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Canonical decomposition of linear transformations of two independent Brownian motions¹

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Abstract

Motivated by the Kyle-Back model of “insider trading”, we consider certain classes of linear transformations of two independent Brownian motions and study their canonical decomposition as semimartingales in their own filtration. In particular we characterize those transformations which generate again a Brownian motion.

Keywords: Brownian motion, canonical decomposition, enlargement of filtration, insider trading, stochastic filtering theory, Sturm-Liouville equation, Volterra kernels.

JEL classification: D82, G14.

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1 Introduction

Consider two independent Brownian motions $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$. We study solutions X of stochastic differential equations

$$dX_t = dW_t + Y_t dt \quad (1)$$

driven by W , where the drift Y depends linearly on X and \tilde{W} . Our purpose is to derive the canonical decomposition of X as a semimartingale in its own filtration (\mathcal{F}_t^X) and to characterize those cases where X is again a Brownian motion.

As a simple example consider the Brownian bridge from 0 to \tilde{W}_1 defined by (1), where the drift is given by

$$Y_t = \frac{\tilde{W}_1 - X_t}{1 - t}. \quad (2)$$

The process X is a new Brownian motion such that $X_1 = \tilde{W}_1$. This example plays a crucial role in the Kyle-Back model of “insider trading” (see Kyle [14] and Back [2]). The “insider” knows in advance the final value \tilde{W}_1 . He applies the drift (2) in order to modify the original Brownian motion W in such a way that (i) the resulting process X ends up in \tilde{W}_1 , and (ii) the distribution of the process remains unchanged, i.e., X is again a Brownian motion. Condition (i) guarantees that the strategy maximizes the insider’s expected gain; cf. Back [2]. Condition (ii) corresponds to the notion of equilibrium as defined in Back [2].

Let us now modify the example as follows. Suppose that the “insider” cannot anticipate the final value \tilde{W}_1 from the beginning. Instead, his “insider information” consists in observing the second Brownian motion \tilde{W} . This suggests to replace the anticipating drift (2) by the adapted drift

$$Y_t = \frac{\tilde{W}_t - X_t}{1 - t}. \quad (3)$$

The resulting process X converges again to \tilde{W}_1 . But its distribution has changed: X is no longer a Brownian motion. In section 2 we determine explicitly its canonical decomposition as a semimartingale in its own filtration (\mathcal{F}_t^X) .

This example suggests to study a more general class of processes of the form (1) where the drift is given as a time-dependent linear function in X and \tilde{W} , i.e.,

$$Y_t = f(t)\tilde{W}_t + h(t)X_t \quad (4)$$

for functions f and h in $C^1(0, 1)$ satisfying some mild integrability conditions. In section 3, we derive the canonical decomposition of X in its own filtration. To this end, we consider the Radon-Nikodym density D of the law of X with respect to that of W . We express D in terms of (W, \tilde{W}) , and we compute in closed form $E[D|W]$, its conditional expectation with respect to W . Applying one more time the Girsanov transformation we obtain the canonical decomposition of X . In order to formulate the result, we introduce the fundamental solution $\Psi(t)$ of the Sturm-Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t), \quad (5)$$

with boundary conditions $\Psi(0) = 0$ and $\Psi'(0+) = 1$. The canonical decomposition of X is given by

$$X_t = B_t + \int_0^t (f(u)k_u(X_v; v \leq u) + h(u)X_u) du,$$

where (B_t) is an (\mathcal{F}_t^X) -Brownian motion and the functional k_u is given by

$$k_u(X_s; s \leq u) = \frac{1}{\Psi'(u)} \int_0^u \Psi(v)(f(v)dX_v - f(v)h(v)X_v dv).$$

An alternative method consists in applying the stochastic filtering theory for Gaussian processes. This is explained in section 4. There we consider stochastic differential equations (1) where the drift is given by an adapted linear transformation of X and \tilde{W} , i.e.,

$$dX_t = dW_t + \left(\int_0^t F(t, u)d\tilde{W}_u + \int_0^t H(t, u)dX_u \right) dt, \quad (6)$$

with some square-integrable Volterra kernels F and H ; see the definition in section 3.2. Theorem 4.1 shows that the canonical decomposition of X is of the form

$$X_t = B_t + \int_0^t \left(\int_0^s G_F(s, u)dB_u + \int_0^s H(s, u)dX_u \right) ds, \quad (7)$$

where (B_t) is a standard Brownian motion with respect to (\mathcal{F}_t^X) and G_F is the square-integrable Volterra kernel determined by the equation

$$\int_0^s F(s, u)F(t, u)du = G_F(t, s) + \int_0^s G_F(s, u)G_F(t, u)du. \quad (8)$$

In the special cases considered in section 3, the kernel G_F can be identified in terms of the solution of a Sturm-Liouville equation.

In section 5.1 we return to the discussion of properties (i) and (ii) which appeared in our initial example (2). In the general context of equation (6) where the drift Y is a linear functional of the past of X and \tilde{W} , we characterize those cases which satisfy condition (ii), i.e., the resulting process X is again a Brownian motion. The criterion is that

$$H(t, u) = -G_F(t, u),$$

where G_F is given by (8). But it is not possible to obtain at the same time condition (i), i.e., to tie such a Brownian motion X to the endpoint of the Brownian motion \tilde{W} . In fact, the simple argument of Proposition 5.1 shows that there is no adapted drift (Y_t) such that the solution X of (1) is a Brownian motion with endpoint $X_1 = \tilde{W}_1$.

In section 5.2 we point out the following connection to an enlargement of filtration. First we note that a Brownian motion X , given as the solution of equation (7) with $H = -G_F$, can be expressed directly in terms of W and \tilde{W} :

$$dX_t = dW_t + \left\{ \int_0^t L_F(t, u)dW_u + \int_0^t (F(t, u) + \int_u^t L_F(t, v)F(v, u)dv)d\tilde{W}_u \right\} dt, \quad (9)$$

where L_F denotes the resolvent kernel of G_F ; see (67). In the special case $F(t, u) = f(t)$, this can be reduced to

$$W_t = X_t - \int_0^t \frac{f(u)}{\Psi'(u)} \left(\int_0^u \Psi'(v)d\tilde{W}_v - \int_0^u f(v)\Psi(v)dW_v \right) du,$$

where $\Psi(t)$ is the solution of the Sturm-Liouville equation (5). This representation of W in terms of the Brownian motion X can be viewed, after time reversal, as the decomposition of a Brownian motion in some enlarged Gaussian filtration.

2 A bridge between two Brownian motions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(W_t)_{0 \leq t \leq 1}$ be a standard Brownian motion with respect to its canonical filtration $(\mathcal{F}_t^W)_{0 \leq t \leq 1}$. Now let $(\tilde{W}_t)_{0 \leq t \leq 1}$ be another standard Brownian motion on the same probability space which is independent of $(W_t)_{0 \leq t \leq 1}$, and denote by $(\mathcal{F}_t)_{0 \leq t \leq 1}$ the filtration generated by these two Brownian motions.

We know that the solution $(\tilde{X}_t)_{0 \leq t \leq 1}$ of the stochastic differential equation

$$d\tilde{X}_t = dW_t + \frac{\tilde{W}_1 - \tilde{X}_t}{1-t} dt, \quad (10)$$

with initial value $\tilde{X}_0 = 0$, is a standard Brownian motion which converges to the final value \tilde{W}_1 (cf., for example, Jeulin-Yor [8]). Now we look at the process $(X_t)_{0 \leq t \leq 1}$ starting in $X_0 = 0$ which is defined by the stochastic differential equation

$$dX_t = dW_t + \frac{\tilde{W}_t - X_t}{1-t} dt. \quad (11)$$

Clearly, for any $t \in [0, 1]$, X_t is normally distributed, and $\langle X \rangle_t = t$. The following Lemma shows that X_t approaches \tilde{W}_1 as $t \rightarrow 1$. However, we will see that $(X_t)_{0 \leq t \leq 1}$ is no longer a Brownian motion.

Lemma 2.1 $X_t \rightarrow \tilde{W}_1$ as $t \rightarrow 1$.

