

Moen, Espen R.; Yashiv, Eran

**Working Paper**

## Worker matching and firm value

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## **Worker Matching and Firm Value**

Espen R. Moen  
Eran Yashiv

June 2012

# Worker Matching and Firm Value

**Espen R. Moen**

*Norwegian School of Management*

**Eran Yashiv**

*Tel Aviv University  
and IZA*

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IZA

P.O. Box 7240  
53072 Bonn  
Germany

Phone: +49-228-3894-0

Fax: +49-228-3894-180

E-mail: [iza@iza.org](mailto:iza@iza.org)

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## **ABSTRACT**

### **Worker Matching and Firm Value**

This paper studies the value of firms and their hiring and firing decisions in an environment where the productivity of the workers depends on how well they match with their co-workers and the firm acts as a coordinating device. Match quality derives from a production technology whereby workers are randomly located on the Salop circle, and depends negatively on the distance between the workers. It is shown that a worker's contribution in a given firm changes over time in a nontrivial way as co-workers are replaced with new workers. The paper derives optimal hiring and replacement policies, including an optimal stopping rule, and characterizes the resulting equilibrium in terms of employment, wages and distribution of firm values. The paper stresses the role of horizontal differences in worker productivity, as opposed to vertical, assortative matching issues. Simulations of the model show the dynamics of worker replacement policy, the resulting firm value and age distributions, and the connections between them.

JEL Classification: E23, J62, J63

Keywords: firm value, complementarity, worker value, Salop circle, hiring, firing, match quality, optimal stopping

Corresponding author:

Eran Yashiv  
Naftali 052  
Tel Aviv University  
Tel Aviv 69978  
Israel  
E-mail: [yashiv@post.tau.ac.il](mailto:yashiv@post.tau.ac.il)

### 1 Introduction

How does the value of the firm depend on the value of its workers? When one considers firms that have little physical capital – such as IT firms, software development firms, investment banks and the like – the neoclassical model does not seem to provide a reasonable answer. The firm has some value that is not manifest in physical capital. Rather, Prescott and Visscher’s (1980) ‘organization capital’ may be a more relevant concept in this context. One aspect of the latter form of capital, discussed in that paper, is the formation of teams and this is the issue taken up in the current paper. We ask how workers affect each other in production and how this interaction affects firm value. The paper studies the value of firms and their hiring and firing decisions in an environment where the productivity of the workers depends on how well they match with their co-workers and the firm acts as a coordinating device. This role of the firm is what generates value.

In the model, match quality derives from a production technology whereby workers are randomly located on the Salop (1979) circle, and depends negatively on the distance between them. It is shown that a worker’s contribution in a given firm changes over time in a nontrivial way as co-workers are replaced with new workers. The paper derives optimal hiring and replacement policies, including an optimal stopping rule, and characterizes the resulting equilibrium in terms of employment, wages and distribution of firm values.

Key results are the derivation of an optimal worker replacement strategy, based on a productivity threshold that is defined relative to other workers. This strategy, interacted with exogenous worker separation and firm exit shocks, generates rich turnover dynamics. The resulting firm value distributions are found to be – using simulations – non-normal, with negative skewness and negative excess kurtosis. This shape reflects the fact that, as firms mature, there is a process of forming good teams on the one hand and the effects of negative separation and exit shocks on the other hand.

The paper stresses the role of horizontal differences in worker productivity, as opposed to vertical, assortative matching issues. The literature on the

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<sup>1</sup> We thank Russell Cooper, Rani Spiegler and seminar participants at the 2009 annual SED meetings in Istanbul, at the CREI, Barcelona November 2009 search conference, at the 2011 ESSIM meetings of the CEPR, at the 2011 SAM conference in Bristol, at the 2011 NBER RSW group meetings in Aarhus, and at Tel Aviv University for helpful comments on previous versions of the paper, the UCL Department of Economics for its hospitality, and Tanya Baron for excellent research assistance. All errors are our own.

latter – see for example the survey by Li (2008) and the recent contributions by Teulings and Gautier (2004), Shimer (2005) and Eeckhout and Kircher (2010, 2011) – deals with the matching of workers of different types, with key importance given to the vertical or hierarchical ranking of these types. These models are defined by assumptions on the information available to agents about types, the transfer of utility among workers (or other mating agents), and the particular specification of complementarity in production (such as supermodularity of the joint production function). In the current paper, workers are ex-ante homogenous, there is no prior knowledge about their complementarity with other workers before joining the firm, and there are no direct transfers between them.

The paper has some points of contact with other papers in the search literature. It shares some features with the search model of Jovanovic (1979a,b): there is heterogeneity in match productivity and imperfect information ex-ante (before match creation) about it; these features lead to worker turnover, with good matches lasting longer. But it has some important differences: the Jovanovic model stresses the structural dependence of the separation probability on job tenure and market experience. There is growth of firm-specific capital and of the worker’s wage over the life cycle.<sup>2</sup> In the current model the workers do not search themselves and firms do not offer differential rewards to their workers. But the Jovanovic model does not cater for the key issue here, namely that of worker complementarities. Burdett, Imai and Wright (2004) analyze models where agents search for partners to form relationships and may or may not continue searching for different partners while matched. Both unmatched and matched agents have reservation match qualities. A crucial difference with respect to the current set-up is that they focus on the search decisions of both agents in a bi-lateral match and stress the idea that if one partner searches the relationship is less stable, so the other is more inclined to search, potentially making instability a self-fulfilling prophecy. They show that this set-up can generate multiple equilibria. In the current paper we do not allow for the workers themselves to search but rather focus on the main issue, which is optimal team formation through search by firms.

The paper proceeds as follows: in Section 2 we outline the model. We describe the set up and delineate the interaction between workers. In Section 3 we derive the optimal hiring and firing policy and study the implications for firm value. In Section 4 we allow for exogenous worker separation Section 5 presents simulations of the model. Section 6 concludes.

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<sup>2</sup>Pissarides (2000, Chapter 6) incorporates this kind of model into the standard DMP search and matching framework, keeping the matching function and Nash bargaining ingredients, and postulating a reservation wage and reservation productivity for the worker and for the firm, respectively.

## 2 The Model

In this section we first describe the set-up of the firm and the production process (2.1). We then define worker interaction and the emerging state variables (2.2).

### 2.1 The Set-Up

The firm starts off with three workers with given productivity. Workers are located on the Salop (1979) circle, with their placement randomly drawn from a uniform distribution. Any new worker will be located with the same distribution. The worker's contribution to the firm's output depends negatively on the distance between her and the other two workers. Each period the firm faces an exogenous exit probability.

In each period the firm can replace at most one worker. It does so by first firing one of the existing workers without recall, and then sampling – from outside the firm – one worker. Thus, we do not allow the firm to compare the existing and the sampled worker and hire the more productive one. We rationalize this by assuming that it takes a period to learn a worker's productivity. Replacing a worker is costly.

This way of modelling aims at capturing properties that have been found in empirical micro-studies of team production. Hamilton, Nickerson and Owan (2003) find that teamwork benefits from collaborative skills involving communication, leadership, and flexibility to rotate through multiple jobs. Team production may expand production possibilities by utilizing collaborative skills. Turnover declined after the introduction of teams. A very recent study undertaken by MIT's Human Dynamics Laboratory, collected data from electronic badges on individual communications behavior in teams from diverse industries. The study, reported in Pentland (2012), stresses the huge importance of communications between members for team productivity. In describing the results of how team members contribute to a team as a whole, the report actually uses a diagram of a circle (see Pentland (2012, page 64)), with the workers placed near each other contributing the most. The findings are that face to face interactions are the most valuable form of communications, much more than email and texting, thereby emphasizing the role of physical distance.

### 2.2 Workers' Productivity and Interactions

The three workers are located on the unit circle. The one in the middle (out of the three) is the  $j$  worker who satisfies

$$\min_j \sum_{i=1}^3 d_{ij} \quad (1)$$

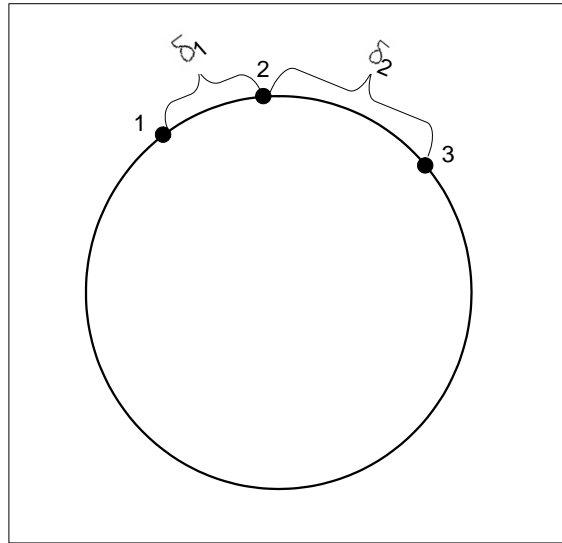
where  $d_{ij}$  is the distance between worker  $i$  and  $j$ . We shall define two state variables  $\delta_1, \delta_2$  as follows:

$$\delta_1 = \min_{i,j} d_{ij} \quad (2)$$

$$\delta_2 = \min_j d_{kj}, k \neq i^*, j^* \quad i^*, j^* = \arg \min_{i,j} d_{ij} \quad (3)$$

The first state variable  $\delta_1$  expresses the distance between the two closest workers. The second state variable  $\delta_2$  expresses the distance between the third worker and the closest of the two others.

