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Abstract

This paper presents a new numerical method for pricing American call options when the volatility of the price of the underlying stock is stochastic. By exploiting a log-linear relationship of the optimal exercise boundary with respect to volatility changes, we derive an integral representation of an American call price and the early exercise premium which holds under stochastic volatility. This representation is used to develop a numerical method for pricing the American options based on an approximation of the optimal exercise boundary by Chebyshev polynomials. Numerical results show that our numerical approach can quickly and accurately price American call options both under stochastic and/or constant volatility.

Keywords: American call option, stochastic volatility, early exercise boundary, Chebyshev polynomials.

JEL Classification: G12, G13, C63

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1 Introduction

Pricing American options is one of the most difficult problems in option pricing literature. The difficulty stems from the fact that, unlike a European, an American call (or put) option has no explicit closed form solution. This happens because the optimal boundary above which the American call option will be exercised is unknown and part of the option price solution. Therefore, efforts have been concentrated on developing numerical approximation schemes which can price the American options accurately and faster than the lattice or simulation based methods, which are time consuming and computationally more demanding. These schemes are based on integral representations of the American option evaluation formula or they exploit the partial differential equation satisfied by the option prices.[1]

The existing approximation schemes for pricing American call (or put) options in the literature are valid only under the assumptions of the Black and Scholes (1973) option pricing model, which claim that the stock price of the underlying stock is log-normally distributed conditional on the current stock price, with constant volatility. However, these assumptions are in contrast to most of the empirical evidence of the option and stock pricing empirical literature which indicates that stocks' prices volatility is stochastic [see Ghysels, Harvey and Renault (1996), for a survey].

The aim of this paper is to develop a new numerical method for pricing American call option prices for the case that the underlying stock's price volatility is stochastic, as it appears to be in reality. The lack of such type of methods in the literature of the American options is primarily due to the fact that, under stochastic volatility, the optimal exercise boundary depends, in addition to time, on the paths of the volatility [see Broadie *et al* (2000)]. This considerably

[1]Examples of such type of numerical methods include the Barone-Adesi and Whaley (1987) analytical approximation method, the approximating methods of Geske-Johnson (1984) and Bunch and Johnson (1992), the Gaussian quadrature method of Sullivan (2000), *inter alia*, and the recently developed exercise boundary approximation methods of Subrahmanyam and Yu (1996), and Ju (1998).

complicates the derivation of a suitable, analytic representation for an American call option price upon which a numerical approximation method can be build up. Our strategy of circumventing this problem is to approximate the optimal exercise boundary function with a log-linear function with respect to volatility changes over different pieces of the maturity interval.[2] Based on this approximation, we derive an analytic, integral representation of the early exercise premium of the American call option price. This representation unbundles the early exercise premium (and hence the American call option price) into a portfolio of Arrow-Debreu type of securities [see Bakshi and Madan (2000), for a European call option price]. The prices of these securities can be calculated based on the joint characteristic function of the stock price and its conditional volatility process which is derived in closed form in the paper. To complete our numerical method for evaluating the American call option under stochastic volatility, we employ Chebyshev polynomials to approximate the logarithm of the exercise boundary function. With these polynomials, we can efficiently approximate any non-linear pattern of the optimal exercise boundary function, over the different pieces of the maturity interval, because we can choose the point with the minimum approximation error to fit a high-degree polynomial approximating function into the true function of the exercise boundary.

To appraise the pricing performance of our method, the paper reports numerical results of the speed and accuracy of the method in comparison with benchmark methods. We also compare the pricing performance of the method for the case that volatility is constant with other numerical approximation methods for the log-normal model, which are frequently used in practice. The results of the numerical evaluations are very encouraging. They show that a very parsimonious, two degree approximating function of the exercise boundary based on Chebyshev polynomials can satisfactorily price American call options for a broad class of stock and exercise prices considered in our numerical experi-

[2] Note that this approach is consistent with recent evidence suggesting that, when volatility is stochastic, the exercise boundary is smooth with respect to volatility changes [see Broadie *et al* (2000)].

ments. This is true both under stochastic and constant volatility. Our results show that the pricing errors of our method are very close to zero, and they are of the same order of magnitude independently on whether the volatility is constant or stochastic. In the constant volatility case, we find that the pricing errors of our method can become substantially smaller in magnitude than the other approximation methods compared with, especially when the curvature of the true optimal exercise boundary function is a high.

The paper is organised as follows. In Section 2, we present the evaluation framework for the American call option price under stochastic volatility and derive an analytic, integral representation of the American call price. In Section 3, we show how to implement Chebyshev polynomials to approximate the optimal exercise boundary function for the lognormal and stochastic volatility models, respectively. In Section 4, we list and discuss numerical results of the performance of our method to price the options. Section 5 summarizes and concludes the paper.

2 Analytic evaluation of American call options under stochastic volatility

In this section, in order to derive an analytic evaluation formula for an American call option we assume that the price of the underlying stock follows a geometric stochastic volatility process. This model of the stock price is known in the literature as the stochastic volatility (SV) model [see Heston (1993), *inter alia*]. The analysis of the section proceeds as follows. First, we present a general evaluation framework for pricing an American call option under stochastic volatility which is in line with that of Broadie *et al* (2000). Based on this framework, we next derive an analytic, integral representation of the American call option price.

2.1 The valuation framework

Consider Heston's (1993) specification of the stochastic volatility (SV) model to characterise the dynamics of the underlying stock's price, denoted P_t, at time t. For analytic convenience, assume that dividends are paid at the constant rate δ and that the riskless interest rate, r, is constant. Then, the SV model implies that the spot stock price should satisfy the following risk-neutralised process

$$\frac{dP_t}{P_t} = (r - \delta)\, dt + \sqrt{V_t} dW_{1,t}, \tag{1}$$

where the instantaneous conditional variance (volatility), V_t, follows the mean reverting square root process

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2,t}, \tag{2}$$

where k is adjusted by the market price of volatility risk, $\{W_{j,t}, t > 0\}$, $j = 1, 2$, are two correlated standard Brownian motion processes, with correlation coefficient given by $Corr(dW_{1,t}; dW_{2,t}) = \rho dt$, $\rho \in (-1, 1)$.

Consider now an American call option contract for the above stock with maturity date T and strike price K, at the exercise time. This contract gives the holder the right of exercising the call option at any time h in the maturity interval $[t, T]$, i.e. $h \in [t, T]$. The critical stock price above which the American call will be exercised is referred to as the optimal exercise boundary. Since the price of the underlying stock depends on the paths of the volatility process V_t, we will hereafter denote the time t, which represents the current time price of the American call option contract (i.e. the American call option price) as $C_A(P_t, V_t, T - t)$, while the optimal exercise boundary will be denoted as $B(V_h, h)$, $\forall\ h \in [t, T]$.

The American call option price $C_A(P_t, V_t, T - t)$ can be calculated by the maximum value of the discounted payoffs from the option where the maximum is taken over all possible stopping (exercise) times, denoted τ, in the maturity interval, $[t,T]$. Define the optimal stopping time as

$$\tau^* = \inf\left\{\tau \in [t,T] : C_A\left(P_t, V_t, T-t\right) = \left(P_t - K\right)_+\right\}. \tag{3}$$

Then American call option pricing problem can be represented by the Snell envelop[3]

$$C_A\left(P_t, V_t, T-t\right) = \sup_{\tau \in \mathcal{S}_{[t,T]}} E^Q\left(e^{-\int_t^\tau r ds}\left(P_\tau - K\right)_+\right), \tag{4}$$

where $\mathcal{S}_{[t,T]}$ is the set of stopping times in the maturity interval, $[t,T]$, E_t^Q denotes the time t conditional expectation under the equivalent martingale measure Q, and $\left(P_\tau - K\right)_+$ is the payoff of the American call option at the stopping time τ.

The following theorem characterises the optimal solution of the problem defined by equation (4).

Theorem 1 *Let the stock price satisfy processes (1) and (2). Then, the American call option price $C_A\left(P_t, V_t, T-t\right)$ can be written as*

$$\begin{aligned} C_A\left(P_t, V_t, T-t\right) &= C_E\left(P_t, V_t, T-t\right) \\ &\quad + E_t^Q \int_t^T e^{-r(s-t)}\left(\delta P_s - rK\right) I_{\{P_s \geqslant B(V_s,s)\}} ds, \end{aligned} \tag{5}$$

where $C_E\left(P_t, V_t, T-t\right)$ is the value of a European call price with maturity date T and strike price K, $B(V_s,s)$ denotes the value of the optimal exercise boundary, at time $s \in [t,T]$, and I_A is the indicator function of the set A, defined as $A = \{P_s : P_s \geqslant B(V_s,s)$ and $V_s \in \mathcal{R}^+\}$, which contains the prices of the stock at which the American call will be exercised. The optimal exercise boundary $B(V_h,h)$ entered into the American call option price formula (5)

[3]See Karatzas (1988), *inter alia.*

should satisfy the following recursive equation

$$
\begin{aligned}
& B\left(V_h, h\right)-K && (6)\\
= \; & C_E\left(B(V_h, h), K, V_h, T-h\right) \\
& +E_h^Q\left[\int_h^T e^{-r(s-h)}\left(\delta P_s-r K\right) I_{\left\{P_s \geq B\left(V_s, s\right)\right\}} ds\right], \quad \forall\ s \geqq\ h \in[t, T],
\end{aligned}
$$

with terminal condition

$$
B\left(V_T, T\right) \text{-} K=\max \{K, r K / \delta\} . \tag{7}
$$

In Appendix A, we give a proof of Theorem 1 based on a decomposition of the optimal stopping problem (4) in terms of the optimal exercise boundary [see Myneni (1992)].

Theorem 1 shows that the American call option price $C_A\left(P_t, V_t, T-t\right)$ can be calculated by (5), once the values of optimal exercise boundary $B\left(V_h, h\right)$ are provided. However, this is not a trivial calculation problem. The recursive nature of $B\left(V_h, h\right)$ reveals that the difficulty in deriving an analytic formula for evaluating the American option is due to the fact that the exercise boundary is determined as part of the solution. This problem becomes more complicated under the SV model rather than the lognormal model because, for the SV model, the optimal exercise boundary function depends, in addition to the time h, on conditional variance (volatility) V_h.