Proof. The explicit solution of (11) is given by

$$X_t = (1-t) \int_0^t \frac{\tilde{W}_s}{(1-s)^2} ds + (1-t) \int_0^t \frac{1}{(1-s)} dW_s. \quad (12)$$

The first term approaches \tilde{W}_1 and the second goes to 0 as $t \rightarrow 1$, and this implies the result. Alternatively, we could note that the process $2^{-\frac{1}{2}}(X - \tilde{W})$ satisfies the equation of a Brownian bridge tied down to the final value 0. \square

Lemma 2.2 For $0 \leq s \leq t < 1$, we have

$$E[X_t \tilde{W}_t] = t + (1-t) \log(1-t), \quad (13)$$

and the covariance function of X is given by

$$E[X_s X_t] = s + 2s(1-t) + (2-s-t) \log(1-s). \quad (14)$$

Proof. Applying the integration by parts formula in the first formula in (12), the solution of (11) is given by

$$X_t = (1-t) \int_0^t \frac{d\bar{W}_s - d\tilde{W}_s}{1-s} + \tilde{W}_t.$$

Since (W_t) and (\tilde{W}_t) are independent, we establish

$$\begin{aligned} E[X_t \tilde{W}_t] &= t + (1-t) E \int_0^t \frac{\tilde{W}_t (dW_s - d\tilde{W}_s)}{1-s} \\ &= t + (1-t) \log(1-t), \end{aligned}$$

and

$$\begin{aligned} E[X_s X_t] &= E[\tilde{W}_s \tilde{W}_t] + (1-t)(1-s) E \left[\left(\int_0^s \frac{dW_u - d\tilde{W}_u}{1-u} \right)^2 \right] \\ &+ (1-s) E \left[\tilde{W}_t \int_0^s \frac{dW_u - d\tilde{W}_u}{1-u} \right] + (1-t) E \left[\tilde{W}_s \int_0^t \frac{dW_u - d\tilde{W}_u}{1-u} \right] \\ &= s + 2s(1-t) + (2-s-t) \log(1-s). \end{aligned}$$

□

This Lemma shows that $(X_t)_{0 \leq t \leq 1}$ is not a Brownian motion, since its covariance function differs from $t \wedge s$. But from (11) we see that it is a semimartingale with respect to $(\mathcal{F}_t)_{0 \leq t \leq 1}$, and therefore it is obviously a semimartingale relative to its natural filtration. A natural question is: what is the explicit form of its canonical decomposition? That is the problem we want to discuss in this section.

Lemma 2.3 *Suppose that the process $(X_t)_{t \geq 0}$ is given by*

$$X_t = W_t + \int_0^t Y_u du,$$

with a Brownian motion $(W_t)_{t \geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ and an (\mathcal{F}_t) -adapted process $(Y_t)_{t \geq 0}$ satisfying $\int_0^t E|Y_u| du < \infty$ for all t .

(1) *The canonical decomposition of X in its natural filtration (\mathcal{F}_t^X) is given by*

$$X_t = B_t + \int_0^t E[Y_u | \mathcal{F}_u^X] du, \quad (15)$$

where the process B defined by (15) is an (\mathcal{F}_t^X) -Brownian motion, which is often called the innovation process of X . In particular, (X_t) is a Brownian motion if and only if

$$E[Y_u | \mathcal{F}_u^X] = 0, \quad d\mathbb{P} \times du - a.s..$$

(2) Furthermore, if the function $s \mapsto Y_s$ is L^1 -continuous on $(0, \infty)$ and if $(X_t)_{t \geq 0}$ is a Gaussian process, then $(X_t)_{t \geq 0}$ is a Brownian motion if and only if

$$E(X_s Y_t) = 0, \quad (16)$$

for all $0 < s \leq t$.

Proof of Lemma 2.3. (1) The first part of the first assertion is the Innovation Theorem by Shiryaev and Kailath, the proof of which can be found in Hida-Hitsuda [5] or Liptser-Shiryaev [15]. The second part is immediate from the uniqueness of the canonical decomposition of X in (\mathcal{F}_t^X) .

(2) As to the second assertion, suppose (X_t) is a Brownian motion. Then

$$E(X_s W_t) = E(X_s W_s) = s + \int_0^s E(Y_u W_s) du.$$

Therefore,

$$\int_0^t E(Y_u X_s) du = s - E(X_s W_t) = - \int_0^s E(Y_u W_s) du.$$

Taking derivatives with respect to t on both sides, we obtain (16). Conversely, from Stricker [18] we know that if X is a Gaussian semimartingale, then its canonical decomposition is Gaussian, i.e., $(X_t, \int_0^t E[Y_u | \mathcal{F}_u^X] du)_{t \geq 0}$ is a Gaussian process. Since $s \mapsto Y_s$ is L^1 -continuous, $(X_t, E[Y_t | \mathcal{F}_t^X])_{t \geq 0}$ is Gaussian. From (16) we have $E[Y_t | \mathcal{F}_t^X] = 0$, and so X is a Brownian motion due to (15). \square

Corollary 2.1 *Let the process $(X_t)_{0 \leq t \leq 1}$ satisfy (11). Then the process B , defined as*

$$B_t := X_t - \int_0^t \frac{E[\tilde{W}_u | \mathcal{F}_u^X] - X_u}{1-u} du,$$

is a Brownian motion relative to $(\mathcal{F}_t^X)_{0 \leq t \leq 1}$.

Proof. Set

$$Y_u = \frac{\tilde{W}_u - X_u}{1 - u},$$

then from the first assertion in Lemma 2.3, we obtain the required result. \square

Therefore, we have only to compute the conditional expectation of \tilde{W}_t relative to \mathcal{F}_t^X , the goal we want to reach in this section.

Lemma 2.4 *Set $A := \frac{1}{2}(1 + \sqrt{5})$ and $B := \frac{1}{2}(1 - \sqrt{5})$. Then for $0 \leq t < 1$,*

$$\begin{aligned} E[\tilde{W}_t | \mathcal{F}_t^X] &= \int_0^t \frac{B(1-t)(1-u)^{-A-1} - A(1-t)(1-u)^{-B-1}}{A(1-t)^A - B(1-t)^B} X_u du \\ &\quad + \frac{(1-t)^B - (1-t)^A}{A(1-t)^A - B(1-t)^B} X_t. \end{aligned} \quad (17)$$

Proof. Let us assume the conditional expectation of \tilde{W}_t with respect to \mathcal{F}_t^X is of the form

$$E[\tilde{W}_t | \mathcal{F}_t^X] = \int_0^t a(u, t) X_u du + b(t) X_t.$$

Apply the projection property of the conditional expectation: $E[X_s(\tilde{W}_t - E[\tilde{W}_t | \mathcal{F}_t^X])] = 0$ for all $0 \leq s \leq t < 1$, as well as the martingale property to obtain

$$E(X_s \tilde{W}_s) - b(t)E(X_s X_t) = \int_0^t a(u, t)E(X_s X_u) du.$$

Using (13), (14) and computing explicitly the left hand side (LHS) and the right hand side (RHS) in this equation, we get

$$\text{LHS} = s[1 - (3 - 2t)b(t)] + \log(1 - s)[1 - (2 - t)b(t)] + s \log(1 - s)[b(t) - 1].$$

$$\begin{aligned} \text{RHS} &= \int_0^s a(u, t)(u + 2u(1 - s) + (2 - s - u) \log(1 - u)) du \\ &\quad + \int_s^t a(u, t)(s + 2s(1 - u) + (2 - s - u) \log(1 - s)) du. \end{aligned}$$

Taking the second derivatives with respect to s on both sides implies

$$\frac{1}{1 - s} - \frac{b(t)(t - s)}{(1 - s)^2} = -a(s, t) + \int_s^t \frac{a(u, t)(u - s)}{(1 - s)^2} du.$$

Multiplication of both sides with $(1-s)^2$ leads to:

$$1-s-b(t)(t-s) = -a(s,t)(1-s)^2 + \int_s^t a(u,t)(u-s)du.$$

Taking two further derivatives with respect to s on both sides we get

$$(1-s)^2 a''(s,t) - 4(1-s)a'(s,t) + a(s,t) = 0.$$

The solution of this differential equation is given by

$$a(s,t) = c_1(t)(1-s)^{(-A-1)} + c_2(t)(1-s)^{(-B-1)}.$$

Substituting this equation in RHS and comparing the coefficients of s , $\log(1-s)$ and $s \log(1-s)$ in LHS and RHS, we derive the desired result. \square

Using Itô's product rule, we can rewrite the conditional expectation in (17) as

$$E[\tilde{W}_t | \mathcal{F}_t^X] = \int_0^t \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-t)^{-B} - B(1-t)^{-A}} dX_s + X_t. \quad (18)$$

Therefore, Corollary 2.1 allows us to conclude:

Proposition 2.1 *The canonical decomposition of X is given by*

$$X_t = B_t + \int_0^t \int_0^u \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-u)^{-A} - B(1-u)^{-B}} dX_s du, \quad (19)$$

for $0 \leq t < 1$.

