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# AGE MATCHING PATTERNS AND SEARCH

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ABSTRACT. Consider a marriage market with continuous-time two-sided search and transferable utility in which the match payoff depends on age. This paper characterizes a set of payoff functions consistent with two salient marriage age patterns: (1) assortative matching by age, and (2) “differential age matching”, a formalization of the age difference at marriage between men and women. The payoff function has to satisfy certain conditions on its slope and curvature in age. However, to achieve sorting it need not be supermodular (submodular) in partners’ ages, unlike in search models where the match payoff depends on the fixed “type” of each individual. (JEL C78, D83, J64)

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## 1. INTRODUCTION

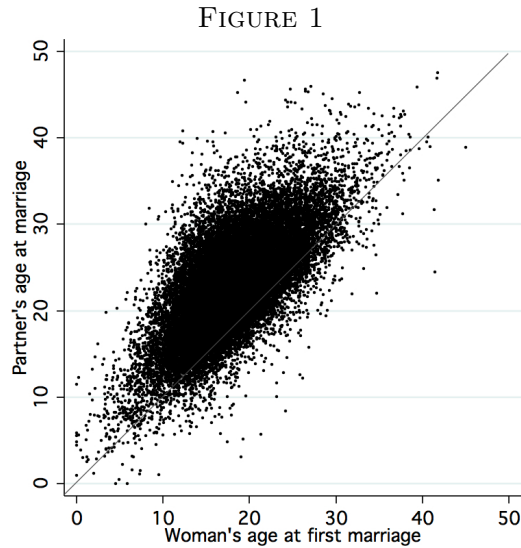
Age patterns of marriage exhibit remarkable regularities across marriage markets. As an example, figure 1 depicts the marriage ages of Indian women and their spouses according to the 2005-06 National Family Health Survey. There is strong sorting by age, and women often marry men considerably older than themselves, while an age difference in the opposite direction is less common and usually quite small. This pattern is not unique to India; Goldman et al. (1984) draw a strikingly similar graph using marriages from the United States in 1976-78. Consistent with these patterns, the average marriage age of men is higher than that of women in almost every country, and spousal age is highly correlated (e.g. United Nations (2000), Choo and Siow (2006b))<sup>1</sup>.

Marriage age patterns are important because they have implications for education, fertility and maternal health, and they affect decision making, for example on savings. They also play a role in the supply and demand of spouses: if the population grows over time, there is a unique (average) age difference at which the numbers of women and men are balanced. Conversely, if women marry younger than men, an increase in population growth causes a surplus of women at the prevailing marriage age. This so-called marriage squeeze has been blamed for the high marriage transfers paid by women's families in India, and ultimately for increased rates of neglect or infanticide of unwanted – expensive – female children (e.g. Bhat and Halli (1999)).

This paper develops a search model with transferable utility in which the joint payoff to a match,  $f(i, j)$ , depends on age  $i$  of the woman and  $j$  of the man at the time of marriage. The marriage payoff function summarizes individuals' preferences over their own and their spouse's age, based for instance on fertility and health considerations, current wealth, or labor market value. A couple will marry if they prefer  $f(i, j)$  over the outside option of staying in the market and meeting another potential partner in the future, when they are both older. The model applies to other settings in which coalitions of two have to form and payoffs depend on time-until-match. Examples are buyers and sellers of perishable goods,

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<sup>1</sup>Coincidentally, the dating website OkCupid (2010) recently documented age preference profiles among its users that are again very similar to figure 1.



*Age at first marriage (jittered) for women born before 1981 and their partners, NFHS 2005-06.*

or professional pairs like an executive and her assistant. There may in principle be both gains or losses to “aging”. For example, the payoff for an architect and an engineer of founding a firm together may at first increase, while they separately acquire valuable skills as employees, but eventually they are past the optimal “age” because the learning curve is flat and their creativity begins to suffer.

Individuals are assumed to enter the market at age zero and to leave it after age  $I$  and  $J$  for women and men, respectively. A matched pair exits the market, and there is no remarriage. The age distributions of singles and the ratio of unmarried men and women in the market are endogenous, a product of market entry at age zero, and exit from the market through marriage or upon reaching the final marriage ages. If for instance women marry earlier than men, there will be overall more single men than women, and the age distribution of unmarried women will be more concentrated at the low ages. In a steady-state equilibrium, entry and exit are balanced and the singles age distributions and sex ratio are constant over time.

In this paper I characterize a set of payoff functions under which there is assortative matching as well as “differential age matching”, a formalization of the difference in marriage

age between men and women. Intuitively, differential age matching requires that whenever an “old” woman and “young” man are observed to marry, then the opposite pairing between an old man and a young woman will lead to marriage as well; or in other words, the partners of any man will be as young or younger than the partners of a woman of the same age.

Sufficient conditions for both marriage age patterns involve the slope and curvature of  $f$  in each partner’s age. To provide a first intuition, it is useful to think of the net surplus of marriage – the payoff  $f(i, j)$  minus the man’s and woman’s outside options – as a function of one partner’s age. For example, for a given age of the man  $j$ , the net surplus from marriage is decreasing in the woman’s age whenever  $f$  itself falls fast enough in  $i$ . But ages  $i$  and  $j$  match with each other exactly if the net surplus is positive, meaning that the set of ages that  $j$  marries is an interval beginning with the youngest women at  $i = 0$  up to a cut-off age where the net surplus from marriage is just equal to zero. Similarly, if  $f$  is “concave enough” in  $i$  (in a sense made precise later), the marriage surplus will be quasi-concave, again leading to  $j$  matching with an interval of ages. Interval matching, or a convex matching set, is a key ingredient for assortative matching and plays a role for differential age matching as well (see section 4 and Shimer and Smith (2000)).

As a motivating example, consider a simple formalization of the payoffs to marriage that arise from an exchange of fertility for protection and status:

**Example:** Let  $I = J$  be the age of death and assume no discounting. Suppose the woman receives a flow payoff  $b$  from the date of marriage to the end of her life. This may reflect the value of the connection to (or protection by) her new family or a gain in status from being a married woman. It is particularly relevant when status is patrilineal, as for example in India, where the wife adopts the husband’s caste. In addition, the couple receives a payoff  $\Pi$  if they have a child. Between the date of marriage and the death of the older partner, the woman has a uniform probability of becoming infertile, modeled by an exponential distribution with parameter  $\gamma$ . At the same time, conditional on being fertile the probability of conceiving is assumed to be time independent, so that the arrival rate of a child is again exponentially distributed (with parameter  $\beta$ ). The payoff function is then

$$f(i, j) = \Pi\beta e^{-\gamma i} \int_0^T e^{-(\gamma+\beta)t} dt + b(I - i)$$

where  $T = I - i$  if  $i > j$ , and  $T = J - j$  if  $j \geq i$ . ◇

It will be shown that this marriage payoff function leads to both positive sorting and differential age matching if the woman's bargaining position is weak.

The theory of two-sided matching and search has long been interested in conditions for assortative matching, that is, the matching of similars or opposites. Usually it is assumed that the match payoff depends on a heterogenous, but fixed characteristic, for example the partners' productivity or what Burdett and Coles (1997) call "pizzazz". Age has a special status in this context: with search frictions, the time that the partners have spent in the market, and therefore their age, is itself a source of heterogeneity. Even if individuals on the same side of the market are ex ante identical, their marriage payoff is subject to change over time, and there will be a distribution of agents at different ages in the market. Correspondingly, the conditions for sorting differ from those in existing search models where heterogeneity comes from a fixed characteristic (see Becker (1973), Shimer and Smith (2000)), and I will show that supermodularity (submodularity) of the payoff function is neither necessary nor sufficient for positive (negative) assortative matching.

After a literature review in section 2, section 3 describes the model and proves existence of a steady-state equilibrium. Section 4 first defines and discusses assortative matching and differential age matching (DAM). I then derive sufficient conditions on the payoff function so that every individual matches with an interval of ages (convex matching sets). This leads to conditions for assortative matching. Finally, I provide a set of conditions which guarantee DAM, again closely related to the conditions for matching set convexity. Section 5 concludes.

## 2. RELATED LITERATURE

Assortative matching has been a major focus of attention in the theory of matching and search. The literature begins with Becker (1973), who analyzed a static, frictionless setting with transferable utility and provided the well-known condition for positive assortative matching (PAM) that the matching partners' types must be complements. This is equivalent to a supermodular payoff function  $f$ , or  $f(x_1, y_1) + f(x_2, y_2) \geq f(x_1, y_2) + f(x_2, y_1)$  for types  $x_2 > x_1$  and  $y_2 > y_1$ . Similarly, negative assortative matching (NAM) obtains if types are

substitutes. Becker also showed that sorting can be sustained under nontransferable utility only if both sides have monotone preferences in their partner’s type.<sup>2</sup>

Subsequent authors have analyzed assortative matching in a setting with search frictions for both the transferable and nontransferable case. The most general result under nontransferability is by Smith (2006), who proves that the production function has to satisfy log supermodularity, or  $f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1)$  (where  $f$  is increasing). Under this condition, the market becomes “segregated”, that is, the type space can be partitioned in intervals which only match among themselves. Similar results for less general production functions have been proven earlier, most notably by Burdett and Coles (1997), who were the first to consider a full equilibrium with endogenous meeting rates (see Smith (2006) for a discussion).

Search with heterogeneity and transferable utility was introduced by Sattinger (1995), who explores the efficiency properties of equilibrium. Lu and McAfee (1996) show that there must be assortative matching if  $f(x, y) = xy$ . Shimer and Smith (2000) generalize this result and show that positive assortative matching requires supermodularity of the production function as well as log supermodularity of  $f_x$  and  $f_{xy}$ . Negative assortative matching is achieved only if all three are (log) submodular (although Atakan (2006) recovers assortative matching under supermodularity alone by introducing an explicit additive search cost into the model).

This paper is closely related to Shimer and Smith’s work. The key difference is that Shimer and Smith assume a given population distribution of productivity types  $x$  and  $y$  and a fixed, symmetric match payoff for each pair of types.<sup>3</sup> In the model here, heterogeneity is the result of individual search spells of different lengths. The production function depends on time in the market, that is, ages  $i$  and  $j$ , and it may be asymmetric. Consequently, an individual’s value of search is a function of age and changes over time. In a model with fixed productivity types the equilibrium value of search is a time-invariant number for each market participant.