The following figure illustrates:



**Figure 1: The State Variables**

The firm's task is to find what we refer to as a common ground for the three workers; in what follows we assume that the firm chooses the middle worker as the focal point and all distances are measured going via the middle worker. This concept of a "middle worker" corresponds to the finding of the afore-cited Pentland (2012) study, whereby the "ideal" team player may be called a 'charismatic connector,' serving to connect team mates with one another.



Every period, each worker works together with both co-workers to produce output. Production  $y_{ij}$  is negatively related to the distance  $d_{ij}$  :

$$y_{ij} = \frac{\tilde{y}}{3} - d_{ij} \quad (4)$$

The firm's total output is then given by the linear additive function:

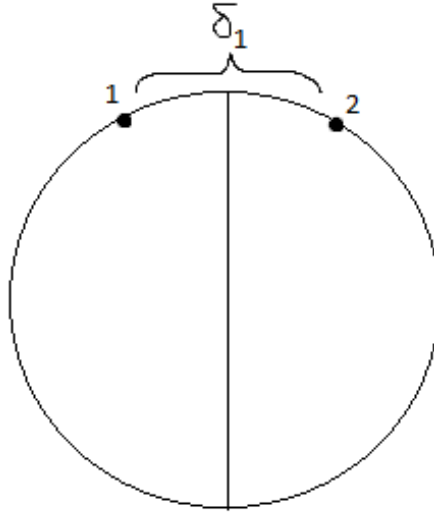
$$\begin{aligned} Y &= y_{12} + y_{13} + y_{23} \\ &= \tilde{y} - \sum_{i=1}^3 d_{ij} \\ &= \tilde{y} - 2(\delta_1 + \delta_2) \end{aligned} \quad (5)$$

We assume that wages are independent of match quality. This is consistent with a competitive market where firms bid for *ex ante* identical workers prior to knowing the match quality. The profits ( $\pi$ ) of the firm are then given by:

$$\begin{aligned} \pi &= Y - W \\ &= \tilde{y} - \sum_{i=1}^3 d_{ij} - W \\ &= y - \sum_{i=1}^3 d_{ij} \end{aligned} \quad (6)$$

where  $W$  is total wage bill and  $y$  is production net of wages ( $\tilde{y} - W$ ).

As already mentioned, the firm can replace up to one worker each period. It replaces the worker who is further away from the middle worker. The new values  $\delta'_1$  and  $\delta'_2$  are random draws from a distribution that depends on  $\delta_1$ . We write  $(\delta'_1, \delta'_2) = \Gamma\delta_1$ . Figure 2 illustrates, how, without loss of generality, workers 1 and 2, who are not replaced, are situated symmetrically around the north pole:



**Figure 2: Incumbent Workers**

From Figure 2 it follows that  $\Gamma$  can be characterized as follows:

1. With probability  $1 - 3\delta_1$ ,  $\delta'_1 = \delta_1$  and  $\delta'_2 \sim \text{unif}[\delta_1, \frac{1-\delta_1}{2}]$
2. With probability  $2\delta_1$ ,  $\delta'_1 \sim \text{unif}[0, \delta_1]$  and  $\delta'_2 = \delta_1$
3. With probability  $\delta_1$ ,  $\delta'_1 \sim \text{unif}[0, \delta_1/2]$  and  $\delta'_2 = \delta_1 - \delta'_1$

### 3 Optimal Hiring and Firing

Our aim in this section is to derive an optimal stopping rule for replacement. We show that an optimal stopping rule can be expressed in terms of the two state variables  $\delta_1$  and  $\delta_2$ .

Before we start, let us make a small detour, and consider the variable  $X = \sum_{i=1}^3 d_{ij}$ , the total distance between the workers. Optimal replacement does not imply a unique cut-off for  $X$ , as  $X$  is not a “sufficient statistic” for the value of replacement. To see why, note that the lowest possible distance after replacement is  $2\delta_1$ . Hence, the value of replacement depends negatively on  $\delta_1$  – the lower is  $\delta_1$ , the higher is the expected gain from one more round of replacement. Hence if a lower  $X$  comes together with a lower  $\delta_1$ , the incentives to replace may actually increase.

Consider instead an optimal stopping rule of the form: “stop searching if  $\delta_2 \leq \bar{\delta}_2(\delta_1)$ ”. Since, by definition,  $\delta_2 > \delta_1$  stopping can only take place if  $\delta_1 <$

$\bar{\delta}_2(\delta_1)$ . At the end of a period, the firm chooses whether or not to replace the more distant worker if  $\delta_2$  is above a threshold that may depend on  $\delta_1$ . We refer to this as the *ex post* stopping rule, when the decision is taken after the period's values of  $\delta_1$  and  $\delta_2$  are realized. An *ex ante* stopping rule is a rule of the form "search until  $\delta_2^t < \bar{\delta}_2^a(\delta_1^{t-1})$ " where  $t$  denotes time. Of course, as the decision is taken *ex post*, an *ex ante* optimal stopping rule may very well not exist. However, sometime it does.

We show this in our first preliminary result.

**Lemma 1** *Suppose  $\bar{\delta}_2(\delta_1)$  is an ex post stopping rule. Suppose  $\bar{\delta}_2(\delta_1)$  is strictly decreasing in  $\delta_1$ . Then for any  $\delta_1^t < \bar{\delta}_2(\delta_1)$ , the ex ante stopping rule "stop if  $\delta_2^{t+1} \leq \bar{\delta}_2^a(\delta_1^t)$ " and the ex post stopping rule "stop if  $\delta_2^{t+1} \leq \bar{\delta}_2(\delta_1^t)$ " gives rise to the same search behavior for all possible realizations of  $\delta_1^{t+1}$  and  $\delta_2^{t+1}$  (where the realizations of  $\delta_1^{t+1}$  and  $\delta_2^{t+1}$  depend on  $\delta_1^t$  as described above in the specification of  $\Gamma$ ).*

The proof is instructive, so it is included in the text.

**Proof.** Suppose  $\delta_1^{t+1} = \delta_1^t$ . Then there is no difference between the *ex ante* and the *ex post* decisions to be made, and they predict the same stopping behaviour. Suppose  $\delta_1^{t+1} \neq \delta_1^t$ . From the specification of  $\Gamma$  it follows that  $\delta_1$  cannot increase, hence  $\delta_1^{t+1} < \delta_1^t$ . Furthermore, it also follows that  $\delta_2^{t+1} \leq \delta_1^t$ . Hence in this case

$$\delta_2^{t+1} \leq \delta_1^t \leq \bar{\delta}_2(\delta_1^t) < \bar{\delta}_2(\delta_1^{t+1})$$

Hence in this case both the *ex ante* and the *ex post* stopping rule implies that the process stops. This completes the proof ■

We want to characterize the optimal stopping function  $\bar{\delta}_2(\delta_1)$  for  $\delta_1 \leq \bar{\delta}_2(\delta_1)$ . We assume that  $\bar{\delta}_2(\delta_1)$  is decreasing in  $\delta_1$ . Let  $\beta = \frac{1}{1+r}$  denote the discount factor and  $r$  the discount rate of the firm, possibly including an exit probability of the firm. Denote the value function of the firm by  $V(\delta_1, \delta_2)$ , and let  $\bar{V}(\delta_1) \equiv EV^{\delta_1, \delta_2'} V(\delta_1', \delta_2') | \delta_1$  (below we simply write this as  $E^{\delta_1} V(\delta_1', \delta_2')$ ). Then

$$\begin{aligned} V(\delta_1, \delta_2) &= \pi(\delta_1, \delta_2) + \beta \max[V(\delta_1, \delta_2), \bar{V}(\delta_1) - c] \\ &= y - 2(\delta_1 + \delta_2) + \max\left[\frac{y - 2(\delta_1 + \delta_2)}{r}, \frac{\bar{V}(\delta_1) - c}{1+r}\right] \end{aligned} \quad (7)$$

It follows directly from proposition 4 in Stokey and Lucas (1989, p. 522) that the value function exists. By definition the optimal stopping rule must satisfy

$$V(\delta_1, \bar{\delta}_2(\delta_1)) = \bar{V}(\delta_1) - c$$

Or (from 7)

$$\frac{y - 2(\delta_1 + \bar{\delta}_2(\delta_1))}{r} = \frac{\bar{V}(\delta_1) - c}{1 + r} \quad (8)$$

According to Lemma 1, the *ex post* stopping rule is also an *ex ante* stopping rule, hence the expected value of replacement is given by:

$$\begin{aligned} \bar{V}(\delta_1) &\equiv E^{|\delta_1}[y - 2(\delta'_1 + \delta'_2) + \Pr(\delta'_2 \leq \bar{\delta}_2(\delta_1))E^{|\delta_1, \delta'_2(\delta_1) \leq \bar{\delta}_2(\delta_1)} \frac{y - 2(\delta'_1 + \delta'_2)}{r} \\ &+ (1 - \Pr(\delta'_2 \leq \bar{\delta}_2(\delta_1))) \frac{\bar{V}(\delta_1) - c}{1 + r}] \end{aligned} \quad (9)$$

(where the dependence of the probabilities on  $\delta_1$  is suppressed). We will show that (9) can be expressed as

$$\begin{aligned} \bar{V}(\delta_1) &= y - \left(\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}\right) \\ &\quad + \frac{(\delta_1 + 2\bar{\delta}_2)y - 2\bar{\delta}_2(2\delta_1 + \bar{\delta}_2) - 2\delta_1^2}{r} \\ &\quad + (1 - \delta_1 - 2\bar{\delta}_2) \frac{\bar{V}(\delta_1) - c}{1 + r} \end{aligned} \quad (10)$$

The derivation goes as follows:

1. First we show that  $E^{|\delta_1}[y - 2(\delta'_1 + \delta'_2)] = y - (\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2})$ . The expected distance between the new worker and one of the existing workers is  $\frac{1 + \delta_1^2}{4}$ . To understand this, consider Figure 2. If the new worker is located on the left side of the circle (where worker 1 is the middle man), the expected distance to worker 1 is  $(1 - \frac{\delta_1}{2})(\frac{1}{2} - \frac{\delta_1}{2})/2 + \delta_1^2/8$ . If the new worker is located on the right side, the expected distance to worker 1 is  $\frac{\delta_1 + 1}{4}$ . The expected distance is

$$\frac{1}{2} \left[ \left(1 - \frac{\delta_1}{2}\right) \left(\frac{1}{2} - \frac{\delta_1}{2}\right) / 2 + \frac{\delta_1^2}{8} + \frac{\delta_1 + 1}{4} \right] = \frac{1 + \delta_1^2}{4}$$

The total expected distance between the workers is therefore  $\frac{1}{2} + \frac{\delta_1^2}{2} + \delta_1$  as claimed.

