

Seidl, Christian

Working Paper

The Petersburg Paradox at 300

Economics Working Paper, No. 2012-10

Provided in Cooperation with:

Christian-Albrechts-University of Kiel, Department of Economics

Suggested Citation: Seidl, Christian (2012) : The Petersburg Paradox at 300, Economics Working Paper, No. 2012-10, Kiel University, Department of Economics, Kiel

This Version is available at:

<https://hdl.handle.net/10419/64811>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

C | A | U

Christian-Albrechts-Universität zu Kiel

Department of Economics

Economics Working Paper
No 2012-10

the petersburg paradox at 300

by Christian Seidl

issn 2193-2476



The Petersburg Paradox at 300

Christian Seidl*, University of Kiel, Germany

August 5th, 2012

Abstract

In 1713 Nicolas Bernoulli sent to de Montmort several mathematical problems, the fifth of which was at odds with the then prevailing belief that the advantage of games of hazard follows from their expected value. In spite of the infinite expected value of this game, no gambler would venture a major stake in this game. In this year, de Montmort published this problem in his *Essay d'analyse sur les jeux de hazard*. By dint of this book the problem became known to the mathematics profession and elicited solution proposals by Gabriel Cramer, Daniel Bernoulli (after whom it became known as the Petersburg Paradox), and Georges de Buffon. Karl Menger was the first to discover that bounded utility is a necessary and sufficient condition to warrant a finite expected value of the Petersburg Paradox. It was, in particular, Menger's article which provided an important cue for the development of expected utility by von Neumann and Morgenstern.

The present paper gives a concise account of the origin of the Petersburg Paradox and its solution proposals. In its third section, it provides a rigorous analysis of the Petersburg Paradox from the uniform methodological vantage point of d'Alembert's ratio text. Moreover, it is shown that appropriate mappings of the winnings or of the probabilities can solve or regain a Petersburg Paradox, where the use of probabilities seems to have been overlooked by the profession.

*I am indebted to Giacomo Corneo, Free University of Berlin, Germany, Lyudmila Egorova, NRU-HSE, Moscow, Russia, Peter J. Hammond, Stanford University, U.S.A., and University of Warwick, U.K., and Kirill Pogorelskiy, Caltech, Pasadena, U.S.A., for most helpful comments. Remaining errors are mine. This paper is a revised and reworked version of Discussion Paper No. 2012-04.

1 The Origin of the Petersburg Paradox

In the French *ancien régime* the aristocrats disposed of large fortunes, but had lost their political influence, in particular after the inauguration of Louis XIV. Hence, many of them spent their time with games of hazard. The more sagacious of them, primarily a certain Chevalier de Méré, brought the famous mathematicians Fermat and Pascal to elaborate for him guidance for optimum gambling. Their suggestions centered around the expected value of the winnings (cf. Samuelson (1977, pp. 37-38) and D. Bernoulli (1738, p. 175)).

Yet the distinguished mathematician Nicolas Bernoulli¹ discovered a rather odd game of hazard, which he told the mathematician Pierre Rémond de Montmort in a letter dated 9th September 1713, as the fifth of several problems: A promises to give a coin to B, if with an ordinary die he achieves 6 points on the first throw, two coins if he achieves 6 points for the first time on the second throw, 4 coins if he achieves 6 points for the first time on the third throw, 8 coins if he achieves 6 points for the first time on the fourth throw, etc. What is the expected value of this game for B?

In his response of 15th November 1713, Montmort expressed opinion that these examples have easy solutions along the lines of geometric progressions, but entered Bernoulli's problems into the second edition of his *Essay d'analyse sur les jeux de hazard* [de Montmort (1713, p. 402)]. It was only in his letter of 20th February 1714, that Nicolas Bernoulli demonstrated the momentousness of his discovery, viz. that this game has an expected value of infinity, to Montmort, who expressed skepticism, but had to admit that he was unable to solve this problem.

Gabriel Cramer, a professor of mathematics at the University of Geneva, read Montmort's book and pondered on N. Bernoulli's fifth problem. In a letter of 21st May 1728 from London, he addressed Bernoulli proposing first a simplification of the gamble by replacing throwing a die by flipping a coin (whereby he reformulated the problem as it had become known since). In addition to that he proposed two solutions of the paradox, viz. by bounding the gambler's perception of winnings from above, or by replacing the winnings by their square root.

In a letter of 27th October 1728, Nicolas Bernoulli communicated the fifth problem in Cramer's simplified version to his cousin Daniel Bernoulli who was at that time a professor of mathematics at the University of St. Petersburg. Subsequently, Daniel had become interested in this paradox and had sent Nicolas a first draft of his later publication. In his response of 5th April 1732, Nicolas thanked Daniel for a copy of his essay and remarked:

¹All citations concerning the correspondence in relation with the Petersburg Paradox are to be found in N. Bernoulli (1713-1732).

“I have read it with pleasure, and I have found your theory most ingenious, but permit me to say to you that it does not solve the knot of the problem in question.” Together with this response, Nicolas sent Daniel a copy of Cramer’s letter, which Daniel appended in the original French version to his Latin manuscript for its final publication (D. Bernoulli (1738, pp. 190-2)).²

2 Attempts at “Solving” the Petersburg Paradox

The challenge to “solve” the Petersburg Paradox has captivated the attention of many prominent scholars, up to the 1950’s mainly mathematicians.