Remark 2.1 *Using Itô's product rule, the representation (19) can be rewritten in the form:*

$$\begin{aligned} X_t = & B_t + \int_0^t \left[\frac{(B+1)(1-s)^{-B} - (A+1)(1-s)^{-A}}{A(1-t)^{-A} - B(1-t)^{-B}} \right. \\ & \left. + \frac{(A+1)(1-s)^{-B} - (B+1)(1-s)^{-A}}{A(1-t)^{-B} - B(1-t)^{-A}} \right] dX_s. \end{aligned}$$

Remark 2.2 Let $(X_t)_{0 \leq t \leq 1}$, a centered Gaussian semimartingale, be of the form

$$X_t = W_t + \int_0^t Y_s ds.$$

A result of Hitsuda [7] states that the conditional expectation $E[Y_s | \mathcal{F}_s^X]$ is equal to the (orthogonal) projection of Y_s to the space $H_s(X)$, which is the L^2 -closure of the set that consists of all stochastic integrals of the form: $\int_0^s f(u) dX_u$ with bounded Borel function f . In our situation (11), the explicit form of this projection is given by (19).

3 Canonical decompositions, Sturm-Liouville equations, and Volterra representations

Now we consider the case where the process $(X_t)_{0 \leq t \leq 1}$ satisfies the stochastic differential equation:

$$dX_t = dW_t + (f(t)\tilde{W}_t + h(t)X_t)dt, \quad (20)$$

with initial value $X_0 = 0$. The functions f and h are assumed to belong to $C^1(0,1) \cap \mathcal{A}(0,1)$, where the space $\mathcal{A}(0,1)$ is defined as the set of all measurable functions ϕ with $\int_0^t s\phi^2(s)ds < \infty$, for all $t < 1$.

From Girsanov's transformation, the law of $(X_s; s \leq t)$, for any $t < 1$, may be written in terms of that of (W, \tilde{W}) via the following formula:

$$E[F(X_s; s \leq t)] = E[F(W_s; s \leq t) \mathcal{E}_t], \quad (21)$$

where F is a measurable functional and

$$\mathcal{E}_t = \exp\left(\int_0^t (f(u)\tilde{W}_u + h(u)W_u)dW_u - \frac{1}{2} \int_0^t (f(u)\tilde{W}_u + h(u)W_u)^2 du\right).$$

Since $(\mathcal{E}_t)_{0 \leq t < 1}$ is a martingale with respect to (\mathcal{F}_t) , the natural filtration generated by $(W_t)_{0 \leq t \leq 1}$ and $(\tilde{W}_t)_{0 \leq t \leq 1}$, we know $\Lambda_t := E(\mathcal{E}_t | \mathcal{F}_t^W)$ is also a martingale with respect to $(\mathcal{F}_t^W)_{0 \leq t < 1}$. Once we have obtained in the next section a closed form of Λ_t , we shall apply again Girsanov's transformation to get the canonical decomposition of $(X_t)_{0 \leq t < 1}$.

3.1 Computation of Λ_t

Obviously, \mathcal{E}_t may be decomposed as: $\mathcal{E}_t = \mathcal{E}_t^{(1)}\mathcal{E}_t^{(2)}$, where

$$\mathcal{E}_t^{(1)} := \exp\left\{\int_0^t h(u)W_u dW_u - \frac{1}{2}\int_0^t h^2(u)W_u^2 du\right\}$$

and

$$\mathcal{E}_t^{(2)} := \exp\left\{\int_0^t f(u)\tilde{W}_u dW_u - \int_0^t f(u)h(u)W_u\tilde{W}_u du - \frac{1}{2}\int_0^t f^2(u)\tilde{W}_u^2 du\right\},$$

so that

$$\Lambda_t = \mathcal{E}_t^{(1)}E[\mathcal{E}_t^{(2)}|\mathcal{F}_t^W].$$

Thus, in order to compute Λ_t , it will suffice to obtain a "robust" formula for $I_{\mu,\nu}(t)$, the expectation of:

$$\mathcal{E}_{\mu,\nu}(t) = \exp\left(\int_0^t \tilde{W}_s \nu(ds) - \frac{1}{2}\int_0^t \tilde{W}_s^2 \mu(ds)\right),$$

with the measure $\mu(ds) := f^2(s)ds$, and $\nu(ds)$ a generic signed measure. Later, we shall justify the replacement of $\nu(ds)$ by the Gaussian "measure":

$$\nu_W(ds) := f(s)dW_s - f(s)h(s)W_s ds.$$

In order to present the next "explicit" formula for $I_{\mu,\nu}(t)$, we need to introduce two fundamental solutions $\Phi(u)$ and $\Psi(u)$ of the Sturm-Liouville equation:

$$\phi''(du) = \mu(du)\phi(u),$$

relative to a measure μ , which are characterized by:

- (i) $\Phi(u)$ is decreasing, and satisfies $\Phi(0) = 1$;
- (ii) $\Psi(u)$ satisfies $\Psi(0) = 0$ and $\Psi'(0+) = 1$.

An important relation between $\Phi(u)$ and $\Psi(u)$ is

$$\Psi(u) = \Phi(u) \int_0^u \frac{1}{\Phi^2(v)} dv. \quad (22)$$

Therefore, we get the important Wronskian relation between Φ and Ψ :

$$\Phi(u)\Psi'(u) - \Psi(u)\Phi'(u) = 1. \quad (23)$$

Throughout this paper, we discuss the case $\mu(ds) = f^2(s)ds$. Therefore, the corresponding Sturm-Liouville equation is given by

$$\phi''(u) = f^2(u)\phi(u). \quad (24)$$

Proposition 3.1 *The expression for $I_{\mu,\nu}(t)$ is equal to*

$$\frac{1}{\sqrt{\Psi'(t)}} \exp\left\{\frac{1}{2} \int_0^t \left(\int_u^t \frac{\Phi(s)}{\Phi(u)} \nu(ds)\right)^2 du - \theta(t) \left(\int_0^t \Psi(s) \nu(ds)\right)^2\right\}, \quad (25)$$

where

$$\theta(t) := \Phi'(t)/(2\Psi'(t)). \quad (26)$$

Proof. The proof simply consists in pushing the computation made in Pitman-Yor [16] a little further. Precisely, changing the Wiener measure with the Radon-Nikodym density:

$$Z_t^\mu := \exp\left\{\frac{1}{2}(F_\mu(t)\tilde{W}_t^2 - \hat{F}_\mu(t)) - \frac{1}{2} \int_0^t \tilde{W}_s^2 \mu(ds)\right\},$$

with $F_\mu(t) = \Phi'(t)/\Phi(t)$ and $\hat{F}_\mu(t) = \int_0^t F_\mu(s)ds = \log \Phi(t)$, it follows that:

$$I_{\mu,\nu}(t) = E^\mu \left[\exp\left\{-\frac{1}{2}(F_\mu(t)\tilde{W}_t^2 - \hat{F}_\mu(t)) + \int_0^t \tilde{W}_s \nu(ds)\right\} \right],$$

where, under the new probability measure P^μ , the process $(\tilde{W}_t)_{0 \leq t \leq 1}$ satisfies:

$$\tilde{W}_t = B_t + \int_0^t F_\mu(s)\tilde{W}_s ds,$$

with a P^μ -Brownian motion $(B_t)_{0 \leq t \leq 1}$. Consequently, $(\tilde{W}_t)_{0 \leq t \leq 1}$ is a centered Gaussian process under P^μ . Now we write

$$I_{\mu,\nu}(t) = \exp\left(\frac{1}{2}\hat{F}_\mu(t)\right) E^\mu \left[\exp\left(\int_0^t \tilde{W}_s \nu(ds) + \sqrt{-F_\mu(t)} N \tilde{W}_t\right) \right],$$

where N is a centered reduced Gaussian random variable, independent of $(\tilde{W}_s; s \leq t)$. Conditioning with respect to N , we are now facing the computation of

$$E^\mu \left[\exp\left(\int_0^t \tilde{W}_s \nu(ds) + c\tilde{W}_t\right) \right] = \exp\left(\frac{1}{2} E^\mu \left[\left(\int_0^t \tilde{W}_s \nu(ds) + c\tilde{W}_t\right)^2 \right]\right).$$

It remains to develop the right-hand side as a second order polynomial with respect to c , and to integrate in c , relatively to the law of $(c =)\sqrt{-F_\mu(t)}N$. A little algebra which hinges in particular upon the elementary formula:

$$E[\exp(\frac{a}{2}N^2 + bN)] = \frac{1}{\sqrt{1-a}} \exp(\frac{b^2}{2(1-a)}), \quad (27)$$

then yields formulas (25) and (26). We use formula (27) to calculate:

$$I_{\mu,\nu}(t) = \exp(\frac{1}{2}\hat{F}(t))E[\exp(\frac{1}{2}(N^2(-F(t))u(t) + 2N\sqrt{-F(t)}v(t) + w(t)))] ,$$

hence, $a = -F(t)u(t)$, $b = \sqrt{-F(t)}v(t)$ with the following values for u , v , w :

$$u(t) = \Phi(t)\Psi(t),$$

$$v(t) = \Phi(t) \int_0^t \Psi(s)\nu(ds),$$

$$w(t) = \int_0^t \frac{1}{\Phi^2(u)} (\int_u^t \Phi(s)\nu(ds))^2.$$

□

Corollary 3.1 *With the notation:*

$$\nu_W(ds) = f(s)dW_s - f(s)h(s)W_s ds,$$

formula (25) with ν changed into ν_W is the conditional expectation of \mathcal{E}_{μ,ν_W} given W .

