Baltensperger, Ernst; Milde, Hellmuth

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Ernst Baltensperger
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D-775 Konstanz
Postfach 733

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I. Introduction

Among the most difficult problems facing a banking firm is the existence of uncertainty concerning various aspects of its activities. One of the main types of uncertainty a bank has to deal with is uncertainty about in - and outflows of cash reserves over the course of a given planning period. This uncertainty arises from two sources: First, the bank does not exactly know in advance what in-and outflows it will experience in its deposit accounts. Second, the bank faces uncertainty about the repayment of (matured) loans (due to the existence of default risk), and possibly also about the granting of new loans (if it gives some of its customers automatic borrowing privileges). This uncertainty induces banks to carry inventories of cash reserves into the period. To hold such inventories reduces the risk of reserve deficiencies during the period and of the costs of portfolio adjustments forced upon the bank by that event. Some authors, (see especially Alchian, 1970) emphasize the economizing on information costs as the main economic function of inventories. A bank can use real resources (including time) in order to acquire more information about its customers and reduce the degree of uncertainty if faces. Existing models of bank reserve management do not consider this additional option of the bank.[1] For an analysis of the relationship between information activity and inventory behavior in a more general context see Baltensperger (1974). For an analysis of some aspects of the relationship between information activity and bank behavior see Aigner and Sprenkle (1968) and Baltensperger (1972).

In most studies in probabilistic microeconomics it is assumed that the degree of information, expressed in the form of a density function over a stochastic variable, is given exogeneously. In the course of a comparative static analysis, the effect of an exogeneous increase in the variance-used as an index of the level of risk-on the optimal solution is then often studied.[2] This paper endogenizes the variance by incorporating information activity and its relation to the degree of uncertainty into the model. The assumption that marginal information costs are infinite, which is implicit in the traditional models with their exogeneous density functions, is given up. Marginal information costs are assumed to be positive and finite.

In Section II we will develope a banking model incorporating information activities. The optimal level of information expenses and the optimal composition of the banks assets between cash reserves, secondary reserves (securities) and loans will be determined simultaneously. The comparative statics of the model are presented in Section III. They show how the optimal asset structure and level of information activity react to parameters changes. Finally, in Section IV we discuss some implications of the results obtained from our model for general equilibrium systems with disaggregated financial sectors.

II. The Model

We consider a bank with exogeneously given funds D. The bank has the choice between three types of assets: loans L, securities S, and cash reserves R. To simplify the presentation, we assume that there are no legal reserve requirements. There are three types of costs related to the bank's portfolio decisions which we will consider:

(a) opportunity costs of reserve inventories

(b) adjustment costs (arising in the event of reserve deficiencies)

(c) information costs

The bank attempts to minimize the sum C of the expected values of these costs:

$$(1)\quad C = rR + (r-i)S + m\int_{R}^{R+S}(u-R)f(u,q)du + \int_{R+S}^{\infty}[mS + n(u-R-S)]f(u,q)du + sq$$

The first two terms on the right hand side of (1) measure the opportunity cost of holding reserves (the interest returns foregone),where the return on loans r is assumed to exceed the rate of return on securities i. An increase in primary reserves R or secondary reserves S increases these opportunity costs. The next two terms measure the expected adjustment costs which the bank incurs in the event of reserve deficiencies. The net outflow or "demand" for cash reserves over the period is denoted by U, a random variable with density function f. The realized value of this random variable is denoted u. If the net reserve demand u is less than the inventory of cash reserves brought into the planning period, there is no deficiency and consequently no corresponding adjustment cost. If the realized value u exceeds R, however, there will be a deficiency, the elimination of which causes adjustment costs. If $R < u < R+S$, the bank will sell securities in the amount of the deficiency (u-R). The third term in (1) represents the expected costs of these transactions, where m is the (constant) marginal cost of converting secondary into primary reserves. We assume $m > i$, since otherwise the optimization problem is trivial.