The first were Gabriel Cramer and Daniel Bernoulli who suggested appropriate concave transformations of the winnings, which Cramer (see D. Bernoulli (1738, p. 192)) called *valeur morale* (moral value) and *esperance morale* (moral hope), and D. Bernoulli (1738, p. 192) called *emolumentum*, which means utility or advantage. While Cramer suggested the square root as a transformation of the winnings, Bernoulli suggested the natural logarithm. D. Bernoulli (1738, pp. 181-2 and 184) indeed went further by adopting also the assumption that the utility of winnings is, in addition, also inversely proportional to the gambler’s wealth. Let w denote a gambler’s wealth and x the winnings, then his or her utility is $u(x + w) = b \ln \frac{(w+x)}{w}$, b being a positive constant. This relationship shows, first, that the marginal utility of a winning is the smaller the larger the gambler’s wealth is, second, that additional constant winnings have decreasing marginal utility, and, third, that the utility of a gain falls short of the disutility of an equivalent amount of money lost.³ These assumptions are in line with modern economics.⁴ Moreover, Bernoulli’s hypothesis was later confirmed by Weber (1834) and Fechner’s (1860) experimental investigations on psychophysics.

D. Bernoulli made good use of the dependence of the marginal utility of winnings and losses on a person’s wealth. In particular, he was able to explain why some hazard should be insured by less wealthy persons while richer persons should better bear the risk of the same hazard by themselves (D. Bernoulli (1738, pp. 184-6)). He also presented a quite

²Note that D. Bernoulli was ignorant of the content of Cramer’s letter to N. Bernoulli before that time. Instead he had concluded the main part of his manuscript before N. Bernoulli had sent him a copy of Cramer’s letter. In this respect, Savage (1972, pp. 92-4) was under misapprehension.

³Savage (1972, p. 94) remarked: “To this day, no other function has been suggested as a better prototype for Everyman’s utility function.”

⁴Hence, Daniel Bernoulli can be considered a precursor of Kahneman and Tversky’s (1979) prospect theory. The same can be said of Buffon (1777/2010, pp. 31-33). Blavatsky (2005) observed that the overweighing of small probabilities in Tversky and Kahneman’s (1992) cumulative prospect theory may well restore a Petersburg Paradox.

modern theory of portfolio selection for risk spreading (D. Bernoulli (1738, pp. 186-7)). Dependence of utility on the gambler's wealth provokes a lot of consequences, e.g. that a gambler's willingness to pay and to accept may diverge, and that the selling price of a gamble will exceed a gambler's willingness to pay.

According to D. Bernoulli's hypothesis, the expected value of a Petersburg game in monetary terms, which was donated to a gambler, is $\exp\{\sum_{i=1}^{\infty} 2^{-i} \ln[w + 2^{i-1}]\} - w$. We may consider it likewise as the selling value of this Petersburg game. If the gambler had paid the stake s for this gamble, its expected value in monetary terms is $\exp\{\sum_{i=1}^{\infty} 2^{-i} \ln[w - s + 2^{i-1}]\} - w$. Since $s > 0$, the latter value is smaller than the former. Bernoulli argues that the latter expression approximates the former one if w is comparatively big in relation to s .⁵

Menger (1934, p. 467-468) objected to the Bernoulli-Cramer solution that a transformation of the winnings which is "sufficiently concave" is only a sufficient, but not a necessary condition for "solving" the Petersburg Paradox. To demonstrate that he proposed to replace $(x_i) = 2^i$ by $x_i = \exp 2^i$. Applying now the transformation $\ln(\cdot)$ to $x_i = \exp 2^i$ regains the Petersburg Paradox. [Replacing $(x_i) = 2^i$ by $(x_i) = (2^i)^2$ regains the Petersburg Paradox in Cramer's version for the transformation $\sqrt{(\cdot)}$.] Samuelson (1977, p. 32) called this *Menger's Super-Petersburg Paradox*. More generally, for each increasing and unbounded utility function an increasing transformation can be defined such that the transformed winnings converge relatively faster to infinity than the probabilities converge to zero.

Menger (1934) was the first to show that boundedness of the utility function is a necessary and sufficient condition for preventing the occurrence of a Petersburg Paradox. He proved that in a somewhat circumstantial way. In his article, he proved sufficiency in Section 6 (pp. 469-471), and necessity with the help of his Super-Petersburg game in Section 5 (pp. 468-469). In the necessity part of his proof he argued that for each unbounded function $u(x_i)$ there exists a related Petersburg game whose expected transformed winnings amount to infinity. The principal contribution by Menger (1934) was his necessary condition, i.e., to have shown that a Petersburg gamble has a finite solution only if the utility of winnings is bounded. Before its 1967 English translation in the *Morgenstern-Festschrift*, Menger's result was widely unknown in the Anglo-Saxon literature, except by Arrow in 1951 and 1965 (reprinted as Essays 1 and 2 in Arrow (1970)). Samuelson frankly acknowledged that Menger's breakthrough was a *quantum jump* in the analysis of the Petersburg Paradox (Samuelson (1977, pp. 32 and 37)). Menger's achievement was

⁵D. Bernoulli (1738, pp. 188-9). For the Cramer proposal, the logarithm should be replaced by the square root, and the exponential function by taking the square.

also acclaimed by other Nobel laureates such as Arrow (1970, pp. 23 and 64-69) and Aumann (1977). In a painstaking analysis, Arrow (1970, pp. 64-69) has shown that a utility function for risky prospects has to be bounded from above and from below when instances of the Petersburg Paradox should be avoided.⁶ This implies that a utility function for risky prospects must have at least one Friedman-Savage inflexion point (Samuelson (1977, p. 50, Footnote 14)) which forms also the central constituent of Kahneman and Tversky's prospect theory.

In view of two centuries' discussions on such a provocative subject like the Petersburg Paradox, it takes wonder why its solution had to wait until Menger's (1934) discovery. This seems to be due to the comprehension of utility as it developed in the course of several centuries. After early contemplations by Thomas Hobbes, Francis Hutcheson, and David Hume, it was, in particular, Bentham (1781/2000) who elevated utility to the central driving power of human actions. Bentham's influential disciple John Stuart Mill engrossed Bentham's ideas and propagated them further. Other early advocates of utility were Auguste Walras and Juvénal Dupuit. These beginnings of utility theory spawned later on the edifice of marginal utility propagated by Heinrich Hermann Gossen, William Stanley Jevons (mainly influenced by Bentham), Léon Walras, Carl Menger, Friedrich von Wieser, Eugen von Böhm-Bawerk, and John Bates Clark.⁷ For all those scholars utility was something palpable and immutable. It was only Pareto who was the first to aver interpersonal noncomparability of utility and it was the Hicks and Allen (1934) ordinal revolution after which the profession had understood that uniqueness of the utility function up to a strictly increasing transformation is sufficient to analyze and rationalize economic decision-making.