Proof. Note that $\int_0^t \tilde{W}_s \nu_W(ds)$ is approximated by

$$\int_0^t \tilde{W}_s \nu_n(ds) = \sum_{\sigma_n} \tilde{W}_{t_i} f(t_i) [(W_{t_{i+1}} - W_{t_i}) - h(t_i)W_{t_i}(t_{i+1} - t_i)],$$

where $(\sigma_n)_{n \geq 0}$ is a sequence of subdivisions of $[0, t]$, whose mesh goes to 0 as $n \rightarrow \infty$, and this approximation holds in the following sense: the limit occurs both in probability, and $\exp(\int_0^t \tilde{W}_s \nu_n(ds))$ converges in any L^p towards $\exp(\int_0^t \tilde{W}_s \nu_W(ds))$. This ensures that formula (25) also holds for ν_W . □

We now go back to our main object, that is, to compute the conditional expectation $\Lambda_t^{(2)} := E[\mathcal{E}_t^{(2)} | \mathcal{F}_t^W]$. It follows from (22), (23), Itô's product rule and Fubini theorem that

$$\begin{aligned}
 & \frac{1}{2} \int_0^t \left(\int_u^t \frac{\Phi(s)}{\Phi(u)} \nu_W(ds) \right)^2 du - \theta(t) \left(\int_0^t \Psi(s) \nu_W(ds) \right)^2 \\
 = & \int_0^t \Psi(s) \left(\int_s^t \Phi(u) \nu_W(du) \right) \nu_W(ds) - \int_0^t \left(\Phi(s) - \frac{1}{\Psi'(s)} \right) \left(\int_0^s \Psi(u) \nu_W(du) \right) \nu_W(ds) \\
 & + \frac{1}{2} \int_0^t \frac{f^2(s)}{(\Psi'(s))^2} \left(\int_0^s \Psi(u) \nu_W(du) \right)^2 ds \\
 = & \int_0^t f(u) k_u(W_s; s \leq u) dW_u - \frac{1}{2} \int_0^t f^2(u) k_u^2(W_s; s \leq u) du \\
 & - \int_0^t f(u) h(u) W_u k_u(W_s; s \leq u) du,
 \end{aligned}$$

where

$$\begin{aligned}
 k_u(W_s; s \leq u) & := \frac{1}{\Psi'(u)} \int_0^u \Psi(v) \nu_W(dv) \\
 & = \frac{1}{\Psi'(u)} \int_0^u \Psi(v) (f(v) dW_v - f(v) h(v) W_v dv). \quad (28)
 \end{aligned}$$

Therefore, we derive

$$\begin{aligned}
 \Lambda_t & = \exp \left\{ \int_0^t (f(u) k_u(W_v; v \leq u) + h(u) W_u) dW_u \right. \\
 & \quad \left. - \frac{1}{2} \int_0^t (f(u) k_u(W_v; v \leq u) + h(u) W_u)^2 du \right\} \quad (29)
 \end{aligned}$$

From (21), the definition of Λ_t and Girsanov's transformation, we can get the main result in this section, i.e., the canonical decomposition of X as a semimartingale in its own filtration (\mathcal{F}_t^X) .

Theorem 3.1 *The canonical decomposition of $(X_t)_{0 \leq t \leq 1}$ is given by*

$$X_t = B_t + \int_0^t (f(u) k_u(X_v; v \leq u) + h(u) X_u) du, \quad (30)$$

where $(B_t)_{0 \leq t \leq 1}$ is a standard Brownian motion with respect to $(\mathcal{F}_t^X)_{0 \leq t \leq 1}$, and the functional $(k_t(X_v; v \leq t))_{0 \leq t \leq 1}$ is of the form (28).

Remark 3.1 (1) Comparing (20) and (30) and recalling the first assertion of Lemma 2.3, we see that

$$E[\tilde{W}_t | \mathcal{F}_t^X] = k_t(X_s; s \leq t). \quad (31)$$

In particular, we have obtained the explicit form of the projection appearing in Remark 2.2.

(2) We have obtained that the identification of the conditional expectation in (31) holds for $f, h \in \mathcal{A}(0, 1)$. In fact, it is enough to assume that f and h belong to the set

$$\bar{\mathcal{A}}(0, 1) := \left\{ \varphi : f \text{ is measurable, } \int_0^t \left(\int_u^t \varphi(s) ds \right)^2 du < \infty \text{ for all } 0 \leq t < 1 \right\}.$$

Note that $\mathcal{A}(0, 1) \subset \bar{\mathcal{A}}(0, 1)$, since

$$\int_0^t \left(\int_u^t \varphi(s) ds \right)^2 du \leq t \int_0^t s \varphi^2(s) ds$$

by the Cauchy-Schwarz inequality. It is not difficult to show that for all $s \leq t$,

$$E[X_s(\tilde{W}_t - \frac{1}{\Psi'(t)} \int_0^t \Psi(u)(f(u)dX_u - f(u)h(u)X_u du))] = 0,$$

where $\Psi(t)$ satisfies Sturm-Liouville equation (24). This implies (31).

3.2 Volterra representation and canonical decomposition

Hitsuda [6] shows that the law of a Gaussian process $(X_t)_{0 \leq t \leq 1}$ with $E(X_t) = 0$ is equivalent to the Wiener measure if and only if X_t can be represented in the form:

$$X_t = B_t + \int_0^t \int_0^s l(s, u) dB_u ds, \quad (32)$$

where B is a Brownian motion and $l(s, u)$ is a square-integrable Volterra kernel, i.e., a measurable function on $(0, 1) \times (0, 1)$ such that

(i) $\int_0^t \int_0^s l^2(s, u) dud s < \infty$, for all $t < 1$.

(ii) $l(s, u) = 0$, for $0 < s < u < 1$.

This representation is unique in the sense that if X has another representation

$$X_t = \tilde{B}_t + \int_0^t \int_0^s \tilde{l}(s, u) d\tilde{B}_u ds,$$

then $B = \tilde{B}$ and $l(s, u) = \tilde{l}(s, u)$ for almost all $s, u \in (0, 1)$; see Hida-Hitsuda [5]. We shall call the representation (32) the *Volterra representation* of X .

Proposition 3.2 *The Volterra representation (32) is the canonical decomposition of X as a semimartingale in its own filtration (\mathcal{F}_t^X) . Moreover, we have $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$.*

Proof. Given a square-integrable Volterra kernel l , there is a unique square-integrable Volterra kernel R_l which satisfies the equations

$$\begin{cases} l(t, s) + R_l(t, s) + \int_s^t l(t, u) R_l(u, s) du = 0, \\ l(t, s) + R_l(t, s) + \int_s^t R_l(t, u) l(u, s) du = 0; \end{cases} \quad (33)$$

for almost all $s \leq t$; We call R_l the resolvent kernel of l ; see Yosida [21] Chapter 4 or Hida-Hitsuda [5] p.136-137, we can now use the kernel R_l in order to reconstruct B in terms of X :

$$\begin{aligned} & dX_t + \int_0^t R_l(t, u) dX_u dt \\ &= dB_t + \int_0^t B(t, u) dB_u dt + \int_0^t R_l(t, u) \left[dB_u + \int_0^u l(u, v) dB_v du \right] dt \\ &= dB_t + \int_0^t (l(t, u) + R_l(t, u) + \int_u^t R_l(t, v) l(v, u) dv) dB_u dt \\ &= dB_t, \end{aligned}$$

i.e., we have

$$X_t = B_t + \int_0^t \int_0^s l(s, u) dB_u ds, \quad (34)$$

and

$$B_t = X_t + \int_0^t \int_0^s R_l(s, u) dX_u ds. \quad (35)$$

Thus, X and B have the same filtration. Hence (32) is the canonical decomposition of X in its own filtration. \square

