get another contradiction. Since $f(p) < 1$ and $f(x) < l(x)$ for $x \in (0,1)$, we have

$$l(f(p)) > f(f(p)) = p. \tag{16}$$

Thus, combining (15) and (16) together with the inequality $x > l^2(x)$ for $x > 1$, we finally obtain

$$p > l^2(p) > l(f(p)) > f^2(p) = p,$$

a contradiction. This completes the proof of part (i).

(ii): See (iv) in Proposition 1. $\square$

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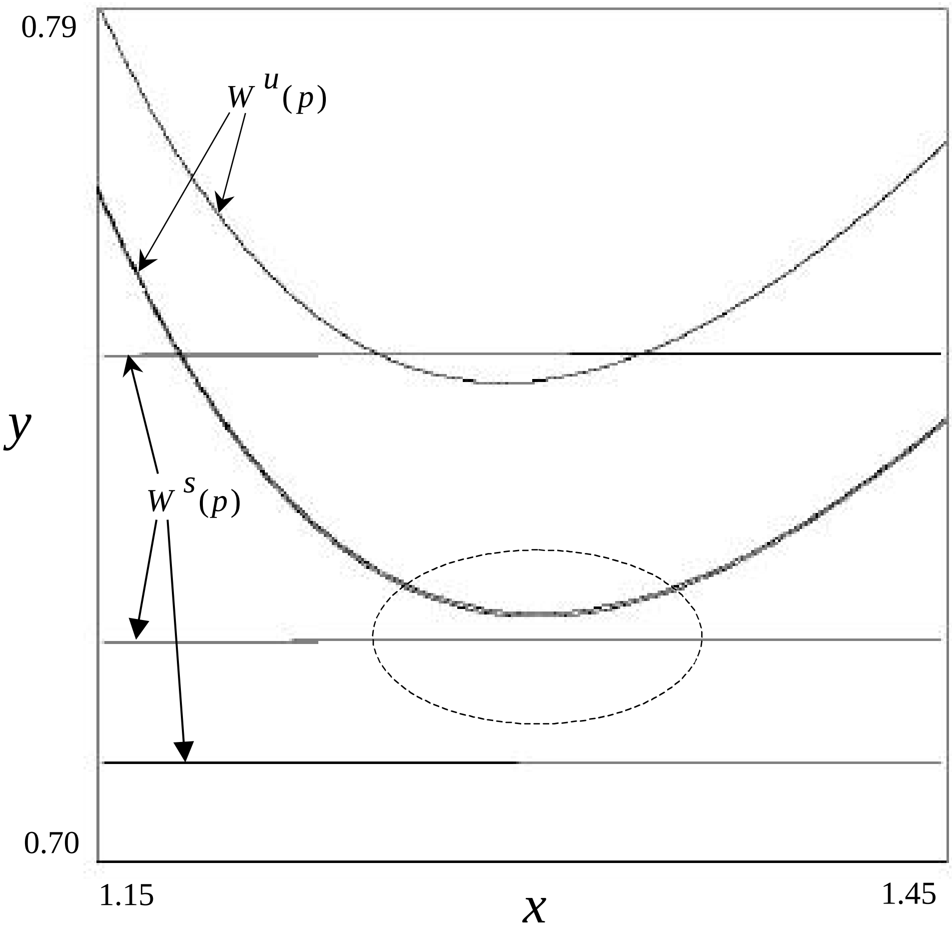
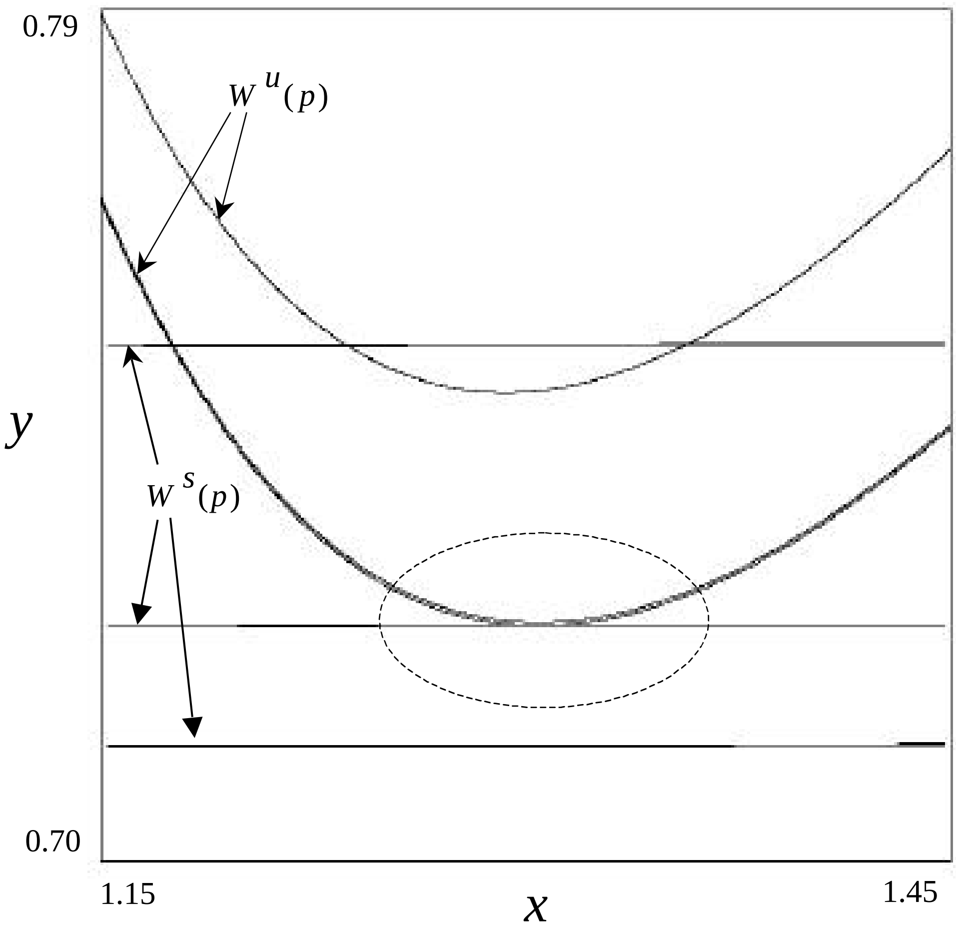