2. Then we show that

$$\Pr(\delta'_2 \leq \bar{\delta}_2(\delta_1)) E^{|\delta_1, \delta'_2(\delta_1) \leq \bar{\delta}_2(\delta_1)} \frac{y - 2(\delta'_1 + \delta'_2)}{r} = \frac{(\delta_1 + 2\bar{\delta}_2)y - 2\bar{\delta}_2(2\delta_1 + \bar{\delta}_2) - 2\delta_1^2}{r}$$

(i) With probability  $\delta_1 + 2\bar{\delta}_2$  the new worker is below the  $\bar{\delta}_2$  threshold. With probability  $\delta_1$  the new worker is between the two existing workers, in which case the total distance between the workers is  $2\delta_1$  and output is  $y - 2\delta_1$ .

(ii) With probability  $2\bar{\delta}_2$  the new worker is below the threshold but not between the two existing workers. The firm has a distance of  $\delta_1$  between existing workers and expects a distance (on average) of  $\frac{\bar{\delta}_2}{2}$  to the closest and  $\delta_1 + \frac{\bar{\delta}_2}{2}$  to the more distant of the existing workers, respectively. Output is thus  $y - 2\delta_1 - \bar{\delta}_2$ .

The per period expected gain from stopping is thus

$$\begin{aligned} & \delta_1(y - 2\delta_1) + 2\bar{\delta}_2(y - 2\delta_1 - \bar{\delta}_2) \\ &= (\delta_1 + 2\bar{\delta}_2)y - 2\bar{\delta}_2(2\delta_1 + \bar{\delta}_2) - 2\delta_1^2 \end{aligned}$$

Dividing by the discount factor gives the expression in (10).

3. Finally we show that

$$(1 - \Pr(\delta'_2 \leq \bar{\delta}_2(\delta_1))) \frac{\bar{V}(\delta_1) - c}{1+r} = (1 - \delta_1 - 2\bar{\delta}_2) \frac{\bar{V}(\delta_1) - c}{1+r}$$

This comes from the fact that with probability  $(1 - \delta_1 - 2\bar{\delta}_2)$  the new worker is above the  $\bar{\delta}_2$  threshold. The firm will keep replacing and pay the cost  $c$  again.

We have thus fully derived equation (10).

Let us write:

$$\begin{aligned} & (\delta_1 + 2\bar{\delta}_2)y - 2\bar{\delta}_2(2\delta_1 + \bar{\delta}_2) - 2\delta_1^2 \\ &= (\delta_1 + 2\bar{\delta}_2)(y - 2(\delta_1 + \bar{\delta}_2)) + 2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2 \end{aligned}$$

Hence we can re-write (10) as follows:

$$\begin{aligned} \bar{V}(\delta_1) &= y - \left(\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}\right) \\ &+ \frac{(\delta_1 + 2\bar{\delta}_2)(y - 2(\delta_1 + \bar{\delta}_2)) + 2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2}{r} \\ &+ (1 - \delta_1 - 2\bar{\delta}_2) \frac{\bar{V}(\delta_1) - c}{1+r} \end{aligned} \tag{11}$$

Substituting out  $\bar{V}(\delta_1)$  and using (8), gives the rule (see Appendix A for details):

$$c + \frac{1}{2} + \frac{\delta_1^2}{2} - \delta_1 - 2\bar{\delta}_2 = \frac{2\delta_1\bar{\delta}_2 + 2\bar{\delta}_2^2}{r} \tag{12}$$

This cut-off rule has a very intuitive interpretation:

*The LHS* of (12) represents net costs of replacing, evaluated at the threshold  $(\bar{\delta}_2)$ . If not replacing the worker, the total distance is given by  $2(\delta_1 + \bar{\delta}_2)$ . When replacing the worker, the firm expects to have a distance of  $\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}$ , because the expected distance of the new worker is  $\frac{1}{2} + \frac{\delta_1^2}{2}$  and between the existing workers it is  $\delta_1$  (this was explained in detail when deriving equation 10). The firm pays  $c$  when replacing the worker. So the net costs are  $c +$  the expected total distance with replacement less the total distance without replacement. The net costs are thus

$$c + \frac{1}{2} + \frac{\delta_1^2}{2} + \delta_1 - 2(\delta_1 + \bar{\delta}_2) = c + \frac{1}{2} + \frac{\delta_1^2}{2} - \delta_1 - 2\bar{\delta}_2$$

which is the LHS of (12).

*The RHS* of (12) represents the gains from replacement, given by the option value of continuing search associated with low costs in all future periods if the draw is good.

With probability  $\delta_1$  the new worker will be between the two existing workers who have a distance of  $\delta_1$  between them. The total distance between the three workers is  $2\delta_1$ . Existing total distance is  $2(\delta_1 + \bar{\delta}_2)$ , and the savings in distance is thus  $2\bar{\delta}_2$ . Multiplying this with the probability of the event,  $\delta_1$ , gives the first term in the nominator of the RHS of (12).

With probability  $2\bar{\delta}_2$  the worker is not between the existing workers but within a distance of  $\bar{\delta}_2$  from one of them. The expected distance of the new worker to the nearest existing worker is  $\bar{\delta}_2/2$  and to the other existing worker it is  $\delta_1 + \bar{\delta}_2/2$ . The per period cost savings is thus

$$2(\delta_1 + \bar{\delta}_2) - [\delta_1 + \frac{\bar{\delta}_2}{2} + (\delta_1 + \frac{\bar{\delta}_2}{2})] = \bar{\delta}_2$$

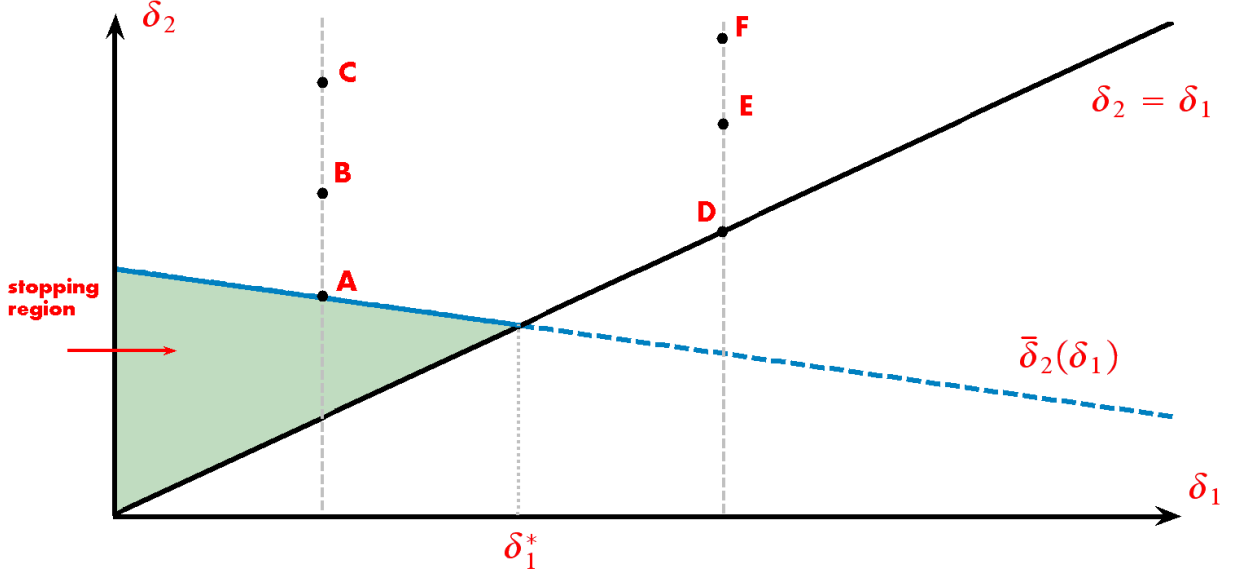
Multiplying this with the probability of the event  $2\bar{\delta}_2$  gives the second term of the RHS of (12).

We see from equation (12) that an increase in  $\delta_1$  reduces the net cost of replacing (reduces the left-hand side) and increases the gain of replacement (the right-hand side) This means that the higher is  $\delta_1$  the worse is the team and the more the firm is willing to replace. Thus  $\bar{\delta}_2(\delta_1)$  is declining. Note also that

$$\frac{\partial \bar{V}(\delta_1)}{\partial \delta_1} = -\frac{1+r}{r} \left[ 1 + \frac{\delta_1(1+r)}{\delta_1 + 2\bar{\delta}_2 + r} \right] < 0 \quad (13)$$

**Proposition 2** For  $\delta_1 \leq \delta^*$ , the optimal stopping rule  $\bar{\delta}_2(\delta_1)$  is uniquely defined by (12), where  $\delta^*$  solves (12) for  $\delta_1 = \bar{\delta}_2(\delta_1)$ .

