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On the smoothness of value functions

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Discussion Paper #1542

On the Smoothness of Value Functions

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On the Smoothness of Value Functions

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Abstract

We prove that under standard Lipschitz and growth conditions, the value function of all optimal control problems for one-dimensional diffusions is twice differentiable, as long as the control space is compact and the volatility is uniformly bounded below, away from zero. Under similar conditions, the value function of any optimal stopping problem is differentiable.

1 Control Problem

We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ that satisfies the usual conditions.[1]

A process $\{X_t\}_{t\in\mathbb{R}_+}$ that is controlled by some other process $\{A_t\}_{t\in\mathbb{R}_+}$, such that X_t takes values in some nonempty closed interval $\mathcal{X}$ of $\mathbb{R}$ with (possibly infinite) endpoints $\underline{x} < \bar{x}$, and follows the dynamic equation

$$\begin{aligned} dX_t &= \mu(X_t, A_t)\,dt + \sigma(X_t, A_t)\,dB_t \\ x_0 &= x \end{aligned} \tag{1}$$

where B_t is the standard Brownian Motion.

Assumption 1 *There exists*[2] *a compact subset* $\mathcal{K}$ *of* $\mathbb{R}$ *such that* $A_t \in \mathcal{K}$ *for all* t.

A control process satisfying Assumption 1 and such that (1) has a unique strong solution is said to be *admissible*. The set of admissible control processes is denoted by $\mathcal{A}$.

[1] We refer the reader to Karatzas and Shreve (1998) for the standard concepts used in this paper.

[2] The analysis can be extended to the case where the control set depends on x. For simplicity we focus on a constant domain.

Given an admissible control A, the agent's expected payoff is given by

$$v(x, A) = E^{x,A}\left(\int_0^{\kappa} e^{-rt} f(X_t, A_t)\, dt + e^{-r\kappa} g(X_\kappa)\right) \tag{2}$$

where $f(x_t, a_t)$ is the flow payoff at time t and $E^{x,A}$ is the expectation operator given starting value x and control process A, and κ is the first exit time of the process X_t from $\mathcal{X}$.

The (optimal) *value function*[3] of the problem starting at x, denoted $v(x)$, is defined by

$$v(x) = \sup_{A \in \mathcal{A}} v(x, A). \tag{3}$$

An admissible control is said to be *optimal* if $v(x, A) = v(x)$.

We make the following Lipschitz and linear growth assumptions on μ, σ, and f.

Assumption 2 *There exists $K > 0$ such that for all a and x*

$$\begin{aligned} \left|\mu(x,a) - \mu(x',a)\right| &\leq K|x - x'| \\ \left|\sigma(x,a) - \sigma(x',a)\right| &\leq K|x - x'| \\ \left|f(x,a) - f(x',a)\right| &\leq K|x - x'|. \end{aligned}$$

Moreover, the functions $\mu(x,\cdot)$, $\sigma(x,\cdot)$, $f(x,\cdot)$ are continuous in a, for all x.[4]

Assumption 3 *For all a, there exist strictly positive constants K_1^μ, K_2^μ, K^σ, K^f and ε such that*

$$\begin{aligned} |\mu(x,a)| &\leq K_1^\mu + K_2^\mu |x| \\ |\sigma(x,a)| &\leq K^\sigma (1 + |x|) \\ |f(x,a)| &\leq K^f (1 + |x|) \\ |\sigma(x,a)| &\geq \varepsilon \\ K_2^\mu &< r. \end{aligned}$$

2 Twice Differentiability of the Value Function

Our objective is to prove that the value function is twice differentiable in the interior of $\mathcal{X}$ and solves the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{a \in \mathcal{K}(x)} -rv(x) + f(x,a) + \mu(x,a)\, v' + \frac{1}{2}\sigma^2(x,a) v''(x) \tag{4}$$

[3] To avoid confusion, we reserved the expression "value function" to the *optimal* expected payoff, and use "(expected) payoff" when the control is arbitrary.

[4] When $\mathcal{A}$ is finite, the continuity assumption is vacuous.

with given boundary conditions

$$v(\underline{x}) = \underline{v} = g(\underline{x}) \text{ if } \underline{x} \text{ is finite, and } v(\bar{x}) = \bar{v} = g(\bar{x}) \text{ if } \bar{x} \text{ is finite.} \tag{5}$$

THEOREM 1 *Under Assumptions 1–3, the value function is twice continuously differentiable and and is the unique solution to the HJB equation* (4) *at all* $x \in \mathcal{X}$.

Our proof consists of the following steps: 1) Prove the existence of a solution, w, to the HJB equation; 2) Construct a control process based on this solution; 3) Prove that the solution is the value function of the problem, and that either the control constructed in 2) is optimal, or that it can be approximated by a sequence of admissible controls.

