

Brekke, Kjell Arne; Rege, Mari

Working Paper

Advertising as distortion of learning in markets with network externalities

Memorandum, No. 2006,24

Provided in Cooperation with:

Department of Economics, University of Oslo

Suggested Citation: Brekke, Kjell Arne; Rege, Mari (2006) : Advertising as distortion of learning in markets with network externalities, Memorandum, No. 2006,24, University of Oslo, Department of Economics, Oslo

This Version is available at:

<https://hdl.handle.net/10419/63104>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

MEMORANDUM

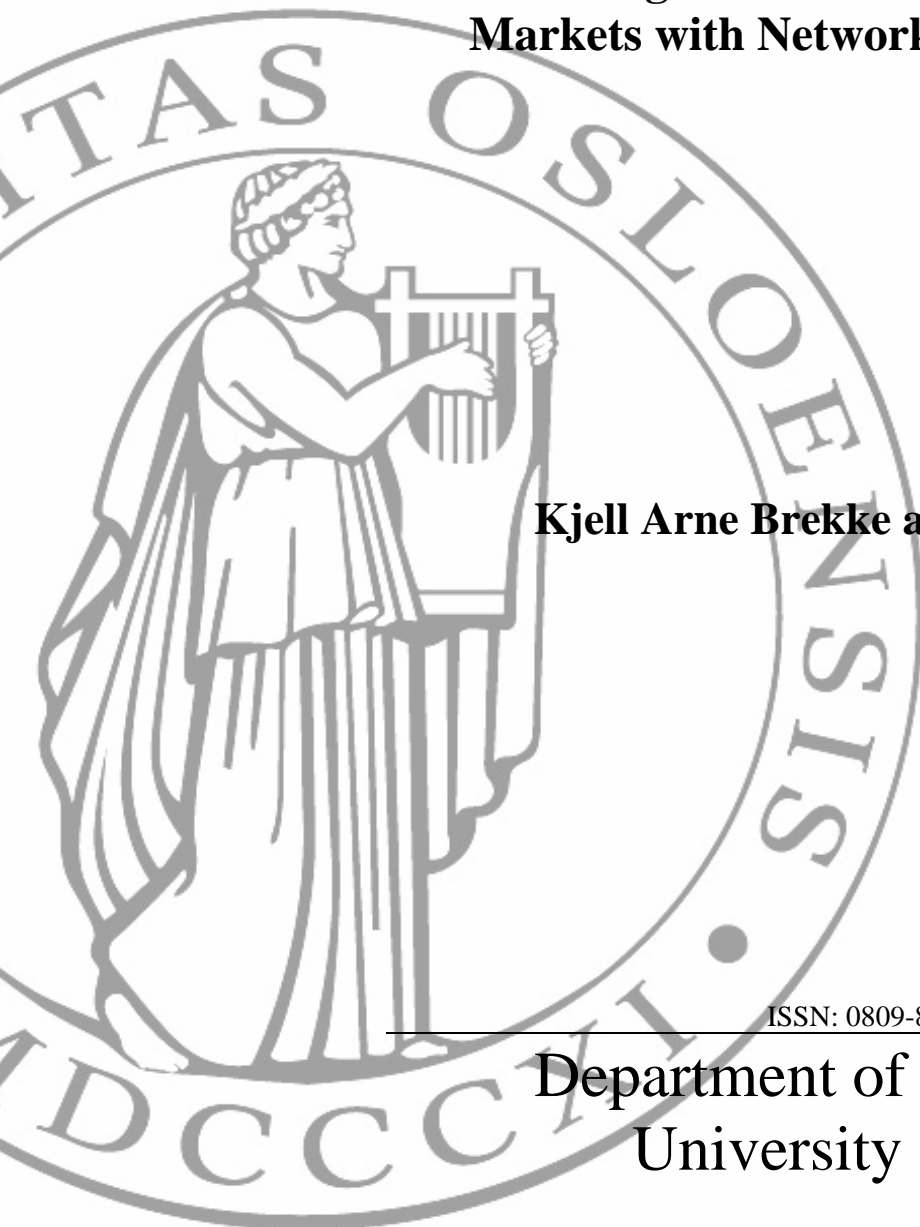
No 24/2006

Advertising as Distortion of Learning in Markets with Network Externalities

Kjell Arne Brekke and Mari Rege

ISSN: 0809-8786

Department of Economics
University of Oslo



This series is published by the
University of Oslo
Department of Economics

P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: <http://www.oekonomi.uio.no/>
e-mail: econdep@econ.uio.no

In co-operation with
**The Frisch Centre for Economic
Research**

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
Internet: <http://www.frisch.uio.no/>
e-mail: frisch@frisch.uio.no

List of the last 10 Memoranda:

No 23	Advertising as a Distortion of Social Learning Kjell Arne Brekke and Mari Rege. 22 pp.
No 22	Line Smart Bakken The Golden Age of Retirement. 44 pp.
No 21	Rolf Golombek and Michael Hoel Climate agreements:emission quotas versus technology policies. 34 pp.
No 20	John K. Dagsvik, Marilena Locatelli and Steinar Strøm Simulating labor supply behavior when workers have preferences for job opportunities and face nonlinear budget constraints. 46 pp.
No 19	Cathrine Hagem Clean development mechanism (CDM) vs. international permit trading – the impact on technological change. 27 pp.
No 18	Geir B. Asheim, Tapan Mitra and Bertil Tungodden Sustainable recursive social welfare functions. 32 pp.
No 17	Geir B. Asheim, Walter Bossert, Yves Sprumont and Kotaro Suzumura Infinite-horizon choice functions. 24 pp.
No 16	Erik Hernæs, Fedor Iskhakov and Steinar Strøm Early Retirement and Company Characteristics. 37 pp.
No 15	Erik Hernæs, Zhiyang Jia and Steinar Strøm Retirement in Non-Cooperative and Cooperative Families. 38 pp.
No 14	Hilde Bojer Income Capability and Child Care. 23 pp.

A complete list of this memo-series is available in a PDF® format at:
<http://www.oekonomi.uio.no/memo/>

Advertising as Distortion of Learning in Markets with Network Externalities

Kjell Arne Brekke and Mari Rege*

November 23, 2006

Abstract

We present a theory of how advertising can break a lock-in by distorting beliefs about market shares in markets with network externalities. On the background of the availability heuristic we assume that people learn about market shares by observing product adoption of others, but are not able to fully distinguish between observations of real people and fictitious characters in advertisements. We look at a game between an incumbent and an entrant producing close substitutes. Our analysis shows that if the entrant's product is of sufficiently high quality, then the entrant will use advertising in order to break the lock-in and the incumbent will not advertise at all. However, if the quality differential between the two products is small, then the incumbent may advertise and make it unprofitable for the entrant to break the lock-in.

JEL Classification Numbers: D21, L10, M37

Keywords: Advertising, availability heuristic, herding behavior, information, lock-in.

*We are grateful to Tore Nilssen for helpful comments. *Authors' contact information:* Kjell Arne Brekke: Frisch Centre of Economic Research, Gaustadalleen 21, 0349 Oslo, Norway. E-mail: k.a.brekke@frisch.uio.no. Mari Rege: Universitetet i Stavanger, 4036 Stavanger, Norway. E-mail: mari.rege@uis.no

1 Introduction

In many markets, the benefits of choosing one product over the others will depend on the share of adopters. In particular, it is often the case that a consumer's benefit from choosing a product increases with the number of other consumers using the product. This is known as network externalities (Katz and Shapiro 1985). An important example is fuel cell cars. Most consumers prefer a traditional gasoline car to a fuel cell car as long as most others own a gasoline car. This is because an infrastructure providing easy access to refill of fuel cells will not be provided unless there is a sufficiently large demand.

In markets with network externalities, even if many consumers believe an alternative product is inherently better, these consumers may not adopt the alternative product unless they expect a sufficiently large share of other people to also adopt this product. Thus, the market can be locked-in on the particular product currently dominating the market, even if this product is inferior to alternatives. Clearly, consumers' beliefs about market shares are crucial as to whether the market will be locked-in on an inferior product (Farrell and Saloner 1985 and 1986, Katz and Shapiro 1985 and 1992). In this paper we will present a theory of how advertising can possibly break a lock-in by distorting beliefs about market shares in markets with network externalities.