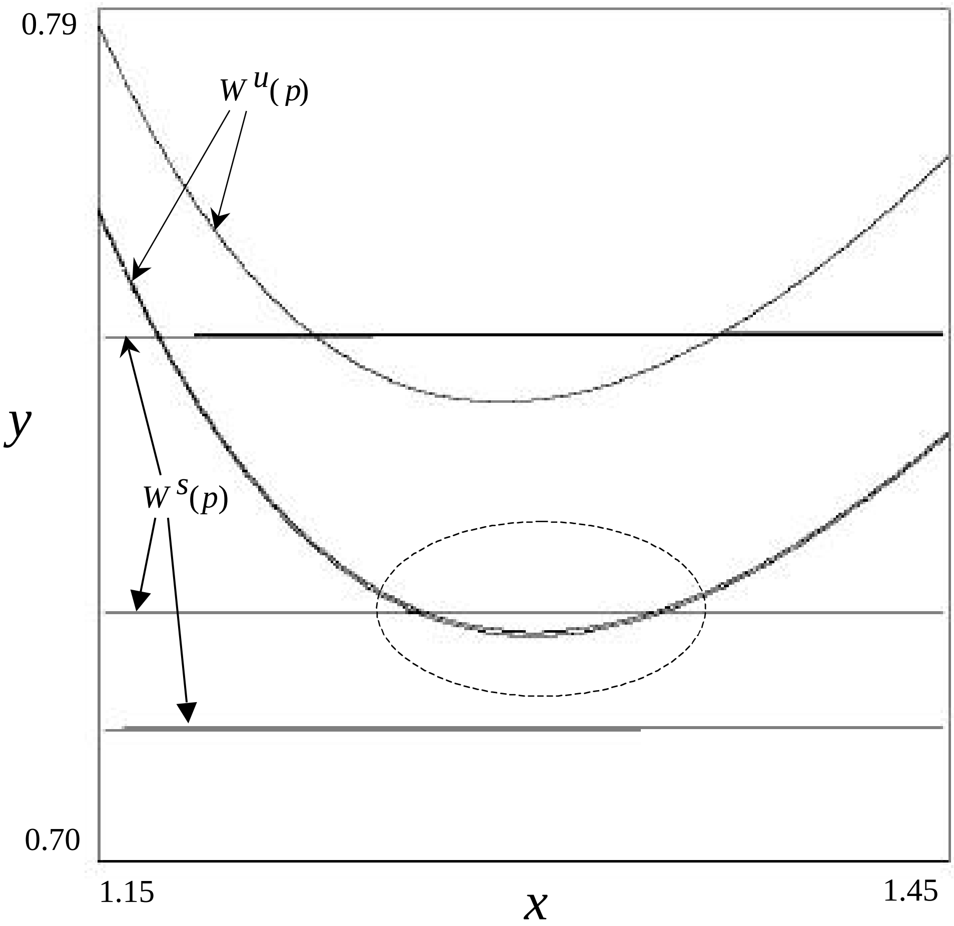
(a) Case of no intersection:

$\alpha = 0.5$ and $m = 0.97$; $\sigma = 6.22$.



(b) Case of homoclinic tangency:

$\alpha = 0.5$ and $m = 0.97$; $\sigma = 6.26$.



(c) Case of transverse intersections:

$\alpha = 0.5$ and $m = 0.97$; $\sigma = 6.30$.

Fig. 1. Relations between stable and unstable manifolds as a function of σ.