The proof is given in Appendix B.  
The following figure illustrates this optimal behavior:



**Figure 3: Optimal Policy**

The space of the figure is that of the two state variables,  $\delta_1$  and  $\delta_2$ . The feasible region is above the 45 degree as  $\delta_2 \geq \delta_1$  by definition. The downward sloping line shows the optimal replacement threshold  $\bar{\delta}_2$  as a function of  $\delta_1$ . Beyond the  $\delta_1^* = \bar{\delta}_2(\delta_1^*)$  point, the firm replaces according to the 45 degree line.

With the replacement of a worker, the firm may move up and down a vertical line for any given value of  $\delta_1$  (such as movement between A, B and C or between D, E and F). This is what happens till the firm gets into the absorbing state of no further replacement in the triangle formed by the  $\delta_1^* = \bar{\delta}_2(\delta_1^*)$  point, the intersection of  $\bar{\delta}_2(\delta_1)$  line with the vertical axis, and the origin ( $\delta_1 = \delta_2 = 0$ ).

The following properties of turnover dynamics emerge from this figure and analysis:

(i) At the NE part of the  $\delta_1 - \delta_2$  space,  $\delta_1, \delta_2$  are relatively high, output is low, the firm value is low. Hence the firm keeps replacing and there is high turnover. Note that some workers may stay for more than one period in the firm when in this region. The dynamics are leftwards, with  $\delta_1$  declining, but  $\delta_2$  may move up and down.

(ii) Above the  $\bar{\delta}_2(\delta_1)$  threshold, left of  $\delta_1^*$  there is a lot of firing of the last recruits but veteran workers are kept.

(iii) In the stopping region there is concentration at a location which is random, with a flavor of New Economic Geography agglomeration models. Thus firms specialize in the sense of having similar workers. There is no turnover, and output and firm values are high.

(iv) Policy may affect the regions in  $\delta_1 - \delta_2$  space via its effect on  $c$ . The discount rate affects the regions as well.

(v) These replacement dynamics imply that the degree of complementarity between existing workers may change. This feature is unlike the contributions to the match of the agents in the assortative matching literature, where they are of fixed types.

Finally, the model is closed by imposing a zero profit condition on firms. There are costs  $K \geq 3c$  to open a firm. A zero profit condition pins down the wage ( $w = \frac{W}{3}$ ):

$$E^{\delta_1 \delta_2} V(\delta_1, \delta_2; w; \tilde{y}, c) = K \quad (14)$$

As we have seen, the hiring rule is independent of  $w$  (since it is independent of  $y$ ). If  $y$  is sufficiently large relative to  $K$ , we know that  $E^{\delta_1 \delta_2} V(\delta_1, \delta_2; w; \tilde{y}, c) > K$ . Trivially, then there exists a wage  $w^*$  that satisfies (14).

Note that there are no externalities associated with the hiring process, and wages are set in a competitive manner. Hence there are no externalities associated with entry, and the equilibrium is efficient.

## 4 Exogenous Replacement

We now allow, with probability  $\lambda$ , for one worker to be thrown out of the relationship at the end of every period. If the worker is thrown out, the firm is forced to search in the next period. This can be interpreted as a quit or as a change of position on the circle of one worker, due to learning to work better with other workers or, the opposite, the “souring” of relations. Thus, if the shock hits, one of the workers, chosen at random, has to be replaced. If the shock does not hit, the firm may choose to replace one of its workers or not.

Suppose one worker is replaced by the firm as above. Then the transition probability for  $(\delta_1, \delta_2)$ ,  $q : [0, 1]^2 \rightarrow [0, 1]^2$  is of the form

$$q(\delta_1) \rightarrow \delta_1, \delta_2 \quad (15)$$

We refer to this as the basic transition probability.

The forced transition probabilities are the transition probabilities which occur when one worker is forced to leave, to be denoted by  $q^F(\delta_1, \delta_2)$ . Which



of the three incumbent workers leaves is random: with probability  $1/3$  the least well located worker leaves, in which case the transition probability is  $q(\delta_1)$ ; with probability  $1/3$ , the second best located worker leaves, in which case the transition probability is  $q(\delta_2)$ ; with probability  $1/3$ , the best located worker leaves, in which case the distance between the two remaining workers is  $\min[\delta_1 + \delta_2, 1 - \delta_1 - \delta_2]$ . It follows that the forced transition probabilities can be written as

$$q^F(\delta_1, \delta_2) = \frac{1}{3}q(\delta_1) + \frac{1}{3}q(\delta_2) + \frac{1}{3}q(\min[\delta_1 + \delta_2, 1 - \delta_1 - \delta_2]) \quad (16)$$

The Bellman equation now reads:

$$V(\delta_1, \delta_2) = \pi(\delta_1, \delta_2) + \beta[\lambda \left[ \begin{array}{l} 1/3 \cdot \left( E^{q^F} V_1(\delta'_1, \delta'_2) - c_1 \right) + \\ 1/3 \cdot \left( E^{q^F} V_2(\delta'_1, \delta'_2) - c_2 \right) + \\ 1/3 \cdot \left( E^{q^F} V_{12}(\delta'_1, \delta'_2) - c_{12} \right) \end{array} \right] + (1 - \lambda)E \max[V(\delta_1, \delta_2), EV^q(\delta'_1, \delta'_2) - c]] \quad (17)$$

where 1, 2, and 12 denote the three cases discussed above, respectively, and we allow for search costs or other transition costs (denoted by  $c_i, i = 1, 2, 12$ ).

## 5 Simulations

We simulate the model to get a sense of the implications for worker turnover, firm age, firm value and the connections between them.

### 5.1 The Set-Up

In simulating the model we look at the full model, with both endogenous and exogenous replacement, allowing for exogenous firm exit. Exogenous worker replacement occurs with a probability of  $\lambda$ . If the latter does not occur there is a decision on voluntary replacement. Both occur with a cost  $c$ . The firm exit shock occurs at the end of each period, after production has taken place, at a given rate  $s$ . When a firm is hit by this shock it stops to exist and its value in the next period is zero. Free-entry guarantees that in the next period this firm will be replaced by a new firm, and the latter will pay an entry cost  $K$  in order to get its first random triple of workers and start production. As long as the shock does not hit, the firm operates as above, going through periods of inaction, voluntary or forced replacement. Thus, in a given period, there coexist young and old firms.

The value function is:

$$V(\delta_1, \delta_2) = \pi(\delta_1, \delta_2) + \beta [s \cdot 0 + (1-s) \cdot \left( \begin{array}{c} \lambda \cdot [E^{\Gamma^F} V(\delta'_1, \delta'_2) - c] \\ +(1-\lambda) \cdot E \max[V(\delta_1, \delta_2), E^{\Gamma^F} V(\delta'_1, \delta'_2) - c] \end{array} \right)]$$

where  $c$  is the cost of replacing one worker. This value function can be found by a fixed point algorithm. Appendix C provides full details.

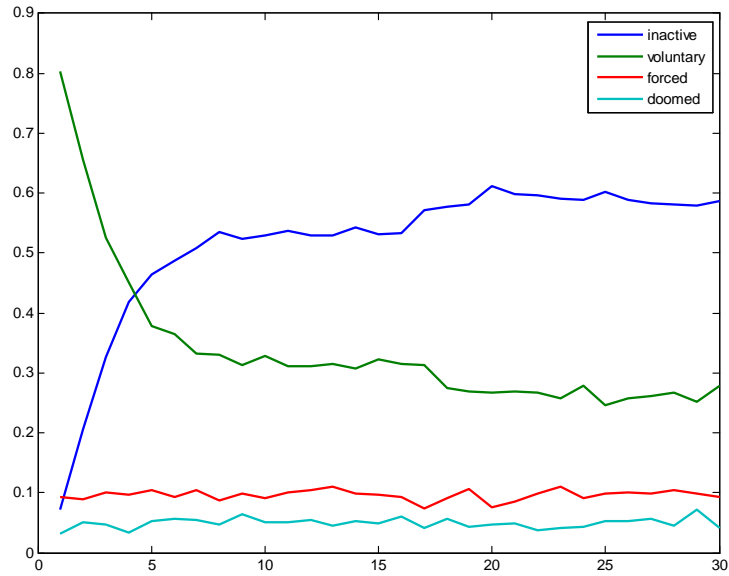
When simulating firms over time, we use the value function formulated above, and subtract from it  $K = 3c$  in case a firm is new-born in a particular period. We simulate 1000 firms over 30 periods, and repeat it 100 times to eliminate run-specific effects. We set:  $y = 1, c = 0.01, r = 0.04, \lambda = 0.1, s = 0.05, K = 0.03$ .

## 5.2 Firms Turnover Dynamics Over Time

In each period, depending upon the realization of the shocks and the optimal hiring decision, a firm might be in one of 4 states:

- Inactive (there was no exogenous separation or firm exit shock, and no voluntary replacement)
- Replacing voluntarily (there was no exogenous separation or firm exit shock and the firm chooses to replace)
- Replacing while forced (there was no firm exit shock, there was an exogenous separation shock)
- Doomed (there is firm exit shock and in the next period a new triple is drawn, a cost  $K$  is paid)

The share of firms in each of above states by periods is shown in the following figure.



**Figure 4: shares of firms in different states**

Figure 4 shows that it takes about 20 periods for the simulated sample to arrive at a regime in which the distribution of firms by states is relatively stable. Before that, there is a reduction in the share of firms engaged in voluntary replacement and an increase in the share of inactive ones, which reflects (temporary) arrival into the absorbing state. After period 20, when almost all firms have already experienced a re-start, as a result of the exit shock occurring at a 5% rate, turnover becomes more stable.