These steps imply that *any* solution to the HJB equation must coincide with the value function of the problem and, therefore, they will show the uniqueness claimed in Theorem 1.

1. Existence of a Solution

Existence is shown by applying Proposition 1 in the appendix.

Equation (4) can be rewritten as

$$v'' + H(x, v, v') = 0, \tag{6}$$

where[5]

$$H(x, u, p) = \max_{a \in \mathcal{K}} \frac{2}{\sigma^2(x, a)}(-ru + f(x, a) + \mu(x, a)p). \tag{7}$$

LEMMA 1 *Under Assumptions 1–3,* $H(x, \cdot, \cdot)$ *satisfies Conditions 1, 2, and 3 in the appendix.*

First, it is easy to check that for all arguments,[6]

$$|H(x, u, p) - H(x, \tilde{u}, \tilde{p})| \leq \max_{a \in \mathcal{K}} \frac{2}{\sigma^2(x, a)} \left|(-ru + f(x, a) + \mu(x, a)p) - (-r\tilde{u} + f(x, a) + \mu(x, a)\tilde{p})\right|. \tag{8}$$

Therefore,

$$|H(x, u, p) - H(x, \tilde{u}, \tilde{p})| \leq \frac{2}{\varepsilon^2}\left(r|u - \tilde{u}| + (K_1^\mu + K_2^\mu|x|)|p - \tilde{p}|\right). \tag{9}$$

The growth condition obtains from the fact that μ is bounded on any compact in support and $\sigma^2 \geq \varepsilon^2$. Condition 2 comes from the fact $r > 0$ and $\sigma^2 > 0$. There remains to check Condition 3.

[5]Because all functions are continuous in a and σ is bounded below away from zero, the supremum is achieved as a maximum.

[6]More generally, if $H(y) = \max_{a \in \mathcal{K}} g(a, y)$, we have $|H(y) - H(\tilde{y})| \leq \max_{a \in \mathcal{K}} |g(a, y) - g(a, \tilde{y})|$. For example, suppose that $a, \tilde{a}$ maximize g at y and $\tilde{y}$, respectively. Then, $H(y) - H(\tilde{y}) = g(a, y) - g(\tilde{a}, \tilde{y}) \leq g(a, y) - g(a, \tilde{y}) \leq \max_{a \in \mathcal{K}} |g(a, y) - g(a, \tilde{y})|$. The other inequality is proved similarly. In our problem, g always has a maximizer, but the result holds even if the supremum of g is not achieved.

To show that $H(x, K_1 + K_2|x|, \varepsilon K_2)$ is negative for large enough K_1, K_2, it suffice to show that

$$-r(K_1 + K_2|x|) + K^f(1 + |x|) + (K_1^\mu + K_2^\mu|x|)K_2, \tag{10}$$

is negative. Since $K_2^\mu < r$, the result holds for all x's, provided that K_1 and K_2 are large enough.

Lemma 1 implies that all conditions of Proposition 1 in the appendix are satisfied, which shows existence of a solution to the HJB equation for arbitrary boundary conditions at $\underline{x}$ and $\bar{x}$, whenever these extreme points are finite.

2. Construction of an Admissible Control

We construct an admissible control based on our solution w. Define $\mathcal{M}(x) \subset \mathcal{K}$ as the set of maximizers in the equation

$$rw(x) = \max_{a \in \mathcal{K}} f(x, a) + \mu(x, a)w'(x) + \frac{1}{2}\sigma(x, a)^2 w''(x) \tag{11}$$

Because f, μ, σ are continuous in a, the function

$$g(x, a) = f(x, a) + \mu(x, a)\, w'(x) + \frac{1}{2}\sigma(x, a)^2\, w''(x) \tag{12}$$

is continuous in a and, therefore, $\mathcal{M}(x)$ is nonempty, closed, and compact in $\mathcal{K}$.

We can therefore apply the measurable maximum theorem,[7], which guarantees that there exists a measurable function $\hat{a}(x)$ such that $\hat{a}(x) \in \mathcal{M}(x)$ for all x.