Menger had written the first draft of his paper in 1923 and had presented it in 1927 to the Economic Society of Vienna. However, it was not particularly understood, let alone, acclaimed, by the audience. It was only in 1934, that Oskar Morgenstern, in his capacity as the managing editor of the *Zeitschrift für Nationalökonomie*, accepted Menger's manuscript for publication in the *Zeitschrift*. Although Menger had not provided an axiomatization of expected utility, his contribution became a crucial step in the development of expected utility in the second edition of Neumann and Morgenstern (1947). Morgenstern (1976, p. 809) remembered in his article on the co-operation with John von

⁶Fishburn (1970, pp. 206-207) was the first to prove that the utility function of Savage's theory is bounded.

⁷For comprehensive surveys on the development of utility theory cf. Viner (1925a,b) and Stigler (1950a,b). Whereas Viner did not mention Bernoulli at all, Stigler (1950b, pp. 373-7) devoted slightly over four pages to Bernoulli's utility function. However, amidst only theories of utility under certainty, this part looks rather alien in the Stigler survey.

Neumann:

...we decided that we would settle on thinking about numerical utility. It did not take us long to construct the axioms on which the present theory is based that gave us a firm utility concept, that of an expected utility, numerical up to a linear transformation. ... Regarding risk, Karl Menger's important paper of 1934 on the St. Petersburg Paradox ... played a great role. ... the construction of axioms of our expected utility came quite naturally. I recall vividly how Johnny rose from our table when we had set down the axioms and called out in astonishment: "Ja hat denn das niemand gesehen?" ("But didn't anyone see that") ... It was largely my doing that this utility theory was developed ...

Hence, Menger's 1934 article marked a decisive step in the development of expected utility theory [see also Kuhn and Tucker (1958, p. 108, Footnote 2)].

Two other proposals at "solving" the Petersburg Paradox is truncation of either the probabilities or the winnings. Among several other solutions, Buffon (1777/2010, pp. 23, 38, and 46) proposed to set all probabilities smaller than $\frac{1}{10.000}$ equal to zero. In this reasoning, Buffon was followed by Fontaine, Poisson, d'Alembert, and Condorcet (Samuelson (1977, pp. 39-40)). Interestingly enough, even Menger (1934, pp. 473 and 476) believed that small probabilities are undervalued by subjects.⁸

As to truncating the winnings, Cramer, in his letter to Nicolas Bernoulli of 21st May 1728, suggested bounding the winnings at 2^{24} coins, because larger sums do not make for more pleasure of the gambler or because the gambler is never able to receive more than this sum (N. Bernoulli (1713-1732)). In one of his proposals to solve the Petersburg Paradox, Buffon (1777/2010, p. 39) had argued in analogy to Cramer. Several other authors justified boundedness of the winnings by referring to the finiteness of the bookmaker's wealth or that the time of playing a Petersburg game cannot be infinitely long. A bouquet of such authors includes Buffon (1777/2010, pp. 39-44), Pringsheim (D. Bernoulli (1896, pp. 46-52)), Fry (1928, pp. 198-199), Shapley (1977a, p. 440) and, although considerably more cautious, Shapley (1977b, pp. 448-449). A similar argument was put forward by Brito

⁸Menger (1934) was unable to adduce empirical evidence for this assumption. Backed by ample empirical research, Kahneman and Tversky (1979) and Tversky and Kahneman (1992) observed overvaluation of small probabilities. Their prospect theory became famous for its accurate forecasts of human behavior. Recently, several experiments were conducted to investigate subjects' behavior in Petersburg-game situations. For details see Neugebauer (2010, pp. 22-31). Neugebauer, too, conducted extensive experiments. His results confirmed Daniel Bernoulli's hypothesis that the willingness to pay for participation in a Petersburg game is an increasing function of wealth, and Nicholas Bernoulli's and Buffon's conjecture that small probabilities are set equal to zero.

(1975) and Cowen and High (1988), viz. that the utility of winnings is finite because a gambler has not enough time to consume that much money involved by extremely high winnings, or because the marginal utility of money becomes zero as the gambler disposes of extremely high sums of money.⁹

Even 300 years after the discovery of the Petersburg Paradox it seems that only half of the story was given attention. All expected values depend on two components, viz. a value component and a probability component, which has ever been the daily routine in statistics (see, e.g., DeGroot (1970, pp. 89-91)). The literature on the Petersburg Paradox has exclusively paid attention to the value component and its transformations and has largely neglected the role of the probabilities.

However, in many cases a Petersburg Paradox can be “solved” by re-defining the probabilities rather than the winnings. As the winnings or the utilities of the winnings go to infinity, we can define other probabilities which shrink relatively faster to zero than the winnings or their utilities grow to infinity. This instrument can also be used in the opposite direction. When the winnings or their utilities grow to infinity, but have a finite expected value, then the probabilities can be replaced by other probabilities which shrink relatively slower to zero so that a Petersburg Paradox emerges or re-emerges.

The next section provides a rigorous analysis of the Petersburg Paradox and its solutions in terms of a uniform treatment using d’Alembert’s ratio test. This device allows us to cover both all traditional solutions of the Petersburg Paradox as well as appropriate transformations of the probabilities.