In the state space, the following scatterplots of Figures 5 a-e show the position of firms, which dynamics were depicted above, in five selected periods.

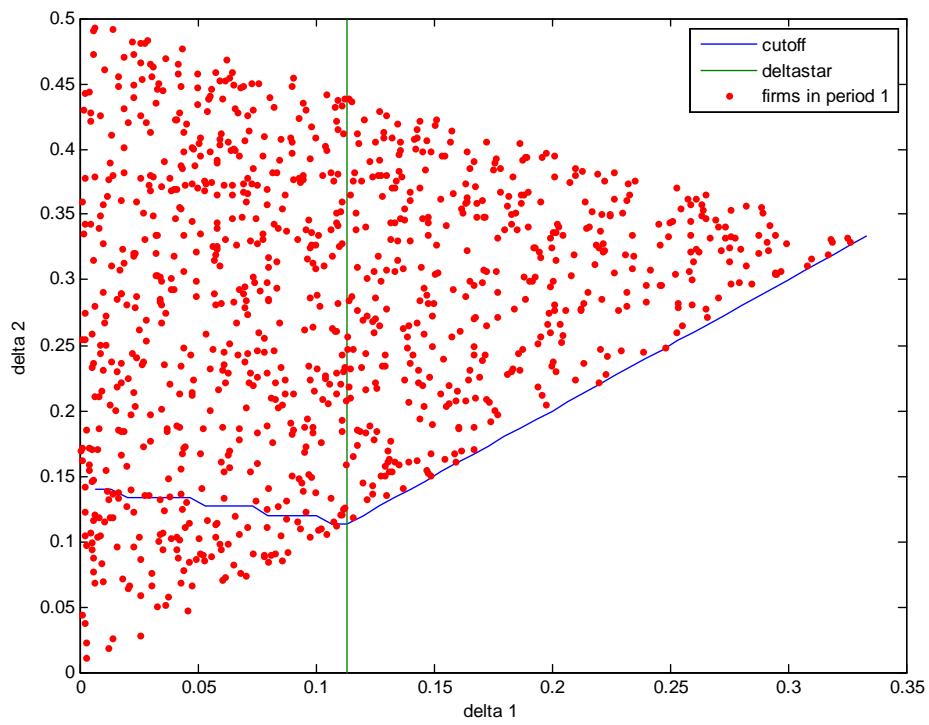
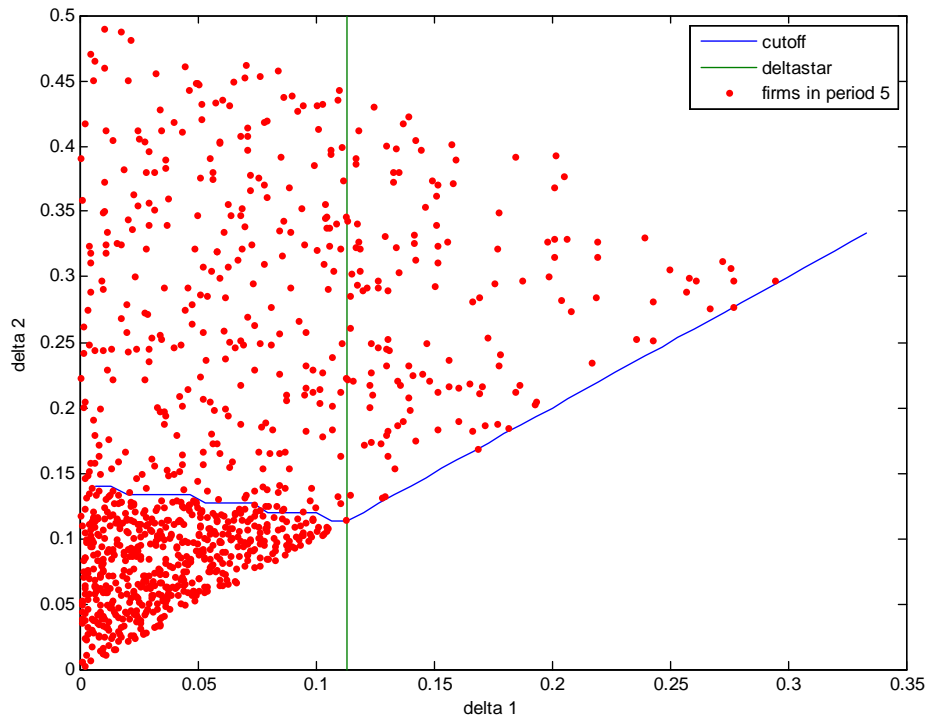


Figure 5a: Period 1



**Figure 5b: Period 5**

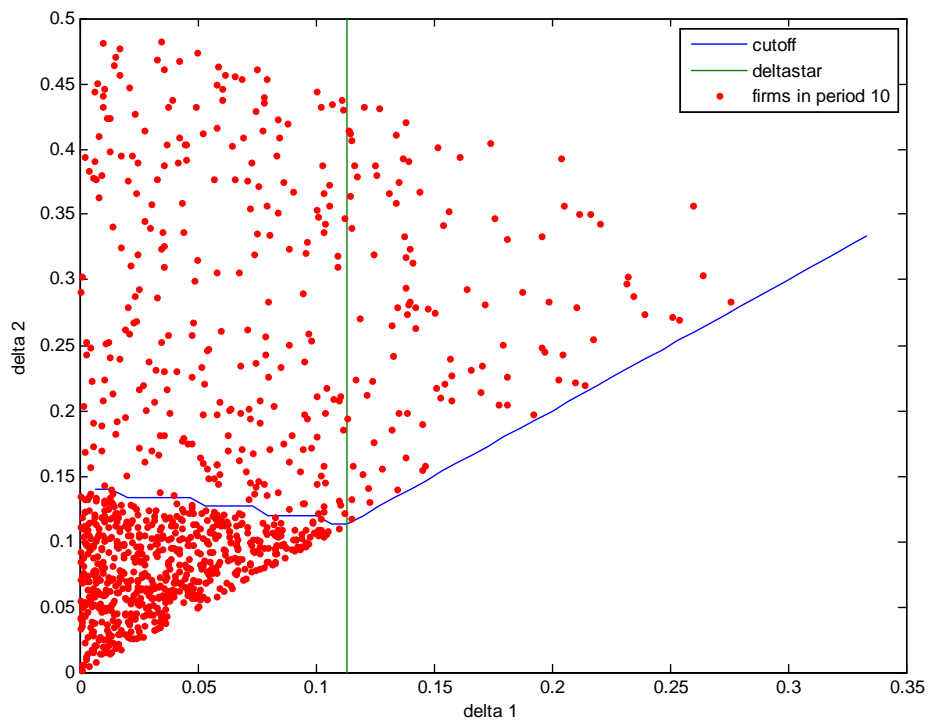


Figure 5c: Period 10

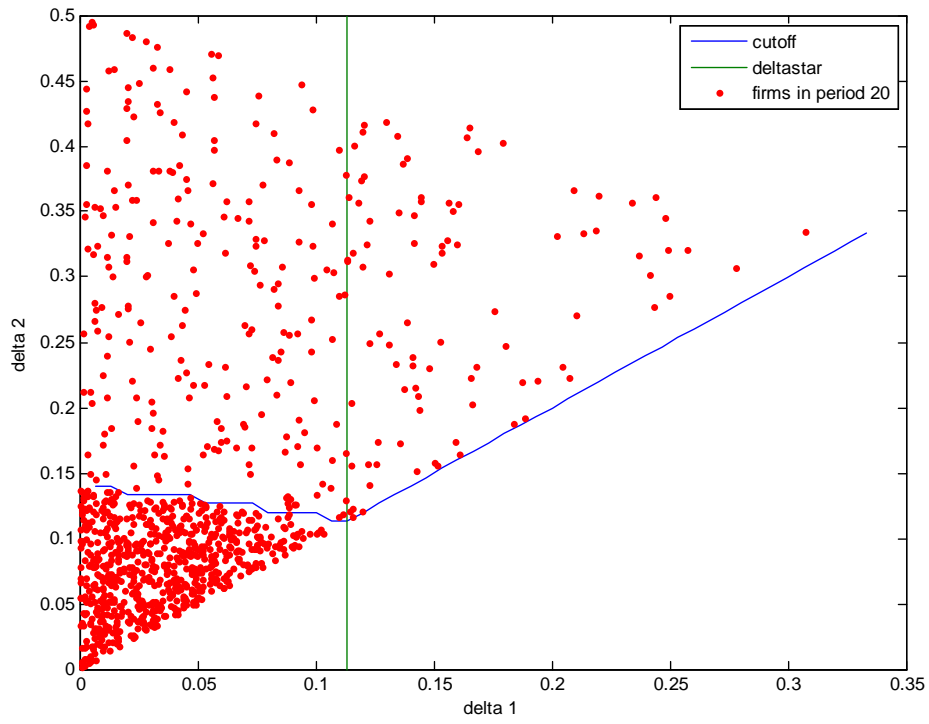
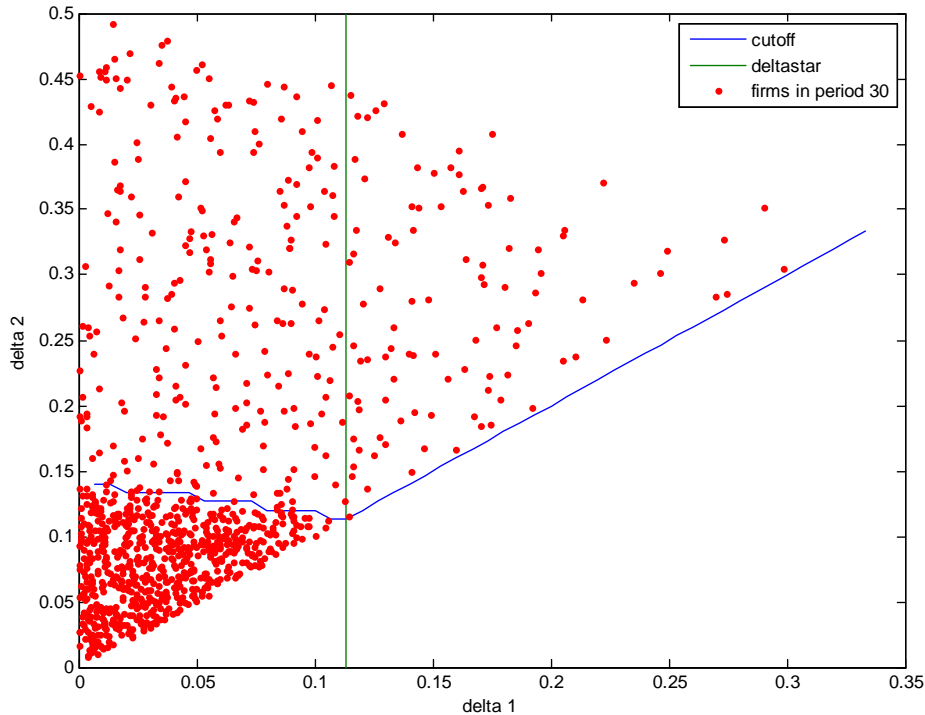


