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Price Taking Equilibrium in Club Economies with Multiple Memberships and Unbounded Club Sizes

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Abstract: This paper develops a model of an economy with clubs where individuals may belong to multiple clubs and where there may be ever increasing returns to club size. Clubs may be large, as large as the total agent set. The main condition required is that sufficient wealth can compensate for memberships in larger and larger clubs. Notions of price taking equilibrium and the core, both with communication costs, are introduced. These notions require that there is a small cost, called a communication cost, of deviating from a given outcome. With some additional standard sorts of assumptions

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on preferences, we demonstrate that, given communication costs parameterized by $\varepsilon > 0$, for all sufficiently large economies, the core is non-empty and contains states of the economy that are in the core of the replicated economy for all replications (Edgeworth states of the economy). Moreover, for any given economy, every state of the economy that is in the core for all replications of that economy can be supported as a price-taking equilibrium with communication costs. Together these two results imply that, given the communication costs, for all sufficiently large economies there exists Edgeworth states of the economy and every Edgeworth state can be supported as a price-taking equilibrium.

JEL Classification Numbers: C 62, D 71, H 41.

1 Motivation

It seems compelling that gains to cooperation by large groups of individuals may be substantial. For example, in economies with public goods, coordination of activities and decreasing costs of providing public goods may provide increasing benefits to ever larger club membership. Consider questions of global pollution, global harmonization of productive activities and memberships in networks. If we wish a model to describe clubs such as the World Trade Organization, the United Nations, the World Environmental Organization, or religions that wish to embrace all people, then a model with bounded club sizes, where clubs become infinitesimal in large economies is not appropriate.² Of course much economic activity is carried out within small clubs – marriages, small firms, and swimming pool clubs for example. It is also clear that a general model should also allow overlapping clubs so that a participant may belong, for example, to a two-person partnership, a dance club, and a world-wide social movement.

In this paper we explore the boundaries of price-taking equilibrium in club economies where clubs may overlap and also may be large. Providing most agents have many close substitutes, if an economy is sufficiently large then an equilibrium with communication costs and possibly some frictions, captured by the presence of an exceptional set of agents, exists and is in the core. Communication costs are parameterized by ε and ε can be allowed to zero as the economy becomes large. Moreover, we demonstrate that in large economies the core with communication costs is nonempty and that an Edgeworth equilibrium exists. The set of Edgeworth equilibria is contained in the set of equilibria with communication costs.

Allowing clubs to be as large as the entire agent set leads to a situation that appears, in essence, to be fundamentally different from a private goods economy, or an economy where small groups of agents can exhaust all gains to coalition formation or a pure public goods economy. Even in large economies, discrimination between otherwise identical individuals can persist.

Recent literature suggests that whenever almost all gains to collective activities can be realized by relatively small groups of participants then when there are many participants diverse economies resemble markets. This includes economies with indivisibilities, nonconvexities, local public goods, and

²In fact, if the economy is essentially superadditive – that is, if an option open to a large club is to divide into smaller clubs – then economies with possibly large clubs can do no worse than those with clubs restricted in size.

club economies with multiple memberships. In particular, under apparently mild conditions – essentially just a mild superadditivity condition, boundedness of average or per capita payoffs, and many close substitutes for most agents – approximate cores are nonempty, approximate cores treat similar people similarly and economies, modeled as games with side payments, generate market games. In addition, analogous of the Laws of Demand and Supply hold.³ Models of games with many agents, however, cannot treat the properties of price-taking economic equilibrium, except for situations where the ‘commodities’ to be priced are types of agents. To obtain richer results on price-taking equilibrium, more detailed economic models are required. Our primary focus is the extent to which increasing returns to club formation in larger and larger economies is consistent with existence of price-taking equilibrium and equivalence of the outcomes of price-taking equilibrium with cooperative outcomes.

Our research grows out of the seminal works of Tiebout [1956] and Buchanan [1965]. Tiebout conjectured that, in large economies with sufficient diversity of communities in terms of their local public good offerings, competitive forces would lead to a ‘market-like outcome.’ Buchanan stressed that there may be congestion so that optimal club sizes may exist; that is, there may exist some finite population at which all gains to membership size would be exhausted. There are now many models showing that large economies with small optimal groups (communities, firms, clubs, jurisdictions, and so on) generate markets; club membership is simply another commodity. For example, think of movie theatres. Movies can be provided by clubs or by profit maximizing entrepreneurs. They tend to be provided by non-market organizations when the demand is small – foreign film clubs, for example – and price discrimination of some sort may be required to cover costs. Most models of such situations rule out large clubs that are few in number, for example, the individual States in the United States. Requiring that optimal clubs be small rules out much interesting economic activity, for example, the formation or break up of nations.

Our paper is one of a few allowing the possibility of large clubs, perhaps as large as the entire population, and the first to study price-taking equilibrium in contexts permitting both overlapping clubs and large clubs. Moreover, we allow a compact metric space of player types so it does not necessarily hold

³We refer the reader to Wooders (1999) for a survey and to Kovalenkov and Wooders (2003a) for more recent developments.

that there are many exact substitutes for any player. Other than some standard conditions such as desirability of private goods, the main assumption of our research is that sufficient wealth, measured in terms of private goods, can compensate for ever larger club sizes. This permits ever-increasing returns to club size while ruling out unbounded increasing returns. A simple example is provided.

An interesting aspect of allowing unbounded club sizes is that, even though we have an Edgeworth sort of equivalence result, our equilibrium concept has personalized prices. Without assumptions further limiting gains to club size and/or multiple memberships, we cannot relax the feature of personalized prices and still obtain our results. Thus, we have a case that is between the private goods case, with anonymous pricing and Edgeworth equivalence, and the pure public good case, with personalized prices and where Edgeworth equivalence does not in general hold.

In the following, Section 2 develops the model, Section 3 introduces games induced by the economy, and states nonemptiness of the core with communication costs. Section 4 introduces the equilibrium concept. Our main Theorems are stated in this section. With one exception, all Theorems are proven in an Appendix. Section 5 relates our results to the literature and Section 6 concludes the main body of the paper and the Appendix follows.

2 A club economy allowing large clubs

2.1 Agents

Let Ω be a compact set of attributes. An element of Ω , typically denoted by ω , is interpreted as a possible description of an agent. Let $F(\Omega)$ denote the set of all pairs (S, α) where S is a finite non-empty set and $\alpha : S \rightarrow \Omega$ is an *attribute function*. In interpretation, S will be a set of agents and $\alpha(i)$, ($i \in S$) describes all relevant characteristics of agent i , including a consumption set, endowment, preferences, productive abilities, crowding attributes, and so on. For $\omega \in \Omega$, the set of agents in S with attributes ω is $S \cap \alpha^{-1}(\omega)$ and $|S \cap \alpha^{-1}(\omega)|$ is their number. An *economy* is a pair $(N, \alpha) \in F(\Omega)$ where $N = \{1, \dots, n\}$ is the set of agents and $\alpha : N \rightarrow \Omega$ is an *attribute function*.

2.2 Clubs and club structures

Let (N, α) be an economy. With each nonempty subset of N there is an associated activity. We call such a subset a *club* and the activity, the *club activity*. This club activity could be consumption of a local public good or some shared activity, such as listening to music or swimming in the pool belonging to the club. We note that a club and a coalition will have distinct interpretations. A *coalition* is simply a nonempty subset of N while a club is a nonempty subset associated with an activity. We will typically denote a club by S_k and a coalition by simply S . Let S be a coalition and let $\{S_k\}$ denote a covering of S (with no repetitions) by clubs.⁴ Such a covering is called a *club structure* of S . Let $C(S)$ denote the set of club structures of S .

Note that an agent may belong to a number of clubs and thus participate in a number of different club activities. For example, an individual may be a member of a marriage, a firm, and a dance club. Observe also that there are no *a priori* restrictions on club size; for any economy (N, α) the total agent set N may constitute a club.

For a club S_k , the production of the club activity requires $z_{S_k} \in -\mathbb{R}_+^L$ *inputs of private goods*. For a purely ‘hedonic’ club – a club where the membership of the club itself is the benefit of the club – with no costs of club formation, the required inputs may be zero.