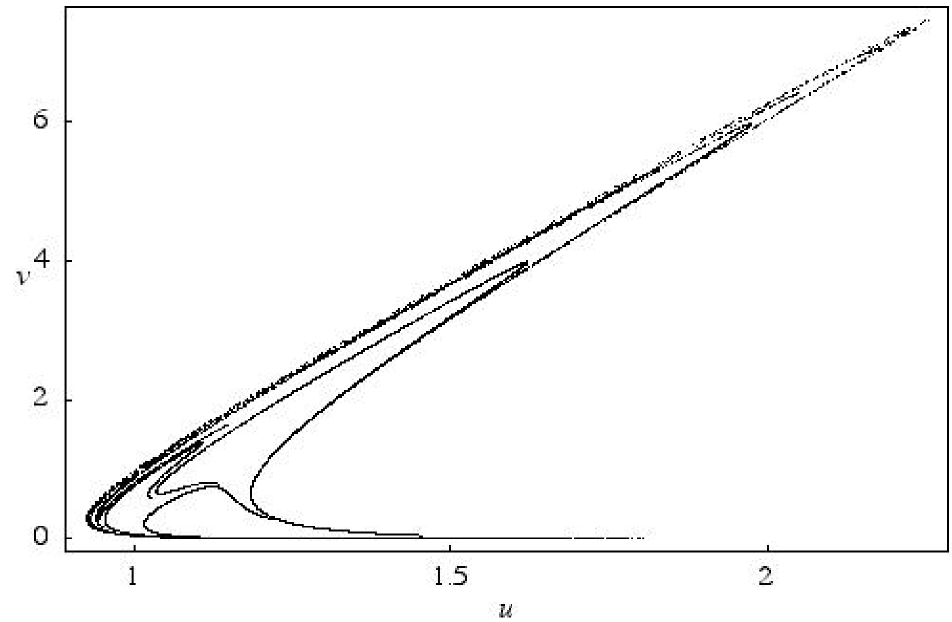


Fig. 2. Strange attractor for Eq. (6) and (7): $\alpha = 0.2$, $\sigma = 8.1$ and $m = 0.8$.

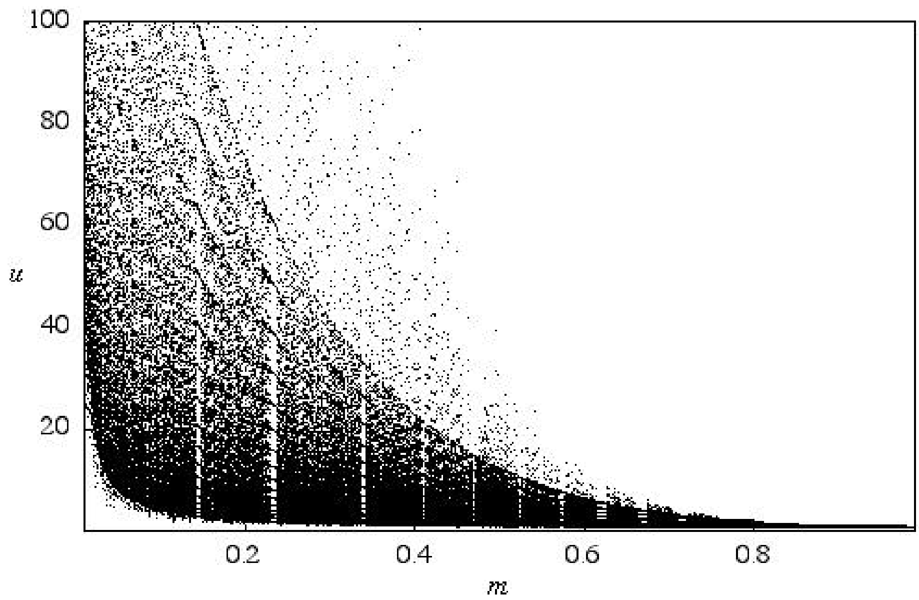


Fig. 3. Bifurcation diagram for Eq. (6) and (7) with respect to *m*:
$\alpha = 0.2$, and $\sigma = 8.0$.

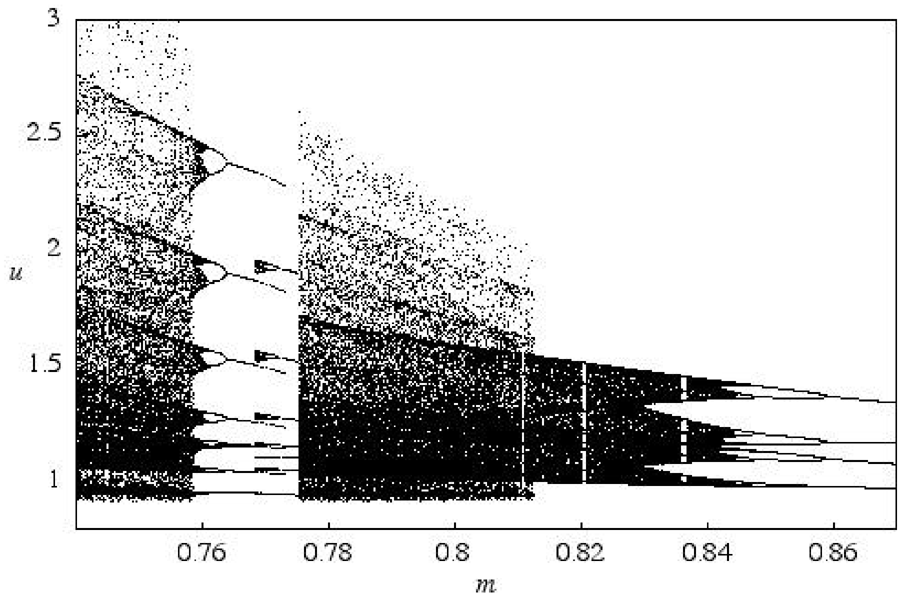


Fig. 4. Expanded bifurcation diagram shown in Fig. 3 with respect to *m*:
$\alpha = 0.2$, and $\sigma = 8.0$.