3 Rigorous Analysis of the Petersburg Paradox

THEOREM 1: Let \mathbb{N} denote the set of natural numbers and let $x : \mathbb{N} \rightarrow X \subset \mathbb{R}_+$ denote a strictly increasing mapping, i.e., $x_j > x_i \ \forall j > i, \ i, j \in \mathbb{N}$. Let $u : X \rightarrow \mathbb{R}_+$ denote a nondecreasing function such that $u(x_i) < \infty \ \forall i < \infty$, and $p : X \rightarrow [0, 1]$, a nonincreasing function such that $\sum_{i=1}^{\infty} p(x_i) = 1$, i.e., a probability distribution. Then the following holds:

Case 1: $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$ if $\exists i^* < \infty$ such that $p(x_i) = 0 \ \forall i \geq i^*$.

⁹Note that Petersburg games with infinite expected value cannot be compared even if one dominates another one for finitely many values. Hence, modern varieties of truncation were developed to come to grips with unbounded winnings and utility functions; see, e.g., DeGroot (1970, p. 112, Assumption U_6), Toulet (1986), Wakker (1993).

Case 2: $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$ if $\exists i^* < \infty$ such that $\sup_{i \geq i^*} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} < 1$.

Case 3: $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$ if $\exists i^* < \infty$ such that $\inf_{i \geq i^*} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} \geq 1$.

Case 4: $\sum_{i=1}^{\infty} u(x_i)p(x_i)$ may converge or diverge if $\lim_{i \rightarrow \infty} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = 1$.

PROOF: Case 1 is immediate. Cases 2, 3, and 4 follow from d'Alembert's ratio test (see, e.g., Stephenson (1973, pp. 75-76)). \square

COROLLARY 2: [Petersburg Case] For each probability distribution $p(x_i) > 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, which is strictly decreasing for all $i \geq i^*$, $i^* < \infty$, there exist functions $u(x_i)$ which are strictly increasing for all $i \geq i^*$, $i^* < \infty$, such that $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$.

PROOF: Applying Case 3 of Theorem 1, we consider functions $u(x_i)$ such that

$$\frac{u(x_{i+1})}{u(x_i)} > \frac{p(x_i)}{p(x_{i+1})} > 1 \quad \forall i \geq i^*,$$

that is, the growth rate of utility exceeds the shrinkage rate of probability¹⁰ for infinitely many items. If the equality sign holds instead of the inequality sign in the above relation, then Case 4 of Theorem 1 holds. \square

REMARK 3: The original Petersburg Paradox assumes $x_i = i$, $u(x_i) = 2^{i-1}$, $p(x_i) = \frac{1}{6} \left(\frac{5}{6}\right)^{i-1}$, which implies $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \sum_{i=1}^{\infty} \frac{1}{6} \left(\frac{10}{6}\right)^{i-1} = \infty$. Since $\frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = \frac{10}{6} > 1$ for all $i \geq 2$, the original Petersburg Paradox comes up to Case 3 of Theorem 1 for which $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$.

The “classical” Petersburg Paradox (in Cramer's version) assumes $x_i = i$, $u(x_i) = 2^{i-1}$, $p(x_i) = 2^{-i}$, which implies $\sum_{i=1}^{\infty} u(x_i)p(x_i) = 2^{-1} \times \infty = \infty$. Note that $\frac{u(x_{i+1})}{u(x_i)} = 2$, $\frac{p(x_i)}{p(x_{i+1})} = 2$, and $\frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = 1$ for all $i \geq 2$. That is, the “classical” Petersburg Paradox comes up to Case 4 of Theorem 1 for which $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$. Hence, the Petersburg Case may hold for some instances of Case 4 and holds quite generally for Case 3 of Theorem 1.

¹⁰ $\frac{p(x_i)}{p(x_{i+1})}$ denotes the shrinkage rate of probabilities. If it is, for instance, equal to 2, this means that $p(x_i)$ is double as high as $p(x_{i+1})$. A shrinkage rate of 4 would indicate that $p(x_i)$ is four times as high as $p(x_{i+1})$.

COROLLARY 4 (Arrow (1974)): Assume a probability distribution for which $E(x) < \infty$. Then for all concave functions $u(\cdot)$ such that $u(x_i) < \infty \forall i < \infty$, we have $E[u(x)] < \infty$.

PROOF: By Jensen's inequality we have $E[u(x)] \leq u[E(x)]$ for concave functions. Since $E(x) < \infty$ and $u(x_i) < \infty$ for $x_i < \infty$, we have $u[E(x)] < \infty$ and, hence, $E[u(x)] < \infty$. \square

REMARK 5: In his proof, Arrow (1974) did not make use of Jensen's inequality. Obviously, this theorem carries over to cases such that $E[u(x)] < \infty$ and all concave functions $\psi[u(x)]$ such that $\psi[u(x_i)] < \infty \forall i < \infty$. Note that the condition $E(x) < \infty$ or $E[u(x)] < \infty$ depend, *inter alia*, both on the probability distribution and on the function $u(\cdot)$

COROLLARY 6 (Bernoulli-Cramer Case): For any nondecreasing function $0 < u(x_i) < \infty, \forall i < \infty$, and a strictly decreasing probability distribution $p(x_i) > 0 \forall i < \infty$, such that $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$ and $\inf_{i \in \mathbb{N}} \frac{p(x_i)}{p(x_{i+1})} > 1$, there exists an increasing transformation of $u(x_i)$, viz. $\psi[u(x_i)]$, such that $\sum_{i=1}^{\infty} \psi[u(x_i)]p(x_i) < \infty$.