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<sup>2</sup>Legros and Newman (2007) extend the analysis of the frictionless model to partially transferable utility.

<sup>3</sup>Moreover, agents are infinitely lived and discount the future at a constant rate. Matches dissolve at an exogenous rate.

Age-dependent match payoffs are not entirely new to the literature. Choo and Siow (2006a) empirically analyze matching by age in a dynamic model without search frictions. Ljungqvist and Sargent (1998) and Coles and Masters (2000) have incorporated time-variant payoffs into labor market search models. Both assume that workers gradually lose skills when unemployed and regain them when employed, either through training or with experience on the job. However, all firms are identical, and the authors are not concerned with assortative matching, but with policy questions and the effects of shocks on the labor market.

### 3. THE SEARCH MODEL AND MARRIAGE MARKET EQUILIBRIUM

The description of the model begins with the steady-state “singles distribution”. For an unmarried individual, the probability of meeting a person of the other sex at a certain age depends on the age composition of the unmarried population. This composition is the result of entry by young cohorts on the one hand, and of exit through marriage or upon reaching the final marriage age on the other. For given marriage behavior, the market is in a steady state when the age distributions of singles at different ages and the relative number of men and women are constant in every period. We will start by deriving the steady-state “singles distribution”, consisting of the men’s and women’s age distributions  $m$  and  $w$  and the singles sex ratio  $r$ .

Let  $i \in [0, I]$  be the age of a woman, and  $j \in [0, J]$  that of a man.<sup>4</sup> Age zero is the age of entry into the marriage market, and  $I$  and  $J$  are the last possible marriage ages. The size of the female singles population at time  $t$  is  $W_t$ ,  $W_t(i)$  denotes the measure of single women at age  $i$ , and  $w_t(i) = \frac{W_t(i)}{W_t}$  describes the age distribution of single women. Equivalently use  $M_t$ ,  $M_t(j)$  and  $m_t(j)$  for single men. The singles sex ratio is  $r_t = \frac{M_t}{W_t}$ . Assume that men and women enter the marriage market in equal numbers at every point in time, so that  $W_t(0) = M_t(0)$  for all  $t$ .

Men and women meet randomly, at a rate that depends on the ratio of single men and women. Equivalently, the arrival rates of a potential match on each side of the market follow

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<sup>4</sup>As distinct from fixed types  $x$  and  $y$  used in many matching models.



a Poisson process with parameters  $\lambda_m(r_t)$  and  $\lambda_w(r_t)$  for men and women, respectively. One-to-one matching requires that the number of men who meet someone in any given period must equal the number of women who do. The Poisson parameters above must therefore satisfy

$$r_t \equiv \frac{M_t}{W_t} = \frac{\lambda_w(r_t)}{\lambda_m(r_t)}$$

for any given  $r_t$ . Assume in addition that the two functions  $\lambda_w$  and  $\lambda_m$  are positive, continuous and bounded, so that there are some minimal frictions of search.

If  $r_t$  increases, there are relatively more men available, and this may plausibly increase the overall meeting probability for women, while it decreases that for men. As an example, there could be symmetric meeting probabilities given by  $\lambda_w(r_t) = r_t^{0.5}$  and  $\lambda_m(r_t) = r_t^{-0.5}$ . However, note that the assumption of bounded meeting rates means that this cannot hold for all possible  $r_t$ . As an example, suppose that the maximal meeting frequency allows two encounters per period on both market sides. In this case we would have  $\lambda_m(r_t) = 2$  for any  $r_t < 0.25$  and therefore  $\lambda_w(r_t) = 2r_t$  (and similarly  $\lambda_m(r_t) = 2r_t^{-1}$  and  $\lambda_w(r_t) = 2$  for  $r_t > 4$ ).

To simplify, I assume proportional conditional meeting probabilities, that is, given an individual meets someone, the probability that this person is of a particular age equals the proportion of that age in the singles population.

Couples decide if they want to marry based on the payoff from their partnership and the outside options both of them have. The marriage decision for a couple  $(i, j)$  is captured by a measurable marriage indicator function  $\alpha : [0, I] \times [0, J] \rightarrow [0, 1]$ .<sup>5</sup> Taking marriage behavior and therefore  $\alpha$  for now as given, the mass of single women at age  $i$  in  $t$  is

$$W_t(i) = e^{-\int_0^i \lambda_w \left( \int_0^J \alpha(x, y) m_{t-i+x}(y) dy \right) dx} W_{t-i}(0),$$

and therefore

$$w_t(i) = \frac{W_t(i)}{W_t} = \frac{e^{-\int_0^i \lambda_w \left( \int_0^J \alpha(x, y) m_{t-i+x}(y) dy \right) dx}}{\int_0^I e^{-\int_0^u \lambda_w \left( \int_0^J \alpha(x, y) m_{t-i+x}(y) dy \right) dx} du}.$$

Clearly the age distribution function is decreasing for all  $i$ : after entering the market, each singles cohort is reduced gradually by those who marry. We can write a similar expression for

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<sup>5</sup> $\alpha$  can lie between zero and one if a couple is using a mixed strategy between marrying and staying single.

the men's market side, and in a steady state, the age distributions are invariant. The time index can therefore be suppressed, and the equations that describe the singles distribution are

$$(3.1) \quad w(i) = \frac{e^{-\int_0^i \lambda_w(r) (\int_0^J \alpha(x,y)m(y)dy) dx}}{\int_0^I e^{-\int_0^u \lambda_w(r) (\int_0^J \alpha(x,y)m(y)dy) dx} du}$$

$$(3.2) \quad m(j) = \frac{e^{-\int_0^j \lambda_m(r) (\int_0^I \alpha(x,y)w(x)dx) dy}}{\int_0^J e^{-\int_0^u \lambda_m(r) (\int_0^I \alpha(x,y)w(x)dx) dy} du},$$

where  $r = \frac{w(0)}{m(0)}$ . The expression for  $r$  follows from the observation that  $w(0) = 1/W$  and  $m(0) = 1/M$ .

Of course, marriage behavior  $\alpha$  is itself a response to market conditions. When a man and a woman meet, they make their marriage decision by comparing the payoff to marrying with the outside value of being single and remaining in the market. This value depends on the relative numbers of men and women at each age available for marriage, in other words, the singles distribution. The description of the market equilibrium is complete with two value functions that describe the individuals' outside options and determine their choice to marry based on  $w$  and  $m$ . I will focus on the women's side of the market, stating the equivalent expressions for the men's side without derivations.

Denote by  $f_w(i, j)$  and  $f_m(i, j)$  the (expected) payoffs of marriage for a woman and a man at age  $i$  and  $j$ , respectively, and let  $d(i, j)$  be the transfer paid by the bride to the groom. A couple will marry upon meeting if they both prefer marrying each other over continuing search. The woman agrees to marry if the net payoff  $f_w(i, j) - d(i, j)$  exceeds the value of searching at age  $i$ ,  $V(i)$  (which will be time-invariant in a steady-state). Equivalently, the man compares search value  $H(j)$  with  $f_m(i, j) + d(i, j)$ .

Following the literature (e.g. Pissarides (1990)), the expected marriage transfer between ages  $i$  and  $j$  is assumed to be the generalized Nash product, given by

$$d(i, j) = \arg \max_d [f_w(i, j) - d - V(i)]^\theta [f_m(i, j) + d - H(j)]^{(1-\theta)}$$

where  $\theta$  describes the woman's bargaining power. This means that the transfer is

$$d(i, j) = (1 - \theta)(f_w(i, j) - V(i)) - \theta(f_m(i, j) - H(j)),$$

and the net payoff to the woman from marrying

$$V(i) + \theta (f_w(i, j) + f_m(i, j) - V(i) - H(j)).$$

In other words, she receives her outside option, plus a share  $\theta$  of the net marriage surplus.<sup>6</sup> As long as this share is positive, she will agree to marry. Similarly, the groom gets his outside option plus a  $(1 - \theta)$  share of the net surplus. This implies that the marriage decisions of a couple  $(i, j)$  coincide and can be described by a single function  $\alpha$  indicating a positive net surplus. For the value function (although not the transfer payment) it is not important from which source the marriage payoff originates, so  $f_w(i, j) + f_m(i, j)$  can be summarized as  $f(i, j)$ . Assume that  $f$  is continuous.

Define the marriage surplus as  $S(i, j) = f(i, j) - V(i) - H(j)$ . The value functions are then described by a system of differential equations (see the appendix for the derivation) where the final conditions are given by the value of being single when leaving the marriage market,  $\underline{V}$  for the woman and  $\underline{H}$  for the man.  $\underline{V}$  and  $\underline{H}$  are assumed to be nonnegative.<sup>7</sup>

$$(3.3) \quad \dot{V}(i) = -\lambda_w(r)\theta \int_0^J \max\{S(i, y), 0\} m(y)dy + \rho V(i)$$

$$(3.4) \quad V(I) = \underline{V}$$

$$(3.5) \quad \dot{H}(j) = -\lambda_m(r)(1 - \theta) \int_0^I \max\{S(x, j), 0\} w(x)dx + \rho H(j)$$

$$(3.6) \quad H(J) = \underline{H}.$$

Note that the integral in  $\dot{V}(i)$  is weakly positive, so  $V(i)$  can be at most increasing by  $\rho V(i)$ .  $V(i)$  and  $H(j)$  are bounded below at zero and above at  $\max_{(i,j)} f(i, j) \equiv \bar{f}$ . It should be emphasized once more that the value of search for an individual is a function of time

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<sup>6</sup>A possible interpretation is that either the man or the woman gets to make an ultimatum proposal for the division of the surplus, where the probability of being the proposer equals the bargaining power.

<sup>7</sup>Implicit in this is the normalization of the payoff of being single in the market to zero. Thus,  $\underline{V}$  and  $\underline{H}$  describe the value to being single, but not searching anymore. Positive values might for example indicate a cost of searching, which is not incurred anymore when the individual leaves the market. A value of zero implies that the individual is indifferent between spending a period in or out of the market.