2.2 An integral representation of the American call option price for the SV model

To circumvent the above difficult calculation problem of an American call option price, in this subsection we present a new strategy for evaluating the option price $C_A\left(P_t, V_t, T-t\right)$. Based on a first-order log-linear approximation of the optimal exercise boundary function around the time t conditional mean of volatility, we derive an integral representation of the American call option price

$C_A(P_t, V_t, T-t)$ upon which we can build up a numerical method for evaluating this price.

Suppose that the logarithm of the optimal exercise boundary function, at time h, denoted $b(V_h, h) \equiv \ln B(V_h, h)$, can be approximated around the conditional mean of volatility $E_t V_h$ by the linear in volatility function

$$b(V_h, h) = b_0(h) + b_1(h)(V_h - E_t V_h). \tag{8}$$

Relationship (8) asserts that, for small changes of V_h around the conditional mean of volatility $E_t V_h$, at time t, the true optimal exercise boundary function is an exponentially smooth surface with respect to volatility changes. This assumption can be justified by recent evidence provided by Broadie *et al* (2000), who recovered the American call option price and the exercise boundary reduced forms from the data following a non parametric statistical approach.

In the next theorem we give a two-dimension integral representation for the price $C_A(P_t, V_t, T-t)$ and its associated optimal exercise boundary recursive equation.

Theorem 2 *For relationship (8), the American call option price $C_A(P_t, V_t, T-t)$ can be calculated as*

$$\begin{aligned} & C_A(P_t, V_t, T-t) \\ = \;& C_E(P_t, V_t, T-t) + \int_t^T \delta P_t e^{-\delta(s-t)} \Pi_1(b_0(s), b_1(s) | P_t, V_t) ds \\ & - \int_t^T rKe^{-r(s-t)} \Pi_2(b_0(s), b_1(s) | P_t, V_t) ds, \end{aligned} \tag{9}$$

where $\Pi_1(.)$ and $\Pi_2(.)$ are defined in the Appendix B. The optimal exercise *boundary* $B(V_h, h)$ *satisfies the following recursive equation*

$$\begin{aligned} & B(V_h, h) - K = C_E(P_h, V_h, T-h) \\ & + \int_h^T \delta B(V_h, h)\, e^{-\delta(s-h)} \Pi_1' \left(b_0(s), b_1(s) | B(V_h, h), V_h\right) ds \\ & - \int_h^T rK e^{-r(s-h)} \Pi_2' \left(b_0(s), b_1(s) | B(V_h, h), V_h\right) ds, \end{aligned} \tag{10}$$

$\forall \;\; s \geqq h \in [t, T]$, *with terminal condition*

$$B(V_T, T) - K = \max\{K, rk/\delta\},$$

where $\Pi_1'(.)$ *and* $\Pi_2'(.)$ *are defined in the Appendix B.*

The proof of the Theorem is given in Appendix B.

The integral representation of the American option price $C_A(P_t, V_t, T-t)$ and its associated exercise boundary recursive equation (10), given by Theorem 2, unbundles the early exercise boundary premium (and hence the American call option) into a portfolio of Arrow-Debreu type of securities. The prices of these securities can be derived by calculating the following risk neutral expectations

$$\Pi_1\left(b_0(s), b_1(s) | P_t, V_t\right) = E_t^Q \left[\frac{P_s I_{\{(P_s, V_s): P_s \geqq B(V_s, s)\}}}{E_t^Q [P_s]} | P_t, V_t \right], \tag{11}$$

or using the transformed measure Q_1 with $\frac{dQ_1}{dQ} = \frac{P_s}{E_h^Q [P_s]}$ as

$$\Pi_1\left(b_0(s), b_1(s) | P_t, V_t\right) = E_t^{Q_1} \left[I_{\{(P_s, V_s): P_s \geqq B(V_s, s)\}} | P_t, V_t \right], \tag{12}$$

and

$$\Pi_2\left(b_0(s), b_1(s) | P_t, V_t\right) = E_t^Q \left[I_{\{(P_s, V_s): P_s \geqq B(V_s, s)\}} | P_t, V_t \right] \tag{13}$$

[see Appendix B].

The above relationships indicate that the prices $\Pi_1\left(b_0(s), b_1(s)|P_t, V_t\right)$ and $\Pi_1\left(b_0(s), b_1(s)|P_t, V_t\right)$ constitute the market prices of a security which pays \$1 in state $\{(P_s, V_s) : P_s \geqq B(V_s, s)\}$ and 0 otherwise under the measures Q_1 and Q, respectively.

The integral representation of the American call option given by Theorem 2 can be reduced to that derived by Kim (2000), for the lognormal model with constant volatility. This can be obtained by setting $k = \theta = \sigma = 0$ in equations (9) and (10) and noticing that, under the assumptions of the log-normal model, the exercise boundary equation (8) becomes the exact relationship $B(V_h, h) = \exp[b_0(h)]$. Then, it can be easily seen that equation (9) reduces to

$$\begin{aligned} C_A\left(P_t, T-t\right) &= C_E\left(P_t, T-t\right) + \int_t^T \delta P_t e^{-\delta(s-t)} \Pi_1(B(s)|P_t) ds \\ &\quad - \int_t^T rKe^{-r(s-t)} \Pi_2\left(B(s)|P_t\right) ds, \ \text{ for } s \geqslant h \in [t, T], \end{aligned} \tag{14}$$

while equation (10) reduces to

$$\begin{aligned} & B\left(h\right) - K \\ = \; & C_E\left(P_h, T-t\right) + \int_t^T \delta B\left(h\right) e^{-\delta(s-h)} \Pi_1'\left(B(s)|B(h)\right) ds \\ & - \int_t^T rKe^{-r(s-h)} \Pi_2'\left(B(s)|B(h)\right) ds, \end{aligned} \tag{15}$$

where now $\Pi_1\left(B(s)|P_t\right)$ and $\Pi_2\left(B(s)|P_t\right)$ are given by

$$\begin{aligned} & \Pi_1\left(B(s)|P_t\right) \\ = \; & \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log B(s)} F(\phi_X - i, 0, s-h| \ln P_t, 0)}{i\phi}\right] d\phi \\ = \; & N\left(\frac{\log(B(s)/P_t) - (r - \delta + \frac{1}{2}\sigma^2)(s-t)}{\sigma\sqrt{s-t}}\right) \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
&\Pi_2\left(B(s)|P_t\right)\\
= &\ \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\log B(s)}F(\phi_X,0,s-h|\ln P_t,0)}{i\phi}\right]d\phi\\
= &\ N\left(\frac{\log(B(s)/P_t)-(r-\delta-\frac{1}{2}\sigma^2)(s-t)}{\sigma\sqrt{s-t}}\right), \qquad (17)
\end{aligned}
$$

respectively.[4] Note that now the prices of the Arrow-Debreu type of securities $\Pi_1\left(B(s)|P_t\right)$ and $\Pi_2\left(B(s)|P_t\right)$ reflect the prices of a security which pays \$1 in the state $\{P_s \geqq B(s)\}$ and 0 otherwise under the measures Q_1 and Q, respectively. As equations (16) and (17) indicate, these prices can be calculated as the probabilities of the standardized normal distribution at the values of $\log(B(s)/P_t)$ adjusted by the quantities $(r-\delta+\frac{1}{2}\sigma^2)(s-t)$ and $(r-\delta-\frac{1}{2}\sigma^2)(s-t)$ under the measures Q_1 and Q, respectively.[5]

3 Numerical evaluation of American call options using Chebyshev polynomial functions to approximate the exercise boundary

The two-dimension integral representation of the American call option price and its associated recursive optimal exercise boundary relationship given by Theorem 2 can be used to build up a numerical approximation method for pricing American options under stochastic volatility. In this section we introduce such a method based on an approximation of the optimal exercise boundary function using Chebyshev polynomials.

[4] The prices $\Pi_1'\left(B(s)|B(h)\right)$ and $\Pi_2'\left(B(s)|B(h)\right)$ can be defined analogously.

[5] Note that the above two quantities differ by σ^2 which reflects the fact that the price of risk under the meassure Q_1 is smaller than under measure Q. This can be attributed to the fact that under measure Q_1 the payoff of the Arrow-Debreu price is scaled by the stock price [see equations (11) and (12)].

Our motivation to implement a numerical approach to approximate the optimal exercise boundary rather than to directly approximate the whole American value formula stems from recent evidence suggesting that this numerical group of methods can considerably increase the computation speed of calculations without losing much in accuracy [see Huang, Subrahmanyam and Yu (1996), and Ju (1998)]. This happens because the boundary approximation methods can separate the estimation problem of the optimal exercise boundary function from that of the American call option. This can increase the computation speed while, simultaneously, avoid accumulating pricing errors through the evaluation steps of the American option risk neutral pricing formula. Our motivation to employ Chebyshev polynomials to approximate the true optimal exercise boundary function stems from the fact that, with these polynomials, we can efficiently approximate any non-linear function by choosing the point with the minimum approximation error to fit a high-degree polynomial approximating function to the true function.[6] Note that the accuracy of this method increases with the number of polynomials terms used in the approximating function.

To better understand how to implement the Chebyshev polynomials method to approximate the optimal exercise boundary, which will be hereafter referred to as the CB method, we first start our analysis with the case of the lognormal model. We next extend the analysis to the SV model.