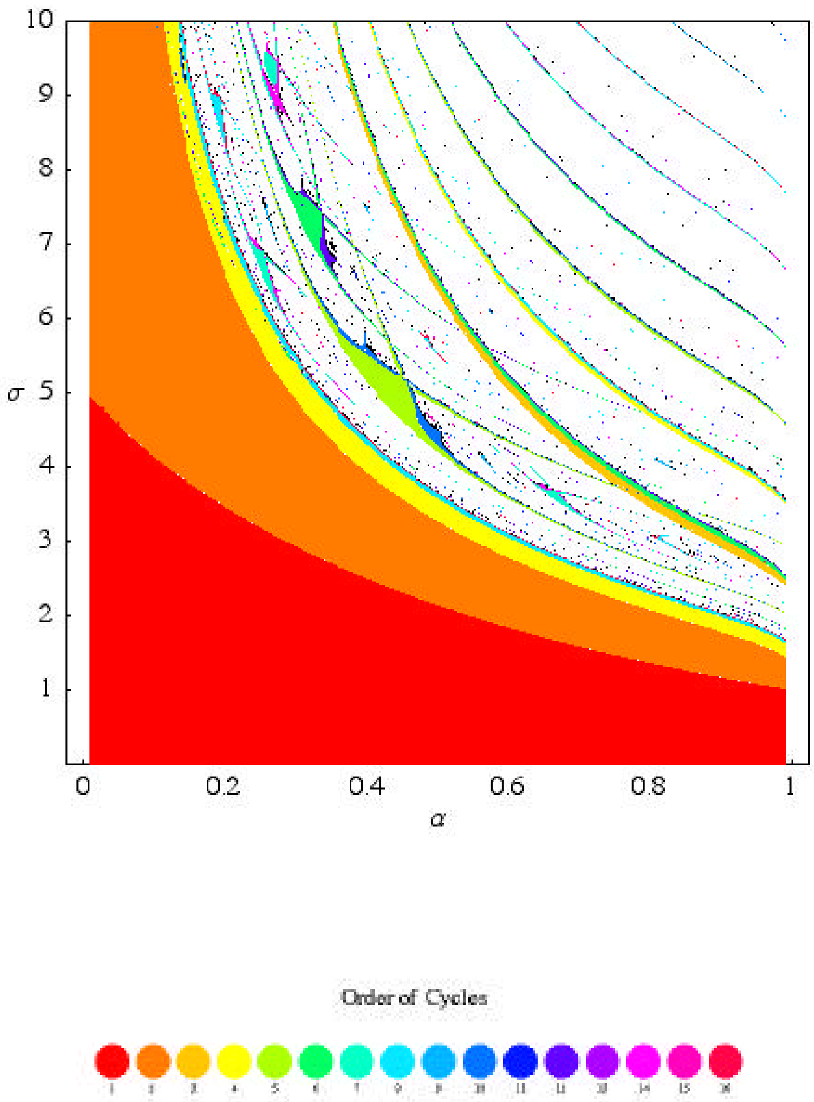


Fig. 5. Bifurcation diagram for Eq. (6) and (7) with respect to (α, σ): $m = 0.8$.

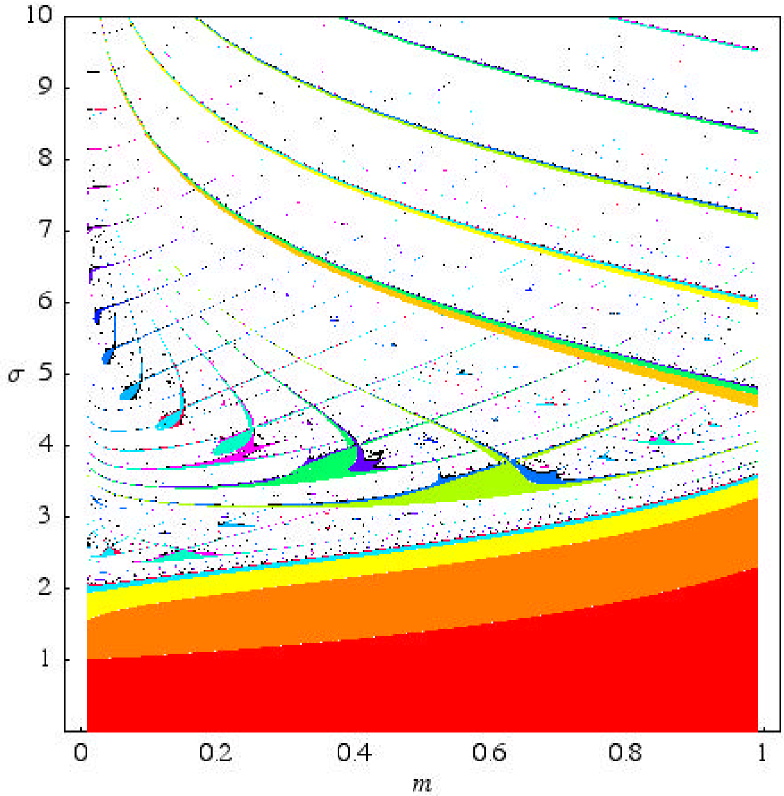


Fig. 6. Bifurcation diagram for Eq. (6) and (7) with respect to (m, σ): $\alpha = 0.6$.