Given (N, α) , $S \subset N$, a club structure $\mathcal{S} = \{S_1, \dots, S_k, \dots, S_K\} \in C(S)$ of S and $i \in S$, let

$$\mathcal{S}[i] = \{S_k \mid S_k \in \mathcal{S} \text{ and } i \in S_k\} \quad (1)$$

denote the set of all profiles of clubs in \mathcal{S} that contain consumer i . The set $\mathcal{S}[i]$ describes the club memberships of agent i with respect to \mathcal{S} . The set $\mathcal{C}[i; S] = \cup_{\{\mathcal{S} \in C(S)\}} \mathcal{S}[i]$, where the union is taken over all club structures $\mathcal{C}(S)$ of S , is called the *club consumption set* relative to S for an agent $i \in N$.

⁴In principle, our techniques allow there to be two (or more) clubs with identical membership offering different activities – repetitions could be allowed. To introduce this formally would significantly increase notational complexity. Also, in principle, some particular clubs may be inadmissible – for example, three-person marriages may be ruled out, at least legally. Inadmissible clubs can be accommodated within our current framework by simply assigning negative utility to such clubs so that being a member of one would not be individually rational.

2.3 Attributes

For each attribute $\omega \in \Omega$ it is assumed that the description of an agent provided by ω includes a positive endowment of each of a finite number L of private goods (and that there are no endowments of club activities). Let e^ω be the *endowment of an agent with attribute ω of the private goods*. For all $\omega \in \Omega$, we assume $e^\omega = (e_1^\omega, \dots, e_\ell^\omega, \dots, e_L^\omega) \in \mathbb{R}_{++}^L$. For an economy (N, α) and $i \in N$ with $\alpha(i) = \omega$, the *endowment of i* , denoted by e^i , equals e^ω .

In any economy (N, α) the utility function of an agent $i \in N$ with attribute ω is denoted by $u^\omega(\cdot, \cdot)$ and maps $X^\omega \times \mathcal{C}[i; N]$ into \mathbb{R} , where X^ω , called the *commodities consumption set* for an agent with attribute ω , is a given nonempty subset of \mathbb{R}_+^L with e^ω in the interior of X^ω .

It is assumed that, given any $\mathcal{S}[i] \in \mathcal{C}[i; S]$ with $\alpha(i) = \omega$, the utility function u^ω satisfies the usual properties of monotonicity, continuity and convexity. Specifically, for any given $i \in N$ satisfying $\alpha(i) = \omega$, for any given club consumption $\mathcal{S}[i]$ for i , the utility function u^ω satisfies:

(a) **Monotonicity:** $u^\omega(\cdot, \mathcal{S}[i])$ is an increasing function, that is, if $x < x'$ then $u^\omega(x, \mathcal{S}[i]) < u^\omega(x', \mathcal{S}[i])$.

(b) **Continuity:** $u^\omega(\cdot, \mathcal{S}[i])$ is a continuous function.

(c) **Convexity:** $u^\omega(\cdot, \mathcal{S}[i])$ is a quasi-concave function.

(d) **Desirability of endowment:** There exists a real number $\tau > 0$ with the property that if $u^\omega(e^\omega - \tau \bar{1}, \{i\}) \leq u^i(x', \mathcal{S}[i])$, then $x' > 0$.⁵

(e) **Boundedness:** The marginal utility of agent i for the L^{th} private good is bounded away from zero.

With the exception of (d), the conditions above are all standard. Condition (d) incorporates the Hammond-Kaneko-Wooders (1989) and Kaneko-Wooders (1989) condition that the endowment is preferred to any outcome which assigns an agent zero of any of the indivisible (club) goods.⁶

⁵This assumption could be weakened but at the cost of more notation and without significant gain in economic understanding.

⁶In the literature of private goods exchange economies, related, more restrictive conditions go back to Broome (1973). For economies with local public goods/clubs an analogous condition was introduced in Wooders (1978,1980). The Hammond-Kaneko-Wooders (1989) condition is less restrictive.

2.4 States of the economy and communication costs

Let (N, α) be an economy, let S be a nonempty subset of N , and let \mathcal{S} be an admissible club structure of S . A *state of the economy for S relative to \mathcal{S}* is an ordered pair (x^S, \mathcal{S}) , where $x^S = (x^i : i \in S)$ is an allocation for S satisfying the property that when $\alpha(i) = \omega$, it holds that $x^i \in X^\omega$. The state (x^S, \mathcal{S}) is *feasible* if

$$\sum_{i \in S} (x^i - e^i) \leq \sum_{S_k \in \mathcal{S}} z_{S_k}.$$

Assume that if a group of agents is to form an alliance – a coalition – then the agents must communicate with each other and possibly reallocate goods among themselves. This motivates the introduction of a communication cost required to form a coalition. Denote the communication cost for coalition S by

$$c(\varepsilon, S) \stackrel{\text{def}}{=} \varepsilon |S| \bar{z}$$

where $\bar{z} \in -\mathbb{R}_{++}^L$ is given and ε is a positive real number. A state (x^S, \mathcal{S}) is *$c(\varepsilon, S)$ -feasible* if

$$\sum_{i \in S} (x^i - e^i) \leq \sum_{S_k \in \mathcal{S}} z_{S_k} + \varepsilon |S| \bar{z}.$$

2.5 The core with communication costs

The following concept of the core can be interpreted as either a notion of an approximate core arising from market frictions or as an exact core relative to communication costs, denoted by $c(\varepsilon)$ and parameterized by ε . Let (x^N, \mathcal{N}) be a state of the economy relative to the club structure \mathcal{N} . A coalition S can *$c(\varepsilon)$ -improve upon* the state (x^N, \mathcal{N}) if there is a club structure \mathcal{S} of S and a $c(\varepsilon, S)$ -feasible state of the economy (x'^S, \mathcal{S}) for S such that for all consumers $i \in \{1, \dots, N\}$ it holds that:

$$u^i(x'^i, \mathcal{S}) > u^i(x^i, \mathcal{N}).$$

A feasible state of the economy (x^N, \mathcal{N}) is in the *$c(\varepsilon)$ -core* (of the economy) if it cannot be $c(\varepsilon)$ -improved upon by any coalition S .

It is clear that when $\varepsilon = 0$ the notion of the $c(\varepsilon)$ -core coincides with the standard notion of the core.

2.6 The communication core with remainders

Given the composition of a population N it may be that some subset of agents cannot be accommodated in their preferred clubs. If this set relatively small, then a solution concept ignoring an exceptional set of agents may provide reasonable approximations to outcomes of an exact solution. Thus, we weaken our notion of the $c(\varepsilon)$ -core to take account of these observations.

An ε_1 -remainder $c(\varepsilon_0)$ -core state of the economy is a feasible state of the economy (x^N, \mathcal{N}) satisfying the property that for some subset $N^0 \subset N$ with $\frac{|N \setminus N^0|}{|N|} < \varepsilon_1$, (x^{N^0}, \mathcal{N}^0) is an $c(\varepsilon_0)$ -core state of the economy as defined above. An ε_1 -remainder $c(\varepsilon_0)$ -core state of the economy simply ignores an exceptional set of agents.

3 The game derived from an economy

Given an economy, we associate a parameterized collection of games with the economy, where the parameter depends on the communication costs. We first select a real number $\varepsilon_0 > 0$ sufficiently small so that for every $\varepsilon \in [0, \varepsilon_0]$ it holds that $\varepsilon \bar{z} \geq -\tau \bar{1}$.

Given an economy $(N, \alpha) \in F(\Omega)$ and $\varepsilon \in [0, \varepsilon_0]$ we denote the game induced by the economy by $(N, V_\alpha^\varepsilon)$, where V_α^ε is a correspondence mapping subsets S of N into \mathbb{R}^N . For each subset S of N , define V_α^ε as the set of vectors $v \in \mathbb{R}^N$ with the property that for some club structure \mathcal{S} of S and some $c(\varepsilon, \mathcal{S})$ -feasible state with associated allocation $(x^{\mathcal{S}}, \mathcal{S})$ we have $v^i \leq u^i(x^i, \mathcal{S}[i])$ for each $i \in S$. When $\varepsilon = 0$, we denote V_α^ε simply by V_α .