3.1 The case of the lognormal model

To implement the CB method for the lognormal model, notice that the optimal exercise boundary equation (15) can be reduced to the one-dimension integral relationship:

[6] A brief discription of the Chebyshev function approximation is given in Appendix C.

$$
\begin{aligned}
&B(h) - K \\
&= C_E(P_h, T-h) - B(h)e^{-\delta(T-h)}N\left(d_1\left(B(h), B(T), T-h\right)\right) + B(h)N(\xi) \\
&+ Ke^{-r(T-h)}N\left(d_1\left(B(h), B(T), T-h\right)\right) - KN(\xi) \\
&+ \int_t^T B(h)e^{-\delta(s-h)}n\left(d_1\left(B(h), B(s), s-h\right)\right)\frac{\partial d_1\left(B(h), B(s), s-h\right)}{\partial s} \\
&- \int_h^T Ke^{-r(s-h)}n\left(d_2\left(B(h), B(T), s-h\right)\right)\frac{\partial d_1\left(B(h), B(s), s-h\right)}{\partial s}ds, \qquad (18)
\end{aligned}
$$

where $\xi = \lim_{s\to h}\frac{\ln(B_h)-\ln(B(s))}{\sigma\sqrt{s-h}}$, $N(\cdot)$ and $n(\cdot)$ denote the cumulative not standard normal distribution and its associated probability density function, respectively.

Let $\widetilde{b}(h)$ denote an approximating function of the logarithm of the optimal exercise boundary which consists of ν-Chebyshev polynomials terms. The functional form of $\widetilde{b}(h)$ is given in Appendix C. Substituting $\widetilde{b}(h)$ into equation (18) implies the following system of equations

$$
\begin{aligned}
&\widetilde{B}(h) - K \\
&= C_E(P_h, T-h) - \widetilde{B}(h)e^{-\delta(T-h)}N\left(d_1\left(\widetilde{B}(h), \widetilde{B}(T), T-h\right)\right) + \frac{1}{2}\widetilde{B}(h) \\
&+ Ke^{-r(T-h)}N\left(d_1\left(\widetilde{B}(h), \widetilde{B}(T), T-h\right)\right) - \frac{1}{2}K \\
&+ \int_t^T \widetilde{B}(h)e^{-\delta(s-h)}n\left(d_1\left(\widetilde{B}(h), \widetilde{B}(s), s-h\right)\right)\left(\sum_{i=0}^{\nu-2}\alpha_i\frac{s^i}{\sqrt{s-h}}\right)ds \\
&- \int_h^T Ke^{-r(s-h)}n\left(d_1\left(\widetilde{B}(h), \widetilde{B}(s), s-h\right)\right)\left(\sum_{i=0}^{\nu-2}\gamma_i\frac{s^i}{\sqrt{s-h}}\right)ds, \qquad (19)
\end{aligned}
$$

where $\widetilde{B}(h) \equiv e^{\widetilde{b}(h)}$, and a_i and γ_i satisfy the following recursive equations

$$\alpha_i = \gamma_i = \left(2(i+1)c_{i+1} - \sum_{j=i+1}^{\nu} c_j\right)/2\sigma, \qquad \text{for } i = 1, 2, ..., \nu - 2,$$

and $\alpha_0 = \frac{1}{2\sigma}\left[2\,(i+1)\,c_{i+1} - \sum_{j=i+1}^{\nu} c_j + r - \delta + 0.5\sigma^2\right]$ and $\gamma_0 = \alpha_0 - \frac{1}{2\sigma}$, for $i = 0$, where c_{i+1} (or c_j) are the coefficients of the logarithm of the exercise boundary approximating function $\widetilde{b}(h)$ [see Appendix C].

The system of equations defined by (19) consists of ν-nonlinear equations with ν-unknown c_{i+1}, for $i = 0, 1, 2, ..., \nu - 2$, coefficients. Based on the minmax criterion, we can solve out this system for c_{i+1}, and determine the optimal exercise boundary approximating function, $\widetilde{B}(h)$. The above numerical approach guarantees that $\widetilde{B}(h)$ converges to its true value, $B(h)$, as the number of the polynomial terms (ν) of the approximating function increases. This happens because, according to the minmax criterion, $\widetilde{B}(h)$ is chosen so that to be equal to the true function $B(h)$ at ν-zero points, where $\widetilde{B}(h)$ cuts off $B(h)$. As ν increases, $\widetilde{B}(h)$ converges to $B(h)$ by Weierstrass theorem.

To increase the computation speed of the CB method without significantly losing in accuracy, we can employ Richardson's extrapolation scheme [see Ju (1988), *inter alia*]. According to this scheme, we need to calculate the optimal exercise boundary approximating function $\widetilde{B}(h)$ over the whole maturity interval, which is divided into $\lambda = 1, .., \Lambda$ pieces (points), where Λ denotes the maximum number of pieces. The values of the American price corresponding to the maturity interval with λ pieces will be hereafter denoted as $C_{A,\lambda}(P_t, \lambda(T-t))$. Below, we introduce all necessary notation in order to show how to calculate the American call price $C_{A,\lambda}(P_t, \lambda(T-t))$.

Let $\widetilde{B}_{\lambda l}(h)$, where $l = 1, 2, ..., \lambda$, denote the value of $\widetilde{B}(h)$ over the $l^{th}-$ subinterval of the λ pieces maturity interval. Denote by $\widetilde{B}_{\lambda l}(z_j)$, for $j = 1, 2..., \nu$, the ν-zero points of $\widetilde{B}_{\lambda l}(h)$ and by Δ the fraction of the maturity interval $\Delta = \frac{T-t}{\lambda}$. Then, system (19) evaluated at the $\nu-$zero points implies the following $\nu \times \Lambda$ dimension system of equations

$$
\begin{aligned}
&\widetilde{B}_{\lambda l}\left(z_j\right)-K=C_E\left(\widetilde{B}_{\lambda l}\left(z_j\right),T-z_j,z_j\right)\\
&-\widetilde{B}_{\lambda\lambda}\left(T\right)N\left(d_1\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda\lambda}\left(T\right),T-z_j\right)\right)\\
&+\frac{1}{2}\widetilde{B}_{\lambda\lambda}\left(T\right)+KN\left(d_2\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda\lambda}\left(T\right),T-z_j\right)\right)-\frac{1}{2}K\\
&+\int\limits_{z_j}^{t+l\Delta}\widetilde{B}_{\lambda l}\left(z_j\right)e^{-\delta(s-z_j)}n\left(d_1\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda l}\left(s\right),s-z_j\right)\sum_{i=0}^{\nu-2}\alpha_i\frac{s}{\sqrt{s-z_i}}ds\right)\\
&+\int\limits_{z_j}^{t+l\Delta}Ke^{-r(s-z_j)}n\left(d_2\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda l}\left(s\right),s-z_j\right)\sum_{i=0}^{\nu-2}\gamma_i\frac{s}{\sqrt{s-z_i}}ds\right)\\
&+\sum_{h=l+1}^{\lambda}\int\limits_{t+(h-1)\Delta}^{t+h\Delta}\widetilde{B}_{\lambda l}\left(z_j\right)e^{-\delta(s-z_j)}n\left(d_1\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda l}\left(s\right),s-z_j\right)\sum_{i=0}^{v-2}\alpha_i\frac{s}{\sqrt{s-z_j}}ds\right)\\
&-\sum_{h=l+1}^{\lambda}\int\limits_{t+(h-1)\Delta}^{t+h\Delta}Ke^{-r(s-z_j)}n\left(d_2\left(\widetilde{B}_{\lambda l}\left(z_j\right),\widetilde{B}_{\lambda l}\left(s\right),s-z_j\right)\sum_{i=0}^{\nu-2}\gamma_i\frac{s}{\sqrt{s-z_j}}ds\right),
\end{aligned}
\tag{20}
$$

for $j=1,2,...,\nu$ and $l=1,2,...,\lambda$. The above system can be solved out in the same way as system (19) in order to determine the optimal exercise boundary approximating function $\widetilde{B}_{\lambda l}(h)$, corresponding to the maturity interval with the λ pieces. The American call option price $C_{A,\lambda}(P_t,\lambda(T-t))$, used by Richardson's extrapolation scheme, can be then calculated as

$$
\begin{aligned}
&C_{A,\lambda}(P_t, \lambda(T-t)) = C_E(P_t, \lambda(T-t)) \\
&+ \sum_{l=1}^{\lambda} \int\limits_{t+(l-1)\Delta}^{t+l\Delta} \delta P_t e^{-\delta(s-t)} N\left(d_1\left(P_t, \widetilde{B}_{\lambda l}\left(s\right), s-t\right)\right) ds \\
&- \sum_{l=1}^{\lambda} \int\limits_{t+(l-1)\Delta}^{t+l\Delta} rKe^{-r(s-t)} N\left(d_2\left(P_t, \widetilde{B}_{\lambda l}\left(s\right), s-t\right)\right) ds. \qquad (21)
\end{aligned}
$$

3.2 The case of the SV model

The implementation of the CB method to the stochastic volatility case is slightly more complicated than the constant volatility case, described in the previous subsection. This happens because the optimal exercise boundary now is a function in two dimensions: the time and volatility. According to equation ((8), this means that we need to approximate the functional forms of the two coefficients $b_0(h)$ and $b_1(h)$ in order to approximate the optimal exercise boundary function $B\left(V_h, h\right)$.

Let us denote the approximating functional forms of these coefficients as $\tilde{b}_0(h)$ and $\tilde{b}_1(h)$, respectively. Then, equation (8) implies that $\tilde{b}_0(h)$ and $\tilde{b}_1(h)$ can be determined once two distinct values of the conditional variance (say $V_{h,0}$ and $V_{h,1}$) are provided. Denote the approximating boundary function by the CB method at the above two values of the conditional variance as $\widetilde{B}\left(V_{h,i}, h\right)$, $i = 0, 1$, respectively. Then, the coefficients $\tilde{b}_0(h)$ and $\tilde{b}_1(h)$ can be calculated as

$$
\tilde{b}_1(h) = \frac{\ln[\widetilde{B}\left(V_{h,1}, h\right) \diagup \widetilde{B}\left(V_{h,0}, h\right)]}{V_{h,1} - V_{h,0}} \qquad (22)
$$

and

$$
\tilde{b}_0(h) = \frac{\ln[V_{h,0}\widetilde{B}\left(V_{h,1}, h\right) \diagup \widetilde{B}\left(V_{h,0}, h\right) V_{h,1}]}{V_{h,0} - V_{h,1}}, \qquad (23)
$$

respectively. For an American call option with maturity date T, natural choices of $V_{h,0}$ and $V_{h,1}$ can be taken to be the time t expected values of the conditional variance $E_t V_h$ and $E_t V_T$, respectively. These constitute the values of the conditional variance around which the future values of the conditional variance over the maturity horizon $[h,T]$ are expected to fluctuate.