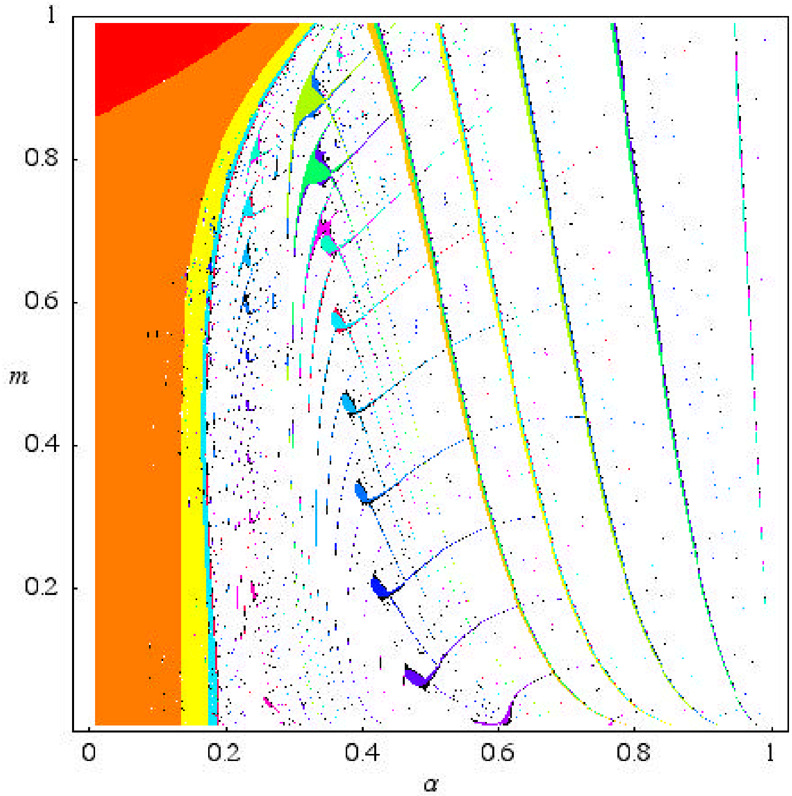


Fig. 7. Bifurcation diagram for Eq. (6) and (7) with respect to (α, m): $\sigma = 0.7$.

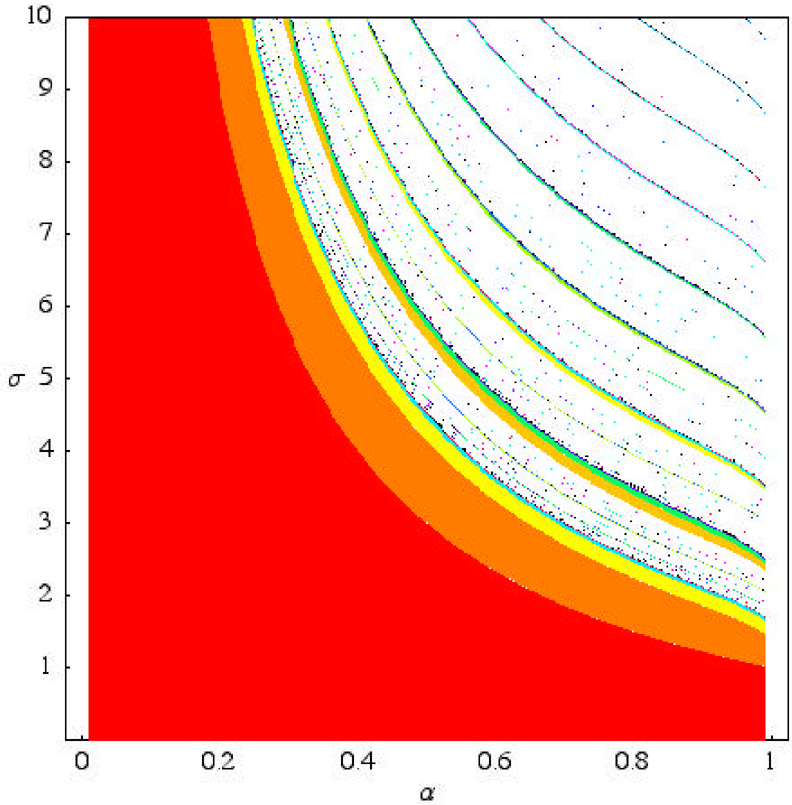


Fig. 8. Bifurcation diagram of the reduced model (5).