We will assume throughout that the following notion of continuity on $F(\Omega)$ holds.⁷

Mean continuity: Given any $\varepsilon \in [0, \varepsilon_0]$, for every $\rho > 0$ there exists a real number $\theta > 0$ such that if $(N, \alpha), (N, \beta) \in F(\Omega)$ and $d(\alpha(i), \beta(i)) < \theta$ for all $i \in N$, then $H(V_\alpha^\varepsilon(N), V_\beta^\varepsilon(N)) < \rho$, where H is the Hausdorff distance.

⁷This assumption could be obtained as a consequence of continuity assumptions on attributes. Agents whose attributes are ‘close’ are intended to be approximate substitutes – that is, in the metric on attribute space, their utility functions, crowding types, endowments and so on are close and they are perceived as similar by other agents. It is worth noticing that if Ω is a finite set, we have a finite number of types of agents and this assumption is satisfied automatically.

3.1 Replica games

Our central results depend on extending the replication case (with a fixed distribution of agents on attribute space) to the case of a compact metric space of attributes. Since one of our main assumptions, ensuring ‘per capita boundedness’ of payoff sets of derived games, is required only for replication sequences, we now turn to this case.

Given $(N, \alpha) \in F(\Omega)$, for each positive integer r we define the r^{th} replica economy, denoted by $(N_r, r\alpha) \in F(\Omega)$ as the economy with agent set

$$N_r = \{(i, q) : i = 1, \dots, N \text{ and } q = 1, \dots, r\},$$

and attribute function $r\alpha : N_r \rightarrow \Omega$ where $r\alpha(i, q) = \alpha(i)$, $q = 1, \dots, r$ (i.e all agents (i, q) , (i, q') , are identical in terms of attributes). The agent (i, q) is called the q^{th} agent of type i . To replicate a state of the economy, in addition to replicating the consumer set we also replicate the club structure and consumptions so that all replicas of an individual consumer are in clubs with identical profiles, and are allocated identical consumptions.

Let $\mathcal{N} = \{J_1, \dots, J_g, \dots, J_G\}$ be a club structure of N and let r be a positive integer. Let \mathcal{N}_r be a club structure of N_r containing rG clubs and denoted by:

$$\mathcal{N}_r = \{J_{gj} : j = 1, \dots, r \text{ and } g = 1, \dots, G\},$$

where for each $j = 1, \dots, r$ and each $g = 1, \dots, G$ the profile of J_{gj} equals the profile of J_g . Then \mathcal{N}_r is the r^{th} replication of \mathcal{N} .

Let (x, \mathcal{N}) be a state of the economy (N, α) . A state of the replicated economy $(N_r, r\alpha)$, denoted by (x^{N_r}, \mathcal{N}_r) , is an r^{th} replication of (x, \mathcal{N}) if

- (a) for each $g = 1, \dots, G$ and each $j = 1, \dots, r$,

$$z_{J_{gj}} = z_{J_g};$$

(b) for each consumer $i \in N$ there are r consumers (i, q) where $q = 1, \dots, r$, in the replicated agent set N_r who are allocated the same private goods bundle as i .

A state of the economy (x^N, \mathcal{N}) is in the $c(\varepsilon)$ -core for all replications if, for each positive integer r , it holds that an r^{th} replication of (x, \mathcal{N}) is in the $c(\varepsilon)$ -core of the r^{th} replication of the economy.

We will also require some minimal assumption on the economy to ensure that equal-treatment utilities derived from the economy do not become infinite. To this purpose we introduce the following assumption:

Desirability of wealth: Assume that there is a bundle of private goods, x^* and a replication number r^* , such that for some club structure \mathcal{N}_{r^*} of the r^* th economy, it holds that, for any r , for each $i \in N$,

$$u^i(x^i + x^*, \mathcal{N}_{r^*}[i]) \geq u^i(x^i, \mathcal{N}_r[i])$$

for any x^i and any club structure \mathcal{N}_r of the r th economy.

Informally, this assumption ensures that wealth, in terms of private goods, can substitute for ‘large’ clubs, no matter how large the economy. Because of the possibility of ever-increasing returns to club size, due to public goods for example, in our model agents may derive more and more utility from larger and larger clubs. Informally, desirability of wealth dictates that *if* an individual were sufficiently wealthy, however, he could provide club goods for himself and just a few friends (no more than r^* of each type that appears in the economy) and achieve a preferred outcome. Note that x^* is independent of the type of the agent; this is for simplicity of statement. Also, note that x^* may not be feasible for the r^* th economy. Desirability of wealth is considerably weaker than bounding club sizes.

Example 1. As a simple example, suppose individuals derive utility only from money and from sharing some common activity with other individuals. Suppose that the utility function of a representative agent can be described by

$$u(\xi, m) = \xi - \frac{1}{m}$$

where m is the size of the club to which he belongs and ξ is money. Possible values for r^* and x^* , in the definition of desirability of wealth, are $r^* = 10$ and $x^* = 1$. For all club sizes m it holds that

$$u(\xi + x^*, r^*) = \xi + 1 - \frac{1}{10} \geq u(\xi, m) = \xi - \frac{1}{m}.$$

We highlight that desirability of wealth is satisfied and the feasible per capita utility level as a function of the economy size does not achieve a maximum – desirability of wealth does not imply the existence of an optimal club size.

3.2 Nonemptiness of communication cost cores with remainders

Our first Theorem, showing nonemptiness of communication cost cores with remainders, is central. To demonstrate existence of equilibrium we will then demonstrate the existence of prices supporting states of the economy in an ‘Edgeworth core’.

Theorem 1. Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that: if N is a set of agents with $|N| > n(\varepsilon_1, \varepsilon_0)$ then for any attribute function $\alpha : N \rightarrow \Omega$, the ε_1 -remainder $c(\varepsilon_0)$ -core of (N, α) is nonempty.

Indeed, we prove more. First, let us define an $(\varepsilon_1, \varepsilon_0)$ *Edgeworth state of the economy* (N, α) as a state of the economy with the property that the state is in the ε_1 -remainder $c(\varepsilon_0)$ -core and, for all replications $(N_r, r\alpha)$ of that state, there is a subset of agents $N_r^0 \subset N_r$ such that no coalition $S \subset N_r^0$ can $c(\varepsilon_0)$ -improve upon the replicated state. Moreover, we can take $N_r^0 = rN^0$ where

$$|N \setminus N^0| < \varepsilon_1 |N|$$

or, equivalently, for each positive integer r ,

$$|N_r \setminus N_r^0| < \varepsilon_1 |N_r|.$$

Less formally, for any economy (N, α) with sufficiently many agents an $(\varepsilon_1, \varepsilon_0)$ *Edgeworth state of the economy* (N, α) is in an approximate core (the ε_1 -remainder $c(\varepsilon_0)$ -core) for all replications of the economy.

Theorem 2. Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an $n(\varepsilon_1, \varepsilon_0)$ such that: for any set of agents N , if $|N| > n(\varepsilon_1, \varepsilon_0)$ then for any attribute function $\alpha : N \rightarrow \Omega$ there exists an $(\varepsilon_1, \varepsilon_0)$ *Edgeworth state of the economy* (N, α) .

Note that neither of the above Theorems depend on replication.

4 Equilibrium with communication costs

In this section we first define a communication cost equilibrium, called the $c(\varepsilon_0)$ -equilibrium, and then state our existence and ‘Edgeworth equivalence’ results.

A *price system for private goods* is a vector $p \in \mathbb{R}_+^L$. A *participation price system* is a set

$$\Pi = \{\pi^i(S_k) \in \mathbb{R} : S_k \subset N \text{ and } i \in S_k\},$$

stating a participation price, positive, negative, or zero, for each agent in each club S_k .