(age), even though the market is in a steady-state. The distribution of ages is invariant, but an individual's search value is not.<sup>8</sup>

To summarize, the singles distribution depends on marriage behavior, expressed by the marriage indicator function  $\alpha$ . In turn, this behavior based on marriage payoffs and the singles distribution. In a steady-state equilibrium, marriage choices and the singles distribution are mutually consistent. Men and women's expectations about meeting probabilities later in life are correct and identical to current market conditions. Formally, define:

**Definition 3.1** (Steady-State Matching Equilibrium). For a given marriage payoff function  $f$  and Poisson meeting rate functions  $\lambda_w$  and  $\lambda_m$ , a matching equilibrium is given by an indicator function  $\alpha$ , value functions  $V$  and  $H$ , and singles distributions  $m$  and  $w$  such that

- (1)  $\alpha(i, j) = 1$  if  $S(i, j) > 0$ ,  $\alpha(i, j) = 0$  if  $S(i, j) < 0$  and  $\alpha(i, j) \in [0, 1]$  otherwise,
- (2)  $m$  and  $w$  are steady-state distributions for  $\alpha$  satisfying equations (3.1) and (3.2),  
and
- (3)  $V(i)$  and  $H(j)$  are the value functions for  $m$  and  $w$  given by (3.3)–(3.6).

To show that such an equilibrium exists I construct a self-map  $\mathcal{T}$  on the set of value functions and singles distributions whose fixed points coincide with the matching equilibria. Existence of a fixed point is then proved by applying the Eilenberg-Montgomery fixed point theorem.

**Theorem 3.2** (Existence of a Steady-State Matching Equilibrium). *Suppose  $\lambda_w$  and  $\lambda_m$  are positive, bounded and continuous and satisfy  $r\lambda_m(r) = \lambda_w(r)$ . Then there exists a steady-state matching equilibrium for this market.*

The proof is in the appendix, as are all following proofs if omitted from the main text. Observe that there may in general be more than one equilibrium. A change to who matches with whom (i.e. the  $\alpha$ ) affects the entire singles distribution and therefore each agent's value of search, leading to potential multiplicity.

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<sup>8</sup>In a typical search model with fixed productivity types,  $V$  and  $H$  themselves, not their derivatives, would be a function of the net surplus of matching. Unlike in these models the equations here cannot easily be solved for  $V$  and  $H$  (compare e.g. with equations (6) and (8) in Shimer and Smith (2000)).

## 4. THE SHAPE OF THE EQUILIBRIUM MATCHING SET

Our discussion of equilibrium matching patterns begins with the definitions for assortative matching and differential age matching.

Shimer and Smith (2000) were the first to provide a definition of assortative matching for the search context. Denote the matching set of a man of age  $j$  by  $\mathbb{I}(j) = \{i | \alpha(i, j) = 1\}$  and that of a woman at age  $i$  by  $\mathbb{J}(i) = \{j | \alpha(i, j) = 1\}$ . For later reference define  $\mathbb{M}$  to be the set of couples  $(i, j)$  in  $\mathbb{R}^2$  who match with each other, i.e.  $\mathbb{M} = \{(i, j) | \alpha(i, j) = 1\}$ . It will be convenient to split the definition of positive assortative matching into two parts.

**Definition 4.1** (Positive assortative matching). Consider any  $t, s > 0$ . There is upper positive assortative matching (PAM) if for any  $i$  and  $j$  such that  $i \in \mathbb{I}(j+t)$  and  $i+s \in \mathbb{I}(j)$ , we have  $i+s \in \mathbb{I}(j+t)$ . There is lower PAM if  $i \in \mathbb{I}(j)$ . If both hold there is (full) PAM.  $\diamond$

The left panel of figure 2 illustrates PAM. Positive assortative matching is most common in the marriage market, but since assortative matching is of interest in other contexts as well, I will also discuss negative assortative matching.

**Definition 4.2** (Negative assortative matching). Let  $t, s > 0$ . There is negative assortative matching (NAM) if for any  $i$  and  $j$  such that  $i \in \mathbb{I}(j)$  and  $i+s \in \mathbb{I}(j+t)$ , we have  $i+s \in \mathbb{I}(j)$  and  $i \in \mathbb{I}(j+t)$ .  $\diamond$

The first step in the next subsection will be to define sets of conditions under which each age  $i$  matches with a closed, convex set of ages  $j$  and vice versa. Some of those conditions guarantee that the matching sets always include the boundaries (0 or  $I/J$ ). I will show that these conditions deliver PAM/NAM directly. For an alternative approach I will follow Shimer and Smith (2000), who list four preliminaries for assortative matching: the matching sets  $\mathbb{I}(\cdot)$  and  $\mathbb{J}(\cdot)$  need to be convex and nonempty,  $\mathbb{M}$  has to be a closed set<sup>9</sup>, and there must be “matching in the corners”, in other words, either the couples  $(0, 0)$  and  $(I, J)$  or couples  $(0, J)$  and  $(I, 0)$  marry (for PAM and NAM respectively). To provide a self-contained argument, I make use of the following lemma.

<sup>9</sup>This is equivalent to upper hemicontinuity of the matching correspondence  $\mathbb{I}(j)$  or  $\mathbb{J}(i)$ , since the domain is compact.

**Lemma 4.3** (Connected matching set and assortative matching). *Suppose all matching sets  $\mathbb{I}$  and  $\mathbb{J}$  are convex, and  $\mathbb{M}$  is connected. If ages  $(I, J)$  (ages  $(0, 0)$ ) match, there is upper (lower) PAM. If  $(0, J)$  and  $(I, 0)$  match there is NAM.*

*Proof:* Suppose by contradiction that there exist  $i' > i$  and  $j' > j$  such that  $(i, j') \in \mathbb{M}$  and  $(i', j) \in \mathbb{M}$ , but  $(i', j') \notin \mathbb{M}$ . By convexity of  $\mathbb{J}(i')$  and  $\mathbb{I}(j')$ , for all  $j'' > j'$ ,  $(i', j'') \notin \mathbb{M}$  and for  $i'' > i'$ ,  $(i'', j') \notin \mathbb{M}$ . But this means that  $\mathbb{M}$  can be covered by two disjoint open sets,  $(i', I] \times (j', J]$  and  $([0, I] \times [0, J]) \setminus ([i', I] \times [j', J])$ , contradicting that it is connected. The proofs for lower PAM and for NAM are identical.  $\square$

This is equivalent to Shimer and Smith's approach: since  $S(i, j)$  in the present model is continuous, the preimage of  $\mathbb{R}_0^+$  under  $S$ , that is  $\mathbb{M}$ , is closed. If all matching sets are closed and convex, a connected  $\mathbb{M}$  with matching in the corners  $((0, 0)$  and  $(I, J)$  or  $(0, J)$  and  $(I, 0)$ ) is equivalent to nonempty  $\mathbb{I}$  and  $\mathbb{J}$ . But it will become clear shortly that nonemptiness/connectedness is here an actual restriction, unlike in Shimer and Smith (where symmetry ensures that all types match at least with themselves).

The second pattern of interest is the difference in marriage age between men and women. In a typical marriage market, a man's wife may be his age or considerably younger than him, but is rarely much older, whereas the opposite is true for women and their spouses. Consequently, the bulk of marriages in figure 1 lie to the upper left of the 45-degree line. The following definition is meant to capture this asymmetry (see also figure 2, center panel). A market in which women tend to marry younger men can be modeled by simply relabeling the two market sides.

**Definition 4.4** (Differential age matching). Consider ages  $a$  and  $b$  such that  $0 \leq a \leq b \leq I, J$ . There is differential age matching (DAM) if  $b \in \mathbb{I}(a)$  implies  $b \in \mathbb{J}(a)$ .  $\diamond$

When the matching sets are convex and there is positive assortative matching, differential age matching means that a woman at age  $a$  is considering an older set of men acceptable for marriage than a man at age  $a$ .

**Lemma 4.5.** *Suppose there is PAM and each matching set is an interval. Denote  $\mathbb{I}(a) = [i_1, i_2]$  and  $\mathbb{J}(a) = [j_1, j_2]$  for some  $0 \leq a \leq I, J$ . Then there is DAM if and only if  $j_1 \geq i_1$  and  $j_2 \geq i_2$  for any  $a$ , that is, the woman's set of partners is "older" than the man's.*

*Proof:* Assume first that  $j_2 < a$ . Then DAM implies  $a \in \mathbb{I}(j_2)$ . But PAM means that then  $(a, a)$  marry as well, so  $a \in \mathbb{J}(a)$  and therefore  $a \leq j_2$ . Now suppose  $i_2 > j_2$ . Since  $i_2 \geq a$  and  $i_2 \notin \mathbb{J}(a)$ , DAM implies that  $i_2 \notin \mathbb{I}(a)$  either, contradiction. Similarly, suppose  $i_1 > a$ . Then DAM and PAM imply  $a \in \mathbb{I}(i_1)$  and therefore  $a \in \mathbb{I}(a)$ , so it must be that  $i_1 \leq a$ . Now let  $i_1 > j_1$ ;  $j_1 < a$  and  $a \in \mathbb{I}(j_1)$  means that  $a \in \mathbb{J}(j_1)$  or  $j_1 \in \mathbb{I}(a) = [i_1, i_2]$ , contradiction (note that convexity is not used in this part of the proof). Conversely, suppose women's partners are older than men's, and consider  $b > a$  such that  $a \in \mathbb{J}(b) = [j_1, j_2]$ . Since  $b > a \geq j_1 \geq i_1$ , PAM implies that  $b \in \mathbb{I}(b)$ , and by convexity  $a \in \mathbb{I}(b)$ .  $\square$

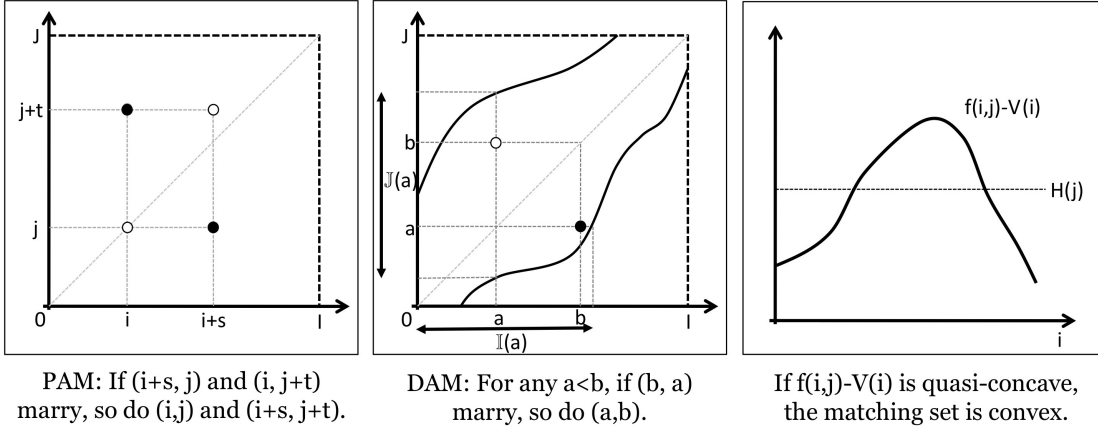
In connection with DAM, it was noted in the introduction that the marriage age of men is in almost all countries higher than that of women. Indeed, DAM is linked with a positive age difference at marriage. For instance, if the distributions  $m$  and  $w$  were identical, men marrying later than women would be equivalent to DAM (but of course the matching asymmetry will typically lead to different  $m$  and  $w$ ). More generally, DAM implies a positive average age gap if  $w(a)m(b) \geq w(b)m(a)$  for all  $a < b$ .<sup>10</sup> However, my definition of differential age matching deliberately does not use average ages. Instead, equivalent to the definition of sorting, I provide a distribution-free characterization that corresponds with observed marriage age patterns (compare with figure 1). But in many cases DAM will also be accompanied by a positive average age difference between spouses.