PROOF: By assumption $\inf_{i \in \mathbb{N}} \frac{p(x_i)}{p(x_{i+1})} = \beta > 1$. Since $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$ we have for all $i > i^*$

$$(1) \quad \frac{u(x_{i+1})}{u(x_i)} > \frac{p(x_i)}{p(x_{i+1})} > \beta,$$

because otherwise the initial series would converge. We construct $\psi(\cdot)$ so that for any $u(\cdot)$ with property (1)

$$(2) \quad \frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} < \beta.$$

To show that note that for any number $\frac{u(x_{i+1})}{u(x_i)} \geq \beta > 1$ there exists a real number $0 < \gamma(x_i, x_{i+1}) < 1$ so that $1 < \left(\frac{u(x_{i+1})}{u(x_i)}\right)^{\gamma(x_i, x_{i+1})} < \beta$. Take $\gamma^* = \inf_{i > i^*} \gamma(x_i, x_{i+1})$, then we can choose $\psi[u(x_i)] = [u(x_i)]^{\gamma^*}$ for $\psi(\cdot)$ in (2). This implies $\frac{\psi[u(x_{i+1})]p(x_{i+1})}{\psi[u(x_i)]p(x_i)} < 1 \forall i > i^*$. Hence, by Case 2 of Theorem 1 we have $\sum_{i=i^*}^{\infty} \psi[u(x_i)]p(x_i) < \infty$. \square

REMARK 7: Note that Bernoulli proposed¹¹ $\psi[u(x_i)] = \ln[u(x_i)] = \ln 2^{i-1}$, and Cramer

¹¹In addition to the gambler's winnings, recall that Bernoulli took also the gambler's wealth into consideration which we neglect in this analysis.

proposed $\psi[u(x_i)] = \sqrt{u(x_i)} = \sqrt{2^{i-1}}$. Since these functions were proposed for $p(x_i) = 2^{-i}$, we have

for the Bernoulli proposal: $\frac{\psi[u(x_{i+1})]p(x_{i+1})}{\psi[u(x_i)]p(x_i)} = \frac{i}{2(i-1)} < 1$ for $i > 2$.

for the Cramer proposal: $\frac{\psi[u(x_{i+1})]p(x_{i+1})}{\psi[u(x_i)]p(x_i)} = \frac{\sqrt{2}}{2} < 1$ for $i \in \mathbb{N}$.

REMARK 8: Note that a Petersburg Paradox is brought about by the winnings *and* the probability distribution.¹² Suppose we replace the probabilities of the Cramer solution by $\tilde{p}(x_i) = \frac{1}{K\sqrt{2^i}}$, where $K = \sum_{i=1}^{\infty} 2^{-\frac{i}{2}} = \frac{1}{\sqrt{2}-1}$ denotes a calibrating constant. Obviously,

$$\sum_{i=1}^{\infty} \tilde{p}(x_i) = \sum_{i=1}^{\infty} \frac{1}{K\sqrt{2^i}} = \sum_{i=1}^{\infty} \frac{1}{\frac{2^{\frac{i}{2}}}{2^{\frac{1}{2}}-1}} = (2^{\frac{1}{2}} - 1) \sum_{i=1}^{\infty} 2^{-\frac{i}{2}} = 1, \text{ and}$$

$$\sum_{i=1}^{\infty} \psi[u(x_i)]\tilde{p}(x_i) = \sum_{i=1}^{\infty} \frac{\sqrt{2^{i-1}}}{\frac{\sqrt{2^i}}{\sqrt{2}-1}} = \sum_{i=1}^{\infty} \left(1 - \frac{1}{\sqrt{2}}\right) = \infty,$$

which re-establishes the Petersburg Paradox.¹³ Hence, a concave transformation of the winnings does not necessarily “solve” the Petersburg Paradox. It is regained if the shrinkage rate of the probabilities is adequately slowed down.

Re-establishing a Petersburg Paradox is somewhat more tricky for the Bernoulli solution.¹⁴ We look for probabilities $\hat{p}(x_i)$ such that $\sum_{i=1}^{\infty} \ln 2^{i-1} \hat{p}(x_i) = \infty$. Since $\frac{\ln 2^i}{\ln 2^{i-1}} = \frac{i}{i-1}$, we have to look for probabilities such that $\lim_{i \rightarrow \infty} \frac{\ln 2^i \hat{p}(x_{i+1})}{\ln 2^{i-1} \hat{p}(x_i)} = 1$. Consider $\frac{\hat{p}(x_{i+1})}{\hat{p}(x_i)} = \frac{i^2}{(i+1)^2}$, which gives us

$$\lim_{i \rightarrow \infty} \frac{u(x_{i+1})\hat{p}(x_{i+1})}{u(x_i)\hat{p}(x_i)} = \lim_{i \rightarrow \infty} \frac{i^3}{(i+1)^2(i-1)} = \lim_{i \rightarrow \infty} \left(1 + \frac{1}{i-1}\right) \frac{1}{(1 + \frac{1}{i})^2} = 1.$$

¹²The equivalent roles of the winnings or their utilities on the one hand and the probability distribution on the other for expected values has ever been daily routine in statistics; see, e.g., DeGroot (1970, pp. 89-91). In the literature on the Petersburg Paradox, the role of the probabilities seems to have been largely neglected.

¹³It is easily checked that $\frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} = \frac{\sqrt{2^i}}{\sqrt{2^{i-1}}} = \sqrt{2}$ and $\frac{\tilde{p}(x_{i+1})}{\tilde{p}(x_i)} = \frac{\frac{1}{\pi\sqrt{2^{i+1}}}}{\frac{1}{\pi\sqrt{2^i}}} = \frac{1}{\sqrt{2}}$. Hence, $\frac{\psi[u(x_{i+1})]\tilde{p}(x_{i+1})}{\psi[u(x_i)]\tilde{p}(x_i)} = 1$. Recall from Theorem 1, Cases 3 and 4, that $\inf_{i \geq i^*} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} > 1$ or $\lim_{i \rightarrow \infty} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = 1$, respectively, may imply a Petersburg Paradox.

¹⁴I owe this example to Lyudmila Egorova.