Equations (22) and (23) indicate that the optimal exercise boundary approximating function $B(V_h,h)$ can be estimated by implementing the CB method to approximate the exercise boundary at the two values of the conditional variance $V_{h,0}$ and $V_{h,1}$, i.e. $\widetilde{B}(V_{h,i},h)$, $i=0,1$, respectively. For a maturity interval with λ pieces, this implies that the following $2(\nu\times\Lambda)$ system of equations

$$
\begin{aligned}
&\widetilde{B}_{\lambda l}(V_{h,i},z_j)-K=C_E\left(\widetilde{B}_{\lambda l}(V_{s,i},z_j),V_i,z_j\right)\\
&+\int_{z_j}^{t+l\Delta}\delta\widetilde{B}_{\lambda l}(V_{s,i},z_j)\,e^{-\delta(s-z_j)}\Pi_1'\left(\widetilde{b}_{\lambda l,0}(s),\widetilde{b}_{\lambda l,1}(s)\,|\widetilde{B}_{\lambda l}(V_{h,i},z_j),V_i\right)ds\\
&-\int_{z_j}^{t+l\Delta}rKe^{-r(s-z_j)}\Pi_2'\left(\widetilde{b}_{\lambda l,0}(s),\widetilde{b}_{\lambda l,1}(s)\,|\widetilde{B}_{\lambda l}(V_{h,i},z_j),V_{h,i}\right)ds\\
&+\sum_{m=l+1}^{\lambda}\int_{t+m\Delta}^{t+m\Delta+\Delta}\delta\widetilde{B}_{\lambda l}(V_{s,i},z_j)\,e^{-\delta(s-z_j)}\Pi_1'\left(\widetilde{b}_{\lambda l,0}(s),\widetilde{b}_{\lambda l,1}(s)\,|\widetilde{B}_{\lambda l}(V_{h,i},z_j),V_{h,i}\right)\\
&-\sum_{m=l+1}^{\lambda}\int_{t+m\Delta}^{t+m\Delta+\Delta}rKe^{-r(s-z_j)}\Pi_2'\left(\widetilde{b}_{\lambda l,0}(s),\widetilde{b}_{\lambda l,1}(s)\,|\widetilde{B}_{\lambda l}(V_{h,i},z_j),V_{h,i}\right)ds,
\end{aligned}
\tag{24}
$$

for $i=0,1$, should be satisfied. Solving out this system with respect to the coefficients of the boundary approximating functions $\widetilde{B}(V_{h,i},h)$, for $i=0,1$, we can estimate the optimal exercise boundary approximating function $\widetilde{B}(V_h,h)$, using relationships (22) and (23). Given $\widetilde{B}(V_h,h)$, then the American call option price corresponding to the maturity interval with λ pieces can be calculated as

$$
\begin{aligned}
&C_{A,\lambda}\left(P_t, V_t, \lambda(T-t)\right) = C_E\left(P_t, V_t, \lambda(T-t)\right) \\
&+\sum_{l=1}^{\lambda} \int\limits_{t+(l-1)\Delta}^{t+l\Delta} \delta P_t e^{-\delta(s-t)} \Pi_1\left(\widetilde{b}_{\lambda l,0}\left(s\right), \widetilde{b}_{\lambda l,1}\left(s\right) | P_t, V_t\right) ds \\
&-\sum_{l=1}^{\lambda} \int\limits_{t+(l-1)\Delta}^{t+l\Delta} rK e^{-r(s-t)} \Pi_2\left(\widetilde{b}_{\lambda l,0}\left(s\right), \widetilde{b}_{\lambda l,1}\left(s\right) | P_t, V_t\right) ds, \qquad (25)
\end{aligned}
$$

and the Richardson's extrapolation scheme can be employed.

4 Numerical results of the Chebyshev approximation method

In this section we report numerical results to evaluate the performance of the CB approximation method of the exercise boundary, developed in the previous section, to price American call options both for the stochastic volatility and lognormal models. The performance of the method is measured in terms of both the speed and accuracy by with which it can price American call options in comparison with benchmark models. For the lognormal model, we compare the method with other existing numerical methods for pricing American call options based on an approximation of the optimal exercise boundary. These are the methods suggested by Huang, Subrahmanyam and Yu (1996) (hereafter HSY-3) and the exponential exercise boundary approximation method suggested by Ju (1998) (hereafter EXP-3). The aim of these comparisons is to investigate whether the CB method can improve upon the other optimal exercise boundary approximation methods, which are available for the lognormal model.[7] The

[7] A detail comparison of the optimal exercise boundary approximating methods with the other numerical methods for pricing American call options, based on the evaluation of the whole American call option risk neutral relationship or the finite difference methods, can be found in Ju (1998). This study clearly shows that the exercise boundary approximation methods are superior both in terms of accuracy and speed.

section has the following order. We present first the numerical results for the lognormal model and, second, for the stochastic volatility model.

4.1 Numerical results for the lognormal model

To assess the ability of the CB method to price American call options satisfactorily, compared with the other two approximation methods of the early exercise boundary function, we calculate the prices of $J = 1250$ American call options, denoted $C_{A,j}(P_t, V_t, T-t)$, $j = 1, 2, ..., J$, based on the above all methods and a benchmark method.[8] The parameters of the lognormal stock price model that we use in calculating the options prices are randomly generated from the uniform distribution over the following intervals: $[85, 115]$ for the current stock price (P_t), $[0.0, 0.10]$ for the dividend (δ) and interest rates (r), $[0.1, 0.6]$ for the volatility ($V_t = \sigma$) and $[0.1, 3.0]$ of years for the maturity interval. The strike price (K) is set as fixed, at the level of $K = 100$. The above intervals of the parameters of the lognormal model cover a set of estimates that have been reported by many studies in the empirical literature of option pricing. As a benchmark model, we use the binomial-tree model of Cox, Ross and Rubinstein (1979) with $N = 10,000$ time steps, denoted as BT. To evaluate the relative performance of the CB method as the degrees of the polynomial approximating function of the optimal exercise boundary increases, we employ the CB method with two and three degrees, denoted CB-2 and CB-3, respectively. For all the numerical methods employed, we evaluate the American call option prices over three-points of the maturity interval. Then, we use the three-point Richardson extrapolation scheme to calculate the American call options prices over the whole maturity interval.

The computational speed of each method is measured by the CPU time (in seconds) required for the calculation of the whole set of the American

[8] Note that in order to implement the HSY-3 method, we have slightly modified the procedure suggested by Huang, Subrahmanyam and Yu (1996). We have only used the HSY method to approximate the exercise boundary. The integral terms of the American call option evaluation formula are calculated numerically, as in our method. We have found that this modification of the HSY method considerably reduces the pricing errors of the method.

call options generated in all ($J = 1250$) experiments. The accuracy of each method compared with the benchmark model is assessed by calculating, over the whole set of generated option prices, the following two measures: the root mean squared error ($RMSE$), which is defined $RMSE = \sqrt{\frac{\sum_{j=1}^{J}(C_{A,j}(\cdot)-\mathrm{BT}_j)^2}{J}}$, and the maximum of the absolute pricing errors (MAE), which is defined as $MAE = \max\{|C_{A,1}(\cdot)-\mathrm{BT}_1|, |C_{A,2}(\cdot)-\mathrm{BT}_2|, ..., |C_{A,J}(\cdot)-\mathrm{BT}_J|\}$. We also calculate the above two measures for the option pricing errors as a percentage of the option prices of the benchmark model, i.e. $100 \cdot \frac{C_{A,j}(\cdot)-\mathrm{BT}_j}{BT_j}$. These measures are denoted as $RMSE\%$ and $MAE\%$. The numerical results of the above all measures and the CPU time can be found in Table 1.

As was expected, the results of the table clearly show that there is a trade off between accuracy and computational speed across all the approximation methods. In terms of accuracy, the CB-2 method can be compared with the EXP-3 method. The estimates of $RMSE$ and MAE measures, as well as of their counterparts for the percentage pricing errors, indicate that both the CB-2 and EXP-3 methods approximate adequately the option prices and clearly outperform the HSY-3 method; with the CB-2 method performing slightly better than the EXP-3 method. The HSY-3 seems to be superior only in terms of computational speed, which is obviously due to its functional simplicity. But this is at the cost of larger pricing errors. Note that accuracy of the CB method increases considerably as the degrees of the polynomial approximation ν increases, which is consistent with the predictions of the Weierstrass' theorem. Comparing the results of the table with those of Ju(1998), we can conclude that the CB-2 and EXP-3 methods perform much better than other numerical methods for pricing American call options based on the approximation of the whole American call option risk neutral relationship, or on the finite difference numerical methods.

The potential gains of CB method, compared with the two other approximation methods of the optimal exercise boundary function, for pricing American call options can be better understood with the help of Figure 1. This figure presents estimates of the optimal exercise boundary function by the CB-2, HSY-3 and EXP-3 methods, as well as those by the benchmark method, for the following set of parameters of the lognormal model: $\{P_t = 100, K = 100, r = 0.03,\ r - \delta = -0.04, \sigma = 0.4 \text{ and } T - t = 0.5\}$. For this set of parameters, we found that the lognormal model can generate a highly concave function of the optimal exercise boundary function with respect to the maturity interval. Inspection of the graphs of the figure indicate that the magnitude of the magnitude of the pricing errors of the CB method are clearly smaller than those of the HSY-3 and EXP-3 methods. The benefits, in terms of accuracy, of the CB method is due to the fact that it achieves a good approximation error of the true optimal exercise boundary. It does this by fitting an approximating polynomial in the neibourhood of the minimum error point. This will have a better pricing performance the more concave the optimal exercise boundary function is. In contrast, the HSY-3 method approximates the optimal exercise boundary function by fitting a straight line within each piece of the maturity interval, while the EXP-3 method uses a tangent line at the initial point of each piece of the interval. This will have as a consequence that the HSY method will result in higher errors compared to the other two methods when the the true optimal exercise boundary function is concave. The pricing errors of the EXP-3 method will depend on the degree of concavity of the optimal exercise boundary function.