3. The solution coincides with the value function, and the constructed control is optimal, whenever admissible

A standard verification argument then implies that $w(x) \geq v(x, A)$ for any x and any admissible control A and, therefore, that $w(x) \geq v(x)$. For completeness, and because a similar argument in the remaining parts of the proof, we reproduce this argument here. For any fixed, admissible control A and time T, Itô's formula, applied to the diffusion X^A controlled by A and starting at x, implies that

$$e^{-rT}w(X_T^A) = w(x) + \int_0^T e^{-rt}\left(-rw(X_t^A) + \mu(X_t^A, A_t)w'(X_t^A) + \frac{1}{2}\sigma^2(X_t^A, A_t)w''(X_t^A)\right)dt + \int_0^T e^{-rt}\sigma(X_t^A, A_t)dB_t. \tag{13}$$

The stochastic integral has zero mean as long as as $|X_t^A|$ grows at a slower rate than r, which is guaranteed by our assumption that $K_2^\mu < r$. This also guarantees that $\lim_{T\to\infty} e^{-rT}w(X_T^A) = 0$. Taking expectations and using (11), we get the inequality

$$\int_0^T e^{-rt} f(X_t^A, A_t) \leq w(x) - e^{-rT}w(X_T^A). \tag{14}$$

[7]See Aliprantis and Border (1999), p. 570.

Taking the limit as T goes to infinity yields

$$v(x, A) \leq w(x). \tag{15}$$

There remains to establish the reverse inequality $w(x) \leq v(x)$. Suppose, first, that the SDE

$$dX_t = \mu(X_t, \hat{a}(X_t)dt + \sigma(X_t, \hat{a}(X_t))dB_t \tag{16}$$

has a unique strong solution, $\{\hat{X}_t\}_{t\in\mathbb{R}_+}$, and let

$$A_t^* = \hat{a}(\hat{X}_t) \tag{17}$$

denote the corresponding control. The control A^* is admissible and, applying the verification argument, this time with an equality, shows that $w(x) = v(x, A^*)$, and therefore, that $w(x) \leq v(x)$.

In general, however, (16) is not guaranteed to have a unique strong solution, because the volatility function $\hat{\sigma}(x) = \sigma(x, \hat{a}(x))$ may jump, violating the standard Lipschitz (or Hölder) continuity conditions that are usually assumed for existence results. A result by Krylov (1980, p. 86) guarantees that, because $\hat{\sigma}$ is bounded away from zero and, hence, elliptic, (16) has a weak solution, unique in the sense of probability law.[8] This would conclude the proof if, instead of defining admissibility in terms of a strong solution, we had only required weak admissibility[9] It is well-known that the verification argument also applies to weak solutions.[10] Weak admissibility is often taken to be "right" concept for control of partially observable systems.

To conclude the proof for the case of strong admissibility, we use a result by Nakao (1972), who proved the existence of a strong solution to (16) as long as as the function $x \mapsto \sigma(x, \hat{a}(x))$ has bounded variation (see also Revuz and Yor, 2001, p.392), in addition to be uniformly bounded away from zero. This condition is satisfied, for example, if the control $\hat{a}(\cdot)$ is continuous or monotonic or, more generally, of bounded variation, or if $\sigma(x, a)$ is in fact independent of a. More generally, it is also satisfied in any "reasonable" problem, where the control $\hat{a}$ does not jump infinitely often on any compact subset of $\mathcal{X}$. Moreover, it is guaranteed to be satisfied if our initial problem has an optimal control. In that case, standard results guarantee that it also has a Markovian optimal control which, which maximizes pointwise the HJB equation, and can be taken without loss of generality to be $\hat{a}$.

We do not need to make any of these assumptions, however, to prove the theorem. We will show that $w(x) \leq v(x) + \varepsilon$ for any constant $\varepsilon > 0$, which will conclude the proof that $v(x) = w(x)$ and,

[8] Our linear growth condition are well-known to rule out explosions of the solution in finite time. See, e.g., Karatzas and Shreve, (1998, p. 342).

[9] A Markovian control $\hat{a}$ is weakly admissible if the SDE (16) has a weak solution, unique in the sense of probability law. See, for example, Yong and Zhou (1999).

[10] See Karatzas and Shreve (1998, Section 5.7)

hence, v is C^2 and solves the HJB equation. Fix any $\varepsilon > 0$ and let $\eta = \varepsilon/K$, where K is the Lipschitz constant included in Assumption 1. Consider a grid of $\mathcal{X}$ with equally spaced intervals of length η, anddefine the Markovian control $\tilde{a}$ by $\tilde{a}(x) = \hat{a}(y(x))$ where $y(x)$ is the element of the grid closest to x (adopting any arbitrary convention when there are two such points). By construction, $\tilde{a}$ is piecewise constant, which guarantees that $x \mapsto \sigma(x, \tilde{a}(x))$ has bounded variation and, hence, that the SDE (16) has a unique strong solution, which we denote $\{\tilde{X}_t\}_{t\in\mathbb{R}_+}$. The control $\{\tilde{A}_t\}_{t\in\mathbb{R}_+}$ defined by $\tilde{A}_t = \tilde{a}(\tilde{X}_t)$ is therefore admissible. Moreover, the Lipschitz property of μ, σ, and f guarantees that $\tilde{a}(x)$, for all x,