Figure 5d: Period 20



**Figure 5e: Period 30**

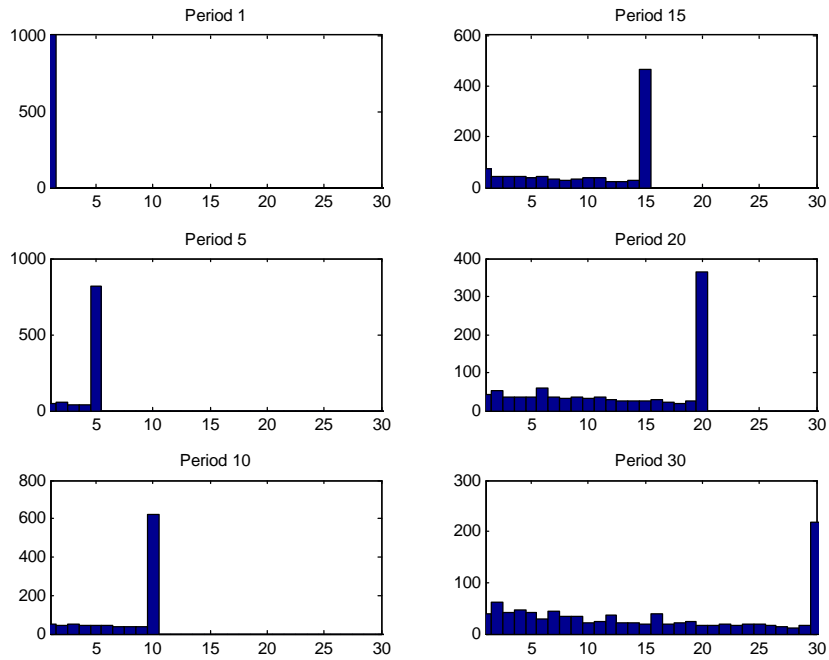
Overall, Figures 4 and 5 indicate that, at first, turnover dynamics are high and firms are spread out in state space, implying disperse productivity and value distributions. Subsequently a more stable turnover pattern is achieved, with most firms staying for some time in the absorbing region. Evidently, due to forced, exogenous separation of workers and to exogenous firm exit, and with the entry of new firms, there is always a group of firms above the cutoff line and beyond the point of  $\delta_1^*$ .

These turnover dynamics of the model, as shown in Figures 4 and 5, are very much in line with the findings in Haltiwanger, Jarmin and Miranda (2010), whereby, for U.S. firms, both job creation and job destruction are high for young firms and decline as firms mature.

### 5.3 The Evolution of the Firm Age Distribution

The presence of a firm exit shock allows us to obtain a non-degenerate distribution of firms age in each period:





**Figure 6a: Firms age distributions, by period**

As time goes by, the population of firms becomes more diverse in terms of age as there is a bunch of long-living survivors and a constant inflow of new-born firms. If we let the simulation run till period 60, we obtain a distribution that is skewed, with most firms being of moderate age and a big group of survivors:

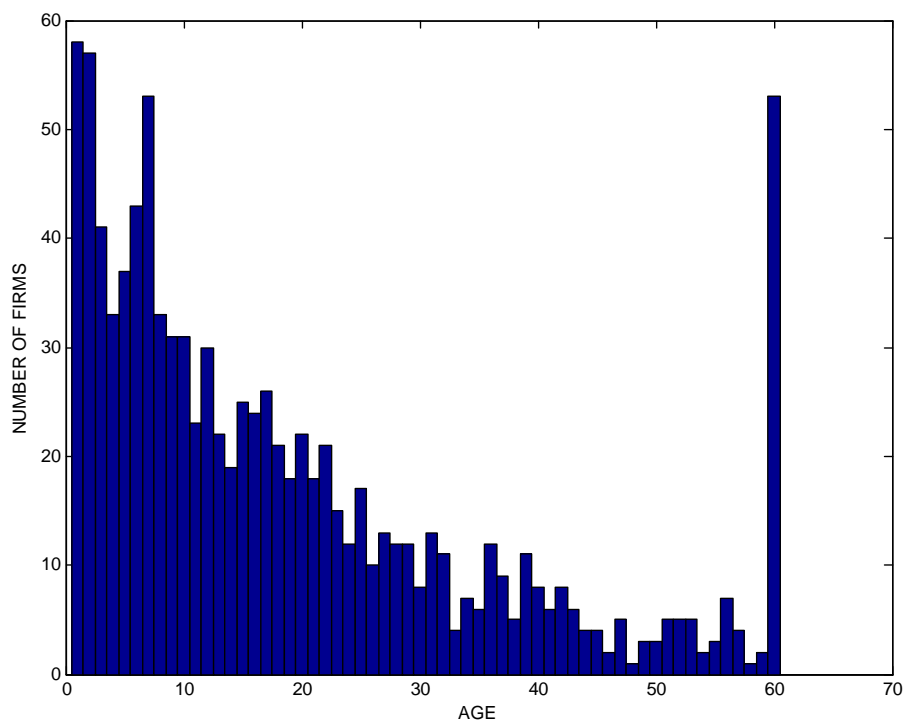
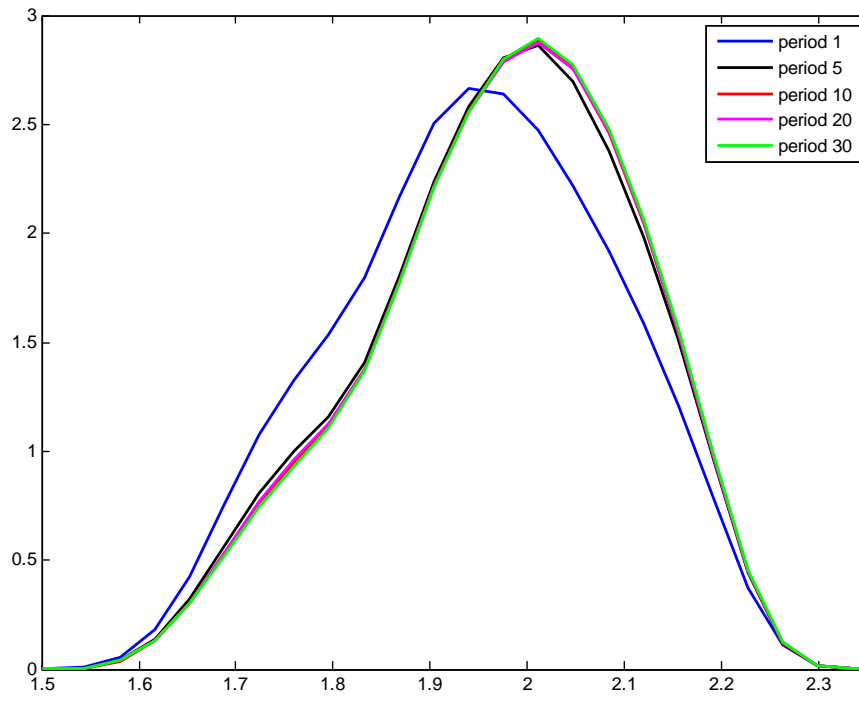


Figure 6b: Firms age distribution in period 60

## 5.4 The Cross-Sectional Distribution of Firm Values

The following figures describe the cross-sectional distributions of firm values (logged), in selected periods, and the evolution of the moments of these distributions over time. Table 1 reports the moments.