Overall, the results of this section indicate that approximating the optimal exercise boundary by the CB-2 has proved to be a very fast and accurate method for pricing American call options for the lognormal model. It can be compared with other efficient approximation methods introduced in the literature, for this model.

4.2 Numerical results for the stochastic volatility model

To assess the performance of the CB method for the SV model, we focus on the CB-2 model which is found to perform very well in the case of the lognormal model. To evaluate the method, we follow steps similar to those in the previous section. We calculate the prices of $J = 1250$ American call option prices by drawing the parameters of the SV model from the uniform distribution over the following intervals: $[90, 110]$ for P_t, $[-1.0, 1.0]$ for the correlation coefficient (ρ), $[0.0, 1.0]$ for r and δ, $[0.1, 3.0]$ for k, $[0.01, 0.2]$ for θ, [0.1,0.5] for σ and $[0.1, 3.0]$ years for $T-t$. As previously, the strike price is assumed to be fixed, $K = 100$, in all experiments. The accuracy and speed performance of the CB-2 method are evaluated based on the $RMSE$ and MAE measures of the options pricing errors (as well as their $RMSE\%$ and $MAE\%$ counterparts for the pricing errors percentages), and the CPU time. To calculate the pricing errors, we use the lattice model suggested by Britten-Jones and Neuberger (2000) with $N = 200$ steps, denoted BJ-N, as benchmark model. In Table 2 we report the results.

The results of the table clearly show that the CB-2 method can be successfully applied to price American call options under the SV model. The $RMSE$ and MAE measures, as well as their $RMSE\%$ and $MAE\%$ counterparts, indicate that the magnitude of the pricing errors is very small. Note that it is almost of the same order as that for the lognormal model. In terms of computation time, the benefits of the CB-2 method are enormous. It only takes 13.35 minutes to calculate the whole set of the American call options. To make these calculations, we need about 6.0 hours by the benchmark model.

The success of the CB-2 method in pricing American call options under stochastic volatility can be attributed to fact that this method successfully approximates the optimal exercise boundary surface. This can also justify the assumption made in deriving Theorem 2 that the optimal exercise boundary surface is smooth with respect to volatility changes. To confirm this, in Figures 2(a)-(b), we present three-dimension graphs of the optimal exercise boundary surface implied by the SV model. This is done for the benchmark and CB-2 methods, respectively, based on the following set of parameters of the SV model: $\{r = 0.03, r - \delta = 0.01, k = 1.0, \theta = 0.03, \rho = 0.00, \sigma = 0.1\}$.[9] In Figure 3, we present a section of the estimated surfaces at the level of volatility $V_t = 0.16$.[10]

[9] This is a set of parameters used by Heston (1993) to calibrate the SV model.

[10] Note that these graphs are indicative. Similar graphs are taken at any other level of the volatility.

Indeed, inspection of the graphs of all the figures leads to the conclusion that a surface of the exercise boundary which is log-linear with respect to volatility changes can adequately approximate the true optimal exercise boundary. This justifies the assumption made in Theorem 2. From these graphs, it can be seen that the success of the CB-2 method in effectively pricing the options prices can be attributed to its ability to efficiently approximate the true optimal exercise boundary for the SV model. As the graphs of Figure 3 indicate, the approximation of the optimal exercise boundary by the CB-2 method under stochastic volatility is as closely as under constant volatility.

5 Conclusions

In this paper we introduced a new numerical method of pricing an American call option under stochastic volatility. The method is based on an approximation of the optimal exercise boundary by Chebyshev polynomials. To implement the method we derived an analytic, integral representation for the American call option price under stochastic volatility employing a log-linear function of the optimal exercise boundary with respect to the volatility changes. This representation unbundles the early exercise premium (and hence the American call option price) into a portfolio of Arrow-Debreu type of securities. The prices of these securities can be calculated by the joint characteristic function of the price of the underlying stock and its conditional variance. The analytic form of this function is derived in closed form in the paper. The paper presented a set of numerical results which show that our method can approximate American call option prices very quickly and efficiently both under stochastic and constant volatility. The numerical results show that our method is very efficient even for cases cases where the curvature of the true optimal exercise boundary function is high.

A Appendix (Proof of Theorem 1)

In this appendix, we prove Theorem 1.

Proof. To prove the theorem we follow similar steps with Myneni (1992), who decomposed the optimal problem (4) for an American put option under the assumptions of the lognormal model in terms of the exercise (stopping) boundary. To this end, notice that (4) implies

$$
\begin{aligned}
C_A\left(P_t, V_t, T-t\right) &= \sup_{\tau \in \mathcal{S}_{[t,T]}} E_t^Q\left(e^{-\int_t^{\tau} r ds}\left(P_\tau - K\right)_+\right) \qquad (26)\\
&= E_t^Q\left(e^{-\int_t^{\tau_t^*} r ds}\left(P_{\tau_t^*} - K\right)_+\right)\\
&= E_t^Q\left(e^{-\int_t^{T} r ds}\left(P_T - K\right)_+\right)\\
&\quad +E_t^Q\left(e^{-\int_t^{\tau_t^*} r ds}\left(P_{\tau^*} - K\right)_+ - e^{-\int_t^{T} r ds}\left(P_T - K\right)_+\right)\\
&= E_t^Q\left(e^{-\int_t^{T} r ds}\left(P_T - K\right)_+\right)\\
&\quad -E_t^Q\left(\int_{\tau^*}^{T} d\left(e^{-\int_t^{s} r du}\left(P_s - K\right)_+\right)\right),
\end{aligned}
$$

where $d(\cdot)$ is the differential operator. Note that first term in the last equation represents the value European option, while the second term constitutes the value of the early exercise premium. Using differentiation rules, the integral term of the early exercise premium term can be written as

$$
\begin{aligned}
&\int_{\tau_t^*}^{T} d\left(e^{-\int_t^{s} r du}\left(P_s - K\right)_+\right)\\
&= \int_{\tau_t^*}^{T} e^{-\int_t^{s} r du} d\left(P_s - K\right)_+ - \int_{\tau_t^*}^{T} r e^{-\int_t^{s} r du}\left(P_s - K\right)_+ ds. \qquad (27)
\end{aligned}
$$

Using Tanaka's formula and *local time* for Brownian motion at the point K, the differential $d\left(P_\tau - K\right)_+$ can be written as

$$d\left(P_s - K\right)_+ = dL_s^P\left(K\right) + I_{(P_s \geqslant K)} dP_s, \tag{28}$$

where $L_s^P\left(K\right)$ is the *local time* for Brownian motion at the value K of the stock price P_s and I_A is the indicator function of the set A, defined in Theorem 1. Using (28) and applying Ito's Lemma, equation (27) can be decomposed as follows

$$\begin{aligned}
& \int_{\tau_t^*}^{T} d\left(e^{-\int_t^s r du}\left(P_s - K\right)_+\right) \\
= & \int_{\tau_t^*}^{T} e^{-\int_t^s r du} dL_s^P\left(K\right) + \int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)} dP_s - \int_{\tau_t^*}^{T} r e^{-\int_t^s r du}\left(P_s - K\right)_+ ds \\
= & \int_{\tau_t^*}^{T} e^{-\int_t^s r du} dL_s^P\left(K\right) + \int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)}\left(r - \delta\right) P_s ds \\
& + \int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)} \sqrt{V_s} P_s dW_{1,s} - \int_{\tau_t^*}^{T} r e^{-\int_t^s r du}\left(P_s - K\right)_+ ds \\
= & \int_{\tau_t^*}^{T} e^{-\int_t^s r du} dL_s^P\left(K\right) + \int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)} \sqrt{V_s} P_s dW_{1,s} \\
& + \int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)} (rK - \delta P_s) ds.
\end{aligned} \tag{29}$$

Taking the conditional expectation of the last equation with respect the measure Q yields

$$E_t^Q\left(\int_{\tau_t^*}^{T} d\left(e^{-\int_t^s r du}\left(P_s - K\right)_+\right)\right) = E_t^Q\left(\int_{\tau_t^*}^{T} e^{-\int_t^s r du} I_{(P_s \geqslant K)} (rK - \delta P_s) du\right), \tag{30}$$

since $E_t^Q\left(dL_s^P\left(K\right)\right) = 0$ and $E_t^Q\left(dW_{1,s}\right) = 0$.

Noticing that, by the un-connected property of the optimal exercise boundary [see Broadie *et al* (2000), for a proof] the optimal exercise time τ^* [see equation (3)] can be defined as

$$\tau^* = \inf\left\{\tau \in [t,T] : P_s \geq B(V_s,s)\right\}, \forall\ s \in [t,T], \tag{31}$$

equation (30) can be written as

$$\begin{aligned} & E_t^Q\left(\int_{\tau_t^*}^{T} d\left(e^{-\int_t^\tau r ds}\left(P_\tau - K\right)_+\right)\right) \\ = \quad & E_t^Q\left(\int_t^T e^{-\int_t^s r du} I_{(P_s \geqslant K)} I_{(P_s \geqslant B(V_s,s))}(rK - \delta P_s)du\right). \end{aligned} \tag{32}$$

By the property of the exercise boundary that $B\left(V_s,s\right) > K$, $\forall\ s \in [t,T]$, the last equation implies

$$\begin{aligned} & E_t^Q\left(\int_{\tau_t^*}^{T} d\left(e^{-\int_t^\tau r ds}\left(P_\tau - K\right)_+\right)\right) \\ = \quad & E_t^Q\left(\int_t^T e^{-\int_t^s r du} I_{(P_s \geqslant B(V_s,s))}(rK - \delta P_s)ds\right). \end{aligned} \tag{33}$$

Substituting equation (33) into (26) proves the result of equation (5), given by Theorem 1. The optimal exercise boundary recursive equation (6) can be derived by (5) based on the arbitrage condition $C_A\left(B(V_h,h), V_h, T-h\right) = B(V_h,h) - K$, $\forall\ h \in [t,T]$. ■

B Appendix (Proof of Theorem 2)

In this appendix, we prove Theorem 2 of the paper. To this end, we first derive the joint conditional characteristic function (CF) of the logarithm of the stock price $\ln P_s$ adjusted by the term $(r-\delta)(s-h)$, $\forall\ s \in [t,T]$, and the variance V_s conditional on the values of $\ln P_h$ and V_h, for $s \geqslant h \in [t,T]$. This is given in the following Lemma.