Bank loans are basically illiquid. If however, a bank's net demand for reserves in a given period exceeds its inventory of total reserves (primary plus secondary), $u > R+S$, the bank is in a situation where, even after selling all its securities, it has to convert part of its portfolio of outstanding loans into cash. The (constant) marginal cost n of doing this is assumed to clearly exceed the cost of converting securities into cash: $n > m$. The fourth term in (1) summarizes the expected adjustment costs in the event where $u > R+S$. Analogous to the parameters constellation for securities, we assume $n > r$. It is easily seen that the expected costs of reserve deficiencies fall with increasing values of R and S.

The bank can reduce its reserve inventories without increasing the risk of reserve deficiencies if, by collecting more information, it can reduce the uncertainty about the net reserve demand for the period. The collection of information is, however, not costless, but requires the use of time and other real resources. The fifth term in (1) represents these informative costs, where q denotes the quantity of resource units (e.g. time) used in the production of information, and s the cost per resource unit.

The density function f is not given exogeneously, but functionally dependent on the information input q. Since a density function is characterized by its moments, the level of information must be reflected in the magnitude of its moments. We will, for simplicity, assume that U is normally distributed. Then f is completely characterized by the first two moments. We define an increase in uncertainty as the stretching of the bank's density function around a constant mean, and consequently, interpret the standard deviation $\sigma(U)$ as a measure of the degree of uncertainty faced by the bank. In our model, the bank can achieve a reduction of $\sigma(U)$ via an increase in the information input q; i.e. we introduce an informative function with the following property:

(2) $\sigma(U) = a(q)$, with $a' < 0$.

With a low information level, a traditionally conservative banker will, for precautionary reasons, attach relatively high subjective probabilities to even extreme realizations u of net reserve outflows, reflected in a relatively high value of $\sigma(U)$. An increase in the information level will induce him to reduce the probabilities attached to extreme realizations, leading to a reduction of the dispersion of U around the expected value. As noted in the introduction, uncertainty about changes in cash reserves can be due to either uncertainty about deposit fluctuations, or to uncertainty about the repayment of matured loans (due to default risk) and the granting of new loans. It is with respect to both of these reasons that the banker can attempt to increase his information level. The bank will be interested in information concerning general market conditions (the general environment within which it operates), as well as specific information about its individual customers. In this last respect, the ability of the bank to differentiate between its customers is of particular importance. A bank's deposit and loan customers are heterogeneous groups, with different expected withdrawal and default rates. By learning about these differences, the bank can effectively reduce the amount of uncertainty it faces. To illustrate, consider an extreme case where each customer's behavior is perfectly determinate, but there is more than one group of customers, characterized by different (and certain) withdrawal or default rates. Then, if the bank can precisely classify all of its customers as to the group they belong too, it faces no uncertainty at all, we would have a deterministic world. However, if the bank is not able to classify its customers correctly, it effectively faces uncertainty. By collecting information about its customers which enables it to better differentiate between them, the bank thus can indeed reduce the degree of uncertainty it faces.

We proceed by standardizing the random variable U, which allows us to explicitly introduce $\sigma(U)$ into the bank's objective function (1). To simplify the presentation, we assume that $E(U) = 0$. The standardized random variable $V \equiv U/\sigma(U)$ thus follows the standard normal distribution g (and is, of course independent of q). We define

(3) $v \equiv u/\sigma(U)$

(4) $\beta \equiv R/\sigma(U)$

(5) $\gamma \equiv S/\sigma(U)$

and rewrite the bank's objective function (also employing (2)) as

$$(6)\quad \begin{aligned} C = {} & r\beta\, a(q) + (r-i)\gamma\, a(q) \\ & + a(q)\, m \int_{\beta}^{\beta+\gamma} (v-\beta)\, g(v)\, dv \\ & + a(q) \int_{\beta+\gamma}^{\infty} [m\gamma + n(v-\beta-\gamma)]\, g(v) dv \\ & + sq \end{aligned}$$