**Figure 7a: Cross-sectional firm values (logged)**

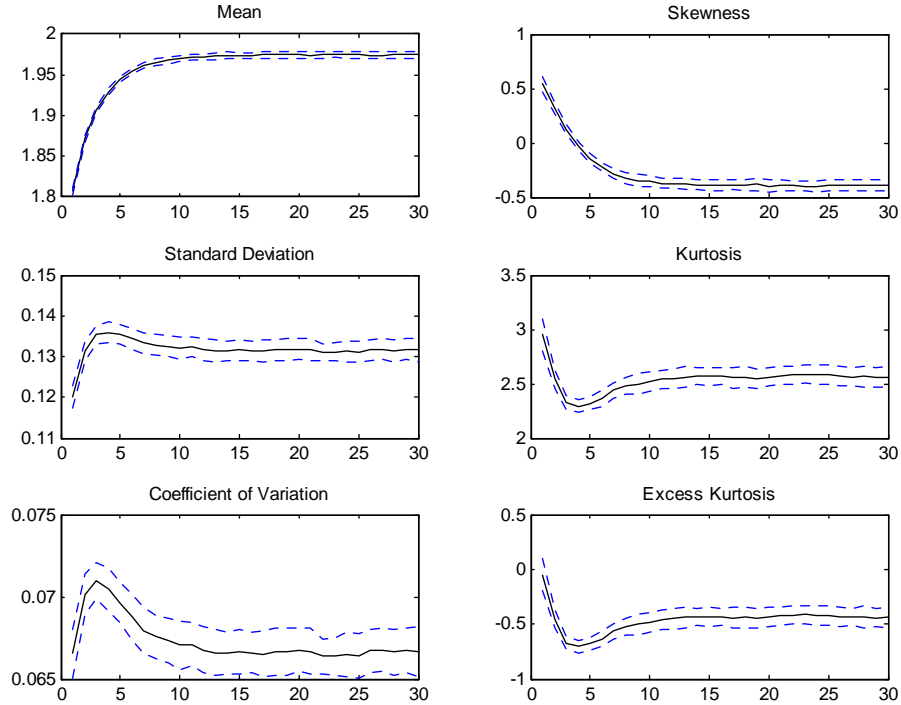


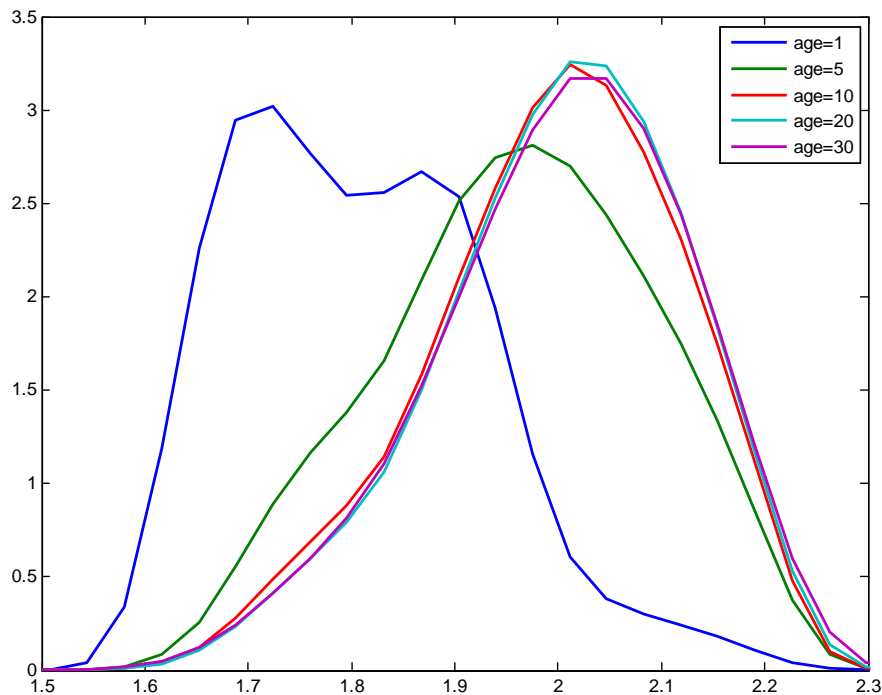
Figure 7b: Moments of cross-sectional, logged firm values distributions

Table 1a  
The Moments of Cross-Sectional Distributions  
of Logged Firm Values, by Periods

	Period 1	Period 5	Period 10	Period 20	Period 30
<b>mean</b>	1.81	1.94	1.97	1.97	1.97
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
<b>std.</b>	0.12	0.14	0.13	0.13	0.13
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
<b>coef. of variation</b>	0.07	0.07	0.07	0.07	0.07
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
<b>skewness</b>	0.55	-0.14	-0.37	-0.39	-0.40
	(0.07)	(0.05)	(0.04)	(0.05)	(0.05)
<b>excess kurtosis</b>	-0.06	-0.68	-0.48	-0.45	-0.42
	(0.15)	(0.07)	(0.07)	(0.10)	(0.09)

The figures and the table indicate that the mean firm value rises and volatility rises and then declines in the early periods. Not much changes after period 10. This is consistent with the movement of firms towards the SW corner of the state space  $\delta_1 - \delta_2$  as shown in Figures 5a-e. For the same reason, the third and fourth moments change so that the left tail becomes thinner and more spread out, i.e., skewness turns more strongly negative and kurtosis declines in absolute value. This reflects the positions of the less valuable firms (right and above the SW corner) in Figures 5 a-e.

We now repeat the above statistics but define them over firm age rather than time. To construct the distributions of firm value by age we looked for all periods, and all firms, when each particular age was observed. For example, due to a firm exit shock and entry of new firms, age 1 will be observed not only for all firms in the first period, but also in all periods when a firm exogenously left and was replaced by a new entrant. In this manner we gathered observations of values for all ages, from 1 to 30, and built the corresponding distributions.



**Figure 7c: Cross-sectional firm values by age (logged)**

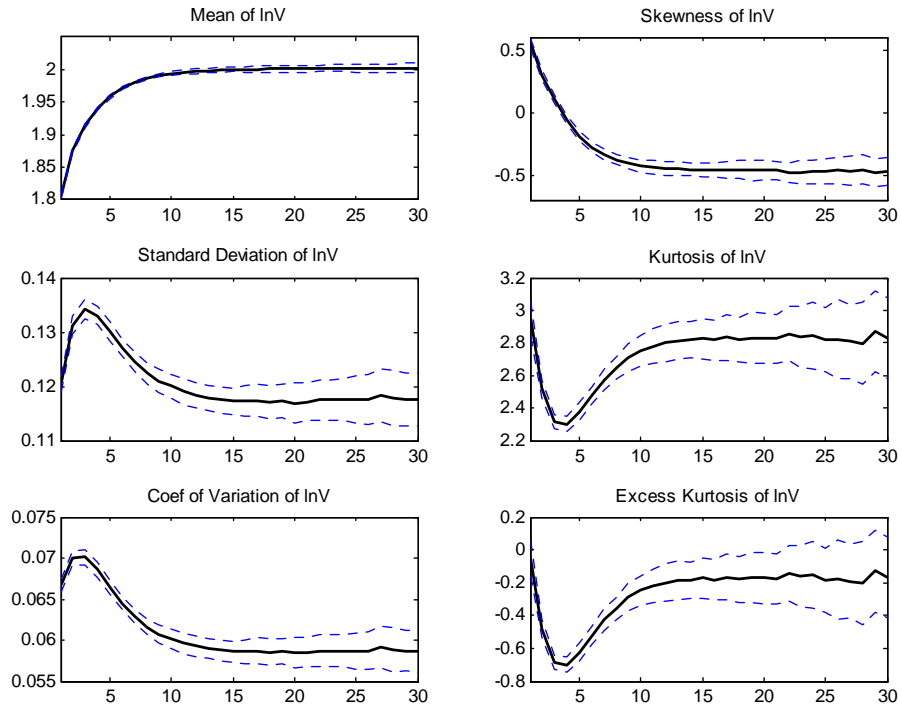


Figure 7d: Moments of firm logged values distributions, by firm age

Table 1b  
The Moments of Firm logged Values Distributions,  
selected ages

	Age = 1	Age = 5	Age = 10	Age = 20	Age = 30
<b>mean</b>	1.81	1.96	1.99	2.00	2.00
	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)
<b>std.</b>	0.12	0.13	0.12	0.12	0.12
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
<b>coef. of variation</b>	0.07	0.07	0.06	0.06	0.06
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
<b>skewness</b>	0.55	-0.19	-0.43	-0.46	-0.47
	(0.04)	(0.03)	(0.05)	(0.08)	(0.11)
<b>excess kurtosis</b>	-0.06	-0.62	-0.25	-0.17	-0.17
	(0.11)	(0.06)	(0.09)	(0.15)	(0.25)

The patterns are essentially the same as in the above statistics relating to time. The value of the firm grows with age as a result of team quality improvements, while the standard deviation is rather stable. As firms mature, more of them enter the absorbing state, with relatively high values, and at the same time there are always unlucky firms that do not manage to improve their teams sufficiently, or which have been hit by a forced separation shock. Therefore the distribution becomes more and more skewed over time.

To further check the connection of firm value with age, we regress the former on a linear-quadratic function of the latter. We obtain the following results.

**Table 2**  
**The Cross-Sectional Relation Between Firm Value and Age**  
**Regression Results of Simulated Values, 1000 firms**

	in period 20	in period 30
<b>age</b>	0.17 (0.02)	0.12 (0.01)
<b>age<sup>2</sup></b>	-0.005 (0.0008)	-0.003 (0.0004)
$R^2$	0.16	0.10

The relation is positive and hump-shaped. This reflects the operation of two opposing forces: as firms mature they rise in value due to the formation of better teams; but, as time passes, exogenous shocks may cause declines in value.