$$f(x, \tilde{a}(x)) + \mu(x, \tilde{a}(x))w'(x) + \frac{1}{2}\sigma(x, \tilde{a}(x))^2 w''(x) \geq \max_{a\in\mathcal{K}} f(x, a) + \mu(x, a)w'(x) + \frac{1}{2}\sigma(x, a)^2 w''(x) - D\varepsilon \tag{18}$$

for some positive constant D This guarantees that, in the verification argument, the inequality (14) is almost tight, the upper bound being proportional to $D\varepsilon/r$, yielding

$$v(x) \geq v(x, \tilde{A}) \geq w(x) - D\varepsilon/r. \tag{19}$$

Taking the limit as ε goes to zero yields the desired inequality $v(x) \geq w(x)$.

The proof has also established the following results, about the existence of an optimal control and its relation to the HJB equation.

THEOREM 2 *Let $\hat{a}$ be a measurable selection of* (11)*, and suppose that the map $x \mapsto \sigma(x, \hat{a}(x))$ has locally bounded variation. Then, the control $\{A_t^*\}$ is optimal.*

THEOREM 3 *If admissibility is replaced by weak admissibility, the control $\{A_t^*\}$ is optimal.*

THEOREM 4 *If the stochastic control problem* (3) *has an optimal control, then there exists a measurable selection of* (11) *that has a strong solution.*

3 Optimal Stopping and Smooth-Pasting

We now establish the smooth-pasting property for optimal stopping problems: the value function is differentiable at any threshold where stopping becomes optimal. For presentation clarity, we separate optimal control and optimal stopping problems, which imply different smoothness levels. The problems can be combined, however, with an appropriate extension of Theorems 1 and 5.

Consider the optimal stopping problem

$$v(x) = \max_{\tau\in\mathcal{T}} E\left(\int_0^\tau e^{-rt} f(X_t)dt + e^{-r\tau} g(X_\tau)\right), \tag{20}$$

where $\mathcal{T}$ is the set of all stopping times adapted to the initial filtration $\mathcal{F}$, and $\{X_t\}_{t\in\mathbb{R}_+}$ solves

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{21}$$

subject to the initial condition $X_0 = x$. We maintain the same assumption as before on μ, σ and f, which guarantee that the SDE has a unique strong solution.

We make the additional assumption that the termination function g is continuously differentiable.

Assumption 4 *$g(\cdot)$ is C^1.*

Theorem 5 *Under Assumptions 2– 4*[11] *v is differentiable on the interior of $\mathcal{X}$.*

Let $\mathcal{Y}$ denote the subset of $\mathcal{X}$ for which $v(x) = g(x)$. $\mathcal{Y}$ consists of all the states at which it is optimal to stop immediately. By continuity of v and g, $\mathcal{X} \setminus \mathcal{Y}$ consists of disjoint open intervals $\{\mathcal{Z}_i\}_{i\in I}$. Pick any two points $x_1 < x_2$ in such an interval. The BVP result stated in Appendix B guarantees the existence of C^2 solution w to the dynamic equation

$$w''(x) + \frac{1}{\frac{1}{2}\sigma(x)^2}(-rw(x) + f(x) + \mu(x)w'(x)) = 0. \tag{22}$$

with boundary conditions $w(x_1) = v(x_1)$ and $w(x_2) = v(x_2)$. A standard verification argument then shows that v coincides with w on any such interval and, therefore, that v is C^2 on such interval, and, therefore, on any $\mathcal{Z}_i$.

Consider now the boundary of such an interval $\mathcal{Z}_i$, for example the upper boundary, and call it x^*. By construction, $v(x) > g(x)$ for x in a left-neighborhood of x^*, and $v(x^*) = g(x^*)$. In particular, the left derivative of v at x^* must be less than or equal to $g'(x^*)$. We show by contradiction that this inequality must be tight. Consider the solution of IVP to the domain $[x_1, x_2 = x^* + \varepsilon]$ for some x_1 in $\mathcal{Z}_i$ and some small $\varepsilon > 0$, with boundary conditions $w(x_1) = v(x_1)$ and initial slope s^*, such that $w(x^*) = v(x^*)$. By construction, $w(x) < g(x)$ for x in a right neighborhood of x^*, and without loss, on (x^*, x_2). Now, take a slightly higher slope $s' > s^*$ and consider the solution to the IVP with domain $[x_1, x_2]$, with starting slope s', which we label $\hat{w}$. For s' close to s, this solution hits g at some $\hat{x} \in (x_1, x_2)$, because solutions to the IVP are continuous in s. Set $\hat{w}(x) = g(x)$ for all $x > \hat{x}$. Lemma 2 below implies that $\hat{w}(x) > w(x)$ for all $x \in (x_1, \hat{x})$ and in particular at $\hat{w}(x^*) > g(x^*) = v(x^*)$. Moreover, $\hat{w}$ corresponds to the expected payoff if the following stopping strategy is used: starting from x^*, continue until either x_1 or $\hat{x}$ is reached. If $\hat{x}$ is reached first, stop. If x_1 is reached first, follow the initial strategy leading to value $v(x_1)$. This strategy thus strictly improves on the value $v(x^*)$, starting from x^*, a contradiction.