The decision variables of the bank are q, β (its inventory of cash reserves, expressed in standardized units), and γ (its security holdings, relative to σ). First order conditions for a minimum of C can be written:

$$(7)\quad \int_{\beta+\gamma}^{\infty} g(v) dv = \frac{r-i}{n-m}$$

$$(8)\quad \int_{\beta}^{\beta+\gamma} g(v)\, dv = \frac{r}{m} - \frac{n}{m} \cdot \frac{(r-i)}{(n-m)}$$

$$(9)\quad -a'(q)\, [m \int_{\beta}^{\beta+\gamma} v g(v) dv + n \int_{\beta+\gamma}^{\infty} v\, g(v) dv] = s$$

Second orders conditions for a minimum of C are

$$(10)\quad T_1 = (n-m)\ g(\beta+\gamma) > 0$$

$$(11)\quad T_2 = (n-m)\ g(\beta+\gamma)\ m\ \ g(\beta) > 0$$

$$(12)\quad T_3 = -(n-m)g(\beta+\gamma)\ mg(\beta)\ a''s/a' > 0$$

Equation (7) to (9) determine the optimal values for q, β and γ. Given q, (2) determines the optimal value of σ. The optimal values of R and S are then obtained by using (4) and (5). The bank's optimal volume of loans is then implicitly determined by the balance sheet constraint:

$$L = D - R - S.$$

From (7) and (8) we obtain:

$$(13)\quad r - n \int_{\beta+\gamma}^{\infty} g(v)\ dv = i - m \int_{\beta+\gamma}^{\infty} g(v)\ dv$$

$$(14)\quad i - m \int_{\beta}^{\infty} g(v)\ dv = 0.$$

(13) requires that the bank allocates its total assets between loans and total reserves (cash and secondary) such that the expected marginal profit of extending loans is equalized with the expected marginal profit of holding reserves. Since cash reserves do not yield positive interest nor cause adjustment costs, (14) requires that the bank's total reserves, determined in (13), are divided between cash and secondary reserves such that the expected marginal profit of secondary reserves is equalized with the marginal profit of cash reserves, which is zero. The relative magnitudes of the various cost and return parameters must satisfy the following conditions, if the optimal portfolio is to include positive amounts of all three assets:

(a) $r - i < n - m$; (b) $r/i < n/m$.

According to (9), the expected marginal profit of information production (given on the left hand side) must be equalized with the marginal cost s of producing information. If we define $\hat{C}$ as total costs C minus information costs sq we can rewrite condition (9) as follows:

$$(9') \quad \frac{\partial \hat{C}}{\partial \sigma} \cdot \frac{\partial \sigma}{\sigma q} = s.$$

$-(\frac{\partial \hat{C}}{\partial \sigma})$ measures the marginal change in profit caused by an increment in the level of risk σ.[3]

The second order conditions (10) to (12) impose some restrictions on the density function g and the information function a(q). The densities $g(\beta)$ and $g(\beta+\gamma)$ must both be strictly positive. The information function must have the property $a''>0$ (since, we have already assumed that $a'<0$), i.e., the reduction in σ must become smaller and smaller with increasing information input q.

III. Comparative Statics

The optimality conditions (7) to (9) contain the various cost and return paramters of the model. Changes in these parameters change the optimal values of q, β and γ, and thus R, S and L. Of particular interest are the following selected results (complete comparative static results are presented in the Appendix):

$$(15) \quad \frac{dq}{ds} = -\frac{a'}{a'' s} < 0$$

$$(16) \quad \frac{dq}{di} = -\frac{(a')^2 \gamma}{a'' s} < 0$$

$$(17) \quad \frac{dq}{dr} = \frac{(a')^2(\beta+\gamma)}{a'' s} > 0$$

(15) is the well known price theoretic result that an increase in the resource price s reduces the optimal resource input q. This is only so, of course, as long as σ responds to changes in q, i.e., $a' > 0$. Expressing (15) in