#### 4.1. CONVEX MATCHING SETS

As outlined earlier, the quest for assortative matching and DAM begins with conditions that guarantee convex matching sets. Interval matching is not only a step towards the main results but also an interesting feature of marriage patterns in its own right, possibly useful for example in the econometric identification of marriage behavior. Most of the discussion

<sup>10</sup>To see this note that the difference in average marriage age is a multiple of  $\int \int (j-i)\alpha(i,j)w(i)m(j)di dj$ . This integral is greater than or equal to zero on the set of  $i, j$  s.t.  $\alpha(a,b) = \alpha(b,a)$ . But  $\alpha(a,b) > 0$  and  $\alpha(b,a) = 0$  can occur only if  $a < b$ , so the term as a whole must be positive.

FIGURE 2.



will again focus on only one side of the market, with the understanding that the results hold in the same manner for the other side.

Consider a man at age  $j$ . The key observation is that, since his outside option  $H(j)$  is fixed, the shape of the matching set depends on  $f(i, j) - V(i)$  as a function of  $i$ . If  $f - V$  is greater than  $H$  on an interval of ages  $i$ , the matching set of  $j$  will be that interval. This is the case if the net surplus of marriage  $S(i, j) = f(i, j) - V(i) - H(j)$  is quasi-concave in  $i$  (see figure 2, right panel). The following conditions (A $\cdot$ ) and (X $\cdot$ ) provide a weak version of concavity/convexity of  $S(i, j)$  and will be used in several different places in what follows. Throughout this section, I assume that  $f$  is twice differentiable almost everywhere.

$$(A1) \quad f_{ii}(i, j) \leq \lambda_w \theta \min_y \{f_i(i, j) - f_i(i, y)\} + \rho f_i(i, j)$$

$$(A2) \quad f_{jj}(i, j) \leq \lambda_m (1 - \theta) \min_x \{f_j(i, j) - f_j(x, j)\} + \rho f_j(i, j)$$

**Claim 1:** Suppose that for all  $i$  on an interval  $(i_1, i_2)$ , (A1) holds for any  $\lambda_w$ , and  $f_i(i, j) - \dot{V}(i) \leq 0$ . Then  $f_i(i_2, j) - \dot{V}(i_2) \leq f_i(i_1, j) - \dot{V}(i_1)$ .

Equation (A1) is an assumption on the curvature of  $f$  in  $i$ . The first term of the right-hand side depends on the variation of  $f_i$  in  $j$ . It must be (weakly) negative, so that for  $\rho \rightarrow 0$  the condition is a strengthening of concavity of  $f$  in  $i$ . The claim essentially says that  $f - V$  will be concave in  $i$  whenever it is decreasing, as long as  $f$  itself is “concave



enough” in  $i$ . This should be understood in relative terms:  $f_{ii}$  may be positive if  $\rho$  and  $f_i$  are large and  $f_i$  varies only little in  $j$ . (A2) is the equivalent of (A1) for the men’s side.

One can similarly define

$$(X1) \quad f_{ii}(i, j) \geq \lambda_w \theta \max_y \{f_i(i, j) - f_i(i, y)\} + \rho f_i(i, j)$$

$$(X2) \quad f_{jj}(i, j) \geq \lambda_m (1 - \theta) \max_x \{f_j(i, j) - f_j(x, j)\} + \rho f_j(i, j)$$

**Claim 2:** Suppose that for all  $i$  on an interval  $(i_1, i_2)$ , (X1) holds for any  $\lambda_w$ , and  $f_i(i, j) - \dot{V}(i) \geq 0$ . Then  $f_i(i_2, j) - \dot{V}(i_2) \geq f_i(i_1, j) - \dot{V}(i_1)$ .

(X1) is the counterpart to (A1). Now strong enough *convexity* of  $f$  ensures that  $f - V$  is *convex* (with the same caveat as before, since  $f_{ii}$  may still be negative). (X2) is again the same condition for the men’s side.<sup>11</sup> The next two propositions use (A1) and (X1) as part of conditions under which  $f(i, j) - V(i)$  is either increasing or decreasing for all  $i$ , respectively, so that the matching sets are intervals including the boundary 0 or  $I$ .

**Proposition 4.6** (Increasing  $S(i, j)$ ). *Suppose that*

$$(IN1) \quad f_i(i, j) \geq \rho \max\{0, \max_{[i, I] \times [0, J]} f(x, y)\}$$

*holds for  $I$ , and either (A1) or (IN1) holds almost everywhere on  $[0, I]$ . Then  $j$ ’s matching set is a closed interval including  $I$ . Similarly, if*

$$(IN2) \quad f_j(i, j) \geq \rho \max\{0, \max_{[0, I] \times [j, J]} f(x, y)\}$$

*is true for  $J$  and (A2) or (IN2) holds almost everywhere on  $[0, J]$ , then  $i$ ’s matching set is a closed interval including  $J$ .*

*Proof:* Fix  $j$ . The right-hand side of equation (IN1) is an upper bound for  $\dot{V}(i)$ , and therefore  $f - V$  is increasing wherever (IN1) holds, including at  $I$ . Assume by contradiction that  $f(i, j) - V(i)$  is not increasing everywhere and let  $i_1$  be such that  $f_i(i_1, j) - \dot{V}(i_1) < 0$ . By continuity there must be  $i > i_1$  such that  $f_i(i, j) - \dot{V}(i) = 0$ . Let  $i_2$  be the smallest such  $i$ ; for all  $i \in [i_1, i_2)$   $f_i(i_1, j) - \dot{V}(i_1) < 0$ , so that (A1) holds and claim 1 applies. Thus we have  $f_i(i_1, j) - \dot{V}(i_1) \geq f_i(i_2, j) - \dot{V}(i_2) = 0$ .  $\square$

<sup>11</sup>The equation labels are a mnemonic for concAve and conveX.

(IN1) “anchors”  $f_i - \dot{V}$  below zero at  $I$ , and (IN1)/(A1) restrict its sign or slope so that it can never be positive for smaller ages  $i$ . Together they imply that the payoff function  $f$  is increasing in  $i$  everywhere. Both (IN1) or (A1) may hold while the other is violated, as long as  $\rho$  is positive. However, as  $\rho \rightarrow 0$ , (A1) is more restrictive, because in the limit (IN1) is satisfied whenever (A1) holds (since then  $f_{ii} \leq 0$  and  $f_i(I, j) \geq 0$ ).

**Proposition 4.7** (Decreasing  $S(i, j)$ ). *Suppose that*

$$(DE1) \quad f_i(i, j) \leq -\lambda_w \theta \max\{0, \max_y [f(i, y)]\}$$

*holds for  $I$  and either (DE1) or (X1) holds almost everywhere on  $[0, I]$ . Then an age  $j$  man’s matching set is a closed interval starting at 0. If*

$$(DE2) \quad f_j(i, j) \leq -\lambda_m (1 - \theta) \max\{0, \max_x [f(x, j)]\},$$

*holds for  $J$  and either (DE2) or (X2) holds almost everywhere on  $j \in [0, J]$ , then the matching set of age  $i$  is a closed interval starting at 0.*

The proof works like the one for the previous proposition, but using claim 2. By (X1)  $f$  is convex whenever it is weakly increasing, and (DE1) anchors  $f_i$  below zero, so the payoff function must be falling everywhere when these conditions hold. Again, (X1) is most interesting when  $\rho$  is bounded away from zero.

The next proposition uses (A1) and (A2) in a much more important role, by combining them with proposition 4.7 to arrive at conditions for convex matching sets where the age boundaries are not necessarily included.

**Proposition 4.8** (Single-peaked  $S(i, j)$ ). *Suppose that there exist  $i_1$  and  $i_2$  so that (A1) is true for all  $i \in [i_1, i_2]$ , (IN1) holds below  $i_1$ , and the conditions of proposition 4.7 for the women’s side apply for  $(i_2, I]$ . Then  $j$ ’s matching set is a closed interval. Equivalently, if (IN2) holds below some  $j_1$ , (A2) holds on  $[j_1, j_2]$ , and the conditions of 4.7 for the men’s side apply on  $(j_2, J]$ , then  $i$ ’s matching set is a closed interval.*

Above  $i_2$ , this proposition simply repeats proposition 4.7, so that  $S(i, j)$  is decreasing. Below  $i_1$ , the marriage surplus must be increasing thanks to (IN1). In between, claim 1

can be applied to show that whenever  $f(i, j) - V(i)$  is strictly decreasing, it will also be decreasing at all consecutive ages up to  $i_2$ , implying that  $S(i, j)$  is a single-peaked function in  $i$ . Here (A1) unfolds its true power, as it is used independently of (IN1) and  $f$  may be increasing or decreasing in  $i$ .