[11] Here, the control set $\mathcal{A}$ is reduced to a singleton.

We have proved so far that v is differentiable at x, if either x lies in the interior of some interval $\mathcal{Z}_i$, or if it connects to intervals $\mathcal{Z}_i$ and $\mathcal{Z}_j$ (i.e., it is the upper bound of one interval, and the lower bound of the other), or if it is a bound of some interval $\mathcal{Z}_i$ and $v(y) = g(y)$ on the other side of x, and in the last two cases, $v'(x) = g'(x)$.

To conclude the proof, we need to show the result when x is such that $v(x) = g(x)$, but x is an accumulation point of stopping and non stopping regions, on either its right side or its left side, or both. Without loss of generality, we set $x = 0$ and establish that $v_r(0) = g'(0)$, where v_r is the right derivative of v at 0. We wish to show that $\lim_{h\downarrow 0}(v(h) - v(0))/h$ converges to $g'(0)$. Consider any $h > 0$. The difference $v(h) - v(0)$ is either equal to $g(h) - g(0)$, if h belongs to $\mathcal{Y}$, or else h belongs to some interval $\mathcal{Z}_i$ close to 0. Let y denote the lower bound of $\mathcal{Z}_i$. By right differentiability of v at y, and because the right derivative is equal to $g'(y)$, we have $v(h) = v(y) + g'(y)(h - y) + o(h - y)$. Since $v(y) = g(y)$, we can rewrite the right-hand side as $g(h) + o(h - y)$. Thus, either way, we have

$$\frac{v(h) - v(0)}{h} = \frac{g(h) - g(0)}{h} + o(h). \tag{23}$$

Taking the limit as h goes to zero yields the result.

Lemma 2 *Consider $v_{s'}$ and v_s, two solutions to the IVP with starting slopes $s' > s$ on an interval $[x_1, x_2]$ which both satisfy $v_{s'}(x_1) = v_s(x_1) = v_1$. Then, $v_{s'}(x) > v_s(x)$ for all $x \in (x_1, x2]$.*

Proof. Let $\hat{x} = \inf\{x : v_{s'}.(x) \leq v_s'(x)\}$. Note that $\hat{x} > \underline{x}$ because $v_{s'}'(\underline{x}) > v_s'(\underline{x})$ and both $v_{s'}$ and v_s are C^2. We have $v_{s'}(\hat{x}) > v_s(\hat{x})$.Since both solutions satisfy the equation

$$v''(x) + \frac{1}{\frac{1}{2}\sigma(x)^2}\left(-rv(x) + f(x) + \mu(x)v'(x)\right) = 0 \tag{24}$$

we have

$$\begin{aligned} v_{s'}''(\hat{x}) &= \frac{1}{\frac{1}{2}\sigma(\hat{x})^2}\left(rv_{s'}(\hat{x}) + f(\hat{x}) + \mu(\hat{x})v_{s'}'(\hat{x})\right) \\ &> \frac{1}{\frac{1}{2}\sigma(\hat{x})^2}\left(rv_s(\hat{x}) + f(\hat{x}) + \mu(\hat{x})v_s'(\hat{x})\right) \\ &= v_s''(\hat{x}) \end{aligned}$$

Since $v_{s'}'(x)$ must hit $v_s'(x)$ from above as x reaches $\hat{x}$, we obtain a contradiction.

A Results for Initial Value Problems of ODEs

Condition 1 *The function $H : (x, y) \mapsto H(x, y)$ defined on $\mathcal{X} \times \mathbb{R}^n$ satisfies, on any compact subset $[\underline{x}, \bar{x}]$ of $\mathcal{X}$,*

- $|H(x,y)| \leq M(1+|y|)$
- $|H(x,y) - H(x,y')| \leq K|y-y'|$.

LEMMA 3 *For any $x_0 \in \mathcal{X} \subset \mathbb{R}$ and $y_0 \in \mathbb{R}^n$, The ODE*

$$y'(x) = H(x, y(x)) \tag{25}$$

with initial condition $y(x_0) = y_0$ has a unique continuously differentiable solution on $\mathcal{X}$.