Our theory of how advertising distorts beliefs about market shares is based on the "availability heuristic" proposed by Tversky and Kahneman (1973). The availability heuristic says that people infer the prevalence of an event "from the ease with which the event can be recalled or imagined" (Tversky and Kahneman 1973). That is, a person who feels that one product seems more familiar than another, infers that he must have seen the familiar product more often. The availability heuristic has been supported by several experimental studies (see Schwarz and Vaughn 2002) and empirical investigations (Schrum 1999).

The availability heuristic suggests that firms can influence people's perception of market shares by making them more familiar with their products (Brekke and Rege 2006). This can be done by exposing people to images of others using their products in advertisements¹. People learn about market shares

¹This application of the availability heuristics can be seen as a formal model of the "social proof" principle of

by observing product adoption of others, but are not able to fully distinguish between observations of real people and fictitious characters in advertisements. Consequently, a person presented with a choice between two products may feel that one is more familiar than the other. He is, however, not able to fully detect whether this familiarity is due to exposure to advertisements. For illustrative purposes, let us revisit the fuel cell car example: A consumer considering a fuel cell car has to assess the future availability of required infrastructure. His assessment will depend on the share of people buying and planning to buy a fuel cell car. This prevalence assessment can be distorted by advertising.

To study firms' advertising behavior in markets with network externalities, we look at a game between an incumbent and an entrant selling their products to consumers influenced by the availability heuristics. The entrant's product is a close substitute to the incumbent's product. Both products exhibit increasing returns to adoption: The benefit of possessing a product is increasing with the share of people who have adopted the product. Moreover, the entrant's product is inherently better than the incumbent's product. The incumbent has a first mover advantage and decides in the first stage how much to advertise. The entrant observes the incumbent's advertising decision and then decides how much to advertise.

The analysis shows that advertising can break a lock-in if and only if the entry product is of sufficiently higher quality than the incumbent product. In order for the entrant to capture the market, he needs to advertise such that the share of consumers who have adopted the entry product is above a certain threshold level. This threshold is smaller, and thus the necessary advertising expenditures for capturing the market are lower, the higher the entrant's product quality. Thus, if the entrant's product quality is sufficiently high, then his benefits from capturing the market by advertising outweigh the costs. The incumbent realizes that it is a "lost game", and that he is better off not advertising at all.

If the quality differential between the incumbent and entrant is small, then the incumbent may advertise in order to make it unprofitable for the entrant to break the lock-in. We refer to this as preemptive advertising. In this case, the entrant realizes that it is a "lost game", and that he is

marketing, discussed in Cialdini (1993).

better off not advertising at all. For very small product quality differentials the incumbent does not even need to advertise in order to make it unprofitable for the entrant to advertise. In this case the incumbent maintains the lock-in even without advertising.

Economists have long been concerned about how advertising can deter entrance and reduce competition² (see e.g. Williamson 1963, Needham 1976, Salop 1979, Spence 1980, Schmalensee 1983, Bagwell and Ramey 1990). Our paper differs from these treatments in that it analyzes how advertising can affect competition in markets with network externalities. Moreover, our paper suggest an explicit mechanism through which advertising affects behavior in market with network externalities. In this way our paper is related to Pastine and Pastine (2002), who provides an alternative mechanism for advertising in markets with network externalities. The authors demonstrate that in such markets advertising may function as a device to coordinate consumer expectations of other consumers' purchasing decisions. While Pastine and Pastine (2002) look at fully rational individual, we look at boundedly rational individuals who are not fully able to distinguish between observation of real people and observations from advertisements. Thus, our paper adds to the recently emerging economic literature on advertising and bounded rationality (see e.g. Gabaix and Laibson 2004, Shapiro 2005, Loginova 2005, Krämer 2004).

2 Lock-In

A lock-in is a situation in which a product is maintaining the majority of the market despite the presence of a higher quality close substitute. In this section we will present a simple model illustrating conditions under which a lock-in can arise³. This model will allow us later, in section 5, to study firms' advertising behavior in markets with network externalities.

Look at the market for two durable products X and Y , which are close substitutes. At any given

²For an overview of this literature, see Bagwell's (2005) excellent survey of economic analyses of advertising, chapter 7.

³Our model is similar to the formulation in Farrell and Saloner (1986).

time there is a continuum $[0, 1]$ of people possessing either good X or good Y . Let $x(t)$ denote the share of people possessing X at time t . Assume that both product X and Y exhibit increasing returns to adoption: The benefit of possessing X (Y) is higher, the larger the share of people who have adopted X (Y). Moreover, product Y is inherently better than product X , meaning that a person will always prefer Y to X if at least as many people have adopted Y as X . We will capture this by assuming that a person i 's payoff π_i of possessing X or Y at time t is given by:

$$\begin{aligned}\pi_i(X) &= x(t) \\ \pi_i(Y) &= 1 - x(t) + \gamma_i,\end{aligned}\tag{1}$$

where γ_i is uniformly distributed on $[0, a]$ and $a < 1$. Note that for a given $x(t)$, a larger a will increase the share of people who prefer Y to X . Thus, in the following we will refer to a larger a as a larger product quality differential between Y and X .

In every time interval τ a random small share τ of people possessing either X or Y stop using their product, and equally many new people enter the market for X and Y . We will refer to the people entering the market for X and Y at time t as the consumers at time t . These consumers have to decide whether to buy product X or Y . A consumer will choose X if her expected utility of choosing X is higher than her expected utility of choosing Y .

The consumers at any time t have imperfect information about the market share $x(t)$. We will capture this by assuming that with probability $\frac{1}{2}$ a consumer is *informed* and receives perfect information about $x(t)$, while with probability $\frac{1}{2}$ a consumer is *uninformed* and does not receive any direct information about $x(t)$. An uninformed consumer believes a priori that X is uniformly distributed on $[0, 1]$. Before making her choice, however, she observes the choice of one random individual who currently possess X or Y and updates her beliefs about $x(t)$ accordingly. We will denote this observation by $o \in \{X, Y\}$.

Let⁴ $\sigma_I(x)$ denote the share of informed consumers choosing X , given that a share x of the

⁴In the remaining we will often omit t for notational convenience.

population is currently using X . Equation (1) implies that an informed consumer i chooses X if

$$x > 1 - x + \gamma_i$$

Since γ_i is uniformly distributed on $[0, a]$, this implies that

$$\sigma_I(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ \frac{2x-1}{a} & \text{if } \frac{1}{2} < x < \frac{1+a}{2} \\ 1 & \text{if } x > \frac{1+a}{2} \end{cases} \quad (2)$$

Let $\sigma_U(x)$ denote the share of uninformed consumers choosing X , given that a share x of the population is currently using X . If a person observes X , her updated beliefs about x are given by

$$\Pr(x|o = X) = \frac{\Pr(o = X|x)f(x)}{\int_0^1 \Pr(o = X|x)f(x)dx} = \frac{x}{\int_0^1 x dx} = 2x,$$

where $f(x) = 1$ is the prior (uniform) probability distribution. Thus, equation (1) implies that an uninformed consumer i who observes X chooses X if

$$\begin{aligned} \int_0^1 2x^2 dx &> \int_0^1 2x(1-x) dx + \gamma_i \\ \Leftrightarrow \frac{1}{3} &> \gamma_i \end{aligned} \quad (3)$$

This implies that if $a < \frac{1}{3}$, then all uninformed consumers who observe X will choose X . To focus on the most interesting case⁵ we will for the remainder of the paper we assume that $a > \frac{1}{3}$. Now, since γ_i is uniformly distributed on $[0, a]$, it follows from (3) that a share $\frac{1}{3a}$ of the uninformed who observe X choose X . Of course, all uninformed consumers who observe Y choose Y , since Y is inherently better than product X . Thus, since a share x of the uninformed consumer observes X , the share of uninformed consumers choosing X is given by

$$\sigma_U(x) = \frac{x}{3a} \quad (4)$$

Recall that in every time interval τ a random small share τ of people possessing X or Y stop using their product, and equally many new people enter the market for X and Y . This implies the following