A $c(\varepsilon_0)$ -*equilibrium* (for an economy with club goods) is an ordered triple $((x^N, \mathcal{N}), p, \Pi)$ consisting of a state of the economy (x^N, \mathcal{N}) , where $\mathcal{N} = \{J_1, \dots, J_g, \dots, J_G\}$, a price system $p \in \mathbb{R}_+^L \setminus \{0\}$ for private goods, and a participation price system Π , such that:

$$(i) \sum_{i \in N} (x^i - e^i) \leq \sum_{S_k \in \mathcal{N}} z_{S_k} \text{ (feasibility) ;}$$

(ii) for each possible club $S_k \subset N$,

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0$$

(no club makes a positive profit);

(iii) for any agent $i \in N$, any $S \subset N$ with $i \in S$, and any club structure \mathcal{S} of S , if

$$u^i(x^i, \mathcal{S}) > u^i(x^i, \mathcal{N})$$

then

$$p \cdot x^i + \sum_{S_k \in \mathcal{S}[i]} \pi^i(S_k) > p \cdot e^i + \varepsilon p \cdot \bar{z}$$

(maximization given costs of coalition formation)⁸;

$$(iv) \quad \sum_g p \cdot z_{J_g} + \sum_g \sum_{i \in J_g} \pi^i(J_g) \geq \sum_{i \in N} \varepsilon p \cdot \bar{z}, \text{ and}$$

⁸Since any club structure \mathcal{S} of S can be embedded in a club structure of N , say \mathcal{N} , so that $\mathcal{S}[i] = \mathcal{N}[i]$, this condition could also be stated in terms of club structures of the total agent set N .

$$\sum_{i \in N} p \cdot x^i + \sum_{i \in N} \sum_g \pi^i(J_g) \leq \sum_{i \in N} (p \cdot e^i + \varepsilon p \cdot \bar{z}).$$

(agents cannot be significantly far, in aggregate, inside their budget sets and similarly for clubs).

Our notion of $c(\varepsilon_0)$ -equilibrium allows at least some agents to spend less than their entire income at the given prices. This is because of the communication costs, which affect not only the opportunities to change club memberships but also opportunities to purchase difference commodity bundles.

An ε_1 -remainder $c(\varepsilon_0)$ -equilibrium is an ordered triple $((x^N, \mathcal{N}), p, \Pi)$ such that for some subset of agents $N^0 \subset N$ satisfying $\frac{|N \setminus N^0|}{|N|} < \varepsilon_1$ there exists a $c(\varepsilon_0)$ -equilibrium $((x^{N^0}, \mathcal{N}^0), p, \Pi)$ as defined above.

Our notion of ε_1 -remainder $c(\varepsilon_0)$ -equilibrium dictates that all agents in the economy are ‘competitive’ or almost competitive except perhaps small proportions of ‘left over’ agents. Concepts of approximate equilibrium or cores involving left over agents are common in the literature of game theory and economics. The left-overs may have unsatisfied demands. Such situations may arise from imperfections in markets.

In Theorem 3 we demonstrate that an ε_1 -remainder $c(\varepsilon_0)$ -equilibrium state of the economy is in the ε_1 -remainder $c(\varepsilon_0)$ -core.

Theorem 3: Let (N, α) be an economy. An ε_1 -remainder $c(\varepsilon_0)$ -equilibrium state of the economy is in the ε_1 -remainder $c(\varepsilon_0)$ -core.

Proof: Suppose the Theorem is false. Then there exists at least one ε_1 -remainder $c(\varepsilon_0)$ -equilibrium (otherwise the result would be vacuously true). Let $((x^{N^0}, \mathcal{N}^0), p, \Pi)$ be an ε_1 -remainder $c(\varepsilon_0)$ -equilibrium with the properties that $\frac{|N \setminus N^0|}{|N|} < \varepsilon_1$ and the state of the economy (x^{N^0}, \mathcal{N}^0) is not in the ε_1 -remainder $c(\varepsilon_0)$ -core. This means that there is a coalition $S \subset N_r^0$, a club structure \mathcal{S} of S and an allocation (x'^S, \mathcal{S}) such that

$$\sum_{i \in S} (x'^i - e^i) \leq \sum_{S_k \in \mathcal{S}} z_{S_k} + \varepsilon |S| \bar{z}$$

and

$$u^i(x'^i, \mathcal{S}) > u^i(x^i, \mathcal{N}_r^0).$$

From (ii) of the definition of an $c(\varepsilon_0)$ -equilibrium it holds that

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0$$

and from utility maximization, it holds that

$$p \cdot x'^i + \sum_{S_k \in \mathcal{S}[i]} \pi^i(S_k) > p \cdot e^i + \varepsilon p \cdot \bar{z}.$$

Summing up these above inequalities, one will have

$$\sum_{i \in S} p \cdot (x'^i - e^i) > \sum_{S_k \in \mathcal{S}} p \cdot z_{S_k} + p \cdot \varepsilon |S| \bar{z},$$

which is a contradiction.

Theorem 4. Assume desirability of wealth. Let (N, α) be an economy. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that: if $|N| > n(\varepsilon_1, \varepsilon_0)$ then there exists an $(\varepsilon_1, \varepsilon_0)$ -Edgeworth state of the economy. Moreover, any $(\varepsilon_1, \varepsilon_0)$ -Edgeworth state of the economy is an ε_1 -remainder $c(\varepsilon_0)$ -equilibrium for the economy (N, α) .

We highlight that: Theorem 3 states that an ε_1 -remainder $c(\varepsilon_0)$ -equilibrium is in the ε_1 -remainder $c(\varepsilon_0)$ -core; Theorem 4 states that an Edgeworth state of the economy (a state of the economy with the property that all replications of that state are in the ε_1 -remainder $c(\varepsilon_0)$ -core of the replicated economy) exists and any Edgeworth state of the economy is an equilibrium. Note that, unlike the case of a private goods economy as in Debreu and Scarf (1963) and many other papers, because new clubs arise when the numbers of agents in the economy increases, an equilibrium for a larger economy requires a larger set of admission prices.⁹

⁹In the economy with agent set N the only ‘club commodities’ that exist are given by subsets of N . Thus, it suffices to have prices only for admission to these clubs.

5 Relationships to the literature

5.1 The seminal works of Tiebout (1956) and Buchanan (1965)

In his seminal paper, Tiebout (1956) observed that if public goods are local rather than pure and it is optimal or near optimal to have many jurisdictions providing public goods, then the movement of consumers to their preferred jurisdictions will lead to a ‘market-type,’ near-optimal outcome and the free rider problem of economies with pure public goods will not arise, the ‘Tiebout Hypothesis.’ While the general ideas of Tiebout’s paper were quite informally expressed – there were no precise definitions or conjectures – he did describe a ‘severe’ model that could easily be formalized. In his severe model (page 441 of the journal), Tiebout supposes that there exists an infinite number of communities, each offering a different public goods package (implicitly, so that all possible levels of public goods are provided). There is no congestion nor any increasing returns to scale within jurisdictions and per capita costs of providing the public goods on offer within a community are constant, independent of the number of members of a community. In such a situation, the consumer-voter can move to the community where his demands for public goods are *exactly satisfied*. Tiebout himself, as he makes clear, did not view this severe model as a good approximation to reality – for one thing, for the exact satisfaction of consumer demands, the number of (nonempty) communities may well need to be equal to the number of types of consumers – but the model was intended to illustrate how his informally described ideas may work.

Example 2. An example may help make Tiebout’s severe model clearer. Suppose all levels of public goods are possible. Thus, let us suppose that community $x \in \mathbb{R}_+$ offers quantity (or quality) x of the public good. Suppose that the costs of providing x to n consumers is $\$xn$ and that consumer who chooses community x must pay the cost (or tax) x . Each consumer $i \in N$ has a quasi-linear utility function

$$u^i(x, \xi) = f^i(x) + \xi.$$

Also assume $f^i(x)$ is concave. Then consumer i faces the problem:

$$\text{maximize}_x f^i(x) + \xi - x.$$

Since $f^i(x) - x$ is concave, a maximum exists and, with free mobility, the utility maximizing consumer may move to exactly that community which maximizes his preferences (subject to his budget constraint).

Note that in this simple example the size of the population is of no real relevance. It could be finite or (with the technical measure-theoretic conditions) a continuum (with or without atoms – no real problem either way).