Remark 3.2 *Let us make explicit the square-integrable Volterra kernel corresponding to our situation (20) in the special case where $h(t) = 0$, i.e.,*

$$X_t = W_t + \int_0^t f(u) \tilde{W}_u du.$$

We have shown that the canonical decomposition of X is given by

$$X_t = B_t + \int_0^t \frac{f(s)}{\Psi'(s)} \int_0^s f(u) \Psi(u) dX_u ds. \quad (36)$$

Integrating $f(t)\Psi(t)$ on both sides and using Itô's product rule, we obtain

$$\int_0^t f(u) \Psi(u) dX_u = \Psi'(t) \int_0^t \frac{f(v) \Psi(v)}{\Psi'(v)} dB_v.$$

Thus, the representation (36) takes the form

$$X_t = B_t + \int_0^t f(s) \int_0^s \frac{f(u) \Psi(u)}{\Psi'(u)} dB_u ds. \quad (37)$$

This shows that the corresponding square-integrable Volterra kernel is given by

$$l(s, u) = \frac{f(s) f(u) \Psi(u)}{\Psi'(u)},$$

and we read from (36) that

$$R_l(s, u) = -\frac{f(s)}{\Psi'(s)} f(u) \Psi(u). \quad (38)$$

One can verify from (38) that R_l satisfies indeed the two characteristic equations in (33).

Remark 3.3 *The study of the general equation (20) for the pair (f, h) can be reduced to that of $(f, 0)$. Suppose a process (X_t) satisfies (20) and introduce the Gaussian process*

$$\xi_t = W_t + \int_0^t f(u) \tilde{W}_u du.$$

Remark that (X_t) and (ξ_t) have the same filtration, since

$$\xi_t = X_t - \int_0^t h(u) X_u du, \quad (39)$$

and

$$X_t = \int_0^t \exp\left(\int_s^t h(u)du\right)d\xi_s.$$

Using the canonical decomposition for (ξ_t) given by (37), we obtain the canonical decomposition

$$X_t = B_t + \int_0^t (f(s) \int_0^s \frac{f(u)\Psi(u)}{\Psi'(u)} dB_u + h(s)X_s) ds \quad (40)$$

for X . The pathwise solution of this equation leads us to the alternative representation

$$X_t = \int_0^t \exp\left(\int_s^t h(u)du\right) dB_s + \int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)} \int_u^t f(v) \exp\left(\int_v^t h(w)dw\right) dv dB_u \quad (41)$$

as a functional of the Brownian motion B , in analogy to (37). From Allinger-Mitter [1], Davis [4] we know that the σ -algebra \mathcal{F}_t^X coincides with \mathcal{F}_t^B for all t , up to some P null sets. Thus, $\mathcal{F}_t^\xi = \mathcal{F}_t^B = \mathcal{F}_t^X$. In the terminology of Hida-Hitsuda [5], the representation (41) is called “the canonical representation relative to B ”, and B is the innovation process for X .

3.3 A class of path dependent transformations

Using the same method as above, we can extend our result in Theorem 3.1 as follows. Consider a process $(X_t)_{0 \leq t \leq 1}$ satisfying the stochastic functional differential equation

$$dX_t = dW_t + (f(t)\tilde{W}_t + \int_0^t h(t,s)dX_s)dt, \quad (42)$$

for some deterministic function $f \in C^1(0,1) \cap \mathcal{A}(0,1)$, a function $h \in C^{1,1}((0,1) \times (0,1))$ with $\int_0^t \int_0^s h^2(s,u)duds < \infty$ for all $t < 1$.

Theorem 3.2 *The canonical decomposition of $(X_t)_{0 \leq t \leq 1}$ is*

$$dX_t = dB_t + (f(t)k_t(X_s; s \leq t) + \int_0^t h(t,s)dX_s)dt,$$

where $(B_t)_{0 \leq t \leq 1}$ is an (\mathcal{F}_t^X) -Brownian motion, and $k_u(X_s; s \leq u)$ is of the form

$$\frac{1}{\Psi'(u)} \int_0^u \Psi(v)f(v)(dX_v - \int_0^v h(v,r)dX_rdv). \quad (43)$$

Proof. We set

$$\nu_W(ds) := f(s)dW_s - f(s) \int_0^s h(u, s)dW_u ds.$$

The rest of the proof is similar to the proof in section 3.1. \square

3.4 Some examples

In the first two examples we look at some processes whose canonical decompositions take a simple form. And in Example 3 we will discuss a special case of the filtration.

Example 1. Consider a process $(X_t)_{0 \leq t \leq 1}$ satisfying the stochastic differential equation:

$$dX_t = dW_t + \frac{a}{1-t}(\tilde{W}_t - X_t)dt,$$

that is, $f(t) = -h(t) = a/(1-t)$, with a nonzero constant a . Then the corresponding Sturm-Liouville equation is

$$\Phi''(u) = \frac{a^2}{(1-u)^2} \Phi(u), \quad (44)$$

which, arguably perhaps, is one of the simplest cases where the Sturm-Liouville equation has elementary solutions. Indeed, it is immediate to check that a function $(1-u)^\lambda$ solves (44) if and only if $\lambda(\lambda-1) = a^2$, an equation which admits the two solutions: $\lambda_+(a)$ and $\lambda_-(a)$, given by

$$\lambda_\pm(a) := \frac{1}{2} \pm \sqrt{a^2 + \frac{1}{4}}.$$

Clearly, $\lambda_-(a) < 0 < \lambda_+(a)$. Thus, the decreasing solution of (44) is

$$\Phi(u) = (1-u)^{\lambda_+(a)}.$$

And from the definition and the boundary condition of $\Psi(u)$ we can get

$$\Psi(u) = \frac{(1-u)^{\lambda_-(a)} - (1-u)^{\lambda_+(a)}}{\sqrt{1+4a^2}}.$$

From (28), the conditional expectation of \tilde{W}_t relative to \mathcal{F}_t^X is given by

$$\begin{aligned} & k_u(X_s; s \leq u) \\ := & \frac{1}{\Psi'(u)} \int_0^u \Psi(v) \left(\frac{a}{1-v} dX_v + \frac{a^2}{(1-v)^2} X_v dv \right) \\ = & X_u + a \int_0^u \frac{(\lambda_-(a) + 1)(1-v)^{-\lambda_+(a)} - (\lambda_+(a) + 1)(1-v)^{-\lambda_-(a)}}{\lambda_+(a)(1-u)^{-\lambda_-(a)} - \lambda_-(a)(1-u)^{-\lambda_+(a)}} dX_v. \end{aligned}$$

In particular, if $a = 1$, then $\lambda_+(1) = A$ and $\lambda_-(1) = B$, defined as in Lemma 2.4, and we are led to the same result as in section 2. \square

Example 2. Consider the simple example:

$$dX_t = dW_t + a(\tilde{W}_t - X_t)dt,$$

with a nonzero constant a . The desired solution of the corresponding Sturm-Liouville equation is

$$\Psi(t) = \frac{1}{2a}(e^{at} - e^{-at}),$$

and the conditional expectation of \tilde{W}_t with respect to \mathcal{F}_t^X is given by

$$k_t(X_s; s \leq t) = X_t - \frac{2}{e^{at} + e^{-at}} \int_0^t e^{-au} dX_u.$$

Therefore, the canonical decomposition of X_t has the form

$$X_t = B_t - \int_0^t \frac{2a}{e^{au} + e^{-au}} \left(\int_0^u e^{-av} dX_v \right) du,$$

where $(B_t)_{0 \leq t \leq 1}$ is a Brownian motion relative to $(\mathcal{F}_t^X)_{0 \leq t \leq 1}$. \square

Example 3. Let $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$ be two independent Brownian motions, starting from 0. The process $(X_t)_{t \geq 0}$ satisfies the stochastic differential equation (20) with $f(t) = -k/t$ with a constant k and $h \equiv 0$. In section 3.2 we have already discussed a few results about the case with $h \equiv 0$. First, we have to solve

$$\Psi''(s) = \frac{k^2}{s^2} \Psi(s),$$

and then single out a solution with $\Psi(0) = 0$. It is easy to check that $\Psi(s) = s^\lambda$ with $\lambda = \frac{1}{2} + \sqrt{k^2 + \frac{1}{4}}$ is the wanted solution. Now formula (37) gives

$$X_t = B_t + \frac{k^2}{\lambda} \int_0^t \frac{B_u}{u} du, \quad (45)$$

with a Brownian motion (B_t) . In Jeulin-Yor ([12], Theorem 9) it has been shown that

$$X_t^\mu := B_t - \mu \int_0^t \frac{B_s}{s} ds \quad (46)$$

has a strictly smaller filtration than the filtration of B iff $\mu > \frac{1}{2}$. Coming back to formula (45), and comparing with (46), we find

$$\mu = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4k^2}.$$

Thus, in our study, μ is always negative, hence $(X_t)_{t \geq 0}$ has the same filtration as (B_t) . \square

4 Application of stochastic filtering theory for Gaussian processes

In section 3 we have computed the canonical representation of our transformations (20) of two Brownian motions by direct methods. As an alternative, we can derive them as corollaries of the stochastic filtering theory for Gaussian processes. At the same time, this allows us to extend our results to a general class of transformations where the drift Y in (1) is given as an adapted linear functional of X and \tilde{W} .