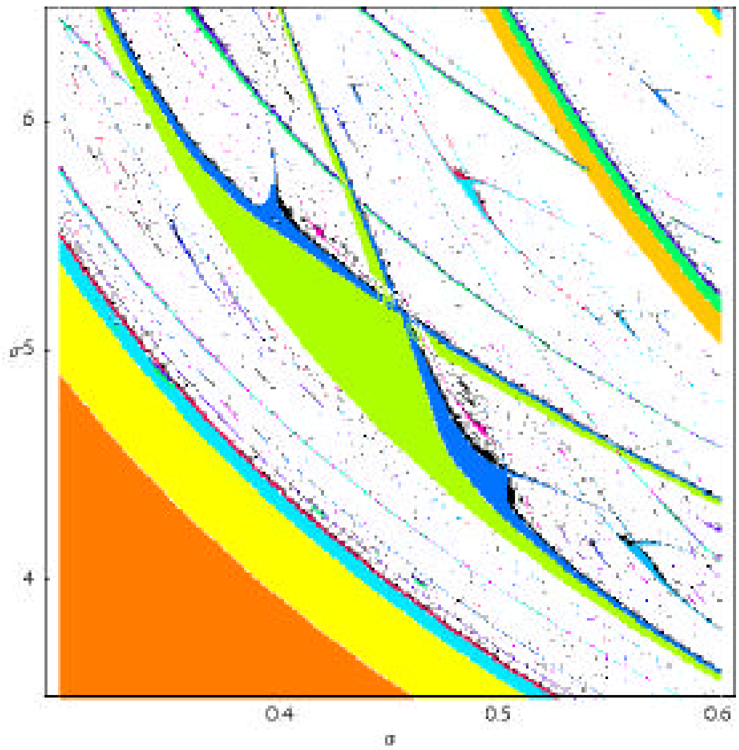


Fig. 9. Partial enlargement of Fig. 8.

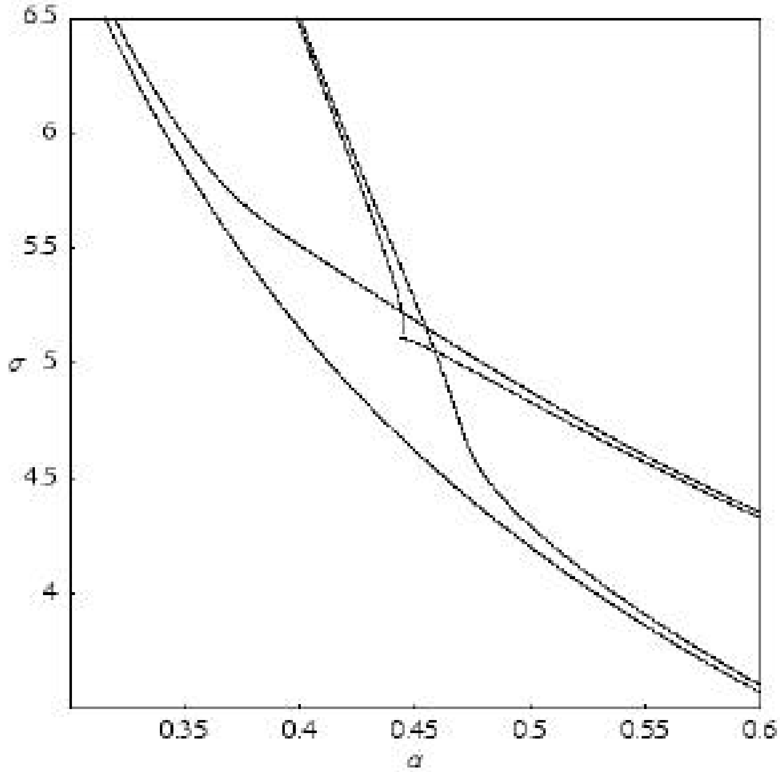


Fig. 10. Bifurcation curves related to a period-5 orbit on the (α, σ)-plane for $m = 0.8$. Thick curves represent fold bifurcations and thin curves represent period-doubling bifurcations.

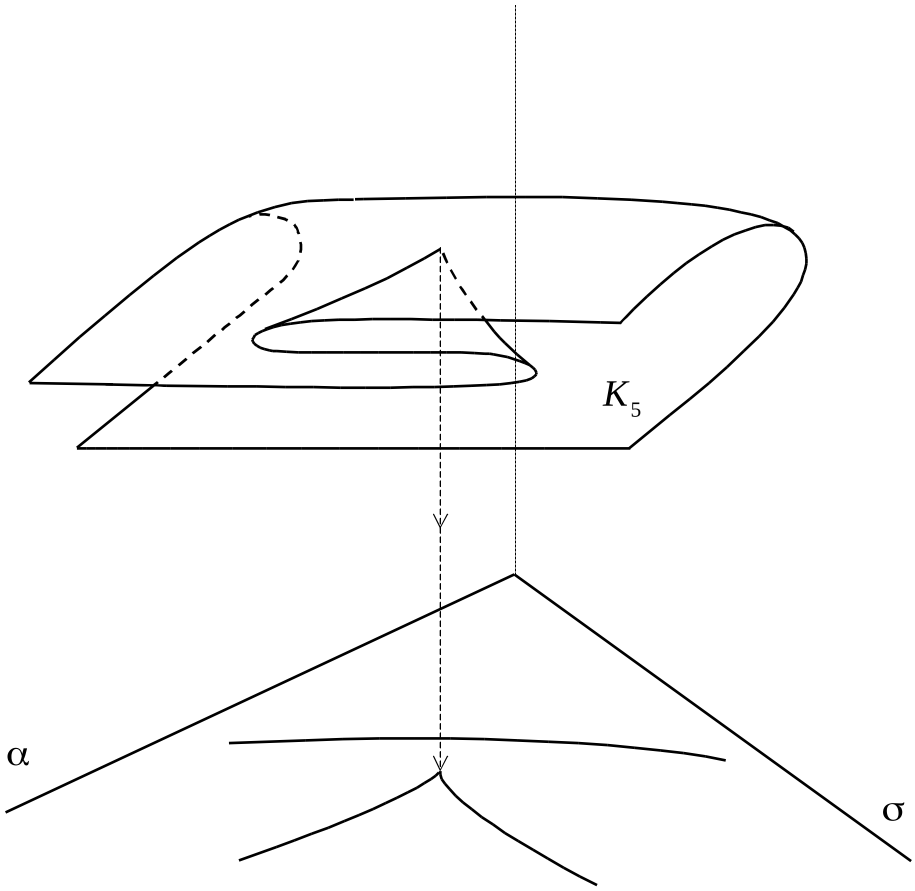


Fig. 11. Periodic point surface K_5 near a cusp point and its projection on the (α, σ)-plane.