Lemma 3 *Let the SV model, defined by processes (1) and (2), hold. Define* $Y_{s,h} = \ln P_s - (r-\delta)(s-h)$, $\forall\ s \geqslant h \in [t,T]$. *Then, the joint characteristic function of* Y_s *and* V_s *conditional on the values of* $Y_{h,h} = \ln P_h$ *and* V_h *is given by*

$$F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h) = e^{g_0(\phi_Y,\phi_V,s-h)+g_1(\phi_Y,\phi_V,s-h)Y_{h,h}+g_2(\phi_Y,\phi_V,s-h)V_h},$$

where

$$\begin{aligned} & g_0(\phi_Y, \phi_V, s-h) \\ = \quad & -\frac{k\theta}{\sigma^2}\left\{(D+B)(s-h) + 2\ln\left[1 - \frac{D+B+\sigma^2 i\phi_V}{2D}\left(1-e^{-D(s-h)}\right)\right]\right\} \end{aligned}$$

$$g_1(\phi_Y, \phi_V, s-h) = i\phi_Y$$

$$g_2(\phi_Y, \phi_V, s-h) = \frac{C(1-e^{-D(s-h)}) + i\phi_V\left[2D-(D-B)(1-e^{-D(s-h)})\right]}{2D-(D+B)(1-e^{-D(s-h)}) - \phi_V\sigma^2(1-e^{-D(s-h)})}$$

and
$A = \frac{1}{2}\sigma^2$, $B = \rho\sigma i\phi_Y - k$, $C = -\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y$ *and* $D = \sqrt{B^2 - 4AC}$.

Proof. By Ito's Lemma, we can write

$$dY_{s,h} = -\frac{1}{2}V_s ds + \sqrt{V_s}dW_{1,s} \tag{34}$$

Denote the joint CF of $Y_{s,h}$ and V_s conditional on the values of $Y_{h,h}$ and V_h, at time h, as $F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)$.

Consider the following general affine solution for $F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)$

$$F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h) = e^{g_0(\phi_Y,\phi_V,s-h)+g_1(\phi_Y,\phi_V,s-h)Y_{h,h}+g_2(\phi_Y,\phi_V,s-h)V_h}. \tag{35}$$

Then, $F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)$ should satisfy the following partial differential equation (PDE)

$$\begin{aligned}
& \frac{\partial F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial h} + \frac{1}{2}V_h \frac{\partial^2 F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial Y_{h,h}^2} \\
& + \rho\sigma V_h \frac{\partial^2 F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial Y_{h,h}\partial V_h} + \frac{1}{2}\sigma^2 V_h \frac{\partial^2 F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial V_h^2} \\
& - \frac{1}{2}V_h \frac{\partial F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial Y_{h,h}} + k(\theta - V_h)\frac{\partial F(\phi_Y, \phi_{V,} s-h|Y_{h,h}, V_h)}{\partial V_h} \\
= {} & 0
\end{aligned} \tag{36}$$

Substituting (35) into (36) yields

$$\begin{aligned}
& V_h[\frac{\partial g_2(\phi_Y, \phi_V, s-h)}{\partial h} + \frac{1}{2}g_1^2(\phi_Y, \phi_V, s-h) \\
& + \rho\sigma g_1(\phi_Y, \phi_V, s-h) g_2(\phi_Y, \phi_V, s-h) \\
& + \frac{1}{2}\sigma^2 g_2^2(\phi_Y, \phi_V, s-h) - \frac{1}{2}g_1(\phi_Y, \phi_V, s-h) - k g_2(\phi_Y, \phi_V, s-h)] \\
& Y_{h,h}\frac{\partial g_1(\phi_Y, \phi_V, s-h)}{\partial h} + [\frac{\partial g_0(\phi_Y, \phi_V, s-h)}{\partial h} + k\theta g_2(\phi_Y, \phi_V, s-h)] \\
= {} & 0
\end{aligned} \tag{37}$$

The CF (35) coefficients $g_0(\phi_Y, \phi_V, s-h)$, $g_1(\phi_Y, \phi_V, s-h)$ and $g_2(\phi_Y, \phi_V, s-h)$ can be derived by solving out the three ordinary differential equations (ODE) implied by the above PDE, i.e.

$$\frac{\partial g_1(\phi_Y, \phi_V, s-h)}{\partial h} = 0, \tag{38}$$

$$
\begin{aligned}
&\frac{\partial g_2(\phi_Y,\phi_V,s-h)}{\partial h}\\
= \quad & -\frac{1}{2}\sigma^2 g_2^2(\phi_Y,\phi_V,s-h) - g_2(\phi_Y,\phi_V,s-h)\left(\rho\sigma g_1(\phi_Y,\phi_V,s-h)-k\right)\\
& -\left(\frac{1}{2}g_1^2(\phi_Y,\phi_V,s-h) - \frac{1}{2}g_1(\phi_Y,\phi_V,s-h)\right), \qquad (39)
\end{aligned}
$$

and

$$
\frac{\partial g_0(\phi_Y,\phi_V,s-h)}{\partial s} = -k\theta g_2(\phi_Y,\phi_V,s-h), \qquad (40)
$$

subject to the following boundary conditions $g_0(\phi_Y,\phi_V,0)=0$, $g_1(\phi_Y,\phi_V,0)=i\phi_Y$, and $g_2(\phi_Y,\phi_V,0)=i\phi_V$.

Solving out ODE (38) for $g_1(\phi_Y,\phi_V,s-h)$ yields

$$
g_1(\phi_Y,\phi_V,s-h) = i\phi_Y. \qquad (41)
$$

To derive the coefficient $g_2(\phi_Y,\phi_V,s-h)$, substitute (41) into (39). This yields

$$
\begin{aligned}
&\frac{\partial g_2(\phi_Y,\phi_V,s-h)}{\partial h}\\
&= -\frac{1}{2}\sigma^2 g_2^2(\phi_Y,\phi_V,s-h) - g_2(\phi_Y,\phi_V,s-h)\left(\rho\sigma i\phi_Y - k\right)\\
&-\left(-\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y\right)\\
&= -\frac{1}{2}\sigma^2\left(g_2(\phi_Y,\phi_V,s-h)-x_1\right)\left(g_2(\phi_Y,\phi_V,s-h)-x_2\right), \qquad (42)
\end{aligned}
$$

where $x_1 = \frac{-B+\sqrt{B^2-4AC}}{2A}$, $x_2 = \frac{-B-\sqrt{B^2-4AC}}{2A}$, $A=\frac{1}{2}\sigma^2$, $B=\rho\sigma i\phi_X - k$, $C=-\frac{1}{2}\phi_Y^2 - \frac{1}{2}i\phi_Y$ and $D=\sqrt{B^2-4AC}$. Rearranging terms in equation (42) and integrating both sides of the resulting equation yields

$$
\frac{1}{D}\int\left(\frac{1}{g_{2(\phi_Y,\phi_V,s-h)}-x_1} - \frac{1}{g_2(\phi_Y,\phi_V,s-h)-x_2}\right)dg_2(\phi_Y,\phi_V,s-h) = \int dh.
$$

Using the boundary conditions of the CF's coefficients, the last equation implies that the closed form solution for $g_2(\phi_Y,\phi_V,s-h)$ is given by

$$g_2(\phi_Y, \phi_V, s-h) = \frac{C(1-e^{-D(s-h)}) + i\phi_V \left[2D - (D-B)\left(1-e^{-D(s-h)}\right)\right]}{2D - (D+B)\left(1-e^{-D(s-h)}\right) - \phi_V \sigma^2 (1-e^{-D(s-h)})}. \tag{43}$$

Substituting the closed form solutions of the coefficients $g_1(\phi_Y, \phi_V, s-h)$ and $g_2(\phi_Y, \phi_V, s-h)$, given by equations (41) and (43), respectively, into ODE (40) and integrating gives the closed form solution for the coefficient $g_0(\phi_Y, \phi_V, s-h)$:

$$\begin{aligned} & g_0(\phi_Y, \phi_V, s-h) \\ = \quad & -\frac{k\theta}{\sigma^2}\left\{(D+B)(s-h) + 2\ln\left[1 - \frac{D+B+\sigma^2 i\phi_V}{2D}\left(1-e^{-D(s-h)}\right)\right]\right\}. \end{aligned}$$

■

Having derived the closed form solution of the CF $F(\phi_Y, \phi_{V,} s-h | Y_{h,h}, V_h)$, we next prove Theorem 2.

Proof. (Proof of Theorem 2). To prove the theorem, we need to derive an integral representation of the early exercise premium

$$E_h^Q\left[\int_h^T e^{-r(s-h)}\left(\delta P_s - rK\right) I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}} ds \mid B(V_h,h), V_h\right], \tag{44}$$

defined in equation (6). This can be done as follows.