Suppose that the process X satisfies a stochastic differential equation of the form

$$dX_t = dW_t + \left(\int_0^t F(t, u) d\tilde{W}_u + \int_0^t H(t, u) dX_u \right) dt, \quad (47)$$

with $X_0 = 0$, where $(\tilde{W}_t)_{0 \leq t \leq 1}$, $(W_t)_{0 \leq t \leq 1}$ are two independent Brownian motions, and F, H are square-integrable Volterra kernels on $(0, 1) \times (0, 1)$, i.e., they satisfy the conditions in section 3.2. Note that the processes considered in (20) belong to this class.

Lemma 4.1 *There is a unique Brownian motion B and a unique square-integrable Volterra kernel G_F such that*

$$W_t + \int_0^t \int_0^s F(s, u) d\tilde{W}_u ds = B_t + \int_0^t \int_0^s G_F(s, u) dB_u ds.$$

The kernel G_F is determined by the equation

$$\int_0^s F(t, v) F(s, v) dv = G_F(t, s) + \int_0^s G_F(t, v) G_F(s, v) dv. \quad (48)$$

Moreover the natural filtration of B is identical to that of the left-hand side.

Proof. (1) Let us denote by Z the “signal process”

$$Z_t = \int_0^t F(t, u) d\tilde{W}_u,$$

and by ξ the “observation process”

$$\xi_t = W_t + \int_0^t Z_s ds = W_t + \int_0^t \int_0^s F(s, u) d\tilde{W}_u ds. \quad (49)$$

From Lemma 2.3 we know that ξ_t can be written as

$$\xi_t = B_t + \int_0^t E[Z_s | \mathcal{F}_s^\xi] ds,$$

where (B_t) is an (\mathcal{F}_t^ξ) -Brownian motion. Since $E[Z_t | \mathcal{F}_t^\xi](\omega)$ can be chosen (t, ω) -measurable and (\mathcal{F}_t^ξ) -adapted, we can write

$$E[Z_t | \mathcal{F}_t^\xi] = \gamma(t, \xi(\omega)),$$

where γ is a non-anticipative functional in the sense of Kallianpur [13], Definition 5.1.1. Furthermore, it follows from

$$E \int_0^u (E[Z_u | \mathcal{F}_u^\xi])^2 du \leq E \int_0^t Z_u^2 du = \int_0^t \int_0^s (F(s, u))^2 duds < \infty$$

that

$$\int_0^t \gamma^2(s, \xi(\omega)) ds < \infty, \quad \mathbb{P} - \text{a.s.},$$

for all $t < 1$. By Kallianpur [13], Theorem 9.4.1 and section 3.2, there is a unique square-integrable Volterra kernel G_F such that the Gaussian process ξ has the representation

$$\xi_t = B_t + \int_0^t \int_0^s G_F(s, u) dB_u ds. \quad (50)$$

Moreover we have $(\mathcal{F}_t^\xi) = (\mathcal{F}_t^B)$, due to Proposition 3.2.

(2) Since W and \tilde{W} are two independent Brownian motions, we get from (49), for $s \leq t$,

$$\begin{aligned} E[\xi_s \xi_t] &= E[W_s W_t] + E\left[\int_0^s \int_0^u F(u, r) d\tilde{W}_r du \int_0^t \int_0^v F(v, q) d\tilde{W}_q dv\right] \\ &= s + 2 \int_0^s \int_0^u \int_0^v F(u, r) F(v, r) dr dv du \\ &\quad + \int_s^t \int_0^s \int_0^v F(u, r) F(v, r) dr dv du. \end{aligned} \quad (51)$$

We can also compute the covariance of ξ from (50):

$$\begin{aligned} E[\xi_s \xi_t] &= E\left[\left(B_s + \int_0^s \int_0^u G_F(u, r) dB_r du\right) \left(B_t + \int_0^t \int_0^v G_F(v, q) dB_q dv\right)\right] \\ &= s + 2 \int_0^s \int_0^u G_F(u, v) dv du + \int_s^t \int_0^s G_F(u, v) dv du \\ &\quad + 2 \int_0^s \int_0^u \int_0^v G_F(u, r) G_F(v, r) dr dv du \\ &\quad + \int_s^t \int_0^s \int_0^v G_F(u, r) G_F(v, r) dr dv du. \end{aligned} \quad (52)$$

Since the right hand sides of these two equations must coincide, differentiating first with respect to t then with respect to s yields (48) for almost every $s \leq t$. \square

Remark 4.1 *In the notation of Kallianpur [13] p.235, equation (48) can be viewed as the factorization $S = (I + G)(I + G^*)$ of the integral operator S defined by $I + FF^*$, where F, G are integral operators with square-integrable Volterra kernels $F(t, s)$ and $G_F(t, s)$, respectively, i.e. for all $f, g \in L^2(0, 1)$,*

$$\langle (I + FF^*)f, g \rangle = \langle (I + G)(I + G^*)f, g \rangle.$$

In order to see this, let $f(u) = I_{(0,s)}(u)$ and $g(u) = I_{(0,t)}(u)$ with $0 \leq s \leq t \leq 1$. Using the properties of Volterra kernels, we have

$$\begin{aligned} & \langle (I + FF^*)f, g \rangle = \langle f, g \rangle + \langle F^*f, F^*g \rangle \\ &= \int_0^1 f(u)g(u)du + \int_0^1 \left(\int_0^1 F(v, u)f(v)dv \right) \left(\int_0^1 F(r, u)g(r)dr \right) du \\ &= s + \int_0^s \left(\int_u^s F(v, u)dv \right) \left(\int_u^t F(r, u)dr \right) du, \end{aligned}$$

which equals to the right-hand side of (51). On the other hand,

$$\begin{aligned} & \langle (I + G)(I + G^*)f, g \rangle = \langle (I + G^*)f, (I + G^*)g \rangle \\ &= \int_0^1 \left(f(u) + \int_0^1 G_F(v, u)f(v)dv \right) \left(g(u) + \int_0^1 G_F(v, u)g(v)dv \right) du \\ &= \int_0^s \left(1 + \int_u^s G_F(v, u)dv \right) \left(1 + \int_u^t G_F(v, u)dv \right) du, \end{aligned}$$

which is exactly the right-hand side of (52).

Now, we look at the canonical decomposition of the process X given by (47).

Theorem 4.1 *The canonical decomposition of X as a semimartingale in its own filtration (\mathcal{F}_t^X) is given by*

$$dX_t = dB_t + \left(\int_0^t G_F(t, u)dB_u + \int_0^t H(t, u)dX_u \right) dt. \quad (53)$$

Moreover we have $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$.

Proof. From (49) and (53), we have

$$dX_t = d\xi_t + \int_0^t H(t, u)dX_u dt. \quad (54)$$

As in (33), let R_{-H} denote the resolvent kernel of the square-integrable Volterra kernel $-H$. Due to equations (34) and (35), we have

$$\xi_t = X_t - \int_0^t \int_0^s H(s, u)dX_u ds,$$

and

$$X_t = \xi_t + \int_0^t \int_0^s R_{-H}(s, u) d\xi_u ds.$$

These two equations together with Lemma 4.1 imply $\mathcal{F}_t^X = \mathcal{F}_t^\xi = \mathcal{F}_t^B$, for all t . Hence, (B_t) is also an (\mathcal{F}_t^X) -Brownian motion. Substituting the representation (50) in the equation (54), we obtain (53). \square

Remark 4.2 Comparing (47), (53) and Lemma 2.3, we see that

$$E\left[\int_0^t F(t, u) d\tilde{W}_u \mid \mathcal{F}_t^X\right] = \int_0^t G_F(t, u) dB_u.$$

Let us now consider the special case when F admits a factorization $F(t, s) = f(t)g(s)$ for some functions $f, g \in C^1(0, 1)$ which satisfy

$$\int_0^t \int_0^u f^2(u)g^2(v)dvdu < \infty, \quad (55)$$

for all $t < 1$.