## 6 Conclusions

The paper has characterized the firm in its role as a coordinating device. Thus, output depends on the interactions between workers. The paper has derived optimal policy, using a threshold on a state variable and allowing endogenous hiring and firing. Firm value emerges from optimal coordination done in this manner and fluctuates as the quality of the interaction between the workers changes. Simulations of the model generate non-normal firm value distributions, with negative skewness and negative excess kurtosis. These moments reflect worker turnover dynamics, whereby a large mass of firms is inactive in replacement, having attained good team formation, while exogenous replacement and firm exit induce dispersion of firms in the region of lower value. Hence there results a hump-shaped connection of firm value

with age, reflecting these opposite effects of maturity. Future work will examine alternative production functions, learning and training processes and wage setting mechanisms within this set-up.



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## 7 Appendix A. Derivation of equation (12)

Substituting (8) into (11) gives

$$\begin{aligned} \frac{y - 2(\delta_1 + \bar{\delta}_2(\delta_1))}{r}(1 + r) + c &= y - \left(\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}\right) \\ &+ \frac{(\delta_1 + 2\bar{\delta}_2)(y - 2(\delta_1 + \bar{\delta}_2)) + 2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2}{r} \\ &+ (1 - \delta_1 - 2\bar{\delta}_2)\frac{y - 2(\delta_1 + \bar{\delta}_2(\delta_1))}{r} \end{aligned} \quad (18)$$

Collecting all terms containing  $y - 2(\delta_1 + \bar{\delta}_2(\delta_1))$  on the left-hand side gives

$$\begin{aligned} &\frac{y - 2(\delta_1 + \bar{\delta}_2(\delta_1))}{r}[1 + r - (\delta_1 + 2\bar{\delta}_2) - (1 - (\delta_1 + 2\bar{\delta}_2))] + c - y \\ &= -\left(\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}\right) + \frac{2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2}{r} \end{aligned}$$

which simplifies to

$$-2(\delta_1 + \bar{\delta}_2(\delta_1)) + c = -\left(\frac{1}{2} + \delta_1 + \frac{\delta_1^2}{2}\right) + \frac{2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2}{r}$$

Collecting terms give

$$\frac{1}{2} + \frac{\delta_1^2}{2} - \delta_1 - 2\bar{\delta}_2(\delta_1) = \frac{2\bar{\delta}_2^2 + 2\delta_1\bar{\delta}_2}{r}$$

which is (12).

## 8 Appendix B. Proof of Proposition 2

We repeat the cut-off equation for convenience

$$c + \frac{1}{2} + \frac{\delta_1^2}{2} - \delta_1 - 2\bar{\delta}_2 = \frac{2\delta_1\bar{\delta}_2 + 2\bar{\delta}_2^2}{r} \quad (19)$$

If  $\bar{\delta}_2 = 0$ , the left-hand side of (19) is strictly positive while the right-hand side is zero (since  $\delta_1 \leq 1/3$  by construction). As  $\bar{\delta}_2 \rightarrow \infty$ , the left-hand side goes to minus infinity and the right-hand side to plus infinity. Hence we know that the equation has a solution. Since the left-hand side is strictly decreasing and the right-hand side strictly increasing in  $\bar{\delta}_2$ , we know that the solution is unique.

In the text we have already shown that  $\bar{\delta}_2(\delta_1)$ , if it exists, is decreasing in  $\delta_1$ . It follows that  $\delta^*$  can be obtained by inserting  $\bar{\delta}_2 = \delta_1 = \delta^*$  in (19). This gives

$$c + \frac{1}{2} + \frac{\delta^{*2}}{2} - \delta^* - 2\delta^* = \frac{2\delta^*\delta^* + 2\delta^{*2}}{r}$$

Hence  $\delta^*$  is the unique positive root to the second order equation

$$c + \frac{1}{2} - \delta^{*2} \frac{8-r}{2r} - 3\delta^* = 0$$

## 9 Appendix C. The Simulation Methodology

The entire simulation is run in Matlab with 100 iterations. In order to account for the variability of simulation output from iteration to iteration, we report the average and the standard deviation of the moments and the probability density functions, as obtained in 100 iterations.

### 9.1 Calculating the Value Function

We find the value function  $V$  numerically for the discretized space  $(\delta_1, \delta_2)$ , using a fixed-point procedure. First we guess the initial value for  $V$  in each and every point of this two-dimensional space; we then mechanically go over all possible events (exit, in which case the value turns zero, forced or voluntary separation, with the subsequent draw of the third worker) to calculate the expected value in the next period, derive the optimal decision at each point  $(\delta_1, \delta_2)$ , given the initial guess  $V$ , and thus compute the RHS of the value function equation below:

$$V(\delta_1, \delta_2) = \pi(\delta_1, \delta_2) + \beta [s \cdot 0 + (1-s) \cdot \left( \begin{array}{c} \lambda \cdot [E^{\Gamma^F} V(\delta'_1, \delta'_2) - c] \\ +(1-\lambda) \cdot E \max[V(\delta_1, \delta_2), E^{\Gamma} V(\delta'_1, \delta'_2) - c] \end{array} \right)] \quad (20)$$

Next, we define the RHS found above as our new  $V$  and repeat the calculations above. We iterate on this procedure till the stage when the discrepancy between the  $V$  on the LHS and the RHS is less than the pre-set tolerance level.

The mechanical steps of the program are the following:

1. We assume that each of  $\delta_1, \delta_2$  can take only a finite number of values between 0 and 1. We call this number of values `BINS_NUMBER` and it may be changed in the program.

2. However, not all the pairs  $(\delta_1, \delta_2)$  are possible, as by definition  $\delta_2 \geq \delta_1$  and  $\delta_2 \leq \frac{1}{2} - \frac{\delta_1}{2}$  (the latter ensures that the distances are measured “correctly” along the circle). We impose the above restriction on the pairs constructed earlier, and so obtain a smaller number of pairs, all of which are feasible. Note that all the distances in the pairs are proportionate to  $1/\text{BINS\_NUMBER}$ .

3. In fact, the expected value of forced and voluntary replacement,  $E^{q^F} V(\delta'_1, \delta'_2)$  and  $EV^q(\delta'_1, \delta'_2)$ , differ in only one respect: when the replacement is voluntary, two remaining workers are those with  $\delta_1$  between them, whereas when the replacement is forced, it might be any of the three:  $\delta_1, \delta_2$  or  $\min((\delta_1 + \delta_2), 1 - (\delta_1 + \delta_2))$ , with equal probabilities. In the general case, if there are two workers at a distance  $\delta$ , and the third worker is drawn

randomly, possible pairs in the following period may be of the following three types: (i)  $\delta$  turns out to be the smaller distance (the third worker falls relatively far outside the arch), (ii)  $\delta$  turns out to be the bigger distance (the third worker falls outside the arch, but relatively close) (iii) the third worker falls inside the arch, in which case the sum of the distances in the next period is  $\delta$ . In the simulation we go over all possible pairs to identify the pairs that conform with (i)-(iii). Note that because all the distances are proportionate to  $1/\text{BINS\_NUMBER}$ , it is easy to identify the pairs of the type (iii) described above. This can be done for any  $\delta$ , whether it is  $\delta_1, \delta_2$  or  $\min((\delta_1 + \delta_2), 1 - (\delta_1 + \delta_2))$

4. Having the guess  $V$ , and given that all possible pairs are equally probable, we are then able to calculate the expected values of the firm when currently there are two workers at a distance  $\delta$ . Call this value  $EV(\delta)$ . Then, if there is a firm with three workers with distances  $(\delta_1, \delta_2)$ , the expected value of voluntary replacement is  $EV(\delta_1)$ , and expected value of forced replacement is  $1/3 \cdot EV(\delta_1) + 1/3 \cdot EV(\delta_2) + 1/3 \cdot EV(\min((\delta_1 + \delta_2), 1 - (\delta_1 + \delta_2)))$ . Thus we are able to calculate the RHS of equation (20) above and compare it to the initial guess  $V$ .

We iterate the process till the biggest quadratic difference in the values of LHS and RHS, over the pairs  $(\delta_1, \delta_2)$ , of equation (20) is less than the tolerance level, which was set at 0.0000001.

## 9.2 Dynamic Simulations

Once the value function is found for all possible points on the grid, the simulation is run as follows.

1. The number of firms ( $N$ ) and the number of periods ( $T$ ) is defined. We use  $N = 1000, T = 30$ .
2. For each firm, three numbers are drawn randomly from a uniform distribution  $U[0, 1]$  using the Matlab function *unifrnd*.
3. The distances between the numbers are calculated, the middle worker is defined, and as a result, for each firm a vector  $(\delta_1, \delta_2)$  is found.
4. For each firm, the actual vector  $(\delta_1, \delta_2)$  is replaced by the closest point on the grid found above  $(\tilde{\delta}_1, \tilde{\delta}_2)$ .
5. According to  $(\tilde{\delta}_1, \tilde{\delta}_2)$ , using the calculations from previous section, we assign to each firm the value and the optimal decision in the current period.

6. It is determined whether an exit shock hits. If it does, instead of the current distances of the firm, a new triple is drawn in the next period. If it does not, it is determined whether a forced separation shock  $\lambda$  hits. If  $\lambda$  hits, a corresponding worker is replaced by a new draw and distances are recalculated in the next period. If it does not, and it is optimal not to replace, the distances are preserved for the firm in the next period, as well as the value. If it is optimal to replace, the worst worker is replaced by a new one, distances are re-calculated in the next period, together with the value.

Steps 4-6 are repeated for each firm over all periods.

As a result, we have a  $T$  by  $N$  matrix of firm values. The whole process is iterated 100 times to eliminate run-specific effects. We also record the events history, in a  $T$  by  $N$  matrix which assigns a value of 0 if a particular firm was inactive in a particular period, 1 if it replaced voluntarily, 2 if it was forced to replace, and 3 if it was hit by an exit shock and ceased to exist from the next period on. We use this matrix to differentiate firms by states and to calculate firms' ages.