⁵This is most interesting because it is harder to maintain a lock-in, the larger the quality differential.

dynamics⁶ for adoption of X :

$$x(t + \tau) - x(t) = -\tau x(t) + \tau \frac{1}{2} (\sigma_I(x(t)) + \sigma_U(x(t))) \quad (5)$$

The first term in equation (5) reflects the number of people who stop using X , while the second term reflects the number of new people adopting X . Equation (5) implies the following dynamics:

$$\begin{aligned} \dot{x} &= \lim_{\tau \rightarrow 0} \frac{x(t + \tau) - x(t)}{\tau} \\ &= \begin{cases} \frac{1}{2} \left(\frac{x}{3a} - 2x \right) & \text{if } x < \frac{1}{2} \\ \frac{1}{2} \left(\frac{x}{3a} - 2x + \frac{2x-1}{a} \right) & \text{if } \frac{1}{2} < x < \frac{1+a}{2} \\ \frac{1}{2} \left(\frac{x}{3a} - 2x + 1 \right) & \text{if } x > \frac{1+a}{2} \end{cases} \quad (6) \end{aligned}$$

The dynamics for adoption of X represented by equation (6) allows us to study the possibility of a lock-in on X . Figure 1 illustrates what is proven in Proposition 1. The figure plots \dot{x} as given in equation (6) for different quality differentials⁷. An increase in the quality differential between the two products shifts the \dot{x} curve downwards. Note that for the lowest quality differential, the \dot{x} curve crosses zero at a high market share x_H . Moreover, in the area close to x_H , $\dot{x} > 0$ for $x < x_H$ and $\dot{x} < 0$ for $x > x_H$. The market share thus locally converges to x_H . Thus, a lock-in at a high market share is possible if the quality differential between the two products is not too big (i.e. $\frac{1}{3} < a < \frac{1}{2}$). For $a = \frac{1}{2}$ the \dot{x} curve is the line just touching zero at x_H . Now, $x = x_H$ is not a stable point since in the area close to x_H , $\dot{x} < 0$ for all $x \neq x_H$. A lock-in is thus not possible if the quality differential is sufficiently large (i.e. $a \geq \frac{1}{2}$). Then, product Y is so much better than X that some new consumers will choose Y even when everybody is using X . These consumers will induce a multiplier effect reinforcing the future consumers' incentives to use Y .

In the case where a lock-in is possible, let \hat{x} denote the market share where the increasing part of the dynamic path crosses zero⁸. We then see from Figure 1 that if Y has a sufficiently large market

⁶This is similar to learning dynamics in evolutionary game theory. See e.g. Gale, Binmore and Samuelson (1995).

⁷Recall that the quality differential between the two product is larger, the larger a . In the figure we have used the values $a = 0.4, 0.5$ and 0.6 .

⁸ \hat{x} is formally defined in the proof of Proposition 1.

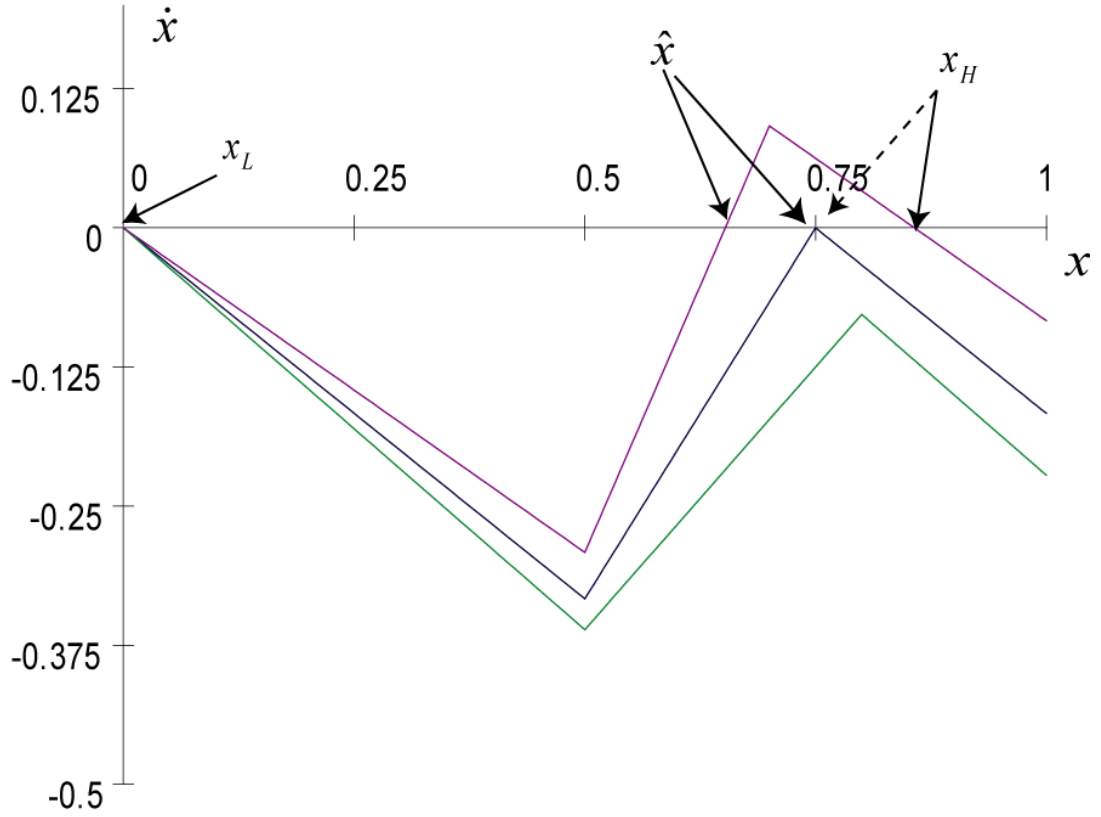


Figure 1: The dynamics of market shares

share (i.e. $x < \hat{x}$) an increasing number of consumers will buy Y ($\dot{x} < 0$). Hence, in order to establish a new product Y in the market, the new product must reach a market share of $1 - \hat{x}$. The better the quality of the new product Y (larger a), the easier it is to establish Y .

Proposition 1

- *There is an asymptotically stable state in which a share $x_L = 0$ (i.e. nobody) consumes the inferior good X .*
- *If $\frac{1}{3} < a < \frac{1}{2}$, then there is an asymptotically stable state in which the market is locked in on the inferior good X . In this look-in a share $x_H = \frac{3a}{6a-1}$ consumes X .*

Proof. See Appendix. ■

3 Availability Heuristic

The model presented in the previous section illustrated how beliefs about market share are a crucial factor as to whether the market will be locked in on the inferior product in markets with network externalities. In these markets, even if many consumers think an alternative product is inherently better, these consumers may not adopt the alternative product unless they expect a sufficiently large share of other people to also adopt this product. In this section we will, based on the “availability heuristic”, proposed by Tversky and Kahneman (1973), make the argument that advertising can affect beliefs about market shares in markets with network externalities.

Tversky and Kahneman (1973) suggest that people infer the prevalence of an event “from the ease with which the event can be recalled or imagined.” That is, if it is easier for a person to imagine H than to imagine L , then the person infers that H happens more frequently than L . The availability heuristic has been detected in several experimental studies (see Schwarz and Vaughn 2002). Moreover, Schrum (1999) argues that the availability heuristic can explain several empirical studies linking television watching to greater perceptions of the prevalence of violent crime, prostitution, alcoholism, drug abuse, divorce, heroic doctors, and private swimming pools. Schrum argues that frequent television watching increases the ease with which a person can imagine these types of events and thus makes him or her overestimate the prevalence of these events.

The availability heuristic suggests that people are not able to fully distinguish between different sources of information. Based on this theory, we will in the following assume that consumers are unable to distinguish between observations of real people and fictitious characters in advertisements⁹. A person presented a choice between H and L may feel that H is more familiar than L . He is, however, not able to detect whether this is due to his exposure to advertisements. This is in line with important findings in cognitive psychology indicating clear functional differences between familiarity

⁹Clearly a more realistic assumption would be that consumers have some ability, however imperfect, to distinguish between observations of real people and fictitious characters. Such an assumption would, however, further complicate our analysis without altering the results.

and recollection¹⁰. When a person sees something that is familiar, the source of that familiarity is often ambiguous.

