

# Department of Economics

## A Note on Covariance Stationarity Conditions for Dynamic Random Coefficient Models

George Kapetanios

Working Paper No. 475

November 2002

ISSN 1473-0278



Queen Mary  
University of London

# A Note on Covariance Stationarity Conditions for Dynamic Random Coefficient Models

George Kapetanios\*  
Queen Mary, University of London

November 2002

## Abstract

In this note we look at sufficient conditions for stationarity of a simple random coefficient model and find that this model is guaranteed to be stationary under strict conditions.

JEL codes: C22

Keywords: Stationarity, Random Coefficient Models

---

\*Department of Economics, Queen Mary, University of London, Mile End Rd., London E1 4NS. Email: G.Kapetanios@qmul.ac.uk

# 1 Introduction

Recently there has been a resurgence in the macroeconomic literature of the investigation of random coefficient models. These models are used to investigate possible shifts in the dynamic evolution of various macroeconomic variables such as inflation. Notable examples of such analyses are Cogley and Sargent (2002) and Benati (2002). These studies start with the presumption that inflation may be nonstationary (not necessarily trending or unit root nonstationary) and model the series using dynamic random coefficient models.

In this note we look at sufficient conditions for stationarity of a very simple random coefficient model and find that this model is guaranteed to be stationary under strict conditions. Section 2 presents the model. Section 3 presents existing relevant work in the literature and section 4 presents the derivation of the sufficient conditions for stationarity.

## 2 The Model

Let the general dynamic random coefficient model be given by

$$x_t = \sum_{i=1}^p a_{t,i} x_{t-i} + \epsilon_t \quad (1)$$

$$\mathbf{a}_t = \boldsymbol{\mu} + \sum_{i=1}^q \boldsymbol{\Gamma}_i \mathbf{a}_{t-i} + \mathbf{u}_t$$

where  $\mathbf{a}_t = (a_{t,1}, a_{t,2}, \dots, a_{t,p})'$ ,  $\mathbf{u}_t = (u_{t,1}, \dots, u_{t,p})'$  and  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)$ . The variance of  $\epsilon_t$  is given by  $\sigma^2$  and the covariance matrix of  $\mathbf{u}_t$  is given by  $\boldsymbol{\Sigma}_u$ . We refer to this as the extended state space representation of the random coefficient model. We will denote this model by  $DRC(p, q)$ . We will

discuss covariance stationarity for  $DRC(1, 1)$  and relate this discussion to existing stationarity conditions for bilinear models.

### 3 Existing Work

A standard bilinear model is obtained from (1) by setting  $q = 0$ . It is easy to see that the  $DRC(p, q)$  model may be written as

$$\mathbf{x}_t = \sum_{i=1}^{\infty} \left( \prod_{j=0}^{i-1} \mathbf{A}'_{t-j} \tilde{\mathbf{I}} \right) \boldsymbol{\epsilon}_{t-i} + \boldsymbol{\epsilon}_t \quad (2)$$

where

$$\mathbf{A}'_t = \begin{pmatrix} a_{t,1} & a_{t,2} & \dots & a_{t,p} & a_{t-1,1} & \dots & a_{t-1,p} & \dots & a_{t-p+1,1} & \dots & a_{t-p+1,p} \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

$$\mathbf{A}_t = \tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\Gamma}} \mathbf{A}_{t-1} + \tilde{\mathbf{u}}_t \quad (3)$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 & \dots & \boldsymbol{\Gamma}_p \\ \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \end{pmatrix} \quad \tilde{\boldsymbol{\Gamma}} = [\boldsymbol{\Gamma} \ \mathbf{0}]$$

$$\tilde{\mathbf{I}} = \begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \dots & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\tilde{\mathbf{u}}'_t = \begin{pmatrix} u_{t,1} & u_{t,2} & \dots & u_{t,p} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \quad \tilde{\boldsymbol{\mu}}' = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_p & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

and  $\boldsymbol{\epsilon}_t = (\epsilon_t, 0, \dots, 0)'$ . Then it follows that the model may be written as

$$\mathbf{x}_t = \sum_{i=1}^{\infty} \left[ \prod_{j=0}^{i-1} \left( \tilde{\boldsymbol{\mu}}' + \sum_{s=0}^{\infty} \tilde{\mathbf{u}}'_{t-s-j} \tilde{\boldsymbol{\Gamma}}'^s \right) \tilde{\mathbf{I}} \right] \boldsymbol{\epsilon}_{t-i} + \boldsymbol{\epsilon}_t \quad (4)$$

where  $\bar{\boldsymbol{\mu}} = [(\mathbf{I} - \boldsymbol{\Gamma})^{-1}(\boldsymbol{\mu} \ 0) \ 0]$ .