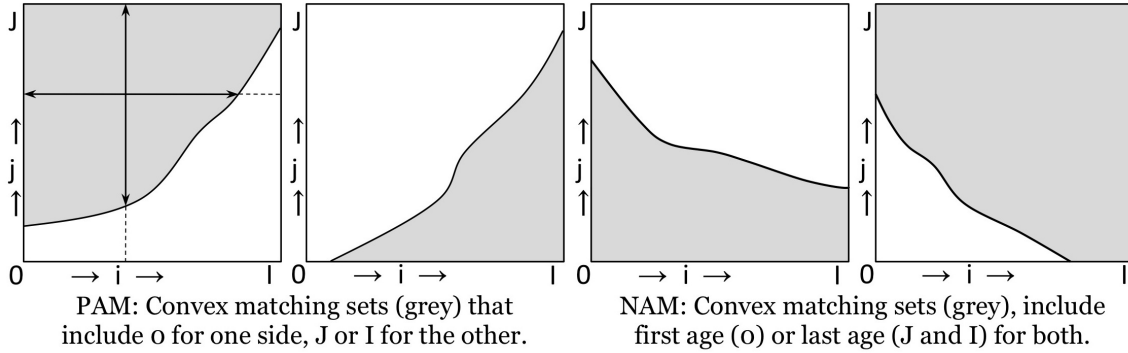
How do the conditions for matching set convexity compare with those in Shimer and Smith (2000), namely supermodularity (submodularity) of  $f$  and log supermodularity (submodularity) of  $f_i$  and  $f_{ij}$ ? Even though the logic of the two models is similar, the conditions turn out to be independent from each other. To illustrate, let  $i, j \in [0, 1]$  and consider the symmetric function  $a(i + j) + b(i^2 + j^2) - cij$  with  $a, b, c > 0$  (ages  $i$  and  $j$  correspond to types  $x$  and  $y$  in Shimer and Smith (2000)). The cross derivative is  $-c$ , so that  $i$  and  $j$  are substitutes and  $f$  is submodular, whereas it is easily verified that  $f_i$  is strictly log supermodular, violating Shimer and Smith. But condition (A1) holds for all ages whenever  $a$  is greater than  $\frac{2}{\rho}b + \left(1 + \frac{\lambda_w \theta}{\rho}\right)c$ , so proposition 4.8 applies.

On the other hand, if  $b < 0$  instead, all three submodularity conditions are satisfied. Now suppose  $a \in (2|b|, 2|b| + c)$ ;  $f$  is increasing in  $i$  for small  $i$  and  $j$ , but at high  $i$  it is falling on a range of  $j$ , so neither proposition 4.7 nor 4.6 applies. Moreover, for small  $i$  and  $j$  the inequality (A1) in proposition 4.8 is violated as well.

Intuitively, the difference between the two models is how other types/ages enter the value function of a given type or age. In Shimer and Smith (2000), different types  $x$  and  $y$  are distinguished *only* by the production function  $f(x, y)$ . For example, with a new production function  $g(x, y) = f(1 - x, 1 - y)$  the economy would still be the same, albeit with relabeled types. Agents of different types are only connected by the fact that they are searching in the same market. The super- or submodularity conditions in Shimer and Smith (2000) are therefore needed to determine the change in the value of search – and thereby in matching behavior – between neighboring types.

By contrast, there is a natural link between different ages. Any  $i' > i$  enters the value function of a woman at age  $i$ , because it is part of her future. The model is fundamentally asymmetric in that old and young cannot be interchanged. Formally, this is what makes  $V$  a differential equation (as opposed to a single number that represents the search value for

FIGURE 3. Propositions 4.6 and 4.7 and assortative matching.



each type), and it makes it possible to derive the shape of  $f(i, j) - V(i)$  from conditions on  $f_i$  and  $f_{ii}$  ( $f_j$  and  $f_{jj}$ ).

#### 4.2. ASSORTATIVE MATCHING

With convexity of the matching sets in place, when will there be sorting by age? First off, observe that the conditions in prop. 4.6 for an increasing match surplus  $S(i, j)$  on both sides of the market guarantee NAM. If ages  $i$  and  $j$  match with each other, then  $j$  must match with  $i + s$  as well and  $i$  must match with  $j + t$ , because their matching sets are intervals ending at  $I$  and  $J$ , respectively (see last panel of figure 3). At the same time, the conditions also imply upper PAM if they hold for just one side of the market: if  $i + s$  has a matching set that includes  $J$  and  $i + s$  and  $j$  match, then so do  $i + s$  and  $j + t$  (and vice versa). A parallel argument applies to the conditions in prop. 4.7; if they hold for both sides there is NAM (third panel of fig. 3), and if they hold for one side there is lower PAM.

Similarly, suppose the surplus function is decreasing in  $i$  only, so that the men's matching sets are convex and include 0 (proposition 4.7 holds for the women's side). Assume that  $f(I, J) \geq \underline{V} + \underline{H}$  and the women's matching sets are convex. Then there will be PAM again, since  $J$  marries  $I$  and therefore all  $i$ , and convexity means that the women's matching sets are intervals including  $J$ . The equivalent argument holds if the women's matching sets are intervals including zero instead.

We could stop here, but the requirements of propositions 4.6 and 4.7 are relatively demanding and allow only for quite specific matching patterns. I would like to take the

approach mapped out earlier and apply lemma 4.3. To do so I need a connected matching set.

**Lemma 4.9** (Connected matching set). *Suppose  $f(i, j)$  is nonincreasing in  $i$  for all  $(i, j)$  and  $f(I, j) - \underline{V} - H(j) \geq 0$  for some  $j$  (so that  $\mathbb{J}(I)$  is nonempty). Then all  $i$  have nonempty matching sets, and  $\mathbb{M}$  is connected. If  $f(i, j)$  is decreasing in  $j$  and  $\mathbb{I}(J)$  is nonempty, then all  $j$  have nonempty matching sets and  $\mathbb{M}$  is connected.*

This result stems from the observation that the surplus from any  $i$ 's best match forms an upper bound to the change in her outside option at that point. All ages in the vicinity of  $i$  will therefore also match with this best match. Their matching sets are nonempty, and the union of those sets is a (path)connected set. Now I can state

**Proposition 4.10** (Positive assortative matching). *There is PAM if*

- (1) *the conditions in proposition 4.6 hold for one market side and those for 4.7 hold for the other; or*
- (2) *the conditions in prop. 4.7 hold for one market side, those in 4.8 hold for the other, and  $f(I, J) - \underline{V} - \underline{H} \geq 0$ , so that  $(I, J)$  match; or*
- (3) *all matching sets are convex,  $f(i, j)$  is decreasing in  $i$  (or  $j$ ),  $(I, J)$  match, and so do  $(0, 0)$ , for example because the conditions of proposition 4.7 hold for  $i = 0$  (or  $j = 0$ ).*

(1) and (2) follow from the earlier discussion. Both imply that one market side's matching sets include age 0, and the other side's include  $I$  or  $J$ ; the two left panels of figure 3 illustrate this situation. (3) is an application of lemmas 4.3 and 4.9.

A similar argument to (2) for NAM is not available, as it would have to be ensured for example that  $(0, J)$  always match, but at the same time  $f$  would have to be increasing in  $i$ . For (3) there is a similar difficulty, but there is a special case: suppose that  $\underline{V} = \underline{H} = 0$ , and  $f(0, J) = f(I, 0) = \bar{f}$ . Since  $V(0)$  and  $H(0)$  are bounded by  $\bar{f}$ , there is matching in the corners.

**Proposition 4.11** (Negative assortative matching). *There is NAM if*

- (1) *the conditions in propositions 4.6 or 4.7 are satisfied for both market sides (so that all matching sets include 0, or they all include J or I, as in panels 3 and 4 of figure 3); or*
- (2) *all matching sets are convex,  $f(i, j)$  is weakly decreasing in  $i$  or  $j$ ,  $\underline{V} = \underline{H} = 0$ , and  $f(0, J) = f(I, 0) = \bar{f}$ .*

Note that the second set of conditions implies that either  $f(0, j) = \bar{f}$  for all  $j$  or  $f(i, 0) = \bar{f}$  for all  $i$ .

The last part of the proposition shows that the constraints on matching in the corners and the requirement of a connected matching set can be quite demanding in the context of age preferences.<sup>12</sup>

#### 4.3. THE AGE DIFFERENCE BETWEEN COUPLES

Lastly, I would like to find a set of conditions that is sufficient for DAM (and compatible with the conditions for positive assortative matching). Just as before some conditions will immediately give us the desired result: Suppose that the inequalities in 4.7 hold for the women's side (so that men's matching sets are intervals including zero), and either the conditions in 4.6 hold for the men's side or those in 4.8 hold and  $I \in \mathbb{I}(J)$  (so that all women's matching sets are convex and include  $J$ ). It is easy to verify that DAM is satisfied (as is PAM; compare with proposition 4.10 and the first panel of figure 3).

However, there is also another road to DAM. Take any two ages  $0 \leq a < b \leq I, J$ . Differential age matching is guaranteed whenever  $f(b, a) - V(b) - H(a) \geq 0$  implies  $f(a, b) - V(a) - H(b) \geq 0$ . That is the case if  $f(a, b) - f(b, a) - (V(a) - V(b)) \geq H(b) - H(a)$ , or equivalently either

$$\int_a^b f_j(b, y) - \dot{H}(y) dy \geq \int_a^b f_i(x, b) - \dot{V}(x) dx \quad \text{for all } x, y < b, \text{ or}$$

$$\int_a^b f_j(a, y) - \dot{H}(y) dy \geq \int_a^b f_i(x, a) - \dot{V}(x) dx \quad \text{for all } x, y > a.$$

---

<sup>12</sup>Compare this once again to the symmetric setup of Shimer and Smith (2000), where each type  $x$  always matches with itself (or with  $1 - x$  in the negative sorting case). This means that there is automatically matching in the corners, and the matching set is always connected.