In fact, Bewley (1981) provides a formulation of Tiebout’s severe model but with the condition of *free entry* replacing the assumption of an infinite number of communities all providing different public goods packages.¹⁰ The free entry condition is that, given prices for private goods, no subset of consumers could provide a preferred level of public goods for themselves. Bewley reaches the same conclusion as Tiebout does for his severe model. Unless there are as many communities as types of consumers, an equilibrium may not exist. Moreover, as Tiebout, Bewley concludes that this severe model does not make much sense.¹¹

We now leave Tiebout aside for the moment and turn to Buchanan (1965), which highlighted the fact that (local) public goods may become congested and there may be optimal, finite club (community, or jurisdiction) sizes. Unlike Tiebout, Buchanan did not appear to have in mind economies with many participants but instead considered first order conditions to characterize optimal club size. But note that there is really not much difference between a Buchanan club and a Tiebout jurisdiction, at least at this point in the development of the literature.

5.2 Decentralizing core outcomes as price taking equilibrium

The analysis of economies with clubs via cooperative game theory appears to have been initiated by Pauly, cf., his 1972 paper. There Pauly considered a model with essentially identical consumers and an optimal club size. Pauly

¹⁰The free entry condition also appears in Pauly (1972), Wooders (1978) and other more recent papers.

¹¹Unlike Tiebout, however, Bewley did not apparently appreciate the importance of a large population and a potentially large number of jurisdictions. Bewley collects a number of examples from the literature demonstrating problems in defining an appropriate notion of equilibrium yielding both existence and optimality but none of these examples treat the possibility of a large number of jurisdictions.

argued that unless the total payer set could be partitioned into clubs of optimal sizes, the core would be empty. Wooders (1978,1980) allows several types of agents and nondifferentiated (or anonymous) crowding.¹² In Conley and Wooders (1996,1997), and Cole and Prescott (1997) the model is further developed in that the crowding types of agents (external characteristics) are separated from taste types and it is shown that first best prices can be defined to depend only on the crowding types of agents; no private information is required.¹³ A short survey of the vast literature is provided in Wooders (1999).

In the economic models of those papers listed above each agent may belong to only one jurisdiction. Allowing multiple memberships in clubs, Shubik and Wooders (1982) demonstrated nonemptiness of approximate cores of economies with many agents but price-taking equilibrium was not studied. Kovalenkov and Wooders (2003a) demonstrated conditions under which large finite games and economies with clubs and permitting multiple memberships have nonempty approximate cores. Subsequently, Ellickson et al (2001) introduced a model of an economy with multiple memberships and obtained approximate versions of existence of equilibrium and equivalence of the core and the set of equilibrium outcomes. Their model is more restrictive than the prior model of Shubik and Wooders (1982) and Kovalenkov and Wooders in the sense that Ellickson et al allow only a bounded number of distinct sorts of clubs; thus clubs become negligible as the economy grows large. Following Conley and Wooders (1996,1997) and Cole and Prescott (1997), Ellickson et al. make a distinction between the crowding types (in their language, ‘external characteristics’) of agents and their taste (and endowment) types.

Our approach in this paper is in part based on earlier research, especially Wooders (1983), showing that under apparently mild restrictions – boundedness of per capita payoffs in utility space – approximate cores of growing

¹²In Pauly (1972), the two types of agents may appear to differ but, as shown in Wooders (1976), the fact that both types of agents make the identical marginal contributions to coalitions implies that only the size and not the composition of coalitions is relevant.

¹³When crowding is anonymous – that is, individuals care only about the numbers of agents in the same jurisdiction and not their characteristics – then prices are anonymous (Wooders 1978). With differentiated (also known as nonanonymous) crowding, until the works of Conley and Wooders (1996,1997) and Cole and Prescott (1997), prices were also nonanonymous, depending on tastes. Conley and Wooders (1996,1997) and Cole and Prescott (1997) separated taste types from crowding types and showed that first best prices can be defined to depend only on crowding types or external characteristics of agents.

games with a fixed distribution of agent types are nonempty and on Shubik and Wooders (1982) who introduce the study of the core in economies with clubs and multiple memberships.¹⁴ A *crucial* innovation in the current paper is our construction of the commodity space. Part of this innovation is in extending and further developing the Foley (1970)-Wooders (1985) proof technique of defining ‘preferred sets of allocations of private goods’ for coalitions. To ensure that the games derived from the economies satisfy per capita boundedness – simply boundedness of the set of equal treatment payoffs – we make an assumption of ‘desirability of wealth’. Informally, this assumption dictates that there is some level of wealth, measured in terms of a bundle of private goods, such that an individual would prefer that level of wealth and membership in some bounded number of clubs, all bounded in size, to any feasible equal-treatment outcome in any economy, no matter how large. Loosely, desirability of wealth implies that private goods can compensate for membership in large clubs.

In the literature on approximate cores of games and economies with collective activities and clubs, there are a number of models in the literature permitting ever-increasing gains to coalition and club sizes (Wooders 1983,1994 and Kovalenkov and Wooders 2001a,b, 2003a,b). These models permit games derived from economies where individuals may belong to overlapping clubs and where there may be ever-increasing gains to club size. In addition, following Shubik and Wooders (1982), Kovalenkov and Wooders (2003a) explicitly allow an individual to belong to multiple clubs. There have also been a number of papers demonstrating that states of the economy in approximate cores of economies with clubs can be supported as price-taking equilibrium outcomes.¹⁵ None of these papers studying price-taking equilibrium, however, allow equilibrium clubs to be large and individuals to belong to overlapping clubs. Indeed, except for sequences of economies with a fixed distribution of agent types, none of these papers allow large clubs. In contrast, we allow all agents to differ in their crowding types; the set of player types is a compact metric space.

In view of the prior literature on large games and large economies one might hope for approximate equivalence in large finite economies even with

¹⁴Note that in Wooders (1983), the set of players is replicated but the payoff set to any coalition of players may increase as the size of the total population increases.

¹⁵See, for example, Conley and Wooders (1997,2001), Ellickson et al. (2002), and Wooders (1989,1996). Except for some results in the last two papers, all these papers bound club sizes.

multiple memberships in clubs and with potentially ever-increasing returns to club size. The crucial restriction appears to be that *almost all gains* to collective activities are realized by groups bounded in size; that is, small groups are effective. In the case of one-private-good, the restrictions of Ellickson et al. transform the economy into an essentially private goods economy with indivisibilities and a consistency condition on club memberships that yield an appropriate feasibility condition. Our research demonstrates an asymptotic equivalence when arbitrarily large clubs and ever increasing returns to club size are allowed.

To place our model and proof techniques in the literature, our research builds on the research of Debreu and Scarf (1963), Foley (1970) and Wooders (1985). Recall that, given a state of the economy that is in the core for all replications of the total agent set, Debreu and Scarf (1963) define the set of preferred net trades of each agent in the economy and show that the convex hull of union of these sets can be separated from the origin. For an economy with pure public goods, Foley (1970) extends the commodity space to make the public good a separate good for each consumer. Wooders (1985) further extends the commodity space to make local public goods for each consumer in each possible jurisdiction separate commodities. In this paper, we build on these three approaches. Precisely, we extend the public good space so that each club and its membership is a different commodity for each agent in the club. Having done so, extensions of the techniques of Debreu and Scarf (1963) can be applied. We also introduce a virtual production set. Even though we have no production in the current paper, our virtual production set plays a similar role to the extended production sets in Foley (1970) and Wooders (1985). In particular, the feasibility requirements ensuring the club choices are consistent are imposed on the virtual production set.

6 Conclusions

The major economic importance of our research is that equilibrium clubs may be unbounded – they do not necessarily become infinitesimal as the economy grows large. This aspect of our modeling is especially relevant for questions of political economy, for example, and to issues of regulation of large firms, such as multinationals. We hope to study these issues, as well as other issues relating to labor markets in economies with large firms/jurisdictions in future research.

7 Appendix

Now, we state and prove our first result.