This model then resembles the form of the general bilinear model dealt with by Liu and Brockwell (1988), (see also Tong (1990)). However the crucial difference is that the model considered in Liu and Brockwell (1988) was of the form

$$\mathbf{x}_t = \sum_{i=1}^{\infty} \left[ \prod_{j=0}^{i-1} \left( \bar{\boldsymbol{\mu}}' + \sum_{s=0}^{s_1} \tilde{\mathbf{u}}'_{t-s-j} \tilde{\boldsymbol{\Gamma}}'^s \right) \tilde{\mathbf{I}} \right] \boldsymbol{\epsilon}_{t-i} + \boldsymbol{\epsilon}_t \quad (5)$$

It can be seen from the analysis of Liu and Brockwell (1988) that the stationarity of the model depends crucially on  $s_1$ . Liu and Brockwell (1988) provide an explicit condition for the case  $s_1 = 2$  and discuss how to generalise this to  $s_1 > 2$ . The condition they propose is in the spirit of the drift condition of Tweedie (1975) which states that a Markov chain,  $x_t$ , is strictly stationary if the following two conditions hold for some  $B > 0$  and  $r < 1$

$$E(g(x_t)|x_{t-1} = x) < rg(x), \quad x \notin C \quad (6)$$

$$E(g(x_t)|x_{t-1} = x) < B, \quad x \in C \quad (7)$$

where  $g(x) > 0$  and  $C$  is a bounded set. In the multivariate case the drift condition involves verifying that the spectral radius of a matrix is less than one. For the bilinear model the dimension of the matrix whose spectral radius needs to be confirmed depends on  $s_1$  as it contains the higher moments of the process  $\mathbf{u}_t$ . As we will see later these moments are crucial for the stationarity of the *DRC*(1, 1) model. Before concluding this section on existing work, we mention the work of Pourahmadi (1988) who provides stationarity conditions for a dynamic random coefficient model where the logarithm of the absolute value of the random coefficient follows a general linear process. This work is related to ours but, of course, deals with different dependence structure in the random coefficient.

## 4 Covariance Stationarity

Following the above the drift condition of Tweedie (1975) does not appear as a promising avenue for our analysis. We therefore go to first principles for deriving sufficient conditions for the covariance stationarity of the  $DRC(1, 1)$  model given by:

$$x_t = a_t x_{t-1} + \epsilon_t \quad (8)$$

$$a_t = \mu + \gamma a_{t-1} + u_t \quad (9)$$

We need to show that  $E(x_t)$ ,  $Var(x_t)$  and  $Cov(x_t, x_{t-s})$  are finite and do not depend on  $t$ . From the representation of the model given in (4) we can easily see that the mean of the process will be zero. We now investigate the conditions under which  $Var(x_t)$  exists and does not depend on  $t$ .

For a  $DRC(1, 1)$  we have that

$$E(x_t^2) = E(\epsilon_t + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} a_{t-j} \epsilon_{t-i})^2 \quad (10)$$

Note that only terms involving squares of  $\epsilon_i$  will contribute to this expectation. Therefore

$$E(x_t^2) = \sigma^2 \left(1 + \sum_{i=1}^{\infty} E\left(\prod_{j=0}^{i-1} a_{t-j}^2\right)\right) \quad (11)$$

We therefore want to derive the behaviour of  $E(\prod_{j=0}^n a_{t-j}^2)$  as  $n$  goes to infinity. Define  $\tilde{\mu} = \mu/(1 - \gamma^2)$ .

$$E\left(\prod_{j=0}^n a_{t-j}^2\right) = E\left(\left(\tilde{\mu} + \sum_{j=0}^{\infty} \gamma^j u_{t-j}\right)^2 \left(\tilde{\mu} + \sum_{j=0}^{\infty} \gamma^j u_{t-j-1}\right)^2 \dots \left(\tilde{\mu} + \sum_{j=0}^{\infty} \gamma^j u_{t-j-n}\right)^2\right) \quad (12)$$

Schematically, this takes the form

$$\begin{bmatrix} \tilde{\mu}^2 & 2\tilde{\mu}u_t & 2\tilde{\mu}u_{t-1} & \dots & \dots \\ & u_t^2 & 2\gamma u_t u_{t-1} & 2\gamma^2 u_t u_{t-2} & \dots \\ & & \gamma^2 u_{t-1}^2 & 2\gamma^2 u_{t-1} u_{t-2} & \dots \\ & & & \gamma^4 u_{t-2}^2 & \dots \end{bmatrix} \times \begin{bmatrix} \tilde{\mu}^2 & 2\tilde{\mu}u_{t-1} & 2\tilde{\mu}u_{t-2} & \dots & \dots \\ & u_{t-1}^2 & 2\gamma u_{t-1} u_{t-2} & 2\gamma^2 u_{t-1} u_{t-3} & \dots \\ & & \gamma^2 u_{t-2}^2 & 2\gamma^2 u_{t-2} u_{t-3} & \dots \\ & & & \gamma^4 u_{t-3}^2 & \dots \end{bmatrix} \times \dots$$