Using the law of iterated expectations, write equation (44) as

$$\int_h^T E_h^Q\left[e^{-r(s-h)}\left(\delta P_s - rK\right) I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}} | B(V_h,h), V_h\right] ds, \tag{45}$$

where $E_h^Q\left[e^{-r(s-h)}\left(\delta P_s - rK\right) I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}} | B(V_h,h), V_h\right]$ represents the present value of the risk neutral continuous payoff of the early exercise at time h. This value can be decomposed as follows

$$E_h^Q\left[e^{-r(s-h)}\left(\delta P_s - rK\right)I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right] =$$

$$\delta e^{-r(s-h)}E_h^Q\left[P_s\right]E_h^Q\left[\frac{P_s I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}}{E_h^Q\left[P_s\right]}|B(V_h,h),V_h\right]$$

$$-rKe^{-r(s-h)}E_h^Q\left[I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right]. \tag{46}$$

Equation (46) decomposes the present value of the risk neutral continuous payoff of the early exercise time t can be unbundled into a portfolio of the Arrow-Debreu type of securities [see Bakshi and Madan (2000)]. The prices of these securities are defined as

$$E_h^Q\left[\frac{P_s I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}}{E_h^Q\left[P_s\right]}|B(V_h,h),V_h\right] \tag{47}$$

and

$$E_h^Q\left[I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right], \tag{48}$$

respectively. Below, we derive analytic, integral representations (solutions) of these prices based on the closed form solution of the CF $F(\phi_Y,\phi_{V,}s-h|Y_{h,h},V_h)$, given by Lemma 3. Substituting these solutions into equation (46) gives the integral representation of the optimal exercise premium. Since our early exercise premium (44) is considered at the optimal exercise boundary price, $B(V_h,h)$, in applying the results of Lemma 3 we assume that now $Y_{s,h}$ is defined as $Y_{s,h} = \ln(B(V_s,s)) + (r-\delta)(s-h)$ and $Y_{h,h}$ as $Y_{h,h} = \ln(B(V_h,h))$.

To derive an analytic, integral representation of the security price defined by equation (47), write $E_h^Q\left[P_s I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right]$ as

$$E_h^Q\left[P_s I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right]=$$
$$\int\limits_{-\infty}^{\infty} dV_s \int_{\log(B(V_s,s))-(r-\delta)(s-h)}^{\infty} e^{(r-\delta)(s-h)+Y_{s,h}}\pi(Y_{s,h},V_s|\ln(B(V_h,h)),V_h)dY_{s,h}, \tag{49}$$

where $\pi(Y_{s,h},V_s|\ln(B(V_h,h)),V_h)$ is the joint probability density function of $Y_{s,h}$ and V_s conditional on the variables $Y_{h,h}=\ln(B(V_h,h))$ and V_h. Denote the marginal characteristic function of $F(\phi_Y,\phi_{V,}s-h|\ln(B(V_h,h)),V_h)$ with respect to V_h as $F_V\left(\phi_Y,V_s|\ln(B(V_h,h)),V_h\right)$, defined

$$F_V\left(\phi_Y,V_s|\ln(B(V_h,h)),V_h\right)=\int\limits_{-\infty}^{\infty} e^{i\phi_Y Y_{s,h}}\pi(Y_{s,h},V_s|\ln(B(V_h,h)),V_h)dY_{s,h}.$$

Then, the exercise boundary relationship (8) implies that equation (49) can be written in terms of one-dimension integrals as

$$E_h^Q\left[P_s I_{\{(P_s,V_s):P_s\geqq B(V_s,s)\}}|B(V_h,h),V_h\right]=$$
$$\int\limits_{-\infty}^{\infty} e^{(r-\delta)(s-h)}dV_s\left\{\frac{1}{2}\pi_V(V_s|\ln(B(V_h,h)),V_h)+\left(\frac{1}{2\pi}\right)\right.$$
$$\left.\int\limits_{-\infty}^{\infty}\mathrm{Re}\left(\frac{e^{-i\phi_Y[b_0(s)+b_1(s)V_s-(r-\delta)(s-h)]}F_V\left(\phi_Y-i,V_s|\ln(B(V_h,h)),V_h\right)}{i\phi_Y}\right)d\phi_Y\right\}, \tag{50}$$

where $\pi_V(V_s|\ln(B(V_h,h)),V_h)$ is the marginal density function of joint probability density $\pi(Y_{s,h},V_s|\ln(B(V_h,h)),V_h)$ with respect to V_h, i.e.

$$\pi_V(V_s|\ln(B(V_h,h)),V_h)=\int\limits_{-\infty}^{\infty}\pi(Y_{s,h},V_s|\ln(B(V_h,h)),V_h)dY_{s,h}.$$

Noticing that the CF $F(\phi_Y, \phi_V, s-h|\ln(B(V_h,h)), V_h)$ and its marginal CF $F_V(\phi_Y, V_s|\ln(B(V_h,h)), V_h)$ are linked through the relationship

$$F(\phi_Y, \phi_V, s-h|\ln(B(V_h,h)), V_h) = \int_{-\infty}^{\infty} e^{i\phi_V V_s} F_V(\phi_Y, V_s|\ln(B(V_h,h)), V_h)\, dV_s,$$

equation (50) can be expressed in terms of one-dimension integrals as

$$\begin{aligned}
& E_h^Q\left[P_s I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}}|B(V_h,h), V_h\right] \\
= \; & \frac{1}{2}e^{(r-\delta)(s-h)} + \left(\frac{e^{(r-\delta)(s-h)}}{2\pi}\right) \\
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{-i\phi_Y[b_0(s)+b_1(s)V_s-(r-\delta)(s-h)]}F_V(\phi_Y - i, V_s|\ln(B(V_h,h)), V_h)}{i\phi_Y}\right) dV_s d\phi_Y \\
= \; & \frac{1}{2}e^{(r-\delta)(s-h)} + \left(\frac{e^{(r-\delta)(s-h)}}{2\pi}\right) \\
& \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{-i\phi_Y[b_0(s)-(r-\delta)(s-h)]}F(\phi_Y - i, -b_1(s)\phi_Y, s-h|\ln(B(V_h,h)), V_h)}{i\phi_Y}\right) d\phi_Y.
\end{aligned} \tag{51}$$

Having derived an integral representation of $E_h^Q\left[P_s I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}}|B(V_h,h), V_h\right]$, the state price defined by equation (47) can be calculated once a closed form solution for $E_h^Q[P_s]$ is derived. This can be done by setting $\phi_Y = -i$ and $\phi_V = 0$ in $F(\phi_Y, \phi_V, s-h|Y_{h,h}, V_h)$, yielding

$$\begin{aligned}
E_h^Q[P_s] &= e^{(r-\delta)(s-h)}F(-i, 0, s-h|\ln(B(V_h,h)), V_h) \\
&= e^{(r-\delta)(s-h)}B(V_h,h).
\end{aligned} \tag{52}$$

Substituting equations (51) and (52) into (47) yields the integral representation of the security price defined by (47):

$$\begin{aligned}
&\Pi_1^{'}\left(b_0(s), b_1(s) | B(V_h, h), V_h\right) \\
\equiv\ & E_h^Q\left[\frac{P_s I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}}}{E_h^Q\left[P_s\right]} | B(V_h,h), V_h\right] \\
=\ & \frac{1}{2B(V_h,h)} + \left(\frac{1}{2\pi B(V_h,h)}\right) \\
& \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{-i\phi_Y[b_0(s)-(r-\delta)(s-h)]} F(\phi_Y - i, -b_1(s)\phi_Y, s-h | \ln(B(V_h,h)), V_h)}{i\phi_Y}\right) d\phi_Y.
\end{aligned} \tag{53}$$

Following similar steps with above, we can derive the following integral representation of the security price defined by equation (48):

$$\begin{aligned}
&\Pi_2^{'}\left(b_0(s), b_1(s) | B(V_h, h), V_h\right) \\
\equiv\ & E_h^Q\left[I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}} | B(V_h,h), V_h\right] = \\
& \frac{1}{2} + \left(\frac{1}{2\pi}\right) \\
& \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{-i\phi_Y[b_0(s)-(r-\delta)(s-h)]} F(\phi_Y, -b_1(s)\phi_Y, s-h | \ln(B(V_h,h)), V_h)}{i\phi_Y}\right) d\phi_Y.
\end{aligned} \tag{54}$$

Substituting (53) and (54) into (46) proves the boundary recursive equation (10), given by Theorem 2.

The closed form solutions of the Arrow-Debreu security prices which enter into the American call option price evaluation formula (9) can be derived by defining $Y_{s,t} = \ln P_s + (r-\delta)(s-t)$ and $Y_{t,t} = \ln P_t$, assuming that $h = t$. This will give us the following

$$
\begin{aligned}
& \Pi_1\left(b_0(s), b_1(s) | P_t, V_t\right) \\
\equiv \quad & E_t^Q\left[\frac{P_s I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}}}{E_t^Q\left[P_s\right]} | P_t, V_t\right] \\
= \quad & \frac{1}{2P_h} + \left(\frac{1}{2\pi P_h}\right) \\
& \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{e^{-i\phi_Y[b_0(s)-(r-\delta)(s-t)]} F(\phi_Y - i, -b_1(s)\phi_Y, s-t | \ln P_t, V_t)}{i\phi_Y}\right) d\phi_Y
\end{aligned}
$$

and

$$
\begin{aligned}
& \Pi_2\left(b_0(s), b_1(s) | P_t, V_t\right) \\
\equiv \quad & E_t^Q\left[I_{\{(P_s,V_s):P_s \geqq B(V_s,s)\}} | P_t, V_t\right] \\
= \quad & \frac{1}{2} + \left(\frac{1}{2\pi}\right) \\
& \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{e^{-i\phi_Y[b_0(s)-(r-\delta)(s-t)]} F(\phi_Y, -b_1(s)\phi_Y, s-t | \ln P_h, V_h)}{i\phi_Y}\right) d\phi_Y,
\end{aligned}
$$

at time t. ■

C Appendix (Chebyshev approximation)

According to the CB method, any continuous function $b(x)$, where $x \in [-1, 1]$, can be approximated by a linear combination of ν-Chebyshev polynomials, denoted $w_j(x)$, as follows

$$\tilde{b}(x) = \sum_{j=1}^{\nu} q_j w_j(x), \tag{55}$$

where $w_j(x)$ denotes the i^{th} Chebyshev polynomial, defined as

$$w_j(x) = \cos\left(j \arccos\left(x\right)\right), \tag{56}$$

with $w_j(x)$ satisfying the recurrence

$$w_{j+1}\left(x\right) = 2xw_j\left(x\right) - w_{j-1}\left(x\right), \tag{57}$$

with $w_0 = 1$ and $w_1 = x$.