Theorem 2. Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an $n(\varepsilon_1, \varepsilon_0)$ such that: for any set of agents N , if $|N| > n(\varepsilon_1, \varepsilon_0)$ then for any attribute function $\alpha : N \rightarrow \Omega$ there exists an $(\varepsilon_1, \varepsilon_0)$ *Edgeworth state of the economy* (N, α) .

Proof of Theorem 2. The proof is divided into two steps.

STEP 1.

Suppose the claim of the Theorem is not true. Then there exists $\varepsilon_1, \varepsilon_0 > 0$ and a sequence of economies $(N_\nu, \alpha^\nu)_{\nu=1}^\infty$ such that for every ν we have $|N_\nu| > \nu$ and the ε_1 -remainder $c(\varepsilon_0)$ -core of (N_ν, α^ν) is empty. From Assumption (e), boundedness of marginal utilities with respect to at least one commodity, there is a positive number $\rho > 0$ such that for every club structure \mathcal{N} , for each agent $i \in N$ and every consumption $x^i \in X^i$ we obtain

$$u^i(x^i - \varepsilon_0 \bar{z}, \mathcal{N}[i]) \geq u^i(x^i, \mathcal{N}[i]) + \rho.$$

From the mean continuity assumption, there exists θ such that for all $(N, \alpha), (N, \beta) \in F(\Omega)$, where $d(\alpha(i), \beta(i)) < \theta$ for all i , it holds that

$$H(V_\alpha^{\varepsilon_0}(N), V_\beta^{\varepsilon_0}(N)) < \frac{\rho}{3}.$$

Let $\Omega_1, \dots, \Omega_T$ be a partition of Ω such that if $\omega, \omega' \in \Omega_i$ then $d(\omega, \omega') < \theta$. For each $i = 1, \dots, T$ select arbitrarily $\omega_i \in \Omega_i$. For each (N_ν, α^ν) define another pair (N_ν, γ^ν) where the attribute function γ^ν is defined by $\gamma^\nu(i) = \omega_i$ whenever $\alpha^\nu(i) \in \Omega_i$. We notice that the range of all the γ^ν is finite and therefore one may represent this sequence of economies as

$$N_\nu = \{(i, q) : i = 1, \dots, T \text{ and } q = 1, \dots, n_i^\nu\}$$

where all agents (i, q) and (i', q') where $i = i'$ are substitutes for each other – that is, they have the same attributes.

The following Lemma is used to approximate the sequence of economies (N_ν, γ^ν) by a sequence of replication economies.

Lemma 1 (Wooders, 1992 Lemma 1). Let $\{N_\nu\}$ be a sequence of sets of agents where

$$N_\nu = \{(i, q) : i = 1, \dots, T \text{ and } q = 1, \dots, n_i^\nu\}$$

for some integers n_i^ν , $i = 1, \dots, T$. Suppose that, for each $i = 1, \dots, T$,

$$\frac{|N_i^\nu|}{|N^\nu|} \text{ converges to a limit } n_i,$$

where $N_i^\nu \stackrel{\text{def}}{=} \{(i, q) : q = 1, \dots, n_i^\nu\}$, the subset of agents in N^ν of type i . Then, given $\varepsilon_1 > 0$ there exists a vector of integers, $\bar{n} = (\bar{n}_1, \dots, \bar{n}_T)$, such that for all ν sufficiently large, for some $r^\nu \in \mathbb{Z}_+$ and $\ell^\nu \in \mathbb{Z}_+^T$

$$n^\nu = (n_1^\nu, \dots, n_T^\nu) = r^\nu \bar{n} + \ell^\nu$$

and

$$\frac{\|\ell^\nu\|}{\|n^\nu\|} < \varepsilon_1$$

where, for any vector n , $\|n\| \stackrel{\text{def}}{=} \sum_t n_t$. (Observe that $|N^\nu| = \|n^\nu\|$.)

Now let us consider an economy \bar{N} with profile \bar{n} . We will need the following definition and Lemma for replication sequences \bar{N}_r of \bar{N} .

A state of the economy (x^N, \mathcal{N}) satisfies the *equal treatment property in utility space* whenever, for all $i, i' \in \{1, \dots, N\}$, if $\alpha(i) = \alpha(i')$ it holds that :

$$u^{i'}(x^{i'}, \mathcal{N}[i']) = u^i(x^i, \mathcal{N}[i]).$$

We call (x^N, \mathcal{N}) a *utility equal treatment feasible state of the economy*.

Lemma 2 Assume desirability of wealth. Then there is a positive real number K such that for any replication number r and for any utility equal treatment feasible state of the r ih economy,

$$\sup u^i(x^i, \bar{N}_r[i]) < K.$$

(per capita boundedness).

Proof. First, define $(\bar{N}_r, V_r^\varepsilon)$ as the game induced by the r^{th} replication of the economy with agent set \bar{N} . To show per-capita boundedness of $(\bar{N}_r, V_r^\varepsilon)_{r=1}^\infty$ we construct a sequence of *-economies and consider the sequence of games, denoted by $(\bar{N}_r, V_r^*)_{r=1}^\infty$, derived from the sequences of *-economies. To obtain the conclusion of the Lemma we construct the sequence of *-economies so that $V_r^\varepsilon(\bar{N}_r) \subset V_r^*(\bar{N}_r)$ and show that $(\bar{N}_r, V_r^*)_{r=1}^\infty$ satisfies per-capita boundedness.

For the *-economy, let the utility function of agent i be defined by

$$u^{*i}(x^i) = \max_{x^* \leq r^*} u^i(x^i + x^*, \bar{N}_r[i]),$$

where r^* satisfies desirability of wealth.

The utility functions u^{*i} are well defined and are quasi-concave. Also. it is clear that given any $(x^i, \bar{N}_r[i])$ we have

$$u^{*i}(x^i) \geq u^i(x^i, \bar{N}_r[i]).$$

For each r , the allocation $(x^{\bar{N}_r})$, is **-feasible* if

$$\sum_{iq \in \bar{N}_r} (x^{iq} - e^{iq}) \leq 0.$$

The set of all *-feasible allocations is denoted by A_r^* . Let K be a real number such that

$$K > \sup_{x_i \in A_1^*} u^{*i}(x^i).$$

From the closeness of A_1^* and quasi concavity there is a such real number. Obviously, since $V_r^\varepsilon(\bar{N}_r) \subset V_r^*(\bar{N}_r)$, K is a per-capita bound for the original sequence of games. \square

To proceed, we introduce another notion of an approximate core, the $s(\rho)$ -core for the game (\bar{N}, V^ε) , based on satisficing behavior rather than on communication costs. A payoff $v \in V^\varepsilon(\bar{N})$ is in the $s(\rho)$ -core of (\bar{N}, V^ε) if for all non-empty subsets S of \bar{N} we have $v + \bar{1}\rho \notin \text{int}V^\varepsilon(S)$.

Lemma 3 *There exists a $v \in \mathbb{R}^{\bar{N}}$ such that v_r , the r th replica of v , is in the $s(\frac{\rho}{3})$ -core of $(\bar{N}_r, V_r^\varepsilon)$ for all sufficiently large r .*

Proof of Lemma 3. This follows from per-capita boundedness and Theorem 2 in Wooders (1983). The convexity assumption of $V_r(\bar{N}_r)$ is replaced by the

overriding desirability of the private goods as in Wooders (1988) (Assumption (e) here).

STEP 2.

Now let us consider the sequence of economies $(\bar{N}_{r\nu}, \bar{\alpha}^{\nu})$ where $\bar{\alpha}^{\nu}$ is the restriction of α^{ν} to $\bar{N}_{r\nu}$. We claim that there exist a ν^* and an allocation in $(\bar{N}_{r\nu^*}, \bar{\alpha}^{\nu^*})$ that is in the $c(\varepsilon_0)$ -core of all the replications. The proof is divided into several Lemmas.