elasticity terms shows that the demand for information input q depends on the properties of the information function (2) only:

$$(15') \quad \varepsilon(q,s) = 1/\varepsilon(a',q) < 0.$$

According to (16) and (17), the optimal q reacts differently to changes in the two interest rates i and r. An increase in r makes loans more attractive, i.e., raises the opportunity cost of holding reserves (of either kind), and thus induces a more intensive use of the alternative method of lowering the risk of reserve deficiencies which the model allows, namely via reducing σ through increases in q. An increase in i, on the other hand, lowers the opportunity cost of holding secondary reserves, i.e., makes reserves overall more attractive and thus induces a less intensive use of the alternative method of lowering liquidity risk, resulting in a reduction of q (in addition to changing the structure of total reserves in favor of secondary reserves).

Rewriting (16) and (17) in elasticity form,

$$(16') \quad \varepsilon(q,i) = -\varepsilon(a,q)\,\varepsilon(q,s)\,\frac{iS}{sq} < 0,$$

$$(17') \quad \varepsilon(q,r) = \varepsilon(a,q)\,\varepsilon(q,s)\,\frac{r(R+S)}{sq} > 0,$$

shows that in an optimum the ratio between opportunity costs and information costs must satisfy the following elasticity expression:

$$\frac{rR + (r-i)S}{sq} = \frac{\varepsilon(q,r) + \varepsilon(q,i)}{\varepsilon(a,q)\,\varepsilon(q,s)}$$

Comparison of (16) and (17) shows that the (positive) reaction of q to increases in r exceeds the (negative) reactions of q to increases in i. That is, if r and i increase at the same time, (an increase in the general level of interest rates), q will increase:

$$(18) \quad \frac{dq}{di} + \frac{dq}{dr} = \frac{(a')^2\beta}{a''\,s} > 0.$$

Computation of the reaction coefficients for q, β and γ, and employment of (2), (4) and (5) (see Appendix) yields the following comparative static results for cash reserves R and secondary reserves S:

$$(19)\quad \frac{dR}{di} = - \frac{a}{mg(\beta)} - \frac{\beta\gamma(a')^3}{a''s} \gtrless 0$$

$$(20)\quad \frac{dS}{di} = \frac{a}{mg(\beta)} + \frac{a}{(n-m)g(\beta+\gamma)} - \frac{\gamma^2(a')^3}{a''s} > 0$$

$$(21)\quad \frac{dR}{dr} = \frac{\beta(\beta+\gamma)(a')^3}{a''s} < 0$$

$$(22)\quad \frac{dS}{dr} = - \frac{a}{(n-m)g(\beta+\gamma)} + \frac{\gamma(\beta+\gamma)(a')^3}{a''\ s} < 0$$

Of particular interest is the result summarized by equation (19). It indicates that the reaction of cash reserves to changes in the security rate i is ambiguous in our model. The usual assumption that an increase in the rate of return on secondary reserves unambiguously reduces the optimal inventory of cash reserves [4] is thus not obtained from our model. This traditional result is based on an analysis of bank behavior without endogeneous information activity. Indeed, if we assume a' = 0, we get a special case where dR/di is unambiguously negative (reflecting the pure cross price effect):

$$(23)\quad \left(\frac{dR}{di}\right)^{a'=0} = - \frac{a}{m\ g(\beta)} < 0$$

The ambiguity in the sign of (19) is thus caused by the information effect (represented by the second term on the right hand side): According to (16), an increase in i lowers the optimal q, which increases σ and leads, ceteris paribus, to increases in the bank's reserve inventories.

As far as the effect on secondary reserves S is concerned, however, the pure price effect (obtained by setting a' = 0) and the information effect operate in the same direction and thus reinforce each other, as can be easily seen from (20). The (positive) reaction of S to an increase in i thus is larger in our model than it would be in a model without endogeneous information activity.