multiplying the non diagonal terms in the above schematic will, on taking the expectation, give zeros for any given term for high enough values of  $n$ . So only the products involving just diagonal terms from the above array matter for the asymptotic (with respect to  $n$ ) behaviour of  $E(\prod_{j=1}^n a_{t-j}^2)$ . In other words we have simplified the problem down to the following expectation

$$E\left(\prod_{j=0}^{\infty} a_{t-j}^2\right) = E\left(\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i}^2\right)\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i-1}^2\right)\dots\right) \quad (13)$$

We now work inductively. Let us determine

$$E\left(\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i}^2\right)\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i-1}^2\right)\right)$$

multiplying the first summand of the second term with the whole first term, taking expectations and adding up gives  $\tilde{\mu}^4 + \tilde{\mu}^2 \sigma_u^2 / (1 - \gamma^2)$  if  $\gamma^2 < 1$ . multiplying the second summand of the second term with the whole first term, adding up and taking expectations gives  $\sigma_u^2 \tilde{\mu}^2 + \sigma_u^4 / (1 - \gamma^2) + \gamma^2 (\tau^4 - \sigma_u^4)$  where  $\tau^i$  is the  $i$ -th moment of  $u_t$ . Doing similar operations for the rest of the terms gives an overall sum of

$$\begin{aligned} \tilde{\mu}^4 + 2\tilde{\mu}^2 \sigma_u^2 / (1 - \gamma^2) + f_{1,1}(\tau^4 - \sigma_u^4) + \sigma_u^4 / (1 - \gamma^2) + \gamma^2 \sigma_u^4 / (1 - \gamma^2) + \gamma^4 \sigma_u^4 / (1 - \gamma^2) + \dots = \\ \tilde{\mu}^4 + 2\tilde{\mu}^2 \sigma_u^2 / (1 - \gamma^2) + f_{1,1}(\tau^4 - \sigma_u^4) + \sigma_u^4 / (1 - \gamma^2)^2 \end{aligned}$$

where

$$f_{1,1}(\tau^4 - \sigma_u^4) = \sum_{i=0}^{\infty} \gamma^{2+4i} (\tau^4 - \sigma_u^4) = \gamma^2 (\tau^4 - \sigma_u^4) / (1 - \gamma^4)$$

We now move to determine

$$E\left(\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i}^2\right)\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i-1}^2\right)\left(\tilde{\mu}^2 + \sum_{i=0}^{\infty} \gamma^{2i} u_{t-i-2}^2\right)\right)$$

Proceeding as before gives an expectation of

$$\tilde{\mu}^6 + 3\tilde{\mu}^4 \sigma_u^2 / (1 - \gamma^2) + 3\tilde{\mu}^2 \sigma_u^4 / (1 - \gamma^2)^2 + \sigma_u^6 / (1 - \gamma^2)^3 + f_{2,1}(\tau^6 - \sigma_u^6) + f_{2,2}(\tau^4 - \sigma_u^4)$$

where the terms involving the fourth and sixth moments of  $u_t$  are given by

$$f_{2,1}(\tau^6 - \sigma_u^6) = \gamma^4(\tau^6 - \sigma_u^6)/(1 - \gamma^6)$$

and

$$f_{2,2}(\tau^4 - \sigma_u^4) = [\gamma^2\sigma^2(\tau^4 - \sigma_u^4)]/[(1 - \gamma^4)(1 - \gamma^2)] + [\tilde{\mu}^2\gamma^2(\tau^4 - \sigma_u^4)]/[(1 - \gamma^4)]$$

Continuing in a similar fashion gives an expectation for  $E(\prod_{j=1}^n a_{t-j}^2)$  equal to  $(\tilde{\mu}^2 + \sigma_u^2/(1 - \gamma^2))^n + M(n)$  where the term  $M(n)$  is a sum of terms which are products of  $\tau^i - \sigma^i$ ,  $\tilde{\mu}^i$  and  $\gamma^i$ ,  $i = 1, \dots, 2n$ . There is a multiple of  $n$  such terms in  $E(\prod_{j=1}^n a_{t-j}^2)$ . Clearly convergence of  $(\tilde{\mu}^2 + \sigma_u^2/(1 - \gamma^2))^n$  requires that  $\tilde{\mu}^2 + \sigma^2/(1 - \gamma^2) < 1$ . Now if the following condition also holds