Lemma 4 *There exists $\beta \in \mathbb{R}^{\bar{N}}$ such that for some positive integer ν^* we have $\beta_{r\nu^*}$ is in the $s(\rho)$ -core of all the replications of $(\bar{N}_{r\nu^*}, \bar{\alpha}^{\nu^*})$.*

Proof of Lemma 4. First, from Lemma 3, one can choose ν^* such that $v_{r\nu^*}$, the r^{th} replica of v , is in the $s(\frac{\rho}{3})$ -core of $(\bar{N}_{r\nu^*}, V_{r\nu^*}^{\varepsilon})$ for all replications. It follows from the mean continuity assumption that $v_{r\nu^*} \in V_{r\nu^*}^{\varepsilon}(\bar{N}_{r\nu^*}) \subset V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(\bar{N}_{r\nu^*}) + \frac{\rho}{3}\bar{1}$. Thus, by taking $\beta_{r\nu^*} = v_{r\nu^*} - \frac{\rho}{3}\bar{1}$ one obtains $\beta_{r\nu^*} \in V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(\bar{N}_{r\nu^*})$. Also, by the mean continuity assumption one obtains $V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(S) \subset V_{r\nu^*}^{\varepsilon}(S) + \frac{\rho}{3}\bar{1}$, for all $S \subset \bar{N}_{r\nu^*}$. By Lemma 3 one has $v_{r\nu^*} + \frac{\rho}{3}\bar{1} \notin \text{int}V_{r\nu^*}^{\varepsilon}(S)$. Rearranging terms, one obtains $\beta_{r\nu^*} + \rho = v_{r\nu^*} + \frac{2\rho}{3}\bar{1} \notin \text{int}V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(S)$. This ends the proof.

Lemma 5 *There exists ν^* and a state of the economy for $(\bar{N}_{r\nu^*}, \bar{\alpha}^{\nu^*})$ in the $c(\varepsilon_0)$ -core for all replications.*

It follows from Lemma 4 that there exists ν^* such that $\beta_{r\nu^*}$ is in the $s(\rho)$ -core for all replications of $(\bar{N}_{r\nu^*}, \bar{\alpha}^{\nu^*})$. Therefore it follows that $\beta_{r\nu^*} + \rho \in V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(\bar{N}_{r\nu^*}) + \rho\bar{1} \subset V_{\bar{\alpha}^{\nu^*}}^{\varepsilon}(\bar{N}_{r\nu^*})$. Now let us consider a feasible allocation $(x^{\bar{N}_{r\nu^*}}, \bar{N}_{r\nu^*})$ such that $u^i(x^i, \bar{N}_{r\nu^*}[i]) \geq \beta^i + \rho$. Clearly, $(x^{\bar{N}_{r\nu^*}}, \bar{N}_{r\nu^*})$ is in the $c(\varepsilon_0)$ -core of any replication of the economy since $\beta_{r\nu^*} + \rho\bar{1} \notin \text{int}V_{n\bar{\alpha}^{\nu^*}}^{\varepsilon}$ for any $S \subset \bar{N}_{r\nu^*}$ and any replication number n . This is a contradiction.

Theorem 4. Assume desirability of wealth. Let (N, α) be an economy. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that: if $|N| > n(\varepsilon_1, \varepsilon_0)$ then there exists an $(\varepsilon_1, \varepsilon_0)$ -Edgeworth state of the economy. Moreover, any $(\varepsilon_1, \varepsilon_0)$ -Edgeworth state of the economy is an ε_1 -remainder $c(\varepsilon_0)$ -equilibrium for the economy (N, α)

Proof of Theorem 4

The proof of the Theorem is an extension of proofs of convergence of the core to equilibrium states due to Debreu and Scarf (1963) and existence proof of Foley (1970) and Wooders (1989). From Theorem 2 there exists an economy sufficiently large, say (N, α) and an allocation (x^N, \mathcal{N}) in the ϵ_1 -remainder $c(\epsilon_0)$ -core of the economy for all replications of the economy, then (x^N, \mathcal{N}) is an ϵ_1 -remainder $c(\epsilon_0)$ -equilibrium state of the economy. Without any loss of generality we can assume that there exists N^0 such that $\frac{|N \setminus N^0|}{|N|} < \epsilon_1$ and (x^{N^0}, \mathcal{N}^0) in the $c(\epsilon_0)$ -core of the economy for all replications of the economy. Then (x^{N^0}, \mathcal{N}^0) is an $c(\epsilon_0)$ -equilibrium state of the economy. Let $\{S_1, \dots, S_k, \dots, S_K\}$ denote the set of all clubs in N^0 and let $\mathcal{N}^0 = \{J_1, \dots, J_g, \dots, J_G\}$.

Preliminaries: We first consider the following space $A = \mathbb{R}^{N^0 K}$ where N^0 is the number of agents and K is the number of all possible clubs in N^0 . Let $a = (a^1, \dots, a^i, \dots, a^{N^0})$ be a vector where, for each i , $a_i = (a_1^i, \dots, a_k^i, \dots, a_K^i)$ and for each k , $a_k^i \in \mathbb{R}$. Let A_i be the set of elements in \mathbb{R}^K defined by

$$A_i = \{a \in \mathbb{R}^{N^0 K} : a_k^{i'} = 0 \text{ if } i \neq i' \text{ or if } i \notin S_k\}.$$

For a given $\mathcal{S}[i] \in \mathcal{S}[i]$, we represent $\mathcal{S}[i]$ in A_i by a , such that a_k^i equals one if S_k belongs to $\mathcal{S}[i]$ and equals zero otherwise.

We next define a ‘virtual’ production set in the extended commodity space. For each k define $b[k] \in \mathbb{R}^{N^0 K}$ as a vector having the properties that:

- (i) $b[k]_{k'}^i = 0$ if $k \neq k'$ or if $i \notin S_k$
- (ii) for (any) i in S_k , $b[k]_k^i = 1$

Define the virtual production set Y as the convex cone generated by the $\{(z_{S_k}, b[k]) : k = 1, \dots, K\}$, where z_{S_k} is the input required to form the club z_{S_k} . The set Y is precisely the set of all positive linear combinations of $\{(z_{S_k}, b[k]) : k = 1, \dots, K\}$.

Step 1: The sets of preferred allocations Ω_i . Let Ω_i denote the set of members of $(x^i - e^i - \varepsilon \bar{z}, a^i)$ in $X^i \times A_i$ such that, for every club structure \mathcal{S} with the property that $\mathcal{S}[i] = \{S_k \mid a_k^i = 1\}$, we have $u^i(x^i, \mathcal{S}[i]) > u^i(\tilde{x}^i, \mathcal{N}[i])$.

The set $\Omega_i \subset \mathbb{R}^{L+N^0 K}$ describes the set of net trades of private goods and club memberships for agent i strictly preferred to his allocation in the given state of the economy $(\tilde{x}^{N^0}, \mathcal{N}^0)$. It is clear that Ω_i is not convex.

Step 2: The preferred set Ω . Let Ω denote the convex hull of the union of the sets Ω_i , $i = 1, \dots, N^0$. We now show, in the remainder of Step 2, that

$$\Omega \cap Y = \emptyset.$$

Suppose, on the contrary, that $(x, a) \in \Omega \cap Y$. Then, by the definition of Ω there exist an integer J and $\lambda \in \mathbb{R}^J$ such that $(x, a) = \sum_{j=1}^J \lambda_j (x^j, a^j)$ with $\lambda_j > 0$, $\sum \lambda_j = 1$.

From the definition of Y there exist a $K' \in \{1, \dots, K\}$ and $\mu \in \mathbb{R}_{++}^{K'}$ such that

$$(x, a) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k]).$$

Let us consider $J[i] = \{j \mid (x^j, a^j) \in \Omega_i\}$. Then, it follows from

$$\sum_{j=1}^J \lambda_j (x^j, a^j) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k])$$

that for each $k \in K'$ and each $i \in S_k$ we have

$$\sum_{j \in J[i]} \lambda_j a_k^{j,i} = \mu_k$$

For a given (x^j, a^j) in Ω_i and a given sequence $\{(\beta^n)\}_n$ of real numbers. Suppose that $\beta^n \geq 1$ for each n and that $(\beta^n x^j, a^j)$ converges to one as n goes to infinity. Then, because of the continuity of preferences, for all n sufficiently large, we have $(\beta^n x^j, a^j)$ is in Ω_i .

We now show that, since we have supposed that $\Omega \cap Y \neq \emptyset$, we can form a blocking coalition for some sufficiently large replication. We will use the following lemma.