4 Advertising as Distortion of Learning

In this section we introduce firms' possibility to advertise into the model presented in Section 2. Assume that firms can expose people to images of others using their product in advertisements. As in Section 2, prior to making his choice of whether to buy product X or Y , a consumer observes the product choice of one other person. Now, however, this person can either be a fictitious character in an advertisement or a real person possessing product X or Y . Let this observation be denoted by $o \in \{X, Y\}$. Based on the evidence of the availability heuristics reviewed in the previous section we will assume that consumers are unable to distinguish between observations of real people and fictitious characters in advertisements.

Let z_i reflect the size of i 's advertising campaign by denoting an individual's likelihood of observing product i in an advertisement. Assume that if a person does not observe product X or Y in an advertisement, then he observes product X with likelihood x , where x denotes the share of people possessing X . Thus, an individual's likelihood of observing either a fictitious character in an advertisement or a real person using X is given by

$$\Pr(o = X|x, z_X, z_Y) = z_X + (1 - z_X - z_Y)x$$

As in the case with no advertising¹¹ a share $\frac{1}{3a}$ of the uninformed consumer who observes X chooses X . Moreover, all uninformed consumers who observe Y choose Y , since Y is inherently better than product X . Then, since a share $z_X + (1 - z_X - z_Y)x$ of the uninformed consumers observe X , the share of uninformed consumers choosing X is given by¹²

¹⁰See Kelley and Jacoby (2000) for a survey of this literature.

¹¹We assume that the consumers are naive, i.e. they do not take advertising into consideration when updating their beliefs about X .

¹²Note that we must always have $z_X + z_Y \leq 1$. We thus assume that $z_X, z_Y \leq 1/2$, and with the below cost

$$\sigma_U(x) = (z_X + (1 - z_X - z_Y)x) \frac{1}{3a}$$

As in the model with no advertising, the share of informed consumers choosing X is given by $\sigma_I(x)$ in equation (2). Plugging in for $\sigma_U(x)$ and $\sigma_I(x)$ in equation (5) gives us the following dynamics for adoption of X :

$$\begin{aligned} \dot{x}(x) &= \lim_{\tau \rightarrow 0} \frac{x(t+\tau) - x(\tau)}{\tau} \\ &= \begin{cases} \frac{1}{2}(z_X + (1 - z_X - z_Y)x) \frac{1}{3a} - x & \text{if } x < \frac{1}{2} \\ \frac{1}{2}(z_X + (1 - z_X - z_Y)x) \frac{1}{3a} - x + \frac{2x-1}{2a} & \text{if } \frac{1}{2} \leq x \leq \frac{1+a}{2} \\ \frac{1}{2}(z_X + (1 - z_X - z_Y)x) \frac{1}{3a} - x + \frac{1}{2} & \text{if } x > \frac{1+a}{2} \end{cases} \end{aligned} \quad (7)$$

Note that for $z_X = z_Y = 0$ (i.e. no advertising) the dynamics in equation (7) is identical to (6). Moreover, increasing z_X will shift the dynamic path upwards, whereas increasing z_Y will shift it downwards. Indeed, if Y 's advertising campaign is sufficiently larger than X 's campaign (i.e. z_Y sufficiently large), then there exist no asymptotically state in which the market is locked in on X . We will denote this critical level of firm Y advertising as $\hat{z}_Y(z_X, a)$. To determine \hat{z}_Y it is useful to establish the following Lemma:

Lemma 2 *The condition $\dot{x}\left(\frac{1+a}{2}\right) = 0$ uniquely determines a function $z_Y^*(z_X, a)$, where $\frac{\partial z_Y^*}{\partial z_X} > 0$ and $\frac{\partial z_Y^*}{\partial a} < 0$ for all z_X and $a < 1$.*

Proof. See Appendix. ■

Lemma 2 allows us to define the critical level of firm Y advertising as

$$\hat{z}_Y(z_X, a) = \max\{0, z_Y^*(z_X, a)\}, \quad (8)$$

where Y will break a lock-in if $z_Y \geq \hat{z}_Y$ (this will be proven in Proposition 3). Let $x_H(z_X, z_Y)$ denote the solution to $\dot{x}(x) = 0$ for $x > \frac{1+a}{2}$ and let $x_L(z_X, z_Y)$ denote the solution to $\dot{x}(x) = 0$ for $x < \frac{1}{2}$.

assumptions, this constraint will be non-binding.

This notation allows us to establish Proposition 3 stating conditions under which the two different types of stable states exist.

Proposition 3

- a) If $z_Y < \hat{z}_Y(z_X, a)$, then there are two asymptotically stable states: On state, $x_H(z_X, z_Y) > \frac{1+a}{2}$, in which the market is locked in on the inferior good X , and another state, $x_L(z_X, z_Y) < \frac{1}{2}$, in which the majority consumes the non-inferior good Y .
- b) If $z_Y \geq \hat{z}_Y(z_X, a)$, then there is a unique asymptotically stable, $x_L(z_X, z_Y) < \frac{1}{2}$, in which the majority consumes the non-inferior good Y .

Proof. See Appendix. ■

Note from Proposition 3 that if $z_Y \geq \hat{z}_Y(z_X, a)$, then x_H is no longer a stable state and the economy will move to the stable state x_L where Y has the majority of the market. This suggests that by advertising s.t. $z_Y \geq \hat{z}_Y(z_X, a)$, Y can break a lock-in on X . However, depending on the cost of advertising, this may not be profitable for Y . In particular, since $\frac{\partial \hat{z}_Y}{\partial z_X} > 0$, the amount of advertising necessary to break a lock-in is increasing in z_X . This may give X a possibility to make it unprofitable for Y to break the lock-in. We will refer to a situation in which X advertises in order to maintain the lock-in as *preemptive advertising*. In the next section we investigate conditions under which Y will break a lock-in and conditions under which there will be preemptive advertising.

5 Breaking the Lock-in and Preemptive Advertising

In this section we will endogenize the sizes of firms and 's advertising campaigns. We will refer to firm X as the incumbent and firm Y as the entrant. We look at a two stage game where the incumbent has a first mover advantage and decides first how much to advertise. The entrant observes the incumbent's advertising decision and then decides how much to advertise. We assume that the resulting market shares are determined by the continuous time model of the product market presented in Section 4.

We assume that this market converges quickly to a new steady state, and that the firms' payoffs are determined by the new steady state.

The following analysis allows us to investigate conditions under which Y will break a lock-in and conditions under which there will be preemptive advertising. Our analysis will show that if Y 's product is of sufficiently high quality, then it is profitable to use advertising in order to break a lock-in. We refer to this as *advertising breaking the lock-in*. However, if the quality differential between Y and X is small, then the incumbent may advertise in order to make it unprofitable for Y to break the lock-in. We refer to this as *preemptive advertising*.

Assume now that X is the incumbent initially having the whole market. Then, another firm tries to enter this market by introducing a product Y that is a close substitute to X . Both product X and Y exhibit increasing returns to adoption: The benefit of possessing X (Y) is higher, the larger the share of people who have adopted X (Y). Moreover, product Y is inherently better than X as described in Section 2. Look at the two stage game illustrated in Figure 2:

The incumbent X has a first mover advantage and decides in the first stage how much to advertise, z_X . The entrant Y observes firm X 's advertising decision, z_X , and then decides how much to advertise, z_Y . Each firm i 's cost of advertising is kz_i . Each firm's payoff π_i is given by the resulting stable steady state market share (derived in Section 4) minus its advertising costs kz_i . We know from Proposition 3 that if Y chooses $z_Y \geq \hat{z}_Y$ in the second stage, then the stable steady state market share of X is $x_L(z_X, z_Y)$, whereas if Y chooses $z_Y < \hat{z}_Y$, then the steady state market share of X is $x_H(z_X, z_Y)$ (since the incumbent initially had the whole market). The game will be solved by backwards induction.