The sum of total reserves (primary and secondary) reacts unambiguously positively to an increase in i. From (19) and (20) we obtain:

$$(24)\quad \frac{dR}{di} + \frac{dS}{di} = - \frac{\gamma(\beta+\gamma)(a')^3}{a'' s} + \frac{a}{(n-m)g(\beta+\gamma)} > 0$$

The pure price effect (obtained from (24) by setting a' = 0) in this case is clearly positive, since the positive direct price effect (obtained from (20) by setting a' = 0) exceeds the negative cross price effect (obtained from (19) by setting a'= 0):

$$(25)\quad \left(\frac{dR}{di} + \frac{dS}{di}\right)^{a'=0} = \frac{a}{(n-m)g(\beta+\gamma)} > 0.$$

The positive information effect, summarized by the first term on the right hand side of (24), further reinforces this (positive) reaction. From the bank's balance sheet constraint it then follows that the optimal volume of loans reacts negatively to an increase in i:

$$(26)\quad \frac{dL}{di} = - \frac{dR}{di} - \frac{dS}{di} < 0.$$

An increase in the loan rate r according to (21) and (22) clearly reduces R as well as S. The information effect here is negative, and the pure price effect is negative in the case of secondary reserves S, and zero in the case of cash reserves R.

From (19) and (21) we see, furthermore, that the response of cash reserves R to an increase in the general level of interest rates (dr = di) is unambiguously negative:

$$(27)\quad \frac{dR}{di} + \frac{dR}{dr} = - \frac{a}{mg(\beta)} + \frac{\beta^2(a')^3}{a'' s} < 0.$$

Combining (20) and (22) we learn, however, that reactions of secondary reserves to an increase in the general level of interest rates are no longer unambigous:

$$(28)\quad \frac{dS}{di} + \frac{dS}{dr} = \frac{a}{mg(\beta)} + \frac{\beta v(a')^3}{a''s} \gtrless 0.$$

The results shown in (19)-(22) would remain fundamentally unchanged, even if we allow simultaneous changes in the penalty rates n and m along with the variations in the corresponding interest rates r and i. For the complete comparative state results, the reader is again referred to the Appendix.

IV. Conclusion

Our paper analyzes the effects of introducing information activity as an endogeneous element into a model of bank portfolio behavior. It is shown that the comparative static effects of parameter changes can conveniently be separated into a "pure price effect" and an "information effect". It is possible that the information effect counteracts the pure price effect and thus brings ambiguity into what would be, in a model without endogeneous information activity, an unambiguous effect, as was discussed in the case of dR/di. In other cases, however, the information effect reinforces the pure price effect. These observations suggest that care must be taken in specifying the reaction of reserve (and other asset) holdings to changes in interest rates and other parameters in general equilibrium models including well developed and disaggregated financial sectors, such as, e.g., those by Tobin or Brunner and Meltzer.[5] Finally, it may be noted that the type of model presented here, by including the notion of real resource inputs into the banking sector (for information production as well as for conducting adjustments made necessary by reserve deficiencies), suggests a link between the real and the financial sector not normally analyzed.

Footnotes

*Ohio State University and University of Konstanz.

[1]See, e.g., Edgeworth (1888), Orr and Mellon (1961), Porter (1961), Morrison (1966), Poole (1968), Frost (1971).

[2]See, e.g., Sandmo (1971), Batra (1974).

[3]Compare the analysis of this kind of derivative in stochastic price theoretic models, e.g. Sandmo (1971), or Batra (1974), where σ is treated as exegeneous variable, and no endogeneous information activity is allowed.

[4]Compare, e.g., Kaufman (1973), p. 114.

[5]Compare, e.g., Tobin (1969), Brunner and Meltzer (1972).