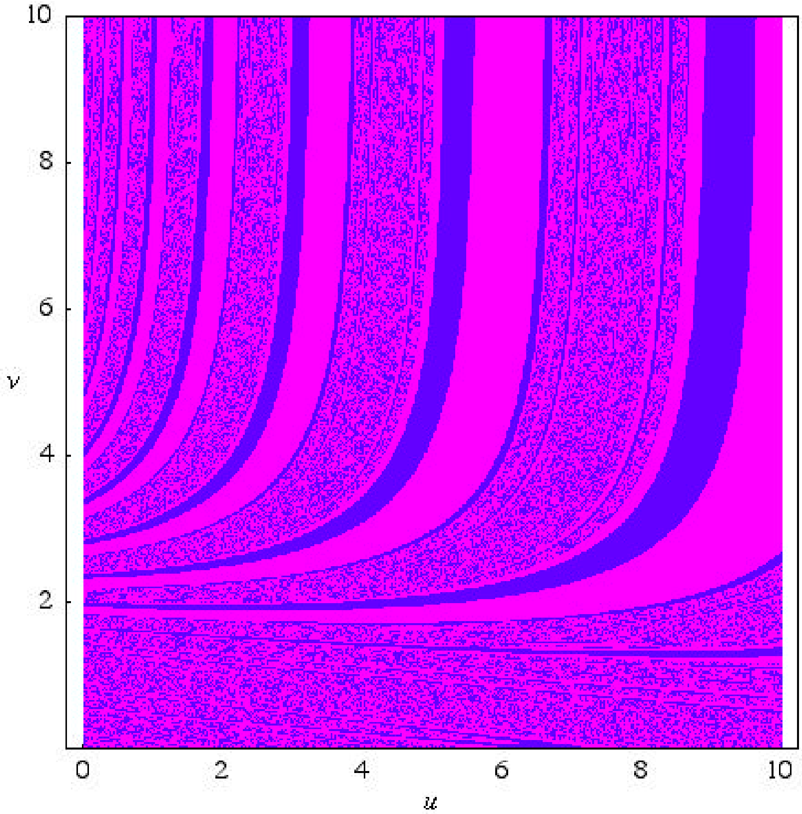


Fig. 12. Basin of attraction for $\alpha = 0.8$, $\sigma = 3.0$ and $m = 0.0915$.
There coexist initial points such that every trajectory starting from them converges to either period-12 or -30 cycles.