Lemma. There exists a sequence of rational numbers $(\lambda_1^n, \dots, \lambda_j^n, \dots, \lambda_J^n)$ converging to $(\lambda_1, \dots, \lambda_j, \dots, \lambda_J)$ and having the properties that:

- (i) $\lambda_j^n \leq \lambda_j$
- (ii) for (any) k , and for any $i, i' \in S_k$ we have:

$$\sum_{j \in J[i]} \lambda_j^n a_k^{j,i} = \sum_{j \in J[i']} \lambda_j^n a_k^{j,i'}.$$

Proof. Let us consider the closed line segment $[0_{\mathbb{R}^J}, \lambda]$ in \mathbb{R}^J . From convexity it follows that, for any $\alpha \in [0_{\mathbb{R}^J}, \lambda]$, for (any) k and for any $i, i' \in S_k$ we have

$$\sum_{j \in J[i]} \alpha_j a_k^{j,i} = \sum_{j \in J[i']} \alpha_j a_k^{j,i'}.$$

But we know that Q^J , where Q is the set of rational number, is dense in \mathbb{R}^J . Hence, $Q^J \cap [0_{\mathbb{R}^J}, \lambda]$ is dense in $[0_{\mathbb{R}^J}, \lambda]$ and therefore we can choose a sequence satisfying (i) and (ii). \square

Let us consider the sequence $(\lambda_1^n, \dots, \lambda_j^n, \dots, \lambda_j^n)$ defined above, and let us select a positive integer n , which will eventually tend to infinity. For each j define $x^{jn} = \frac{\lambda_j}{\lambda_j^n} x^j$. From the concluding paragraph of the last Step, for all n sufficiently large it holds that $(x^{jn}, a^j) \in \Omega_i$. Let n satisfy the property that $(x^{jn}, a^j) \in \Omega_i$ for each i . Recall that λ_j^n is a rational number.

Now, let us define $\mu_k^n = \sum_{j \in J[i]} \lambda_j^n a_n^{j,i'}$. Since $\sum_{j=1}^J \lambda_j^n x^{jn} = \sum_{k \in K'} \mu_k^n z_k$, $\mu_k^n \leq \mu_k$ and $z_k \in -\mathbb{R}_+^L$, it follows that

$$\sum_{j=1}^J \lambda_j^n x^{jn} \leq \sum_{k \in K'} \mu_k^n z_k$$

Let r' be a replication number such that $r' \lambda_j^n$ is an integer for all j . Let $\delta_j = r' \lambda_j^n$ and $\gamma_k = \sum_{j \in J[i]} \delta_j$. It holds that

$$\sum_{j=1}^J \delta_j x^{jn} \geq \sum_{k \in K'} \gamma_k z_k.$$

Let r^* be an integer sufficiently large so that there are γ_k copies of the club S_k , for each k , contained in the set r^{*th} replication $N_{r^*}^0$ of N^0 and so that this does not hold for any $r < r^*$, that is, r^* is minimal. This implies that there is a state of the economy for the coalition S that can $c(\epsilon_0)$ -improve upon the initially given state of the economy. The state of the economy for S described by the consumption plans (x^{jn}, a^j) , for δ_j consumers, for each j is $c(\epsilon_0)$ -feasible and preferred by all members of the replication of the initially given state of the economy. Consequently, S can $c(\epsilon_0)$ -improve upon the r^{*th} replication of $(\tilde{x}^{N^0}, \mathcal{N}^0)$, which is a contradiction. Therefore $\Omega \cap Y = \emptyset$.

Step 3: Prices. From the Minkowski Separating Hyperplane Theorem, there is a hyperplane with normal $(p, \pi) \neq 0$, where p is in the private goods price space, and $\pi \in \mathbb{R}^{N^0 K}$ such that, for some constant C ,

$p \cdot x + \pi \cdot a \geq C$ for all $(x, a) \in \Omega$ and

$p \cdot z + \pi \cdot b \leq C$ for all $(z, b) \in Y$.

Since Y is a closed convex cone with vertex zero, it follows that we can choose $C = 0$. Then, in particular, it follows that for each $(x^i, a^i) \in \Omega_i$

$$p \cdot (x^i - e^i - \epsilon \bar{z}) + \sum_{\{k|a_k^i=1\}} \pi^i(S_k) \geq 0,$$

and for each club $S_k \subset N$ we have

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0.$$

Recall that $(\tilde{x}^{N^0}, \mathcal{N}^0)$ is a $c(\epsilon)$ -core state of the economy relative to the club structure $\mathcal{N}^0 = \{J_1, \dots, J_G\}$ of N^0 . Observe that we can represent the total consumption of each agent i by $(\tilde{x}^i, \tilde{a}^i) \in \mathbb{R}^{L+N^0K}$.

From monotonicity it follows that $p \geq 0$. Suppose that $p = 0$. Therefore, from the separating hyperplane it follows that for each S_k we have

$$\sum_{i \in S_k} \pi^i(S_k) \leq 0,$$

and for each $i \in S_k$ we have $\pi^i(S_k) \geq 0$. Thus $\pi^i(S_k) = 0$, for each S_k and each $i \in S_k$, which is a contradiction to the fact that $(p, \pi) \neq 0$.

Since, for each i , $(\tilde{x}^i - e^i - \epsilon \bar{z}, \tilde{a}^i)$ is in the closure of Ω_i , it holds that

$$p \cdot (\tilde{x}^i - e^i - \epsilon \bar{z}) + \sum_{\{k|\tilde{a}_k^i=1\}} \pi^i(S_k) \geq 0.$$

Moreover, for each club J_g we have

$$p \cdot z_{J_g} + \sum_{i \in J_g} \pi^i(J_g) \leq 0.$$

Summing the above inequalities over consumers one obtains

$$p \cdot \sum_{i \in N} (\tilde{x}^i - e^i - \epsilon \bar{z}) \geq p \cdot \sum_g z_{J_g},$$

and summing over clubs one obtains

$$\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \leq 0.$$

Since $p \in \mathbb{R}_+^L \setminus \{0\}$ and $\sum_{i \in N} (\tilde{x}^i - e^i) \leq \sum_g z_{J_g}$ it follows that

$$p \cdot \sum_{i \in N} (\tilde{x}^i - e^i) \leq p \cdot \sum_g z_{J_g}.$$

Then from the fact that $p \cdot z_{J_g} + \sum_{\{i \in J_g\}} \pi^i(J_g) \leq 0$ it follows that

$$\sum_{i \in N} p \cdot x^i + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) + \sum_{i \in N^0} \epsilon p \cdot \bar{z} \leq \sum_{i \in N} p \cdot e^i,$$

and

$$\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \geq \sum_{i \in N} p \cdot e^i.$$

Now we claim that $((\tilde{x}^{N^0}, \mathcal{N}^0), p, \Pi)$ is a $c(\epsilon)$ -equilibrium. Checking the proof so far, it remains only to show that individual consumers are optimizing, i.e., that the prices p, Π and the state (x^{N^0}, \mathcal{N}^0) satisfy condition (iii) of the definition of an equilibrium.

Suppose that for some consumer i , and some consumption (x^i, a^i) ,

$$\begin{aligned} u^i(x^i, a^i) &> u^i(\tilde{x}^i, \tilde{a}^i) \text{ and} \\ p \cdot (x^i - e^i - \epsilon \bar{z}) + \sum_{\{k | a_k^i = 1\}} \pi^i(S_k) &\leq 0. \end{aligned}$$

From our desirability of endowment assumption, there is a consumption $x^0 \in X^i$ such that

$$p \cdot (x^0 - e^i - \epsilon \bar{z}) + \sum_{\{k | a_k^i = 1\}} \pi^i(S_k) < 0.$$

It follows that for some x^i in the segment $[x^0, x^i]$

$$u^i(x^i, a^i) > u^i(\tilde{x}^i, \tilde{a}^i)$$

and

$$p \cdot (x^i - e^i + \epsilon \bar{z}) + \sum_{\{k | a_k^i = 1\}} \pi^i(S_k) < 0,$$

which is a contradiction.

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