Hence, the probabilities we look for are $\frac{\tilde{K}}{i^2}$, where \tilde{K} denotes a calibrating constant. \tilde{K} is computed by setting $\tilde{K} \sum_{i=1}^{\infty} \frac{1}{i^2}$ equal to 1. By the Basel problem, solved by Leonhard Euler in 1735, we have $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} = \frac{1}{\tilde{K}}$,¹⁵ which gives us

$$\hat{p}(x_i) = \frac{6}{\pi^2} \frac{1}{i^2}, \quad 0 < \hat{p}(x_i) < 1, \quad \sum_{i=1}^{\infty} \hat{p}(x_i) = 1.$$

Finally,

$$\sum_{i=1}^{\infty} \ln 2^{i-1} \hat{p}(x_i) = \frac{6 \ln 2}{\pi^2} \sum_{i=1}^{\infty} \frac{i-1}{i^2} = \frac{6 \ln 2}{\pi^2} \left[\sum_{i=1}^{\infty} \frac{1}{i} - \sum_{i=1}^{\infty} \frac{1}{i^2} \right] = \infty,$$

because the first term in the square brackets is the sum of a harmonic series, which is infinity, and the second term is $-2 < -\frac{\pi^2}{6} < \infty$. Hence, $\sum_{i=1}^{\infty} \ln 2^{i-1} \hat{p}(x_i) = \infty$, which re-establishes a Petersburg Paradox for the Bernoulli solution.

REMARK 9: As a further illustration of Remark 8 suppose we have a Petersburg game with winnings 2^{i-1} at the i -th toss, but the game is somewhat re-defined. Two coins are flipped at the same time. The gambler gets the winning 2^{i-1} if some other result than two heads comes up for the first time at the i -th toss. The respective probabilities are $\left(\frac{1}{4}\right)^{i-1} \left(\frac{3}{4}\right)$ and $\left(\frac{3}{4}\right) \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{i-1} = 1$. This implies $\left(\frac{3}{4}\right) \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{i-1} 2^{i-1} = \frac{3}{2}$. This example demonstrates that the Petersburg Paradox cannot exclusively be “solved” by a transformations of the winnings. Shrinkage rates of probabilities which exceed the growth rates of winnings perform the same job. This other source of solving the Petersburg Paradox was ignored by the literature.

COROLLARY 10: Case 2 of Theorem 1 obtains if the shrinkage rate of probability exceeds the growth rate of utility for infinitely many winnings.

Case 3 of Theorem 1 obtains if the growth rate of utility exceeds the shrinkage rate of probability for infinitely many winnings.

Case 2 can be transformed into Case 3 by an appropriate increasing convex mapping to utility or by an appropriate decreasing convex mapping to probability.

¹⁵ $\frac{\pi^2}{6}$ is approximately equal to 1.6449341... A simple demonstration of the boundedness of $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is

$$\sum_{i=1}^N \frac{1}{i^2} < 1 + \sum_{i=2}^N \frac{1}{i(i-1)} = 1 + \sum_{i=2}^N \left(\frac{1}{i-1} - \frac{1}{i} \right) = 1 + 1 - \frac{1}{N},$$

which converges to 2 as $N \rightarrow \infty$.

Case 3 can be transformed into Case 2 by an appropriate increasing concave mapping to utility or by an appropriate decreasing concave mapping to probability.

PROOF: By Theorem 1 we have for Case 2

$$1 < \frac{u(x_{i+1})}{u(x_i)} < \frac{p(x_i)}{p(x_{i+1})} \quad \forall i > i^*, i^* < \infty$$

and for Case 3

$$\frac{u(x_{i+1})}{u(x_i)} > \frac{p(x_i)}{p(x_{i+1})} > 1 \quad \forall i > i^*, i^* < \infty.$$

Case 2 can be transformed into Case 3 either by applying an appropriate increasing convex mapping $\psi(\cdot)$ to utility [an example being Menger's Super-Petersburg Paradox], or by applying an appropriate decreasing convex mapping $\phi(\cdot)$ to probability [cf. the examples shown in Remark 8]:

$$\frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} > \frac{p(x_i)}{p(x_{i+1})} > 1; \quad \frac{u(x_{i+1})}{u(x_i)} > \frac{\phi[p(x_i)]}{\phi[p(x_{i+1})]} > 1 \quad \forall i > i^*, i^* < \infty.$$

Case 3 can be transformed into Case 2 either by an appropriate increasing concave mapping $\psi(\cdot)$ to utility [examples being the Cramer and Bernoulli solutions to the Petersburg Paradox], or by applying an appropriate decreasing concave mapping to probability [an example being Remark 9]:

$$1 < \frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} < \frac{p(x_i)}{p(x_{i+1})}; \quad 1 < \frac{u(x_{i+1})}{u(x_i)} < \frac{\phi[p(x_i)]}{\phi[p(x_{i+1})]} \quad \forall i > i^*, i^* < \infty.$$

□

REMARK 11: Corollary 10 demonstrates the equivalent role of appropriate mappings of utility and probability for “solving” a Petersburg Paradox or for regaining a Petersburg Paradox.

COROLLARY 12 (Buffon Case): For all nondecreasing functions $u(x_i) < \infty$, $i < \infty$, we have $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$ for all probability distributions with $p(x_i) = 0 \quad \forall i > i^*, i^* < \infty$.

PROOF: Case 1 of Theorem 1.

□

COROLLARY 13 (Menger Case): For all strictly increasing functions $u(x_i) < \infty$, $\forall i < \infty$,

and all probability distributions $p(x_i) \geq 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$ holds if and only if $\exists B < \infty$ such that $u(x_i) \leq B \forall i \in \mathbb{N}$.

PROOF: Obviously $u(x_i) \leq B < \infty \forall i \in \mathbb{N}$ implies $\sum_{i=1}^{\infty} u(x_i)p(x_i) \leq \sum_{i=1}^{\infty} Bp(x_i) = B < \infty$.