Recall that $\hat{z}_Y(z_X, a)$ denotes the minimum advertising level necessary for Y to break a lock-in. We start our analysis by establishing Lemma 4 saying that if the cost of advertising is sufficiently large¹³ (i.e. $k > 2$), then no more than one firm will advertise in equilibrium. If one firm advertises, then it will either be the entrant advertising the minimum amount necessary in order to break the

¹³Since the market size is 1, any advertising cost exceeding 1 will be suboptimal. It follows that $z_X, z_Y \leq \frac{1}{k}$ and hence $k > 2$ ensures that the condition $z_X, z_Y \leq \frac{1}{2}$ is satisfied.

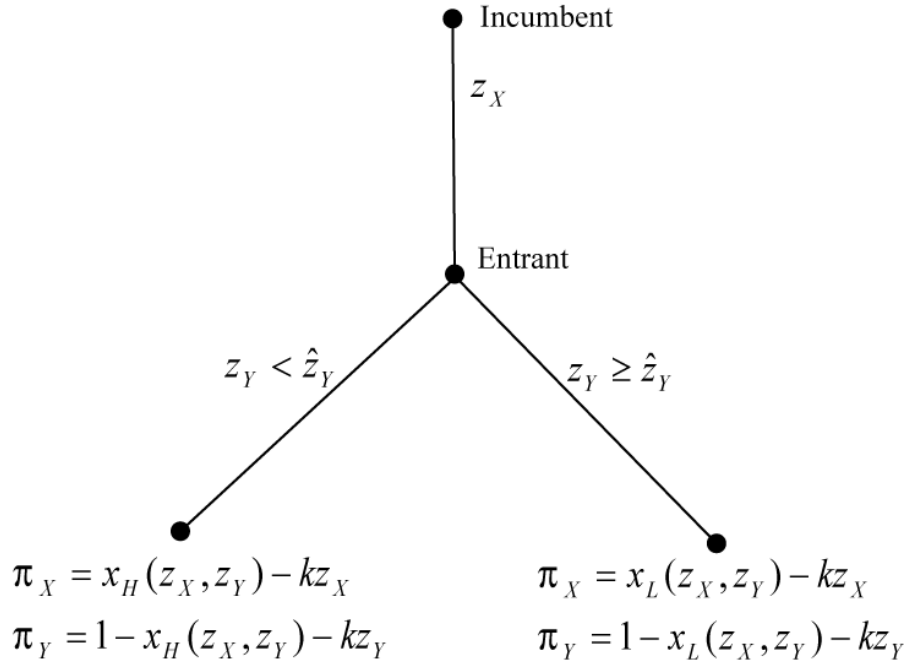


Figure 2: The advertising game

lock-in, or it will be the incumbent advertising the minimum amount necessary in order to maintain the lock-in:

Lemma 4 *Assume that $k > 2$.*

- *In a sub-game perfect equilibrium, Y either chooses to advertise such that $z_Y = \hat{z}_Y$ in order to break the lock-in, or Y does not advertise at all.*
- *In a sub-game perfect equilibrium, X either chooses to maintain the lock-in with preemptive advertising, or X does not advertise at all. If X chooses to preempt, then X will choose the minimum advertising level necessary to achieve this.*

Proof. See Appendix. ■

Lemma 4 implies that Y will choose to break the lock-in if:

$$\begin{aligned}\Delta\pi_Y(z_X) &\equiv \pi_Y(z_X, \hat{z}_Y) - \pi_Y(z_X, 0) \\ &= x_H(z_X, 0) - x_L(z_X, \hat{z}_Y) - k\hat{z}_Y \geq 0\end{aligned}\tag{9}$$

Note that if $\Delta\pi_Y(0) < 0$, then Y will choose not to break the lock-in even if X does not advertise. Thus, in this case X does not have to do anything to maintain the lock-in. In the following Lemma we characterize the conditions under which this applies:

Lemma 5 *Assume $k > 2$. The equation $\Delta\pi_Y(0) = 0$ uniquely determines a strictly increasing function $\tilde{a}(k)$ with $\lim_{k \rightarrow \infty} \tilde{a}(k) = \frac{1}{2}$. In a sub-game perfect equilibrium, the incumbent X will maintain the lock-in without advertising if and only if $a < \tilde{a}(k)$.*

Proof. See Appendix. ■

Consider now the case where $a \geq \tilde{a}(k)$. Then, preemption requires X to advertise such that $z_X \geq \hat{z}_X$, where \hat{z}_X is determined by¹⁴

$$\Delta\pi_Y(\hat{z}_X) = 0.\tag{10}$$

Lemma 4 established that X will either choose to preempt s.t. $z_X = \hat{z}_X$ or he will not advertise at all. Firm X 's payoff from preemption is higher than his payoff from not advertising at all if:

$$\Delta\pi_X \equiv x_H(\hat{z}_X, 0) - k\hat{z}_X - x_L(0, \hat{z}_Y(0)) \geq 0\tag{11}$$

This leads us to establish the following Lemma:

Lemma 6 *Assume $k > 2$. The equation $\Delta\pi_X = 0$ uniquely determines a function $\hat{a}(k)$, s.t. $\hat{a}(k) < \frac{1}{2}$ for all k and $\lim_{k \rightarrow \infty} \hat{a}(k) = \frac{1}{2}$. In a sub-game perfect equilibrium, X will preempt s.t. $z_X = \hat{z}_X$ if and only if $\tilde{a}(k) < a < \hat{a}(k) < \frac{1}{2}$.*

Recall from Proposition 3 that if $a \geq \frac{1}{2}$ then there is no lock-in even if X does not advertise. Since X will advertise only if $a < \hat{a}(k) < \frac{1}{2}$, Y will break the lock-in without advertising when $a > \frac{1}{2}$. The following Proposition follows directly from this observation and from Lemma 4, 5 and 6.

¹⁴ $\Delta\pi_Y$ is defined in equation (9). By lemma 9, $\Delta\pi_Y$ is declining in z_X and hence \hat{z}_X is uniquely defined.

Proposition 7 For $k > 2$

- **Lock-in with no advertising:** If $a < \tilde{a}(k)$, then there is a unique sub-game perfect equilibrium in which neither X nor Y will advertise. X will maintain its lock-in.
- **Lock-in with preemptive advertising:** If $\tilde{a}(k) < a < \hat{a}(k)$, then there is a unique sub-game perfect equilibrium in which X will preempt s.t. $z_X = \hat{z}_X$, and Y will choose not to advertise at all. X will maintain its lock-in.
- **Break lock-in with advertising:** If $\frac{1}{2} > a \geq \hat{a}(k)$, then there is a unique sub-game perfect equilibrium in which X will choose not to advertise at all, and Y will choose to break the lock-in with advertising $z_Y = z_Y^*(0)$.
- **Break lock-in without advertising:** If $a \geq \frac{1}{2}$, then there is a unique sub-game perfect equilibrium in which neither X nor Y advertise, and the lock-in is broken.

Figure 3 illustrates all possible sub-game perfect equilibria stated in Proposition 7. Recall from Proposition 1, that if X has a dominant market share initially, the market will be locked in on a high market share for X whenever $a < \frac{1}{2}$. With advertising however, Y will be able to break the lock-in for $a > \hat{a}(k)$. Thus the cases in which a lock-in would be broken if no advertising was possible (see Proposition 1) are a subset of the cases in which a lock-in would be broken if advertising was possible. Thus, even if opening up for advertising means opening up for preemptive advertising, this does not seem to hinder competition. This allows us to state the following Corollary:

Corollary 8 Advertising increases the likelihood that a lock-in on an inferior product will be broken.