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Appendix

Minimizing (6) with respect to γ, β and q gives the following first order conditions:

$$\frac{\partial C}{\partial \gamma} = (r - i)a + a\int_{\beta+\gamma}^{\infty} (m-n)\; g\; dv = 0$$

$$\frac{\partial C}{\partial \beta} = ra - ma\int_{\beta}^{\beta+\gamma} g\; dv - an\int_{\beta+\gamma}^{\infty} g\; dv = 0$$

$$\frac{\partial C}{\partial q} = a'[r\beta + (r - i)\gamma + m\int_{\beta}^{\beta+\gamma} (v-\beta)\; g\; dv + \int_{\beta+\gamma}^{\infty} [m\gamma + n\,(v-\beta - \gamma]\; g\; dv] + s = 0$$

In differentiated form, this system becomes:

$$\begin{bmatrix} (n-m)g(\beta+\gamma) & (n-m)g(\beta+\gamma) & 0 \\ (n-m)g(\beta+\gamma) & (n-m)g(\beta+\gamma)+mg(\beta) & 0 \\ 0 & 0 & -a''s/a' \end{bmatrix} \begin{bmatrix} d\gamma \\ d\beta \\ dq \end{bmatrix}$$

$$= \begin{bmatrix} -dr + di - dm\int_{\beta+\gamma}^{\infty} g\; dv + dn\int_{\beta+\gamma}^{\infty} g\; dv \\ -\; dr + dm\int_{\beta}^{\beta+\gamma} g\; dv + dn\int_{\beta+\gamma}^{\infty} g\; dv \\ -\; ds - a'(\beta+\gamma)\; dr + a'\gamma di - dm\left\{\int_{\beta}^{\beta+\gamma} (v-\beta) g\; dv + \gamma\int_{\beta+\gamma}^{\infty} g dv\right\} a' \\ -dn\left\{\int_{\beta+\gamma}^{\infty} (v-\beta-\gamma) g\; dv\right\} a' \end{bmatrix}$$

This yields the following comparative static results:

$$\frac{dq}{ds} = \frac{a'}{a'' s} < 0$$

$$\frac{dq}{dr} = \frac{(a')^2(\beta+\gamma)}{a'' s} > 0$$

$$\frac{dq}{di} = - \frac{(a')^2 \gamma}{a'' s} < 0$$

$$\frac{dq}{dm} = \frac{(a')^2}{a'' s} \{ \int_{\beta}^{\beta+\gamma} (v-\beta) g \, dv + \int_{\beta+\gamma}^{\infty} g \, dv \} > 0$$

$$\frac{dq}{dn} = \frac{(a')^2}{a'' s} \{ \int_{\beta+\gamma}^{\infty} (v - \beta - \gamma) \, g \, dv \} > 0$$

$$\frac{d\gamma}{ds} = 0$$

$$\frac{d\gamma}{dr} = - \frac{1}{(n-m)g(\beta+\gamma)} < 0$$

$$\frac{d\gamma}{di} = \frac{1}{mg(\beta)} + \frac{1}{(n-m)g(\beta+\gamma)} > 0$$

$$\frac{d\gamma}{dm} = - \frac{i}{m^2 g(\beta)} - \frac{(r-i)}{(n-m)^2 g(\beta+\gamma)} < 0$$

$$\frac{d\gamma}{dn} = \frac{(r-i)}{(n-m)^2 g\,(\beta+\gamma)} > 0$$

$$\frac{d\beta}{ds} = 0$$

$$\frac{d\beta}{dr} = 0$$

$$\frac{d\beta}{di} = - \frac{1}{m\, g(\beta)} < 0$$

$$\frac{d\beta}{dm} = \frac{i}{m^2 \, g(\beta)} > 0$$

$$\frac{d\beta}{dn} = 0$$

$$\frac{dR}{dx} = a\,\frac{d\beta}{dx} + \beta\, a'\, \frac{dq}{dx}$$

$$\frac{dS}{dx} = a\,\frac{d\gamma}{dx} + \gamma\, a'\, \frac{dq}{dx}$$

where x can be s, r, i, m or n.