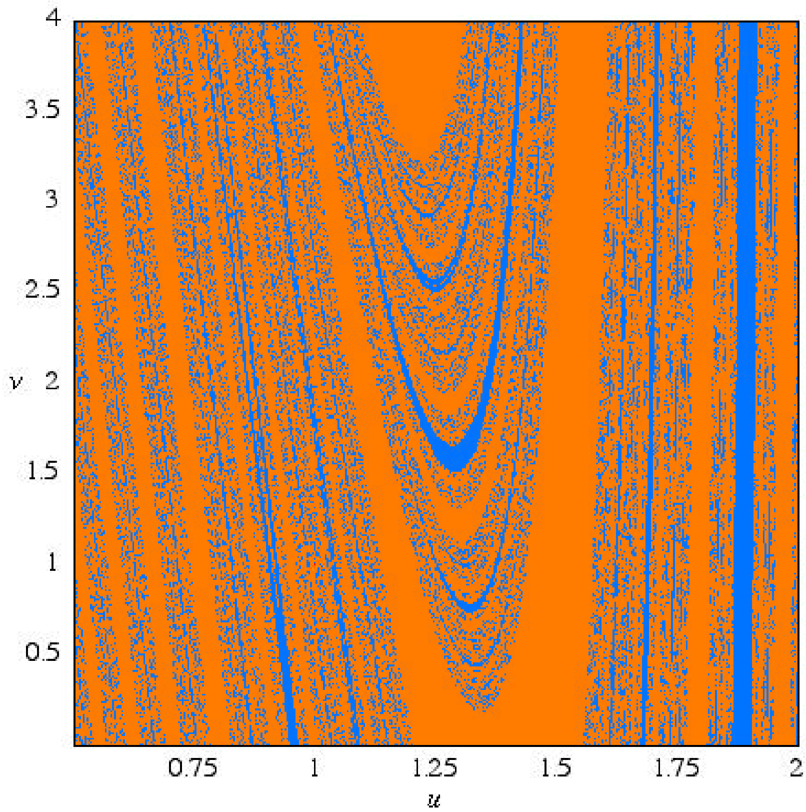
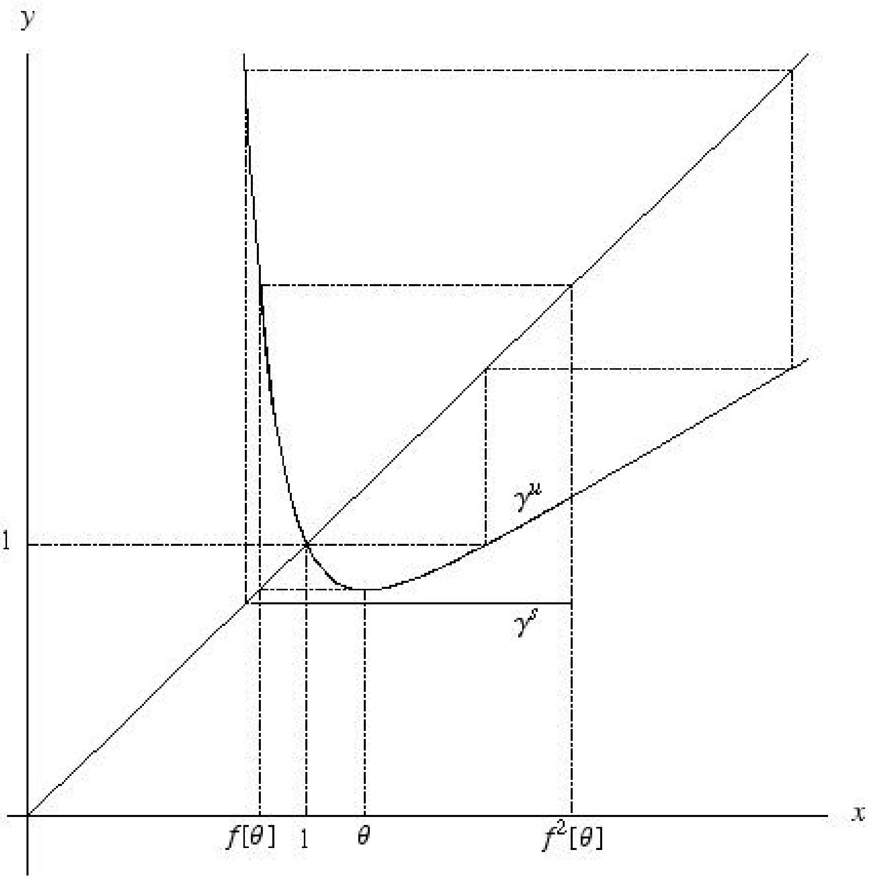
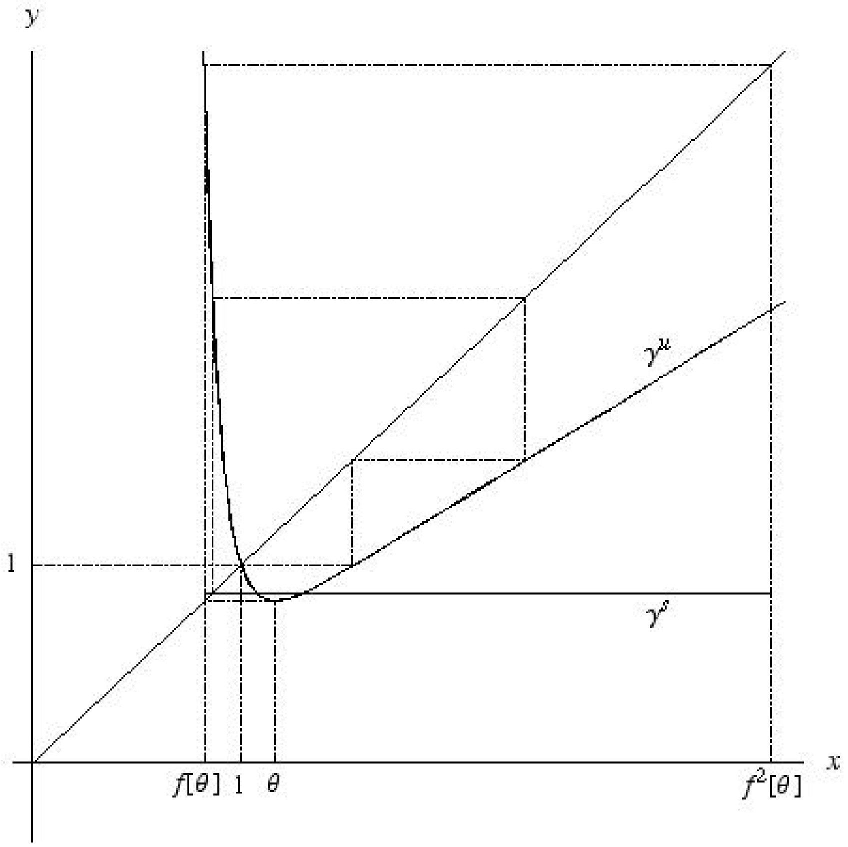


Fig. 13. Basin of attraction for $\alpha = 0.5$, $\sigma = 6.0$ and $m = 0.961$.
There coexist initial points such that every trajectory starting from them converges to either period-10 or -18 cycles.



(a) The unstable arc γ^u and the stable arc γ^s have no intersection.



(b) The unstable arc γ^u and the stable arc γ^s have two transverse intersections.

Fig. 14. One-dimensional inevitable homoclinic tangency with respect to σ