This is intuitive: given a spouse of the same age, it implies that the surplus from marriage falls faster (or rises less fast) in the woman's age than in the man's. Then the surplus between an old man and a young woman is higher than between a young man and old woman at the same ages. Now, if it is known that  $f_j(c, \cdot) - \dot{H}(\cdot) \geq \delta$  and  $f_i(\cdot, c) - \dot{V}(\cdot) \leq \delta$  for some  $\delta$  and all possible  $i, j > c$  (or all  $i, j < c$ ) there must be differential age matching. The next lemma guarantees just that. For given age  $c$ , consider the following conditions:

$$(DE1') \quad f_i(i, c) - \delta \leq -\lambda_w \theta \max\{0, \max_y [f(i, y)]\}$$

$$(X1') \quad f_{ii}(i, c) \geq \max\{0; \lambda_w \theta [f_i(i, c) - \delta - \min_y f_i(i, y)]\} + \rho(f_i(i, c) - \delta)$$

$$(IN2') \quad f_j(c, j) - \delta \geq \rho \max\{0, \max_{[0, I] \times [j, J]} f(x, y)\}$$

$$(A2') \quad f_{jj}(c, j) \leq \min\{0; \lambda_m(1 - \theta)[f_j(c, j) - \min_x f_j(x, j) - \delta]\} + \rho(f_j(c, j) - \delta).$$

Comparing them with (DE1), (X1) etc. shows that they are the same, except for an additional “wedge” of  $\delta$  added to the first derivatives. However,  $\delta$  can be positive or negative, so that the new equations may be more or less demanding than the old ones (and of course each of the conditions needs to hold only on a subset of  $(i, j)$ ). For example, unlike under the original conditions  $f$  may now be increasing in  $i$  or decreasing in  $j$ .

**Lemma 4.12.** *Let  $0 \leq c \leq I, J$ .*

- (1) *Suppose there exists  $\delta_c$  such that (DE1') holds at  $I$  and either (DE1') or (X1') are satisfied for all other  $i \in [c, I]$ ; and (IN2') holds at  $J$  and either (IN2') or (A2') is satisfied for all  $j \in [c, J]$ . Then  $f_j(c, j) - \dot{H}(j) \geq \delta_c \geq f_i(i, c) - \dot{V}(i)$  for all  $i, j \geq c$ .*
- (2) *Alternatively, suppose (DE1') holds at  $i = c$  and either (DE1') or (X1') are satisfied for all other  $i \in [0, c]$ ; and (IN2') holds at  $c$  and either (IN2') or (A2') is satisfied for all  $j \in [0, c]$ . Then  $f_j(c, j) - \dot{H}(j) \geq \delta_c \geq f_i(i, c) - \dot{V}(i)$  for all  $i, j < c$ .*

Note that just as in propositions 4.6 and 4.7, (X1') implies (DE1') on  $[a, I]$  (or  $[0, b)$ ) and (A2') implies (IN2') on  $[a, J]$  (or  $[0, b)$ ) as  $\rho \rightarrow 0$ . Now I can state:

**Proposition 4.13** (Differential age matching). *There is DAM if either*

- (1) *for any age  $c$  there is  $\delta_c$  so that one set of conditions in lemma 4.12 holds, or*

(2) *the inequalities in proposition 4.7 hold for the women's side and either the conditions in 4.6 hold for the men's side or those in 4.8 hold and  $I \in \mathbb{I}(J)$  (so that the women's matching sets include  $J$  and the men's include 0).*

Importantly, both sets of conditions in proposition 4.13 are compatible with PAM. The conditions in (2) are in fact sufficient for PAM (see prop. 4.10 (1) and (2)). Those in (1) may be combined with the conditions in proposition 4.8 to obtain convex matching sets. With  $\delta$  less than zero  $f$  may be decreasing in  $j$ , so that the matching set is connected and every  $\mathbb{I}(j)$  is nonempty.

**Example:** Assume  $\rho = 0$  and  $\underline{V} = \underline{H} = 0$ . Recall the payoff function from the introduction

$$f(i, j) = \Pi\beta e^{-\gamma i} \int_0^T e^{-(\gamma+\beta)t} dt + b(I - i)$$

with  $T = I - i$  if  $i > j$  and  $T = J - j$  if  $j \geq i$ . There is both positive assortative matching and differential age matching if (1)  $\gamma + \beta \geq \bar{\lambda}_m$  and (2)  $\theta \rightarrow 0$ . (1) guarantees that (A2) holds everywhere, so the men's matching sets are convex. When  $\theta$  is close to zero, (DE1) holds as well, and the women's matching sets are convex and include  $j = 0$ . To see this, note that  $f_i \leq -b$  for all possible  $i, j$ , and

$$f_j(i, j) = \begin{cases} 0 & \text{if } i > j \\ -\Pi\beta e^{-\gamma i} e^{-(\gamma+\beta)(J-j)} & \text{if } i \leq j \end{cases} \quad f_{jj}(i, j) = \begin{cases} 0 & \text{if } i > j \\ (\gamma + \beta)f_j(i, j) & \text{if } i \leq j \end{cases}$$

Observe that all propositions continue to hold even though  $f(i, j)$  is not differentiable at  $i = j$ . The marriage surplus is quasi-concave in  $j$  if (A2) holds at all  $i \neq j$ . This is immediate for  $i > j$ , and for  $i > j$  we need

$$(\gamma + \beta)f_j(i, j) \leq \bar{\lambda}_m(1 - \theta)f_j(i, j)$$

or  $\gamma + \beta \geq \bar{\lambda}_m(1 - \theta)$ . Moreover, (DE1) is satisfied if  $b(1 - \bar{\lambda}_w\theta(I - i)) \geq \bar{\lambda}_w\theta\frac{1}{\gamma+\beta}F(i)[1 - e^{-(\gamma+\beta)J}]$ . Finally,  $f(I, J) = 0$ , so that  $I$  and  $J$  marry and case (2) of prop. 4.13 is satisfied.

This payoff function implies that PAM and DAM emerge if the woman's bargaining power is low and fertility rates are relatively high. Another implication is that traditional marriage age patterns may break down as fertility rates fall and bargaining becomes more



equal. Note that I have assumed that a man's and a woman's lifetime income are unaffected by marriage, and that the only payoff to the match comes from fertility and status or protection. Other possible sources of match payoff are companionship, mutual insurance, or gains from specialization within the partnership. The payoff function above is just one of many possible examples.  $\diamond$

## 5. CONCLUSION

This paper lays out a search model in which preferences depend on the age of both partners at the time of matching. This is most relevant for the marriage market, but can also apply e.g. in contexts of skill loss and acquisition.

The first result is a set of sufficient conditions on the slope and curvature of the marriage payoff function under which there is assortative matching by age. It is shown that super- or submodularity are not necessary for sorting by age, but the discussion highlights the link between assortative matching and nonempty matching sets. Nonemptiness turns out to be restrictive when the market is not symmetric. Conditions for assortative matching which do not rely on symmetry and nonempty matching sets remain an open question in the search literature at large.

In addition, I define differential age matching and show that DAM and PAM together imply that men match with a younger set of partners than women of the same age. I then provide conditions under which DAM must occur in equilibrium. These conditions stipulate that the marriage surplus declines faster in the woman's than in the man's age when their partner is the same age.

Together, the conditions for assortative matching and DAM characterize a set of payoff functions consistent with the prevailing age patterns in the marriage market. I provide a simple example which suggests that these patterns could be anchored in a trade-off between protection and fertility, a scenario that may break down as traditional modes of marriage become less important.

## OMITTED PROOFS

## PROOFS SECTION 3

**Derivation of the value function:** Let  $p(n, dt)$  denote the probability of meeting  $n$  potential marriage partners in period  $dt$ , and  $V_n(i+t)$  the expected value of such  $n$  meetings at the time of the first meeting  $i+t$ . We have

$$V_1(i+t) = \int_0^J \theta \max\{S(i+t, y), 0\} m(y) dy + V(i+t).$$

The value of search for a woman at age  $i$  is, for any  $dt > 0$ ,

$$V(i) = p(1, dt)E_t(e^{-\rho t}V_1(i+t)) + \sum_{n=2}^{\infty} p(n, dt)E_t(e^{-\rho t}V_n(i+t)) + \left(1 - \sum_{n=1}^{\infty} p(n, dt)\right) e^{-\rho dt}V(i+dt)$$

where  $\rho$  is the discount rate and the expectation is formed over possible meeting times  $0 < t < dt$ . In a Poisson process with parameter  $\lambda$ ,  $\frac{1}{dt} \sum_{n=2}^{\infty} p(n, dt)$  goes to zero and  $p(1, dt) \rightarrow \lambda$  as  $dt \rightarrow 0$ , so that

$$\dot{V}(i) = \lim_{dt \rightarrow 0} \frac{V(i+dt) - V(i)}{dt} = -\lambda_w(r)\theta \int_0^J \max\{S(i, y), 0\} m(y) dy + \rho V(i).$$

**Proof of Theorem 3.2:** The proof uses the following version of the Eilenberg-Montgomery fixed point theorem (Granas and Dugundji (2003), ch. 19, corollary 7.5):

**Theorem.** *Assume that  $K$  is an absolute retract. Then any compact acyclic map  $\mathcal{T}$  of  $K$  into itself has a fixed point.*

An acyclic map is a correspondence whose values are compact, upper hemi-continuous, and acyclic. The proof constructs a map  $\mathcal{T}$  described by the functions 3.1-3.2 and 3.3 and 3.5 and then verifies the conditions of the theorem.

**Notation** For any functions  $h(x)$  and  $g(x, y)$ , let  $\bar{h} = \sup_x h(x)$  and  $\bar{g} = \sup_{(x,y)} g(x, y)$  (on their respective domains). Similarly, let  $\underline{h} = \inf_x h(x)$  and  $\underline{g} = \inf_{(x,y)} g(x, y)$  respectively. These are taken to mean the max and min whenever they exist.