LEMMA 4 *Under the assumptions of Lemma 3, let $f(x, y_0)$ denote the solution to (25) on $\mathcal{X}$ with initial condition $f(x_0) = y_0$. Then f is continuous in y_0.*

The proofs are standard and hence omitted. See e.g. Hartman (2002), Chapters 2 and 5.

B Boundary Value Problem on Finite Domain

Suppose that $y = (v, v')$ and $H(x, v, v')$ satisfies growth and Lipschitz assumptions of the previous section.

We further assume that:

CONDITION 2 *H is nonincreasing in v.*

CONDITION 3 *For all positive constants K_1, K_2 large enough and $\varepsilon \in \{-1, 1\}$,*

$$H(x, K_1 + K_2|x|, \varepsilon K_2) < 0 \text{ and } H(x, -K_1 - K_2|x|, \varepsilon K_2) > 0. \tag{26}$$

PROPOSITION 1 *Under Conditions 1– 3, the ODE*

$$v'' + H\left(x, v, v'\right) = 0 \tag{27}$$

with boundary conditions

$$\begin{aligned} v\left(\underline{x}\right) &= \underline{v} \\ v\left(\bar{x}\right) &= \bar{v} \end{aligned}$$

has a solution.

The proof is based on the "shooting method." Intuitively, we start from the initial conditions $\left(\underline{x}, \underline{y}\right)$ and consider the solution to the initial value problem (IVP)

$$v'' + H\left(x, v, v'\right) = 0 \tag{28}$$

subject to the initial conditions $v(\underline{x}) = \underline{y}$ and $v'(\underline{x}) = s$. Given our assumption on H, Lemma 3 guarantees that this IVP will have a unique, twice continuously differentiable solution. Lemma 4 guarantees that the solution continuously depends on the starting slope s. We can establish existence to a solution of the boundary value problem (BVP) if we can show that it is always possible to pick the slope s in such a way that at $\bar{x}$, the solution to the IVP will hit $\bar{v}$.

The proof relies on constructing a rectangle of the (x, v) plane, in which the solution must be contained and which has the property that any solution to the IVP will eventually hit the set. We can then define a mapping between the initial slope s and points in the rectangle, and show that the mapping is onto, which establishes our result. Uniqueness follows from a separate argument.

Lemma 5 *For any positive constant K_v large enough, the functions $b_1(x) = -K_v(1 + |x|)$ and $b_2(x) = K_v(1 + |x|)$ satisfy the inequalities*

$$\begin{aligned} b_1'' + H\left(x, b_1, b_1'\right) &> 0 \\ b_2'' + H\left(x, b_2, b_2'\right) &< 0 \end{aligned}$$

for all $x \neq 0$, and the boundary constraints $\underline{v} \in (b_1(\underline{x}), b_2(\underline{x}))$ and $\bar{v} \in (b_1(\bar{x}), b_2(\bar{x}))$.

Proof. We have for $x \neq 0$

$$b_2''(x) + H(x, b_2(x), b_2'(x)) = H(x, K_v + K_v|x|, K_v sgn(x)),$$

which is strictly negative, by Condition 3. The argument for v_1 is analogous.

Lemma 6 *There exist s_1 and s_2 such that the solution to IVP (28) hits b_2 for all initial slopes $s \geq s_2$ and b_1 for all initial slopes $s \leq s_1$.*

Proof. By suitably translating the problem, we can without loss assume that $\underline{x} = \underline{y} = 0$.[12] We wish to show that for high enough initial slopes s, the solution $v(s)$ hits b_2. Consider the auxiliary IVP

$$u'' + Ku' + H(x, u(x), 0) + \varepsilon = 0 \tag{29}$$

subject to $u(0) = 0$ and $u'(0) = s$, where K is the Lipschitz constant of H and ε is a positive constant. We will show that, for s high enough, u is strictly increasing on $[0, \bar{x}]$. For fixed s, let $\tilde{x} > 0$ denote the first time that $u'(x) = 0$. On $[0, \tilde{x}]$, we have $u(x) \geq 0$. By Condition 2, we have $H(x, u(x), 0) \leq H(x, 0, 0)$ on that domain, and

$$u''(x) + Ku'(x) + M \geq 0, \tag{30}$$

[12]The translation is obtained by letting $\bar{v}(x) = v(x - \underline{x}) - \underline{y}$ and $\bar{H}(x, v, v') = H(x - \underline{x}, v + \underline{y}, v')$. $\bar{H}$ inherits the Lipschitz and monotonicity properties of H, as is easily checked.

where $M = \max_{x\in[0,\bar{x}]} H(x,0,0)+\varepsilon$. Applying Grönwall's inequality to the function $g(x) = -u'(x) - M/K$, which satisfies the inequality

$$g'(x) \leq -Kg(x) \tag{31}$$

on $[0,\tilde{x}]$, we conclude that

$$u'(x) \geq [s + M/K]\exp(-Kx) - M/K \tag{32}$$

on that domain, which implies that $\tilde{x}$ is bounded below by[13]

$$\frac{1}{K}\log\left(\frac{s+|M|/K}{|M|/K|}\right), \tag{33}$$

which exceeds $\bar{x}$, for s high enough. Moreover, the lower bound on u' also implies that u hits b_2 for s large enough.