The Chebyshev polynomials satisfy the Weierstrass theorem and meet the minmax criterion. According to this criterion, the Chebyshev approximating function, denoted $\tilde{b}(x)$, is one that equals the true function $b(x)$ at the set of ν zeros values of $w_j(x)$, taken for $x = \cos\left(\pi\left(j - 0.5\right)/v\right)$, $j = 1, 2, ..., v$. The ν zeros values of $w_j(x)$ imply a system of the ν equations with ν unknown coefficients q_j. Solving out this system with respect to q_j can determine the approximating function.

Although the Chebyshev approximating faction $\tilde{b}(x)$ is defined in the finite interval $[-1, 1]$, we can approximate other function $\tilde{b}(h)$, where h is defined in the interval $[t, T]$, by rescaling the values of x to h as $h = \frac{1}{2}\left(\left(T - t\right)x + T + t\right)$. This implies that

$$\tilde{b}(h) = \tilde{b}\left(x = \frac{2h - T - t}{T - t}\right). \tag{58}$$

Substituting $x = \frac{2h-T-t}{T-t}$ into equation (55), the new function $\tilde{b}(h)$ can be written as

$$\tilde{b}(h) = \sum_{i=0}^{\nu} c_i h^i. \tag{59}$$

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TABLE 1: NUMERICAL RESULTS FOR THE LOGNORMAL MODEL

	HSY-3	EXP-3	CB-2	CB-3
RMSE	0.0059	0.0029	0.0026	0.0012
MAE	0.0679	0.0178	0.0163	0.0089
RMSE%	0.0673%	0.0251%	0.0236%	0.0171%
MAE%	0.474%	0.163%	0.142%	0.087%
CPU(secs)	3.17	9.75	9.65	67.71

The table presents the values of the $RMSE$ and MAE measures of accuracy for American call option prices (as well as their percentage errors, denoted by $RMSE\%$ and $MAE\%$, respectively) and the CPU time for the following optimal exercise boundary approximation methods: HSY-3, EXP-3, CB-2 and CB-3, under the assumptions of the lognormal model. $RMSE$ is calculated as $RMSE = \sqrt{\frac{\sum_{j=1}^{J}(C_{A,j}(\cdot)-\mathrm{BT}_j)^2}{J}}$, where BT denotes the American call prices calculated by the benchmark model and $J = 1250$ is the total number of the American call option prices calculated, while $MAE = \max\{|C_{A,1}(\cdot)-\mathrm{BT}_1|,\ |C_{A,2}(\cdot)-\mathrm{BT}_2|, .., |C_{A,J}(\cdot)-\mathrm{BT}_J|\}$. To calculate $RMSE\%$ and $MAE\%$, we use the percentage pricing errors $(100 \cdot \frac{C_{A,j}(\cdot)-\mathrm{BT}_j}{\mathrm{BT}_j})$. The prices of the options are calculated by drawing the parameters of the lognormal model randomly from the uniform distribution over the following intervals: $[85, 115]$ for the current stock price (P_t), $[0.0, 0.10]$ for the dividend (δ) and interest rates (r), $[0.1, 0.6]$ for the volatility $(V_t = \sigma)$ and $[0.1, 3.0]$ of years for the maturity interval. The strike price (K) is set as $K = 100$.

TABLE 2. NUMERICAL RESULTS FOR THE SV MODEL

	RMSE	MAE	RMSE%	MAE%	CPU (secs)
CB-2	0.0035	0.0123	0.063%	0.191%	801.51 (or 13.35 mins)

Notes: The table presents the values of the $RMSE$ and MAE measures (as well as their % counterparts) for pricing American call option prices under the SV model based on the CB-2 approximation method. The estimates of these measures are based on $J = 1250$ American call option prices drawing the parameters of the SV model from the uniform distribution over the following intervals: $[90, 110]$ for P_t, $[-1.0, 1.0]$ for the correlation coefficient (ρ), $[0.0, 1.0]$ for r and δ, $[0.1, 3.0]$ for k, $[0.01, 0.2]$ for θ, [0.1,0.5] for σ and $[0.1, 3.0]$ years for $T - t$. The strike price K is set up as $K = 100$. As benchmark model, we use the model suggested by Britten-Jones and Neuberger (2000), with $N = 200$ steps.

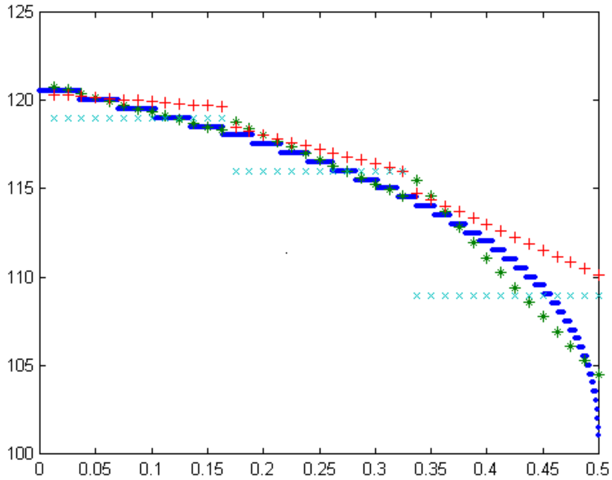

Figure 1:

Figure 1. This figure presents the graphs of the optimal exercise boundary functions for the lognormal model estimated by the benchmark model (...) and the CB-2 (***), HSY-3 (xxx) and EXP-3 (+++) approximating methods, for $K = 100$, $T - t = 0.5$ yrs and the following set of parameters of the BS model $\{r = 0.03,\ r - \delta = -0.04 \text{ and } \sigma = 0.4\}$.

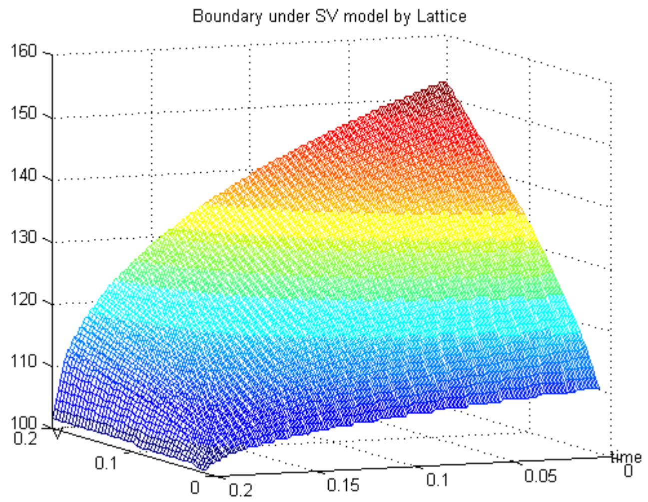

Figure 2:

Figure 2(a): This figure presents the graph of the optimal exercise boundary surface for the SV model estimated by the benchmark model, for $T - t = 0.5$, $K = 100$ and using the following set of the parameters of the SV model $\{r = 0.03, r - \delta = 0.01, k = 1.0, \theta = 0.03, \rho = 0.00, \sigma = 0.1\}$.

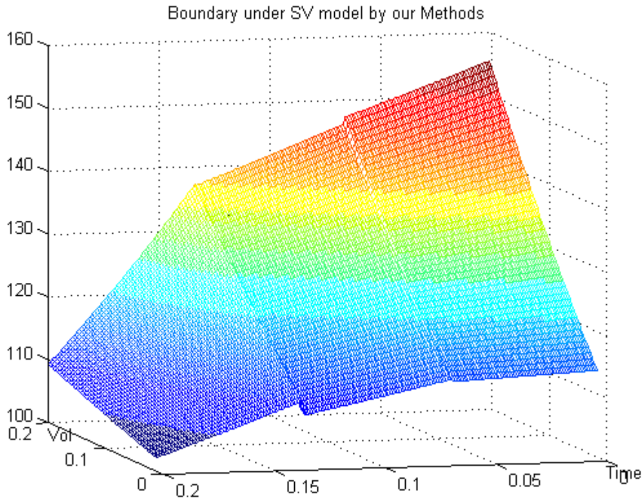

Figure 3:

Figure 2(b): This figure presents the graph of the optimal exercise boundary surface for the SV model estimated by the CB-2 method, for $T-t = 0.5$, $K = 100$ and using the following set of the parameters of the SV model $\{r = 0.03, r-\delta = 0.01, k = 1.0, \theta = 0.03, \rho = 0.00, \sigma = 0.1\}$, as in Figure 2(a).

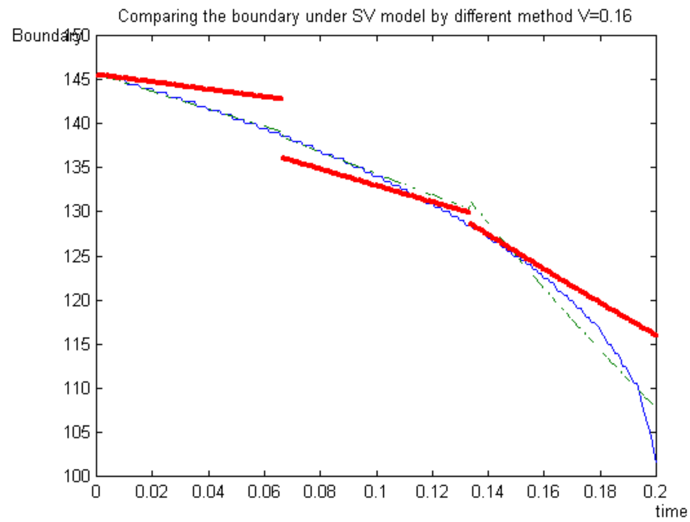

Figure 4:

Figure 3: This figure presents a section of the optimal exercise boundary surface of Figures 2(a)-2(b) estimated by the benchmark model (....) and the CB-2 approximating method (**...**), respectively, at the level of volatility $V_t = 0.16$.

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