Let  $K$  be the product  $\mathcal{V} \times \mathcal{H} \times \mathcal{W} \times \mathcal{M}$  with function tuples  $k = (V, H, w, m)$  as elements.  $\mathcal{V}$  is the set of continuous functions on  $[0, I]$ , bounded below by zero and above by  $\bar{f}$ . Equivalently,  $\mathcal{H}$  are the sets of continuous functions on  $[0, J]$  bounded by zero and  $\bar{f}$ .  $\mathcal{W}$

$(\mathcal{M})$  are the continuous functions on  $[0, I]$  ( $[0, J]$ ), bounded by

$$\begin{aligned}\underline{w} &= \frac{1}{I}e^{-\bar{\lambda}_w I} \text{ and } \bar{\bar{w}} = \frac{\bar{\lambda}_w}{1 - e^{-\bar{\lambda}_w I}} \\ \underline{m} &= \frac{1}{J}e^{-\bar{\lambda}_m J} \text{ and } \bar{\bar{m}} = \frac{\bar{\lambda}_m}{1 - e^{-\bar{\lambda}_m J}}\end{aligned}$$

The mapping  $\mathcal{T}$  is defined as follows. Let  $\mathcal{T}[(V, H, w, m)] = (\hat{V}, \hat{H}, \hat{w}, \hat{m})$ .  $\hat{w}(i)$  and  $\hat{m}(j)$  equal the right-hand sides of (3.1) and (3.2);  $\hat{V}(i)$  and  $\hat{H}(j)$  are given by

$$\begin{aligned}\hat{V}(i) &= \max \left\{ 0, \lambda_w(r)\theta \int_i^I \int_0^J \max \{0, f(x, y) - V(x) - H(y)\} m(y) dy - \rho V(x) dx + \underline{V} \right\} \\ \hat{H}(j) &= \max \left\{ 0, \lambda_m(r)(1 - \theta) \int_j^J \int_0^I \max \{0, f(x, y) - V(x) - H(y)\} w(x) dx - \rho H(y) dy + \underline{H} \right\}\end{aligned}$$

For any given  $k$ , denote the set of  $(i, j)$  where  $f(i, j) - V(i) - H(j) = 0$  by  $A_k$  and let  $\bar{A}_k = [0, I] \times [0, J] \setminus A_k$ . For short, also define  $A_k(i) = A_k \cap ([0, i] \times [0, J])$  and  $\bar{A}_k(i) = \bar{A}_k \cap ([0, i] \times [0, J])$ . Finally, define  $\alpha(i, j)$  to be 0 whenever  $S(i, j) < 0$  and 1 if  $S(i, j) > 0$ , and let  $\alpha(A_k) = c$  be a constant between 0 and 1. In other words,  $\alpha$  describes a particular mixed strategy in which all couples who are indifferent between marrying and not marrying randomize in the same way. Note that  $\mathcal{T}(k)|_{\mathcal{V}} = \hat{V}$  and  $\mathcal{T}(k)|_{\mathcal{H}} = \hat{H}$  are singletons for any  $k$ .  $\mathcal{T}(k)|_{\mathcal{W}}$  and  $\mathcal{T}(k)|_{\mathcal{M}}$ , on the other hand, may be sets of functions  $\hat{w}$  and  $\hat{m}$  if  $A_k$  is a set with nonzero measure: in this case, some couples may choose mixed strategies. These sets are described by varying  $c$  continuously between 0 and 1.

**Step 1** Endow  $K$  with the norm implied by the metric  $d(k_1, k_2) = \max_{(f \in \{V, H, w, m\})} [\sup_x |f_1(x) - f_2(x)|]$ . Each of the function spaces constituting  $K$  is closed and convex, so that  $K$  inherits those properties. A convex subset of a normed linear space is an absolute retract.

**Step 2**  $\mathcal{T}$  is compact if its image is contained in a compact subset of  $K$ . It is sufficient to prove that the image of  $\mathcal{T}$  on each subspace of  $K$  is a bounded, equicontinuous set, and has therefore a compact closure by the Arzelà-Ascoli theorem<sup>13</sup>. The product  $C$  of these closures is compact (Tychonoff), and  $\mathcal{T}(K) \subset C$ . We therefore consider  $\hat{V}(i)$  and

<sup>13</sup>More precisely, by Arzelà-Ascoli a set of functions on a metric space is relatively compact if it is bounded and uniformly equicontinuous, and a relatively compact set has a compact closure. An equicontinuous set of functions is uniformly equicontinuous if its domain is compact (see Ok (2007) p. 262-64).

$\hat{w}(i)$  separately (with the equivalent proofs for  $\hat{H}(j)$  and  $\hat{m}(j)$  omitted). It is easily verified that they lie in  $\mathcal{V}$  and  $\mathcal{W}$  respectively, so we are left with proving equicontinuity. For any  $(V, H, w, m)$  we have

$$\begin{aligned} \left| \hat{V}(i_1) - \hat{V}(i_2) \right| &= \left| \int_{i_1}^{i_2} \lambda_w(r) \theta \int_0^J \max \{0, f(x, y) - V(x) - H(y)\} m(y) dy - \rho V(x) dx \right| \\ &\leq \bar{\lambda}_w \theta \bar{f} |i_1 - i_2| \end{aligned}$$

Therefore  $|i_1 - i_2| < \frac{1}{\bar{\lambda}_w \theta \bar{f}} \epsilon$  implies  $|\hat{V}(i_1) - \hat{V}(i_2)| < \epsilon$ , for all  $\hat{V} \in \mathcal{T}(K) \cap \mathcal{V}$ . Similarly,

$$\begin{aligned} |\hat{w}(i_1) - \hat{w}(i_2)| &= \hat{w}(i_1) \left| 1 - \exp \left( \int_{i_1}^{i_2} \lambda_w(r) \int_0^J \alpha(x, y) m(y) d(y) dy \right) \right| \\ &\leq \bar{w} |1 - \exp(-\bar{\lambda}_w |i_1 - i_2|)| \end{aligned}$$

This implies that  $|\hat{w}(i_1) - \hat{w}(i_2)| < \epsilon$  whenever

$$|i_1 - i_2| < \frac{1}{\bar{\lambda}_w} \ln \left( 1 - \frac{\epsilon}{\bar{w}} \right),$$

for all  $\hat{w} \in \mathcal{T}(K) \cap \mathcal{W}$ .

**Step 3** Consider any  $\hat{w}$  in  $\mathcal{T}(k)|_{\mathcal{W}}$ . There must be an  $\alpha$  and  $c$  such that

$$(A.1) \quad \hat{w}(i) = \frac{e^{-\lambda_w \left( \frac{w(0)}{m(0)} \right)} \left( \int_{\bar{A}_k(i)} \alpha(x, y) m(y) d(x, y) + \int_{A_k(i)} \alpha(x, y) m(y) d(x, y) \right)}{\int_0^I e^{-\lambda_w \left( \frac{w(0)}{m(0)} \right)} \left( \int_{\bar{A}_k(u)} \alpha(x, y) m(y) d(x, y) + \int_{A_k(u)} \alpha(x, y) m(y) d(x, y) \right) du}.$$

Observe that  $\alpha$  is uniquely defined to be either 1 or 0 on  $\bar{A}_k$ , and that  $\alpha(x, y) = c$  on  $A_k$  is unique as well (consider  $\hat{w}(0)$ ).  $\hat{w}$  is continuous in  $c$ . Now let  $g(i, j, t)$  be defined as follows:

$$g(i, j, t) = \begin{cases} (1-t)c + t & \text{if } (i, j) \in A_k \\ \alpha(i, j) & \text{else} \end{cases}$$

and define a function  $F : \mathcal{T}(k)|_{\mathcal{W}} \times [0, 1] \rightarrow \mathcal{T}(k)|_{\mathcal{W}}$  where  $F(\hat{w}, t)(i)$  is given by (A.1), except that  $\alpha(x, y)$  is replaced by  $g(x, y, t)$ . Then  $F(\hat{w}, 0) = \hat{w}$  and  $F(\hat{w}, 1)$  is identical for the entire set  $\mathcal{T}(k)|_{\mathcal{W}}$ . This means  $\mathcal{T}(k)|_{\mathcal{W}}$  is contractible and therefore acyclic, and the same argument can be made for  $\mathcal{T}(k)|_{\mathcal{M}}$ . Together this implies that  $\mathcal{T}$  has acyclic values.

**Step 4**  $\mathcal{T}$  is compact-valued: consider a convergent sequence  $\hat{k}^n \in \mathcal{T}(k)$  with limit  $\hat{k}$ . There exists a sequence  $c^n$  such that  $\alpha(A_k) = c^n$  for each  $\hat{k}^n$  and  $\alpha(A_k) = c$  for  $\hat{k}$ , given  $w, m$ . Moreover,  $c^n \rightarrow c$ . But  $c$  must converge to a point in  $[0, 1]$ , and this implies  $\hat{k} \in \mathcal{T}(k)$ .

**Step 5** Last I have to show that  $\mathcal{T}$  is upper hemicontinuous. Consider a sequence  $k^n \rightarrow k$  in  $K$  and let  $\hat{k}^n \in \mathcal{T}(k^n)$ . Since the closure of  $\mathcal{T}(K)$  is compact,  $\hat{k}^n$  has a convergent subsequence; w.l.o.g. let it be  $\hat{k}^n$ . I need to show that this limit point lies in  $\mathcal{T}(k)$ . First, let  $k^n$  such that  $d(k^n, k) = \delta$  and denote  $\hat{V}^n = \mathcal{T}(k^n)|_{\mathcal{V}}$  and  $\hat{V} = \mathcal{T}(k)|_{\mathcal{V}}$ . Now

$$\begin{aligned} \left| \hat{V}^n(i) - \hat{V}(i) \right| &\leq |\lambda_w(r^n) - \lambda_w(r)| I \theta \bar{f} \\ &\quad + I \bar{\lambda}_w \theta \bar{f} \int_0^J |m^n(y) - m(y)| dy \\ &\quad + \int_i^I \bar{\lambda}_w \theta \int_0^J m_2(y) |V(x) + H(y) - V^n(x) - H^n(y)| dy dx \\ &\quad + \rho \int_i^I |V^n(x) - V(x)| dx \\ &\leq \left| \lambda_w \left( \frac{w^n(0)}{m^n(0)} \right) - \lambda_w \left( \frac{w(0)}{m(0)} \right) \right| I \theta \bar{f} + [IJ \bar{\lambda}_w \theta \bar{f} + 2I \bar{\lambda}_w \theta + \rho I] \delta \end{aligned}$$

Since  $\lambda_w$  is continuous, for any  $\epsilon$  there exists a small enough  $\delta$  such that  $|\hat{V}_1(i) - \hat{V}_2(i)| < \epsilon$ .

Thus,  $\hat{V}^n \rightarrow \hat{V}$  and by a similar argument,  $\hat{H}^n \rightarrow \hat{H}$ .