To conclude the proof, we will show that the IVP solution v is above u for any fixed s. The Lipschitz property of H in its last argument implies that, for all x, u, u',

$$-Ku' \leq H(x,u,0) - H\left(x,u,u'\right) \tag{34}$$

From the definition of u, this implies that

$$u''(x) + H\left(x,u(x),u'(x)\right) \leq -\varepsilon < 0 \tag{35}$$

for all x. This implies that v, the solution to the IVP, lies above u, for the following reason. At $x = 0$, u and v have the same starting values and slopes, but u has a lower second derivative, by at least ε, which implies that $u' < v'$ in a right neighborhood of 0. We will show that $u' < v'$ for all x in $(0,\bar{x}]$ and, therefore, that $u < v$ on that domain. Suppose by contradiction that there exists an $x > 0$ such that $u'(x) = v'(x)$, and let $\tilde{x}$ be the first such point. Necessarily, $u(\tilde{x}) < v(\tilde{x})$. Moreover, we have

$$u''(\tilde{x}) < -H(\tilde{x},u(\tilde{x}),u'(\tilde{x})) \leq -H(\tilde{x},v(\tilde{x}),v'(\tilde{x})) = v''(\tilde{x}), \tag{36}$$

where the second inequality is guaranteed by 2, which yields a contradiction. ■

We can finally prove Proposition 1. Let

$$B = \{(x,y)|b_1(x) = y \text{ or } b_2(x) = y\} \cup [(\bar{x},b_1(\bar{x})),(\bar{x},b_2(\bar{x}))] \subset \mathbb{R}^2. \tag{37}$$

B consists of the graph of the functions b_1 and b_2 on $\mathcal{X}$, along with the vertical segment joining the endpoints of these graphs at $\bar{x}$. We also define the function $\mathcal{H} : [s_1,s_2] \to \mathbb{R}^2$ as the *last* hitting

[13] If $M = 0$, the inequality implies that u' is strictly positive.

point of B for the solution of the IVP with slope s. This function is clearly well defined: if a solution does not crosses b_1 or b_2 before $\bar{x}$, it has to hit the vertical segment joining $b_1(\bar{x})$ and $b_2(\bar{x})$. From Lemma 6, $\mathcal{H}(s)$ is on the graph of b_2 for s large and on the graph of b_1 for s small (for s large, for example, remember that u had a slope greater than s and hence does not cross b_2 again after hitting it once). Moreover, $\mathcal{H}$ cannot jump from the graph of b_2 to the graph of b_1 as s changes, because Lemma 5 implies, for example, that after v crosses b_2, it stays above v_2 for all x beyond the crossing point,[14] and hence cannot hit b_1. Therefore, $\mathcal{H}$ must connect the upper and lower bounds of B as s goes down. Finally, Lemma 4 implies that $\mathcal{H}$ is continuous at any point s for which $\mathcal{H}(s)$ lies on the vertical segment, this implies that that $\mathcal{H}(s)$ must take all values on that segment as it connects the graphs of b_2 and b_1. Since $(\bar{x}, \bar{y})$ belongs to that segment, this proves existence of a solution that solves the BVP.

C Boundary Value Problems on Infinite Domains

We now establish existence of a function v which satisfies

$$v'' = H\left(x, v, v'\right) \tag{38}$$

and

$$|v(x)| \leq K_v(1 + |x|) \tag{39}$$

on $\mathcal{X}$, where K_v is the constant used to construct the bounds b_1, b_2 in Lemma 5. The arguments are based on Bailey (1968).

From the previous section, we know that the BVP will have a unique C^2 solution for any finite interval $[\underline{x}, \bar{x}]$ and boundary conditions $v\left(\underline{x}\right) = \underline{y}$ and $v\left(\bar{x}\right) = \bar{y}$. Further, we know that the solution satisfies $-K_v(1+|x|) \leq v\left(x\right) \leq K_v(1+|x|)$ on $[\underline{x}, \bar{x}]$. The constant K_v depends only on the primitive functions of the problem, and not on the particular interval chosen.