Conversely, assume $u(x_i) \rightarrow \infty$ as $x_i \rightarrow \infty$. We rule out $p(x_i) = 0 \forall i \geq i^*$, since this would lead trivially to Case 1. Because of $\sum_{i=1}^{\infty} p(x_i) = 1$ all feasible probability distributions have to be strictly decreasing for infinitely many x_i 's. Hence, $p(x_i) > 0$ is strictly decreasing for all $i > i^*$, $i^* < \infty$. This implies $\frac{p(x_i)}{p(x_{i+1})} > 1 \forall i > i^*$, $i^* < \infty$. $\sum_{i=1}^{\infty} p(x_i) = 1$ implies $\sup_{i > i^*} \frac{p(x_i)}{p(x_{i+1})} < \infty$. If $\inf_{i > i^*} \frac{u(x_{i+1})}{u(x_i)} > \sup_{i > i^*} \frac{p(x_i)}{p(x_{i+1})}$, we can proceed along this way. If not, we choose a strictly increasing transformation $\psi(\cdot)$ such that $\inf_{i > i^*} \frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} > \sup_{i > i^*} \frac{p(x_i)}{p(x_{i+1})}$. Note that such transformation $\psi(\cdot)$ always exists: a trivial case is $\psi[u(x_i)] = [u(x_i)]^\alpha$, $\alpha > 1$. Since $\frac{u(x_{i+1})}{u(x_i)} > 1$, $\alpha > 1$ can be chosen such that $\inf_{i > i^*} \left[\frac{u(x_{i+1})}{u(x_i)} \right]^\alpha > \sup_{i > i^*} \frac{p(x_i)}{p(x_{i+1})}$. Then we have

$$\frac{\psi[u(x_{i+1})]}{\psi[u(x_i)]} > \frac{p(x_i)}{p(x_{i+1})} \Leftrightarrow \frac{\psi[u(x_{i+1})]p(x_{i+1})}{\psi[u(x_i)]p(x_i)} > 1 \forall i \geq i^*, i^* < \infty,$$

which comes up to Case 3 of Theorem 1 for $i \geq i^*$. Hence, boundedness of $u(\cdot)$ is a necessary and sufficient condition for $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$. \square

REMARK 14: Although the Menger (1934) article decisively influenced the design of von Neumann-Morgenstern expected utility, there are major differences between both concepts. Whereas for the Menger conception all ordinal transformations of a feasible utility function are admissible utility functions to be screened, a utility function according to the von Neumann-Morgenstern conception is cardinal, i.e., unique only up to an increasing linear transformation. Hence, whenever von Neumann-Morgenstern expected utility is finite, then it is finite for all feasible (i.e., increasing linear) transformations. Note that the von Neumann-Morgenstern axioms envisage some calibration of results, so that the best result is usually assigned the utility value of 1, and the worst result the utility value of 0. The utility values of all other results follow from considering them as certainty equivalents of binary lotteries with the best and worst results as the prizes. The equivalent probability of these lotteries of receiving the best prize is then set equal to the utility of the respective intermediate results (cf., e.g., Luce and Raiffa (1957, pp. 21-3)). Since ordinal

transformations of this setting would paralyze the utility function, only increasing linear transformations are allowable under von Neumann-Morgenstern expected utility. Menger (1934) had something different in mind, viz. to state a condition utility functions have to satisfy to rule out the occurrence of a Petersburg Paradox. Neither did he provide an axiomatization of expected utility, nor did he consider the role of probabilities to “solve” a Petersburg Paradox.

4 Conclusion

In the French *ancien régime*, many aristocrats spent their time with games of hazard. For judging the advantage of games of hazard, contemporary mathematicians proposed the expected value of the winnings less the stake of the game. Nicolas Bernoulli formulated a game of hazard with infinite expected value, but no gambler would stake more than a moderate amount for participation in this gamble. Bernoulli’s gamble became known by dint of de Montmort’s book *Essay d’analyse sur les jeux de hazard*. The challenge of this gamble soon provoked attempts at its “solution”. Cramer and Daniel Bernoulli suggested appropriate concave transformations of the winnings, Buffon and others suggested truncation of the probabilities or of the winnings, by setting small probabilities equal to zero or by realizing that no bookmaker can guarantee extremely high winnings. Menger was the first to show that boundedness of utility functions is a necessary and sufficient condition to warrant a finite expected value of a Petersburg gamble. His seminal article paved the way for von Neumann and Morgenstern’s development of expected utility theory.

In third section it is shown that all four possible cases of a Petersburg Paradox can be uniformly treated using d’Alembert’s ratio test. This test compares the growth rates of utility and the shrinkage rates of probability. If the first exceeds the second, we encounter infinite expected utility, if the second exceeds the first, we encounter finite expected utility. If the limit of the ratio of their product is equal to one, we can have either infinite or finite expected utility. A finite expected value can be transformed into an infinite expected value either by applying an appropriate increasing convex mapping to utility, or by applying an appropriate decreasing convex mapping to probability. An infinite expected value can be transformed into a finite expected value by applying an appropriate increasing concave mapping to utility, or by applying an appropriate decreasing concave mapping to probability. Employing probability mappings for transforming Petersburg Paradoxa is a novel feature of analysis.