Corollary 4.1 Suppose the process $(X_t)_{0 \leq t \leq 1}$ satisfies

$$dX_t = dW_t + (f(t) \int_0^t g(u) d\tilde{W}_u + \int_0^t H(t, u) dX_u) dt, \quad (56)$$

with $f, g \in C^1(0, 1)$ satisfying (55), $f \neq 0$ a.s., and a square-integrable Volterra kernel $H(t, s)$. Then the canonical decomposition of X is of the form

$$dX_t = dB_t + (f(t) \int_0^t \alpha(u) dB_u + \int_0^t H(t, u) dX_u) dt, \quad (57)$$

where the function $\alpha(t)$ is the solution of the differential equation

$$\left(\frac{\alpha(t)}{f(t)}\right)' + \alpha^2(t) = g^2(t), \quad (58)$$

with boundary condition $\alpha(0) = 0$.

Proof. We have only to prove that $G_F(t, s) = f(t)\alpha(s)$, where α satisfies (58) and $\alpha(0) = 0$, is the solution of (48). In fact, the right-hand side of (48) is equal to

$$\begin{aligned} & G_F(t, s) + \int_0^s G_F(t, u)G_F(s, u)du \\ &= f(t)\alpha(s) + f(t)f(s) \int_0^s \alpha^2(u)du \\ &= f(t)\alpha(s) + f(t)f(s) \left[\int_0^s g^2(u)du - \frac{\alpha(s)}{f(s)} \right] \\ &= f(t)f(s) \int_0^s g^2(u)du = \int_0^s F(t, u)F(s, u)du, \end{aligned}$$

which is exactly the left-hand side of (48). \square

In order to see the connection with our discussion in the preceding sections, let us set $g \equiv 1$ and $H(t, u) = h(t)$. Then equation (58) can be written as

$$\left(\frac{\alpha(t)}{f(t)} \right)' + \alpha^2(t) = 1,$$

with $\alpha(0) = 0$. The corresponding solution is given by

$$\alpha(t) = \frac{f(t)\Psi(t)}{\Psi'(t)}, \quad (59)$$

where $\Psi(t)$ is the solution of the Sturm-Liouville equation (24). Substituting this result in (57), we see that the result coincides with (40).

Remark 4.3 *The discussion of equation (56) can be reduced to the case $g \equiv 1$. All we need is to consider $H = 0$. Then, we have*

$$g(t)dX_t = g(t)dW_t + f(t)g(t) \int_0^t g(u)d\tilde{W}_u dt.$$

Let us introduce the (time-changed) processes (\hat{X}_u) , (\hat{W}_u) and $(\hat{\tilde{W}}_u)$ defined as:

$$\int_0^t g(u)dX_v = \hat{X}_{G(t)},$$

where

$$G(t) = \int_0^t g^2(u)du,$$

etc. We obtain

$$d\hat{X}_u = d\hat{W}_u + \varphi(u)\hat{W}_u du,$$

where

$$\varphi(u) = \left(\frac{f}{g}\right)(G^{-1}(u)).$$

Consequently, we obtain the canonical decomposition of \hat{X} :

$$\hat{X}_t = \hat{B}_t + \varphi(t) \int_0^t \hat{\alpha}(u) d\hat{B}_u,$$

where

$$\hat{\alpha}(t) = \frac{\varphi(t)\hat{\psi}(t)}{\hat{\psi}'(t)},$$

and $\hat{\psi}$ is our usual notation for the solution of $u'' = \varphi^2 u$. It now remains to undo the time-change, and relate $\hat{\alpha}$ to α , as given in (58).

5 How to get again a Brownian motion

In our initial stochastic differential equation (10), where the final value \tilde{W}_1 was known in advance, the solution $(X_t)_{0 \leq t \leq 1}$ was again a Brownian motion, and it was tied to the final value \tilde{W}_1 . Is it also possible to satisfy both conditions in our modified situation when at time t we only know the past of \tilde{W} ?

5.1 Characterization of Brownian motions

Consider a process $(X_t)_{0 \leq t \leq 1}$ with $X_0 = 0$ which satisfies the stochastic functional differential equation

$$dX_t = dW_t + \left(\int_0^t F(t, u) d\tilde{W}_u + \int_0^t H(t, u) dX_u\right) dt, \quad (60)$$

with square-integrable Volterra kernels F and H .

Theorem 5.1 *A process X satisfying (60) is a Wiener process with respect to its own filtration (\mathcal{F}_t^X) if and only if $H(t, s) = -G_F(t, s)$, where G_F is the*

square-integrable Volterra kernel defined by (48). In other words, X is of the form

$$X_t = W_t + \int_0^t \left(\int_0^s F(s, u) d\tilde{W}_u - \int_0^s G_F(s, u) dX_u \right) ds. \quad (61)$$

Proof. (1) Suppose X is a Wiener process with respect to its own filtration (\mathcal{F}_t^X) . By uniqueness of the Doob-Meyer decomposition in (\mathcal{F}_t^X) , our representation (53) implies $B = X$ and

$$\int_0^t [G_F(t, u) + H(t, u)] dX_u = 0, \quad (62)$$

\mathbb{P} - a.s. for almost all t . But (62) implies

$$G_F(t, u) + H(t, u) = 0,$$

for almost all $u \leq t$, since X is a Brownian motion.

(2) Conversely, assume that X has the form (61). The canonical representation (53) implies

$$X_t = B_t + \int_0^t \left(\int_0^s G_F(s, u) dB_u - \int_0^s G_F(s, u) dX_u \right) ds,$$

i.e.,

$$X_t + \int_0^t \int_0^s G_F(s, u) dX_u ds = B_t + \int_0^t \int_0^s G_F(s, u) dB_u ds.$$

We can now apply the reconstruction argument in section 3.2 in order to conclude $X = B$. In other words, X is a Brownian motion. \square

Remark 5.1 *The last argument shall be taken up again in Lemma 5.1.*

Let us look at the special case considered in Corollary 4.1, where F is of the form $F(t, s) = f(t)g(s)$ for some continuously differentiable functions f and g satisfying (55).

Corollary 5.1 *A process $(X_t)_{0 \leq t \leq 1}$ satisfying (56) is a Brownian motion if and only if*

$$H(t, u) = -f(t)\alpha(u),$$

where $\alpha(t)$ is the solution of (58) with boundary condition $\alpha(0) = 0$. In other words, if $(X_t)_{0 \leq t \leq 1}$ is a Brownian motion with respect to its own filtration, it must be of the form

$$dX_t = dW_t + f(t) \left(\int_0^t g(u) d\tilde{W}_u - \int_0^t \alpha(u) dX_u \right) dt. \quad (63)$$

In particular, if $g \equiv 1$, then (63) can be written as

$$dX_t = dW_t + f(t) \left(\tilde{W}_t - \int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)} dX_u \right) dt, \quad (64)$$

where $\Psi(t)$ is the solution of the Sturm-Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t), \quad (65)$$

with $\Psi(0) = 0$ and $\Psi'(0+) = 1$.

Proof. In order to get the characterization for this class of Brownian motions from Theorem 5.1, we have only to compute $G_F(t, u)$. From the proof in Corollary 4.1, we know that $G_F(t, u) = f(t)\alpha(u)$. Substituting this result in Theorem 5.1, we get the first assertion. As to the case $g \equiv 1$, simply substitute (59) in (63). \square

Corollary 5.2 *Let X be a process satisfying the stochastic differential equation (20) with $f, h \in C^1(0, 1) \cap \mathcal{A}(0, 1)$. If one of the functions $f(t)$ and $h(t)$ is not equal to 0, then X cannot be a Brownian motion.*

Proof. If X satisfying (20) is a Brownian motion, then it follows from Corollary 5.1 that f must be of the form

$$f(t)\Psi(t) = c\Psi'(t), \quad (66)$$

for some non-zero constant c . Substituting (66) in (65), we get

$$\Psi''(t) = cf(t)\Psi'(t).$$

Hence, the corresponding solution of the Sturm-Liouville equation is given by

$$\Psi(t) = \int_0^t \exp\left(c \int_0^u f(v) dv\right) du.$$

Substituting this solution again in (66), and taking derivatives on both sides with respect to t , we have

$$cf'(t) + (1 - c^2)f^2(t) = 0.$$

This implies

$$f(t) = \frac{c}{1 - c^2} \frac{1}{t},$$

which does not belong to $\mathcal{A}(0, 1)$. \square

In the class of adapted linear drift of the form (60), Theorem 5.1 characterizes those cases where the resulting process X in (61) is a new Brownian motion. Let us now return to the question whether such a Brownian motion can be tied to the endpoint \tilde{W}_1 of the Brownian motion \tilde{W} . This turns out to be impossible as long as we insist on an adapted drift, even if we drop linearity.