In the same manner, let  $\hat{w}^n = k^n|_{\mathcal{W}}$  and  $\hat{w} \in \mathcal{T}(k)|_{\mathcal{W}}$ . Calling the numerator of (3.1)  $\phi(i)$  and the denominator  $\Phi$  for short, we have

$$\begin{aligned} |\hat{w}^n(i) - \hat{w}(i)| &= \left| \frac{\phi^n(i)}{\Phi^n \Phi} (\Phi - \Phi^n) + \frac{1}{\Phi} (\phi^n(i) - \phi(i)) \right| \\ &\leq \bar{w}^2 |\Phi - \Phi^n| + \bar{w} |\phi^n(i) - \phi(i)| \\ &\leq \left( \bar{w}^2 I + \bar{w} \right) \max_i |\phi^n(i) - \phi(i)| \end{aligned}$$

But  $\max_i |\phi^n(i) - \phi(i)|$  is at most

$$\max_i \left| 1 - \exp \left( \int_0^i \left( \lambda_w(r) \int_0^J \alpha(x, y) m(y) dy - \lambda_w(r^n) \int_0^J \alpha^n(x, y) m^n(y) dy \right) dx \right) \right|$$

The expression in the exponent is bounded by

$$|\lambda_w(r^n) - \lambda_w(r)| + \bar{\lambda}_w I J |m(y) - m^n(y)| + \int_0^I \int_0^J I_{\alpha(x, y) \neq \alpha^n(x, y)} m(y) dy dx.$$

Thus,  $\hat{w}$  is a limit point of  $\hat{w}^n$ , provided that the size of the set of  $(i, j)$  at which the signs of  $S^n(i, j)$  and  $S(i, j)$  differ shrinks to zero. But note that for any  $(i, j) \in \bar{A}_k$ , since  $S(i, j)^n \rightarrow S(i, j) \neq 0$ , there must be an  $n$  such that  $\text{sign}(S(i, j)^n) = \text{sign}(S(i, j))$ . If  $A_k$  has measure zero, we are done. Otherwise, for any  $(i, j) \in A_k$  there must be a convergent

sequence  $\alpha(i, j)^n \rightarrow \alpha$ , or else  $\hat{w}^n$  could not converge. But then there must be some  $\hat{w} \in \mathcal{T}(k)|_{\mathcal{W}}$  such that  $c = \alpha$  on  $A_k$ , and this  $\hat{w}$  is the limit point of  $\hat{w}^n$ . The same argument holds for  $\hat{\eta}^n$ .  $\square$

## PROOFS SECTION 4

**Proof of Claim 1 (p. 15):** Using the value function equation (3.3),

$$\begin{aligned}
\dot{V}(i_2) - \dot{V}(i_1) &= \lambda_w \theta \int_0^J [\alpha(i_1, y) - \alpha(i_2, y)] [f(i_1, y) - V(i_1) - H(y)] m(y) dy \\
&\quad + \lambda_w \theta \int_0^J \alpha(i_2, y) [V(i_2) - V(i_1) - (f(i_2, y) - f(i_1, y))] m(y) dy + \rho (V(i_2) - V(i_1)) \\
&\geq \lambda_w \theta \int_0^J \alpha(i_2, y) \left[ \int_{i_1}^{i_2} (\dot{V}(x) - f_i(x, y)) dx \right] m(y) dy + \rho \int_{i_1}^{i_2} \dot{V}(x) dx \\
&\geq \lambda_w \theta \int_0^J \alpha(i_2, y) \left[ \int_{i_1}^{i_2} (f_i(x, j) - f_i(x, y)) dx \right] m(y) dy + \rho \int_{i_1}^{i_2} f_i(x, j) dx \\
&\geq \int_{i_1}^{i_2} f_{ii}(x, j) dx = f_i(i_2, j) - f_i(i_1, j).
\end{aligned}$$

The first inequality holds because the terms  $[\alpha(i_1, y) - \alpha(i_2, y)]$  and  $[f(i_1, y) - V(i_1) - H(y)]$ , if not zero, must have the same sign under optimal matching behavior. The second inequality follows since for all  $x \in [i_1, i_2]$ ,  $\dot{V}(x) \geq f_i(x, j)$ . The last step comes from (A1) (note that  $\min_y \{f_i(i, j) - f_i(i, y)\} \leq 0$ ). But this means that  $f_i(i_2, j) - \dot{V}(i_2) \leq f_i(i_1, j) - \dot{V}(i_1)$ .  $\square$

**Proof of Claim 2 (p. 16):**

$$\begin{aligned}
\dot{V}(i_2) - \dot{V}(i_1) &= \lambda_w \theta \int_0^J [\alpha(i_1, y) - \alpha(i_2, y)] [f(i_2, y) - V(i_2) - H(y)] m(y) dy \\
&\quad - \lambda_w \theta \int_0^J \alpha(i_1, y) [f(i_2, y) - f(i_1, y) - (V(i_2) - V(i_1))] m(y) dy + \rho (V(i_2) - V(i_1)) \\
&\leq \lambda_w \theta \int_0^J \alpha(i_1, y) \left[ \int_{i_1}^{i_2} (\dot{V}(x) - f_i(x, y)) dx \right] m(y) dy + \rho \int_{i_1}^{i_2} \dot{V}(x) dx \\
&\leq \int_{i_1}^{i_2} f_{ii}(x, j) dx = f_i(i_2, j) - f_i(i_1, j),
\end{aligned}$$

and therefore  $f_i(i_1, j) - \dot{V}(i_1) \leq f_i(i_2, j) - \dot{V}(i_2)$ .  $\square$

**Proof of Proposition 4.7:**  $f_i(i, j) - \dot{V}(i) \leq 0$  whenever (DE1) holds. Suppose by contradiction that  $f(i, j) - V(i)$  is not decreasing everywhere and let  $i_1$  be such that  $f_i(i_1, j) - \dot{V}(i_1) > 0$ . Let  $i_2$  be the smallest  $i > i_1$  such that  $f_i(i, j) - \dot{V}(i) = 0$ . By claim 2  $f_i(i_1, j) - \dot{V}(i_1) \leq f_i(i_2, j) - \dot{V}(i_2) = 0$ .  $\square$

**Proof of Proposition 4.8:** Suppose  $f_i(i_1, j) - \dot{V}(i_1) < 0$ , but there exists an  $i > i_1$  such that  $f_i(i, j) - \dot{V}(i) > 0$ . There must be a point between  $i_1$  and  $i$  with  $f_i(i, j) - \dot{V}(i) = 0$ ; let  $i_2$  be the smallest of them. By claim 1,  $f_i(i_2, j) - \dot{V}(i_2) \leq f_i(i_1, j) - \dot{V}(i_1) < 0$ .  $\square$

**Proof of Lemma 4.9:** Define a function  $S^*(i) = \max_j f(i, j) - V(i) - H(j)$  for  $i \in [0, I]$  and constant at  $S^*(0)$  for all  $i < 0$ , with the arg max denoted by  $j^*(i)$ . Since  $f$ ,  $V$  and  $H$  are continuous, so is  $S^*$ . Assume that for some  $i''$ ,  $\mathbb{J}(i'')$  is nonempty. Now suppose there exists an  $i < i''$  such that  $S^*(i) < 0$ . There must be an  $i' \in (i, i'')$  such that  $S^*(x) \leq 0$  for all  $x \in [i, i']$  and  $S^*(i') = 0$ . But then

$$\begin{aligned} V(i') - V(i) &= -\lambda_w \theta \int_i^{i'} \int_0^J \alpha(x, y) [f(x, y) - V(x) - H(y)] m(y) dy dx + \rho \int_i^{i'} V(x) dx \\ &\geq -\lambda_w \theta \int_i^{i'} \max\{0, S^*(x)\} dx + \rho \int_i^{i'} V(x) dx \geq 0. \end{aligned}$$

Since  $f$  is decreasing in  $i$ , this means that  $f(i, j^*(i')) - V(i) - H(j^*(i')) \geq f(i', j^*(i')) - V(i') - H(j^*(i')) = S^*(i') = 0$ ; in other words,  $j^*(i')$  must be in the matching set of  $i$  (as well as all  $x$ ). The connecting path for  $\mathbb{M}$  is given by  $S^*(i) \times i$ .  $\square$

**Proof of Lemma 4.12:** Hold  $j$  fixed at  $a$ . (DE1') directly bounds  $f_i(i, a) - \dot{V}(i) \leq \delta$ . Suppose by contradiction that there is an  $i_1$  such that  $f_i(i_1, a) - \dot{V}(i_1) > \delta$ . Let  $i_2$  be the smallest  $i > i_1$  such that  $f_i(i, a) - \dot{V}(i) = \delta$ . Then  $\dot{V}(x) \leq f_i(x, a) - \delta$  for all  $x \in [i_1, i_2]$ . By an analog argument to that for claim 2,

$$\begin{aligned} \dot{V}(i_2) - \dot{V}(i_1) &\leq \lambda_w \theta \int_0^J \alpha(i_1, y) \left[ \int_{i_1}^{i_2} (\dot{V}(x) - f_i(x, y)) dx \right] m(y) dy + \rho \int_{i_1}^{i_2} \dot{V}(x) dx \\ &\leq \lambda_w \theta \int_0^J \alpha(i_1, y) \left[ \int_{i_1}^{i_2} (f_i(x, a) - \delta - f_i(x, y)) dx \right] m(y) dy + \rho \int_{i_1}^{i_2} (f_i(x, a) - \delta) dx \\ &\leq \int_{i_1}^{i_2} f_{ii}(x, a) dx = f_i(i_2, a) - f_i(i_1, a) \end{aligned}$$

and therefore  $f_i(i_1, a) - \dot{V}(i_1) \leq f_i(i_2, a) - \dot{V}(i_2) = \delta$ ; contradiction. Similarly, let  $i = a$ . (IN2') bounds  $f_j - \dot{H}$  above  $\delta$ . Suppose that there is an  $j_1$  such that  $f_j(a, j_1) - \dot{H}(j_1) < \delta$ . Let  $j_2$  be the smallest age above  $j_1$  at which  $f_j(a, j) - \dot{H}(j) = \delta$ . Then by a similar argument

$$\begin{aligned} \dot{H}(j_2) - \dot{H}(j_1) &\geq \lambda_m (1 - \theta) \int_0^I \alpha(x, j_2) \left[ \int_{j_1}^{j_2} (f_j(a, y) - \delta - f_j(x, y)) dy \right] w(x) dx + \rho \int_{j_1}^{j_2} (f_j(a, y) - \delta) dy \\ &\geq \int_{j_1}^{j_2} f_{jj}(x, a) dx = f_j(j_2, a) - f_j(j_1, a) \end{aligned}$$

and  $f_j(j_1, a) - \dot{H}(j_1) \geq f_j(j_2, a) - \dot{H}(j_2) = \delta$ , contradiction.  $\square$

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