References

- Arrow, Kenneth J. 1970. *Essays in the Theory of Risk-Bearing*. Amsterdam and London: North-Holland Publishing Company.
- Arrow, Kenneth J. 1974. "The Use of Unbounded Utility Functions in Expected-Utility Maximization: Response." *The Quarterly Journal of Economics*, 88: 136-8.
- Aumann, Robert J. 1977. "The St. Petersburg Paradox: A Discussion of Some Recent Comments." *Journal of Economic Theory*, 14: 443-5.
- Bentham, Jeremy. 1781/2000. *An Introduction to the Principles of Morals and Legislation*, reprint: Kitchener: Batoche Books.
- Bernoulli, Daniel. 1738. "Specimen theoriae novae de mensura sortis." *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 5: 175-92.
- Bernoulli, Daniel. 1896. *Die Grundlage der modernen Wertlehre: Versuch einer neuen Theorie der Wertbestimmung von Glücksfällen. Aus dem Lateinischen übersetzt und mit Erläuterungen versehen von Professor Dr. Alfred Pringsheim. Mit einer Einleitung von Dr. Ludwig Fick*. Leipzig: Verlag von Duncker & Humblot.
- Bernoulli, Daniel. 1954. "Exposition of a New Theory on the Measurement of Risk." English translation by Louise Sommer, *Econometrica*, 22: 23-36.
- Bernoulli, Nicolas. 1713-1732. *Correspondence of Nicolas Bernoulli concerning the St. Petersburg Game*, translated by Richard J. Pulskamp. [Downloadable from <http://www.cs.xu.edu/math/Sources/Montmort/stpetersburg.pdf>]
- Blavatsky, Pavlo R. 2005. "Back to the St. Petersburg Paradox?" *Management Science*, 51: 677-8.
- Brito, Dagobert L. 1975. "Becker's Theory of the Allocation of Time and the St. Petersburg Paradox." *Journal of Economic Theory*, 10: 123-6.
- Buffon, Georges Louis Marie Leclerc, Comte de. 1777/2010. *Histoire Naturelle, vol. 4, Supplement*, English translation in: John D. Hey, Tibor M. Neugebauer, and Carmen M. Pasca, Université de Luxembourg Research Paper Series No. 10-06. [Downloadable from http://eprints.luiss.it/769/1/10-06_hey_2010.pdf]
- Cowen, Tyler, and Jack High. 1988. "Time, Bounded Utility, and the St. Petersburg Paradox." *Theory and Decision*, 25: 219-33.

- DeGroot, Morris H. 1970. *Optimal Statistical Decisions*. New York etc.: McGraw-Hill Book Company.
- de Montmort, Pierre Rémond. 1713. *Essay d'analyse sur les jeux de hazard. Revue & augmentée de plusieurs Lettres*. 2nd edition, Paris: Jacque Quillau.
- Fechner, Gustav Theodor. 1860. *Elemente der Psychophysik*. Leipzig: Breitkopf und Härtel.
- Fishburn, Peter C. 1970. *Utility Theory for Decision Making*. New York etc.: John Wiley & Sons, Inc.
- Fry, Thornton C. 1928. *Probability and its Engineering Uses*. New York: Van Nostrand.
- Hicks, John Richard and Roy George Douglas Allen. 1934. "A Reconsideration of the Theory of Value." *Economica (N. S.)*, 1: 52-76.
- Kahneman, Daniel and Amos Tversky. 1979. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, 47: 263-91.
- Kuhn, Harold William and Albert William Tucker. 1958. "John von Neumann's Work in the Theory of Games and Mathematical Economics." *Bulletin of the American Mathematical Society*, 64 (3) Part II: 100-22.
- Luce, R. Duncan and Howard Raiffa. 1957. *Games and Decisions. Introduction and Critical Survey*. New York: John Wiley & Sons, Inc.
- Menger, Karl. 1934. "Das Unsicherheitsmoment in der Wertlehre, Betrachtungen im Anschluß an das sogenannte Petersburger Spiel." *Zeitschrift für Nationalökonomie*, 5: 459-85.
- Menger, Karl. 1967. "The Role of Uncertainty in Economics." In: Martin Shubik (ed.), *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, Princeton: Princeton University Press, 211-31.
- Morgenstern, Oskar. 1976. "The Collaboration Between Oskar Morgenstern and John von Neumann on the Theory of Games." *Journal of Economic Literature*, 14: 805-16.
- Neugebauer, Tibor. 2010. "Moral Impossibility in the Petersburg Paradox : A Literature Survey and Experimental Evidence." LSF Research Working Paper Series No. 10-174, Université de Luxembourg.

- Neumann, John von and Oskar Morgenstern. 1947. *The Theory of Games and Economic Behavior*. 2nd edition, Princeton NJ: Princeton University Press.
- Samuelson, Paul A. 1977. "St. Petersburg Paradoxes: Defanged, Dissected, and Historically Described." *Journal of Economic Literature*, 15: 24-55.
- Savage, Leonard J. 1972. *The Foundations of Statistics*. 2nd edition, New York: Dover Publications.
- Shapley, Lloyd S. 1977a. "The St. Petersburg Paradox: A Con Game?" *Journal of Economic Theory*, 14: 439-42.
- Shapley, Lloyd S. 1977b. "Lotteries and Menus: A Comment on Unbounded Utilities." *Journal of Economic Theory*, 14: 446-53.
- Stephenson, G. 1973. *Mathematical Methods for Science Students*. 2nd edition, Harlow (Essex): Longman.
- Stigler, George J. 1950a. "The Development of Utility Theory. I." *Journal of Political Economy*, 58(4): 307-27.
- Stigler, George J. 1950b. "The Development of Utility Theory. II." *Journal of Political Economy*, 58(4): 373-96.
- Toulet, Claude. 1986. "An Axiomatic Model of Unbounded Utility Functions." *Mathematics of Operations Research*, 11(1): 81-94.
- Tversky, Amos and Daniel Kahneman. 1992. "Advances in Prospect Theory: Cumulative Representation of Uncertainty." *Journal of Risk and Uncertainty*, 5: 297-323.
- Viner, Jacob. 1925a. "The Utility Concept in Value Theory and Its Critics." *Journal of Political Economy*, 33(4): 369-87.
- Viner, Jacob. 1925b. "The Utility Concept in Value Theory and Its Critics." *Journal of Political Economy*, 33(6): 638-59.
- Wakker, Peter P. 1993. "Unbounded Utility for Savage's 'Foundations of Statistics', and other Models." *Mathematics of Operations Research*, 18: 446-85.
- Weber, Ernst Heinrich. 1834. *De pulsu, resorptione, auditu et tactu. Annotationes anatomicae et physiologicae*. Leipzig: Koehler.