Proposition 5.1 *Consider any drift $(Y_t)_{0 \leq t \leq 1}$ adapted to (\mathcal{F}_t) such that*

$$X_t = W_t + \int_0^t Y_s ds,$$

is a Brownian motion. If Z is any (\mathcal{F}_t) -Brownian motion such that $X_1 = Z_1$, \mathbb{P} -a.s., then we have $Z_t = X_t = W_t$, and in particular, $Y_t = 0$, $dt \times d\mathbb{P}$ -a.s..

Proof. If $X_1 = Z_1$, then

$$X_t = E[X_1 | \mathcal{F}_t^X] = E[Z_1 | \mathcal{F}_t^X] = E[Z_t | \mathcal{F}_t^X].$$

This implies

$$E[X_t Z_t] = E[X_t^2] = t,$$

hence

$$E[(X_t - Z_t)^2] = E[X_t^2] + E[Z_t^2] - 2E[X_t Z_t] = 0.$$

Hence, $Z_t = X_t$ is an (\mathcal{F}_t) -Brownian motion, and so: $X_t = W_t$. \square

In our situation the Brownian motion $Z = \tilde{W}$ is independent of W , and so the proposition shows that it is impossible to obtain $X_1 = \tilde{W}_1$.

Remark 5.2 (i) Consider a square-integrable (\mathcal{F}_t) -adapted process $(X_t)_{t \leq 1}$ which is a martingale in its own filtration (\mathcal{F}_t^X) . Let (M_t) be a square-integrable (\mathcal{F}_t) -martingale such that $X_1 = M_1$ and $E[X_t^2] = E[M_t^2]$ for every $t \leq 1$. The proof of Proposition 5.1 shows that this implies $X_t = M_t$ for every $t \leq 1$.

(ii) As a special case of (i), assume that (X_t) is a Brownian motion in its own filtration (\mathcal{F}_t^X) . Then (X_t) is an (\mathcal{F}_t) -Brownian motion if and only if $E[X_1 | \mathcal{F}_t]$ is a Brownian motion.

(iii) As an example of (ii), consider a Brownian motion (B_t) with respect to (\mathcal{F}_t) . We know that

$$X_t := B_t - \int_0^t \frac{B_s}{s} ds$$

defines a new Brownian motion $(X_t)_{t \leq 1}$ which is not a Brownian motion with respect to (\mathcal{F}_t) ; cf., e.g., Yor [19]. It follows from (ii) that $(E[X_1 | \mathcal{F}_t])_{t \leq 1}$ cannot be a Brownian motion. In fact, a direct computation shows that

$$\begin{aligned} E[X_1 | \mathcal{F}_t] &= B_t - \int_0^t \frac{B_s}{s} ds - \int_t^1 \frac{E[B_s | \mathcal{F}_t]}{s} ds \\ &= B_t(1 + \log t) - \int_0^t \frac{B_s}{s} ds = \int_0^t (1 + \log s) dB_s. \end{aligned}$$

5.2 Enlargement of a Brownian filtration

Consider a Brownian motion X arising as the solution of the linear functional stochastic differential equation (61). Let us represent X directly in terms of the two Wiener processes W and \tilde{W} . To this end, we introduce the resolvent kernel L_F of G_F , i.e., G_F and L_F satisfy the following relations:

$$\begin{cases} G_F(t, s) + L_F(t, s) + \int_s^t L_F(t, u) G_F(u, s) du = 0, \\ G_F(t, s) + L_F(t, s) + \int_s^t G_F(t, u) L_F(u, s) du = 0. \end{cases} \quad (67)$$

Lemma 5.1 The solution of (61) is given by

$$X_t = W_t + \int_0^t \left\{ \int_0^s L_F(s, u) dW_u + \int_0^s (F(s, u) + \int_u^s L_F(s, v) F(v, u) dv) d\tilde{W}_u \right\} ds. \quad (68)$$

Proof. By (61), the process ξ defined in (49) has the form

$$d\xi_t = dX_t + \int_0^t G_F(t, u) dX_u dt.$$

Using the same argument as in the proof in section 3.2, we obtain the representation

$$dX_t = d\xi_t + \int_0^t L_F(t, u) d\xi_u dt, \quad (69)$$

Substituting (49) in (69), we obtain (68). \square

In the special case $F(t, u) = f(t)g(u)$, we have $G_F(t, s) = f(t)\alpha(s)$, and the solution L_F of (67) is given by

$$L_F(t, s) = -f(t)\alpha(s) \exp\left(-\int_s^t f(v)\alpha(v)dv\right).$$

Thus, the solution of (63) can be written as

$$\begin{aligned} dX_t &= dW_t + f(t) \left(\int_0^t g(u) \exp\left(-\int_u^t f(v)\alpha(v)dv\right) d\tilde{W}_u \right. \\ &\quad \left. - \int_0^t \alpha(u) \exp\left(-\int_u^t f(v)\alpha(v)dv\right) dW_u \right) dt. \end{aligned}$$

In particular, the solution of (64) is given by

$$dX_t = dW_t + \frac{f(t)}{\Psi'(t)} \left(\int_0^t \Psi'(u) d\tilde{W}_u - \int_0^t f(u) \Psi(u) dW_u \right) dt. \quad (70)$$

We are now going to show that the decomposition (70) of W as the sum

$$W_t = X_t + \int_0^t \frac{f(s)}{\Psi'(s)} \left(\int_0^s f(u) \Psi(u) dW_u - \int_0^s \Psi'(u) d\tilde{W}_u \right) ds$$

can be viewed, after time reversal, as the expression of a Brownian motion in an enlarged Gaussian filtration (see, Jeulin-Chaleyat-Yor [10] or Yor [20]).

We consider an n -dimensional Brownian motion $B_t \equiv (B_t^{(1)}, \dots, B_t^{(n)})$, and we enlarge its filtration with $B(\lambda) \stackrel{\text{def}}{=} \int_0^\infty (\lambda(u), dB_u)$. With respect to the enlarged filtration, the canonical decomposition of B is given by

$$B_t = \tilde{B}_t + \int_0^t \frac{\lambda(u) \int_u^\infty (\lambda(s), dB_s)}{\sigma_\lambda^2(u)} du, \quad (71)$$

where \tilde{B} is again a Brownian motion and σ_λ is defined as

$$\sigma_\lambda^2(u) = \int_u^\infty |\lambda(t)|^2 dt \equiv \int_u^\infty \left(\sum_{i=1}^n (\lambda_i(t))^2 \right) dt;$$

see Jeulin-Chaleyat-Yor [10] or Yor [20]. In order to make sure that formula (71) is meaningful (as a semimartingale decomposition), it is necessary and sufficient that

$$\int_0^t \frac{|\lambda(u)|}{\sigma_\lambda(u)} du < \infty.$$

On the other hand, (70) yields

$$B_t^{(1)} = \tilde{B}_t^{(1)} - \int_0^t \frac{f(1-u)}{\Psi'(1-u)} \left(\int_u^1 \Psi'(1-v) dB_v^{(2)} - \int_u^1 f(1-v)\Psi(1-v) dB_v^{(1)} \right) du, \quad (72)$$

with $B_t^{(1)} = W_1 - W_{1-t}$, $\tilde{B}_t^{(1)} = X_1 - X_{1-t}$ and $B_t^{(2)} = \tilde{W}_1 - \tilde{W}_{1-t}$. We want to view the representation (72) as an enlargement formula (71) for a suitable choice of λ . Thus, we would like to find a pair of functions (λ_1, λ_2) such that

$$\begin{cases} \frac{\lambda_1(u)\lambda_1(s)}{\sigma_\lambda^2(u)} = \frac{f(1-u)f(1-s)\Psi(1-s)}{\Psi'(1-u)} \\ \frac{\lambda_1(u)\lambda_2(s)}{\sigma_\lambda^2(u)} = -\frac{f(1-u)\Psi'(1-s)}{\Psi'(1-u)}. \end{cases} \quad (73)$$

The problem is now to retrieve λ_1 and λ_2 from the system (73). The solution is given by

$$\begin{cases} \lambda_1(s) = cf(1-s)\Psi(1-s) \\ \lambda_2(s) = -c\Psi'(1-s), \end{cases}$$

with some nonzero constant c .

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