6 Conclusion

Economists have long been concerned about how advertising can deter entrance and reduce competition (see e.g. Williamson 1963, Needham 1976, Salop 1979, Spence 1980, Schmalensee 1983, Bagwell

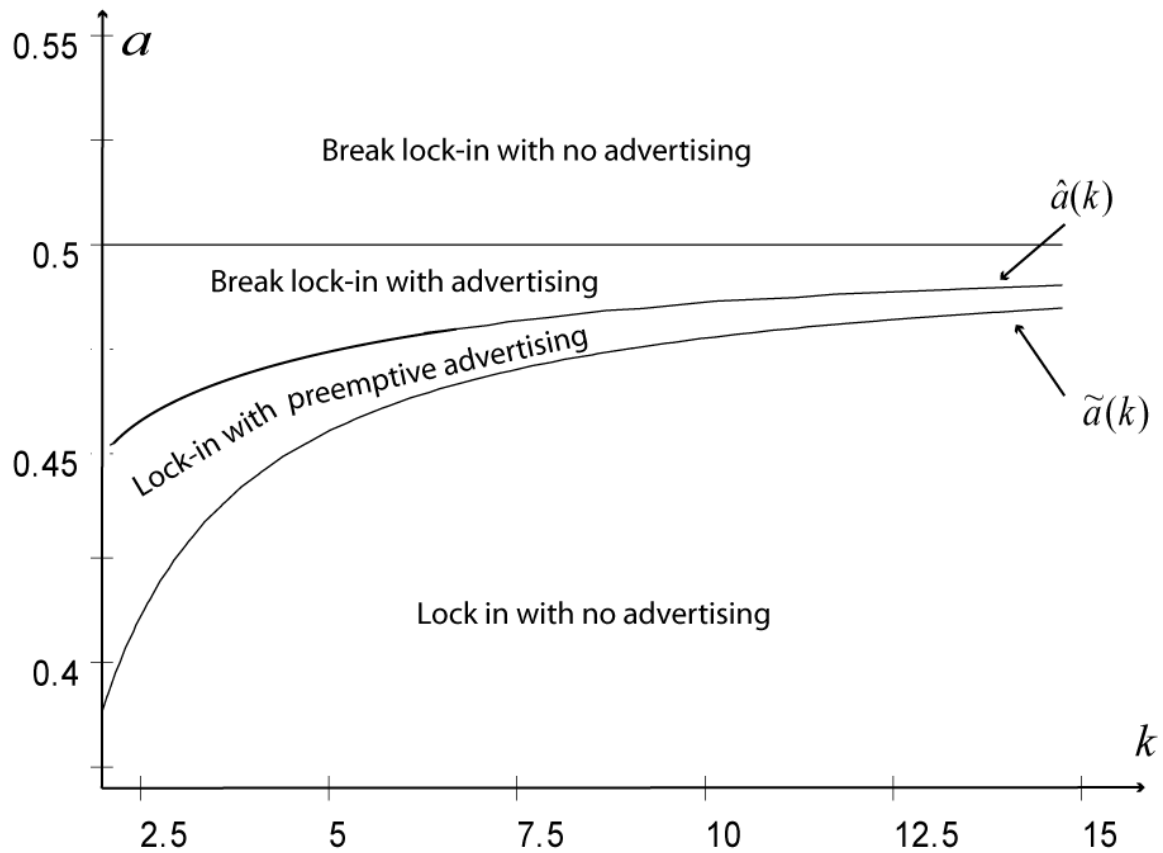


Figure 3: The equilibrium outcome

and Ramey 1990). In order to evaluate the impact of advertising on competition it is important to understand how advertising affects peoples' choices. In this paper we look at how advertising can affect behavior in markets with network externalities. In these markets entry is particularly difficult because the market can be locked-in on the particular product currently dominating the market, even if this product is inferior to alternatives. The lock-in occurs because most consumers prefer the product most others are possessing. Based on the phenomenon of the availability heuristic we suggest that, in markets with network externalities, advertising can affect people's choices by distorting their beliefs about market shares.

Our analysis shows that in markets with network externalities advertising will not deter entrance. In fact, advertising increases the likelihood that a lock-in will be broken. In particular, we show that if the entrant's product is of sufficiently high quality, then it is profitable to use advertising in order to break a lock-in. In this case the incumbent realizes that it is a "lost game", and that he is better off not advertising at all. If the quality differential between the incumbent and the entrant is small, then the incumbent may advertise in order to make it unprofitable for the entrant to break the lock-in. This preemptive advertising does, however, only deter entry when the entrant would not have been able to break the lock-in anyway, even if advertising was impossible. Thus, in this case advertising introduces a deadweight loss in terms of advertising expenditures.

Our model has important policy implications. It suggests that regulating advertising can deter entry in markets with network externalities by not allowing potential entrants with products of substantially better quality to break a lock-in by advertising. There is, however, also a positive effect of regulating advertising. It will reduce the deadweight loss in terms of advertising expenditures among incumbents protecting themselves against potential entrants with products of marginally better quality. However, if the policy goal is to reduce barriers to entry for higher quality products, then advertising should not be restricted.

References

- [1] Bagwell, K., 2005, “The Economic Analysis of Advertising,” Columbia University, mimeo
- [2] Bagwell, K., and G. Ramey, 1990, “Advertising and Pricing to Deter or Accommodate Entry When Demand is Unknown,” *International Journal of Industrial Organization*, 8, 93-113.
- [3] Brekke, K.A., and M. Rege, 2006, “Advertising as a Distortion of Social Learning,” University of Oslo, mimeo.
- [4] Cialdini, R.B., 1993, *Influence*, New York: Quill.
- [5] Farrell, J., and G. Saloner, 1985, “Standardization, compatibility, and innovation,” *Rand Journal of Economics*, 16(1), 70-83.
- [6] Farrell, J., and G. Saloner, 1986, “Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation,” *American Economic Review*, 76(5), 940-955.
- [7] Gabaix, X., and D. Laibson, 2006, “Shrouded Attributes, Consumer Myopia, and Information Suppression in Competitive Markets,” *Quarterly Journal of Economics*, 121(2).
- [8] Gale, J., K. Binmore, and L. Samuelson, 1995, “Learning to be Imperfect: The Ultimatum Game,” *Games and Economic Behavior*, 8, 56-90.
- [9] Katz, M.L., and C. Shapiro, 1985, “Network Externalities, Competition, and Compatibility,” *American Economic Review*, 75(3), 424-440.
- [10] Katz, M.L., and C. Shapiro, 1992, “Product Introduction with Network Externalities,” *The Journal of Industrial Economics*, XL(1), 55-83.
- [11] Kelley, C.M., and L.L. Jacoby, 2000, “Recollection and Familiarity: Process-Dissociation,” in *The Oxford Handbook of Memory*, E. Tulving and F.I.M. Craik (eds.), Oxford University Press.
- [12] Krähmer, D., 2004, “Advertising and consumer memory,” University College London, mimeo.

- [13] Loginova, O., 2005, "Competing for consumers' attention: Advertising when consumers have imperfect memory," University of Missouri-Columbia, mimeo.
- [14] Needham, D., 1976, "Entry Barriers and Non-Price Aspects of Firms' Behavior," *Journal of Industrial Economics*, 25, 29-43.
- [15] Pastine, I., and T. Pastine, 2002, "Consumption Externalities, Coordination, and Advertising," *International Economic Review*, 43(3), 919-943.
- [16] Salop, S., 1979, "Strategic Entry Deterrence," *American Economic Review*, 69, 335-38.
- [17] Schmalensee, R., 1983, "Advertising and Entry Deterrence: An Exploratory Model," *Journal of Political Economy*, 91, 636-53.
- [18] Schrum, L.J., 1999, "Television and Persuasion: Effects of the Programs between the Ads," *Psychology and Marketing*, 16(2), 119-140.
- [19] Schwartz, N and L.A. Vaughn, 2002, "The Availability Heuristic Revisited: Ease of Recall and Content of Recall as Distinct Sources of Information," in *Heuristics and Biases* by Gilovich T., G. Dale and D. Kahnemann (eds.), Cambridge: Cambridge University Press.
- [20] Shapiro, J.M., 2005, "A memory-jamming theory of advertising," University of Chicago, mimeo..
- [21] Spence, M., 1980, "Notes on Advertising, Economies of Scale, and Entry Barriers," *Quarterly Journal of Economics*, 95, 493-507.
- [22] Tversky A. and D. Kahneman, 1973, "Availability: A Heuristic for Judging Frequency and Probability," *Cognitive Psychology*, 5, 207-232.
- [23] Williamson, O. E., 1963, "Selling Expense as a Barrier to Entry," *Quarterly Journal of Economics*, 77, 112-28..