We define a sequence of boundary value problems which satisfy equation (38) on $[\underline{x}_n, \bar{x}_n]$ with boundary conditions $v\left(\underline{x}_n\right) = \underline{y}_n$ and $v\left(\bar{x}_n\right) = \bar{y}_n$ for some values $\underline{y}_n, \bar{y}_n$ in $(b_1(\underline{x}_n), b_2(\underline{x}_n))$ and $(b_1(\bar{x}_n), b_2(\bar{x}_n))$, respectively, and let $\underline{x}_n$ and/or $\bar{x}_n$ tend to infinity. In the following, we use the Arzelà-Ascoli theorem and show that this procedure indeed yields a solution.

Let v_n be the solution to the n'th BVP. We use the following comparison theorem.[15]

[14] The proof of this result is similar to the proof that v stays above u in Lemma 6, showing that $v' \geq b_2'$ after the crossing point, and exploits the inequality $b_2'' + H(x, b_2, b_2') < 0$.

[15] See Hartman (2002), p. 428.

Lemma 7 *Let ϕ denote a nonnegative, continuous function on $\mathbb{R}_+$, such that*

$$\int_0^\infty \frac{s}{\phi(s)} = \infty, \tag{40}$$

and $R > 0$,$\tilde{x} > 0$ denote two constants. Then, there exists a number M such that if $v(x)$ is C^2 on $[0, \bar{x}]$ with $\bar{x} > \tilde{x}$ and satisfies $|v(x)| \leq R$ and $|v''(x)| \leq \phi(|v'(x)|)$, then $|v'(x)| \leq M$ on $[0, \bar{x}]$. The constant M depends only on R, ϕ and $\tilde{x}$.

We have

$$|v''(x)| = |H(x, v(x), v'(x)| \leq |H(x, v(x), 0)| + K|v'(X)| \leq K_1 + K|v'(x)| \tag{41}$$

for any bounded domain $X = [\underline{x}, \bar{x}]$, by the Lipschitz property of H (with constant K), where the constant K_1 is obtained by boundedness of v on X (since it is contained between b_1 and b_2, and continuity of $H(\cdot, \cdot, 0)$ on the corresponding compact domain.

Since $\phi(x) = K_1 + Kx$ satisfies (40), each v_n' is bounded on the compact domain X, and since the bound is uniform for all n. Moreover (41) implies that the second derivative of v is also uniformly bounded.

We now employ the following diagonalization procedure. Consider a finite domain $[\underline{x}_1, \bar{x}_1]$. By the previous argument, for each n, v_n, v_n' and v_n'' are bounded on $[\underline{x}_1, \bar{x}_1]$, and the bounds are uniform in n. By Arzelà-Ascoli, there exists a subsequence such that v_n converges uniformly to a C^1 function $\tilde{v}_1$ on $[\underline{x}_1, \bar{x}_1]$. [16] Moreover, the second derivatives $\{v_n''\}_{n\in\mathbb{N}}$ are also equicontinuous, because they satisfy $v_n''(x) = H(x, v_n(x), v_n'(x))$ with H continuous and v_n and v_n' equicontinuous. This implies that there is a subsequence of v_n that converges uniformly to a C^2 function $\tilde{v}_1$ on $[\underline{x}, \bar{x}]$. This also implies that the limit satisfies $\tilde{v}_1''(x) = -H(x, \tilde{v}_1(x), \tilde{v}_1'(x))$. By construction, $b_1(x) \leq v_n(x) \leq b_2(x)$ on $[\underline{x}_1, \bar{x}_1]$ and, therefore, $\tilde{v}_1$ is also contained between b_1 and b_2.

To conclude, take the finite domain $[\underline{x}_2, \bar{x}_2] \supset [\underline{x}_1, \bar{x}_1]$. Applying Arzelà-Ascoli again, [17] there exists a subsequence of the first subsequence such that v_n converges uniformly to a limit function $\tilde{v}_2$ on $[\underline{x}_2, \bar{x}_2]$. The functions $\tilde{v}_1$ and $\tilde{v}_2$ coincide on $[\underline{x}_1, \bar{x}_1]$. Proceeding iteratively, we can cover the entire domain $\mathcal{X}$. The function v defined by $v(x) = \tilde{v}_k(x)$ for $x \in [\underline{x}_k, \bar{x}_k] \setminus [\underline{x}_{k-1}, \bar{x}_{k-1}]$, solves the BVP and is bounded by b_1 and b_2.

[16] More precisely, we use the following version: if a sequence of C^1 functions have equicontinuous and uniformly bounded derivatives, and is bounded at one point, then it has a subsequence that converges uniformly to a C^1 function. Here equicontinuity of the derivatives is guaranteed by the uniform bound on the second derivative.

[17] Note that the bounds for the domains $[\underline{x}_2, \bar{x}_2]$ and $[\underline{x}_1, \bar{x}_1]$ are different. However, since we are fixing the domain, we are still able to obtain a convergent subsequence.

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