**Assumption 1**  $\tau^{2i} - \sigma_u^{2i} \leq 1$

the term  $M(n)$  is guaranteed to decline geometrically in  $n$ . This is easy to see as setting  $\tau^{2i} - \sigma_u^{2i} = 1$ , for all  $i$ , results in geometrically declining sequences in  $M(n)$ . Therefore,

$$E(x_t^2) = \sigma^2(1 + \sum_{i=1}^{\infty} E(\prod_{j=1}^i a_{t-j}^2)) = \sigma^2(\sum_{i=0}^{\infty} \sigma_u^{2i}/(1 - \gamma^2)^i) + \tilde{M} = \frac{\sigma^2(1 - \gamma^2)}{[1 - \gamma^2 - \sigma_u^2]} + \tilde{M} \quad (14)$$

where  $[\sigma^2(1 - \gamma^2)]/[1 - \gamma^2 - \sigma_u^2] > 0$  by  $\sigma^2/(1 - \gamma^2) < 1$  and  $\tilde{M} = \sum_{n=1}^{\infty} M(n) < \infty$ . A similar treatment proves that the covariances of  $x_t$  are finite and do not depend on  $t$ . Assumption 1 is extremely strict. It essentially implies that the support of  $u_t$  is  $(-1, 1)$ . We can relax somewhat this assumption by setting  $\mu = 0$ . Then by repeating the above analysis it is easy to see that the alternative assumption for the moments of  $u_t$  suffices

**Assumption 2**  $\gamma^{2i}(\tau^{2i} - \sigma_u^{2i}) \sim c^i$ ,  $|c| < 1$



To see this we have that in this case ( $\mu = 0$ ) the term  $M(n)$  contains terms in all even moments of  $u_t$  up to  $2n$ . More specifically,  $M(n)$  will contain terms of the form

$$\frac{\gamma^{2s-2}\sigma^{2(n-s)}(\tau^{2s} - \sigma^{2s})}{(1 - \gamma^{2s})(1 - \gamma^2)^{n-s}}, \quad s = 1, \dots, n$$

The sum of these terms is larger than  $M(n)$  since the construction of some terms involves duplication of expectations of cross products of lags of  $u_t^2$ . Adding up terms made up of  $\tau^{2s} - \sigma^{2s}$  for fixed  $s$  over  $n$  gives terms of the form

$$\frac{\gamma^{2s-2}(\tau^{2s} - \sigma^{2s})}{1 - \gamma^{2s}} \frac{1}{1 - \frac{\sigma^2}{1-\gamma^2}} \quad s = 1, \dots, n$$

These terms make up a geometric series by assumption 2 and therefore are summable giving a finite  $\tilde{M}$ .

However, even this assumption is quite strict. The normal distribution cannot satisfy this condition as  $\tau^n = O(n!)$  if  $u_t$  is normally distributed for even  $n$ . Some truncated distributions satisfy this condition but only truncated distributions with support in  $[-1, 1]$  are guaranteed to satisfy it for all  $|\gamma| < 1$ . To conclude, covariance stationarity of the  $DRC(1, 1)$  model requires quite strict conditions. If  $\mu \neq 0$  we need assumption 1,  $\gamma < 1$  and  $\tilde{\mu}^2 + \sigma^2/(1 - \gamma^2) < 1$ . If  $\mu = 0$ , assumption 1 can be relaxed to assumption 2.

## References

BENATI, L. (2002): "Investigating Inflation Persistence Across Monetary Regimes," Mimeo, Bank of England.

- COGLEY, T., AND T. SARGENT (2002): “Drifts and Volatilities: Monetary Policies and Outcomes in the Post WWII US,” Mimeo, Arizona State University.
- LIU, J., AND P. J. BROCKWELL (1988): “On the General Bilinear Time Series Model,” *Journal of Applied Probability*, 25, 553–564.
- POURAHMADI, M. (1988): “Stationarity of the Solution of  $X_t = A_t X_{t-1} + \epsilon_t$  and Analysis of Non-Gaussian Dependent Random Variables,” *Journal of Time Series Analysis*, 9, 225–239.
- TONG, H. (1990): *Nonlinear time series: A dynamical system approach*. Oxford University Press.
- TWEEDIE, R. L. (1975): “Sufficient Conditions for Ergodicity and Recurrence of Markov Chains on a General State Space,” *Stochastic Processes Appl.*, 3, 385–403.

**This working paper has been produced by  
the Department of Economics at  
Queen Mary, University of London**

**Copyright © 2002 George Kapetanios  
All rights reserved.**

**Department of Economics  
Queen Mary, University of London  
Mile End Road  
London E1 4NS  
Tel: +44 (0)20 7882 5096 or Fax: +44 (0)20 8983 3580  
Email: [j.conner@qmul.ac.uk](mailto:j.conner@qmul.ac.uk)  
Website: [www.econ.qmul.ac.uk/papers/wp.htm](http://www.econ.qmul.ac.uk/papers/wp.htm)**