A Appendix

Proof of Proposition 1. If $a < \frac{1}{2}$, then equation (6) implies that there are three steady states: $x_L = 0$, $\hat{x} = \frac{3}{7-6a}$ and $x_H = \frac{3a}{6a-1}$. Moreover, $\dot{x} < 0$ for $x \in (0, \hat{x}) \cup (x_H, 1)$ and $\dot{x} > 0$ for $x \in (\hat{x}, x_H)$. Hence, there are two asymptotically stable states: $x_L = 0$ and $x_H = \frac{3a}{6a-1}$. If $a = \frac{1}{2}$, then equation (6) implies that there are two steady states: $x_L = 0$ and $\hat{x} = \frac{3}{4}$. Moreover, $\dot{x} < 0$ for $x \in (0, \hat{x}) \cup (\hat{x}, 1)$. Hence, there is one asymptotically stable states: $x_L = 0$. If $a > \frac{1}{2}$, then equation (6) implies that there is one steady states: $x_L = 0$. Moreover, $\dot{x} < 0$ for $x \in (0, 1)$. Hence, there is one asymptotically stable states: $x_L = 0$. ■

Proof of Lemma 2. Setting $\dot{x}(x) = 0$ and $x = \frac{1+a}{2}$ in equation (7) implies

$$z_Y^*(z_X, a) = \frac{1 + z_X(1 - a) + a - 6a^2}{1 + a} \quad (12)$$

Differentiating z_Y^* with respect to z_X and a yields:

$$\begin{aligned} \frac{\partial z_Y^*}{\partial z_X} &= \frac{1 - a}{1 + a} > 0 \\ \frac{\partial z_Y^*}{\partial a} &= -2 \frac{z_X + 6a + 3a^2}{(1 + a)^2} < 0 \end{aligned}$$

■

Proof of Proposition 3. If $z_Y < \hat{z}_Y(z_X, a)$, then equation (7) implies that there are three steady states:

$$x_L(z_X, z_Y) = \frac{z_X}{6a - (1 - z_X - z_Y)} \quad (13)$$

$$\hat{x}(z_X, z_Y) = \frac{3 - z_X}{7 - z_X - z_Y - 6a} \quad (14)$$

$$x_H(z_X, z_Y) = \frac{z_X + 3a}{6a - (1 - z_X - z_Y)} \quad (15)$$

Moreover, $\dot{x} < 0$ for $x \in (0, \hat{x}) \cup (x_H, 1)$ and $\dot{x} > 0$ for $x \in (\hat{x}, x_H)$. Hence, there are two asymptotically stable states: x_L and x_H . If $z_Y = \hat{z}_Y(z_X, a)$, then equation (6) implies that there are two steady

states:

$$x_L(z_X, z_Y) = \frac{z_X}{6a - (1 - z_X - z_Y)}$$

$$\hat{x}(z_X, z_Y) = \frac{1 + a}{2}$$

Moreover, $\dot{x} < 0$ for $x \in (0, \hat{x}) \cup (\hat{x}, 1)$. Hence, there is one asymptotically stable state x_L . If $z_Y < \hat{z}_Y(z_X, a)$, then equation (6) implies that there is one steady state

$$x_L(z_X, z_Y) = \frac{z_X}{6a - (1 - z_X - z_Y)}$$

Moreover, $\dot{x} < 0$ for $x \in (0, 1)$. Hence, there is one asymptotically stable states x_L . ■

Proof of Lemma 4. In order to prove the first statement we must establish that, given that Y breaks the lock-in, then Y cannot increase profits by advertising more, i.e. $-\frac{\partial x_L}{\partial z_Y} < k$. Moreover, we must establish that, given that Y does *not* break the lock-in, then Y cannot increase profits by advertising more, i.e. $-\frac{\partial x_H}{\partial z_Y} < k$. Differentiating equations (13) and (15) with respect to z_Y and using that $a > \frac{1}{3}$ and $k > 2$ yield:

$$-\frac{\partial x_L}{\partial z_Y} = \frac{z_X}{(z_X + z_Y + 6a - 1)^2} < \frac{z_X}{(1 + z_X + z_Y)^2} < 1 < k$$

$$-\frac{\partial x_H}{\partial z_Y} = \frac{z_X + 3a}{(6a - (1 - z_X - z_Y))^2} < \frac{z_X + 3a}{(3a + z_X + z_Y)^2} < 1 < k$$

In order to prove the second statement we must establish that, given that Y will break the lock-in, then X cannot increase profits by advertising more, i.e. $\frac{\partial x_L}{\partial z_X} < k$. Moreover, we must establish that, given that Y will *not* break the lock-in, then X cannot increase profits by advertising more, i.e. $\frac{\partial x_H}{\partial z_X} + \frac{\partial x_H}{\partial z_Y} \frac{\partial z_Y}{\partial z_X} < k$. Differentiating equations (13) and (15) with respect to z_X and using that $a > \frac{1}{3}$ and $k > 2$ yields

$$\frac{\partial x_L}{\partial z_X} = \frac{-1 + z_X + z_Y + 6a - z_X}{(-1 + z_X + z_Y + 6a)^2} = \frac{6a - 1 + z_Y}{(6a - (1 - z_X - z_Y))^2} < 1 < k$$

$$\frac{\partial x_H}{\partial z_X} + \frac{\partial x_H}{\partial z_Y} \frac{\partial z_Y}{\partial z_X} = \frac{6a - (1 - z_X - z_Y) - (z_X + 3a)}{(6a - (1 - z_X - z_Y))^2} + \frac{z_X + 3a}{(6a - (1 - z_X - z_Y))^2} \frac{1 - a}{1 + a}$$

$$< \frac{1}{6a - 1 + z_X + z_Y} < \frac{1}{(1 + z_X + z_Y)} < 1 < k$$

■

Proof of Lemma 5. Recall that $\hat{z}_Y(z_X, a) = \max\{0, z_Y^*\}$. Equation (9), (12), (13) and (15) imply

$$\Delta\pi_Y(0) = \frac{1}{2} \frac{1 + 2a + a^2 - 2k - 2ka + 12ka^2}{1 + a} \quad (16)$$

Thus, equation (16) and the condition $\Delta\pi_Y(0) = 0$ determine:

$$\tilde{a}(k) = \frac{-1 + k + 5k\sqrt{1 - \frac{12}{25k}}}{(1 + 12k)}$$

Differentiating with respect to k yields

$$\frac{\partial \tilde{a}(k)}{\partial k} = \frac{13k\sqrt{\left(\frac{25k-12}{k}\right)} + 97k - 6}{k\sqrt{\left(\frac{25k-12}{k}\right)}(1 + 12k)^2} > 0$$

Moreover,

$$\lim_{k \rightarrow \infty} \tilde{a}(k) = \lim_{k \rightarrow \infty} \frac{\frac{-1}{k} + 1 + 5\sqrt{1 - \frac{12}{25k}}}{\left(\frac{1}{k} + 12\right)} = \frac{1}{2}$$

Differentiating equation (16) with respect to a yields:

$$\frac{\partial}{\partial a} [\Delta\pi_Y(0)] = \frac{1}{2} \frac{1 + 2a + a^2 + 24ka + 12ka^2}{(1 + a)^2} > 0$$

Moreover, by definition $\Delta\pi_Y(0) = 0$ if $a = \tilde{a}(k)$. Thus, $\Delta\pi < 0$ for $a < \tilde{a}(k)$ and $\Delta\pi > 0$ for $a > \tilde{a}(k)$. ■

Proof of Lemma 6. By definition of $\hat{a}(k)$

$$\Delta\pi_X|_{a=\hat{a}(k)} = 0 \quad (17)$$

Thus, if $\frac{d}{da}\Delta\pi_X < 0$, then it follows that $\Delta\pi_X < 0$ for $a > \hat{a}(k)$ and X will not advertise. Moreover, it follows that $\Delta\pi_X > 0$ for $a < \hat{a}(k)$ and X will advertise. In the following we prove that

$$\frac{d}{da}\Delta\pi_X = \frac{\partial\Delta\pi_X}{\partial a} + \frac{\partial\Delta\pi_X}{\partial\hat{z}_X} \frac{\partial\hat{z}_X}{\partial a} < 0$$

Note first that from (11), (13) and (15) we get

$$\begin{aligned} \Delta\pi_X &= x_H(\hat{z}_X, 0) - x_L(0, \hat{z}_Y(0)) - k\hat{z}_X \\ &= \frac{\hat{z}_X + 3a}{6a - 1 + \hat{z}_X} - k\hat{z}_X \end{aligned}$$

and hence

$$\begin{aligned}\frac{\partial \Delta \pi_X}{\partial a} &= -\frac{3(1+z_X)}{(6a-1+\hat{z}_X)^2} < 0 \\ \frac{\partial \Delta \pi_X}{\partial \hat{z}_X} &= \frac{3a-1}{(6a-1+\hat{z}_X)^2} - 1 < 0\end{aligned}\tag{18}$$

Thus, it remains to prove that $\frac{\partial \hat{z}_X}{\partial a} \geq 0$. Recall from (10) that \hat{z}_X is defined by the equation

$$\Delta \pi_Y(\hat{z}_X) = 0$$

Combining this equation with (9) it follows that $\hat{z}_Y(z_X) = \frac{1}{k}(x_H - x_L) > 0$, and the condition of Lemma 6' below is satisfied. Now, since $\Delta \pi_Y(\hat{z}_X) = 0$ for all a ,

$$\frac{d}{da} \Delta \pi_Y(\hat{z}_X) = \frac{\partial \Delta \pi_Y}{\partial a} + \frac{\partial \Delta \pi_Y}{\partial \hat{z}_X} \frac{\partial \hat{z}_X}{\partial a} = 0$$

and from Lemma 6' below we see that

$$\frac{\partial \hat{z}_X}{\partial a} = -\frac{\frac{\partial \Delta \pi_Y}{\partial a}}{\frac{\partial \Delta \pi_Y}{\partial \hat{z}_X}} > 0$$

Next we prove that $\hat{a}(k) < 1/2$ for all k . Since $\hat{a}(k)$ is defined as the value of a where $\Delta \pi_X = 0$ and since $\frac{d}{da} \Delta \pi_X < 0$, the claim that $\hat{a}(k) < 1/2$ for all k follows if we show that $\Delta \pi_X < 0$ for $a = 1/2$ and all k . Note that if $\hat{z}_X > 1/k$, then the advertising cost is $k\hat{z}_X > 1$. Moreover, since $(x_H - x_L) < 1$, we must then have $\Delta \pi_X = x_H - x_L - k\hat{z}_X < 0$. Hence, since $\frac{\partial}{\partial z_X} \Delta \pi_Y(z_X) < 0$ (by Lemma 6') we only need to show that for $a = 1/2$ setting $z_X = 1/k$ is not sufficient to prevent entry, i.e we need to show that $\Delta \pi_Y(1/k) > 0$ when $a = 1/2$.

Now to prove that for $a = 1/2$ setting $z_X = 1/k$ is not sufficient to prevent entry, insert $a = 1/2$ in (12) and (8) to get

$$\hat{z}_Y(z_X) = \frac{z_X}{3}.$$

Hence, from (9), (13) and (15) setting $z_X = 1/k$ we find

$$\begin{aligned}\Delta\pi_Y &= x_H(z_X, 0) - x_L(z_X, \hat{z}_Y(z_X)) - k\hat{z}_Y(z_X) \\ &= \frac{\frac{3}{2} + z_X}{2 + z_X} - \frac{z_X}{2 + \frac{4}{3}z_X} - k\frac{z_X}{3} \\ &> \frac{\frac{3}{2}}{2 + z_X} - k\frac{z_X}{3} > \frac{1}{2} - \frac{1}{3} > 0\end{aligned}$$

This completes the proof that $\hat{a}(k) < 1/2$ for all k .

Similarly, for $a = \tilde{a}(k)$ we know by definition (Lemma 5) that $\Delta\pi_Y(0) = 0$, and thus by definition of \hat{z}_X (se equation 10) it follows that $\hat{z}_X = 0$. Clearly $\Delta\pi_X = x_H - x_L - k\hat{z}_X > 0$, and since we know from (18) that $\frac{\partial\Delta\pi_X}{\partial a} < 0$ it then follows that $\hat{a}(k) > \tilde{a}(k)$. Combining $\tilde{a}(k) < \hat{a}(k) < 1/2$ and $\lim_{k \rightarrow \infty} \tilde{a}(k) = \frac{1}{2}$, we see that $\lim_{k \rightarrow \infty} \hat{a}(k) = \frac{1}{2}$. ■

Lemma 6: $\frac{\partial}{\partial z_X}\Delta\pi_Y(z_X) < 0$. Moreover, if $\hat{z}_Y(z_X) > 0$ then $\frac{\partial}{\partial a}\Delta\pi_Y(z_X) > 0$.

Proof. If $\hat{z}_Y(z_X) > 0$ it follows from (8) that $\hat{z}_Y(z_X) = z_Y^*(z_X)$. Combining (9), (13), (15) and (12) we find

$$\begin{aligned}\Delta\pi_Y(z_X) &= x_H - x_L - k\hat{z}_Y(z_X) \\ &= \frac{z_X + 3a}{6a - 1 + z_X} - \frac{z_X(1 + a)}{2(z_X + 3a)} - k\frac{1 - a}{1 + a}(z_X + 3a) + k(3a - 1).\end{aligned}$$

If $\hat{z}_Y(z_X) = 0$, then $\Delta\pi_Y(z_X) = x_H - x_L$ and the last two terms disappear.

To prove $\frac{\partial}{\partial z_X}\Delta\pi_Y(z_X) < 0$ we consider first the case $\hat{z}_Y(z_X) > 0$:

$$\begin{aligned}\frac{\partial}{\partial z_X}\Delta\pi_Y(z_X) &= \frac{\partial}{\partial z_X} \left[\frac{z_X + 3a}{6a - 1 + z_X} - \frac{z_X(1 + a)}{2(z_X + 3a)} - k\frac{1 - a}{1 + a}(z_X + 3a) + k(3a - 1) \right] \\ &= \frac{3a - 1}{(6a - 1 + z_X)^2} - \frac{3a(a + 1)}{2(z_X + 3a)^2} - k\frac{1 - a}{1 + a} \\ &< \frac{2(3a - 1) - 3a(a + 1)}{2(3a + z_X)^2} - k\frac{1 - a}{1 + a} \\ &= -\frac{2 - 3a(1 - a)}{2(3a + z_X)^2} - k\frac{1 - a}{1 + a} < 0\end{aligned}$$

If $\hat{z}_Y(z_X) = 0$, the last term will be zero, but that will not affect the sign.

Next, we show that $\frac{\partial}{\partial a} \Delta \pi_Y(z_X) > 0$ when $\hat{z}_Y(z_X) > 0$

$$\begin{aligned} \frac{\partial}{\partial a} \Delta \pi_Y(z_X) &= \frac{\partial}{\partial a} \left[\frac{z_X+3a}{6a-1+z_X} - \frac{z_X(1+a)}{2(z_X+3a)} - k \frac{1-a}{1+a} (z_X + 3a) + k(3a-1) \right] \\ &= \frac{6ak}{a+1} - \frac{3(1+z_X)}{(6a+z_X-1)^2} + \frac{z_X(3-z_X)}{2(3a+z_X)^2} + \frac{k(3a+z_X)}{a+1} + \frac{k(3a+z_X)(1-a)}{(a+1)^2} > 0 \end{aligned}$$

To see the last inequality, note first that the expression is increasing in k and hence if it is positive for $k = 2$ it will be positive for all $k > 2$. Setting $k = 2$ we can derive a lower bound for the first term $\frac{6ak}{a+1} > \frac{12}{4} = 3$. The only negative term is the second one, but this term is decreasing in a and hence for $a > 1/3$ we get $\frac{3(1+z_X)}{(6a+z_X-1)^2} < \frac{3(1+z_X)}{(2+z_X-1)^2} < 3$. ■