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Approximation Results for Discontinuous Games with an Application to Equilibrium Refinement*

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Abstract

We provide approximation results for Nash equilibria in possibly discontinuous games when payoffs and strategy sets are perturbed, and compare these conditions to those considered in the related literature. We then prove existence results for a new "finitistic" infinite-game generalization of Selten's [17] notion of perfection, and study some of its properties. The existence results, which rely on the approximation theorems, relate existing notions of perfection to the new specification.

Keywords: infinite normal-form game, discontinuous game, Nash equilibrium correspondence, payoff security, equilibrium refinement, finite approximation, trembling-hand perfect equilibrium, limit-of-finite perfect equilibrium.

JEL classification: C72.

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1 Introduction

In this paper, we provide approximation results for Nash equilibria in possibly discontinuous games when payoffs and strategy sets are perturbed. These conditions are compared to similar conditions from the related literature and are then used to derive existence results for a new "finitistic" infinite-game generalization of Selten's [17] notion of perfection. Some of the properties of the new refinement specification are studied and related to existing formulations of perfection. To introduce the issues that we address, suppose that $G = (X_i, u_i)_{i=1}^N$ is an N-player strategic-form game defined by action sets X_i and payoff functions u_i.

One of our main goals is to identify general conditions under which an approximation result of the following type will hold:

Statement C. If

(i) for each i, (X_i^α) is a net of subsets of X_i and (u_i^α) is a net in the space of payoff functions defined on $X := \times_{i=1}^N X_i$ with limit u_i,

(ii) (x^α) is a net in X with limit $x \in X$ such that, for each α, x^α is an ε^α-Nash equilibrium of the game $G^\alpha = (X_i^\alpha, u_i^\alpha)_{i=1}^N$, and

(iii) $\varepsilon^\alpha \to 0$,

then x is a Nash equilibrium of the game $G = (X_i, u_i)_{i=1}^N$.

This approximation result and some variants of it will be useful when addressing questions of equilibrium refinement. The archetype for all results of this kind is the classic closed graph theorem for the Nash equilibrium correspondence of the mixed extension of a game when the payoff functions are the parameters. This classic result relies on continuity of the payoff functions. Several papers have addressed the more general approximation question identified above in the framework of continuous and discontinuous games. The papers of Lucchetti and Patrone [14] and Gürkan and Pang [13] study the approximation problem when strategy sets are subsets of finite dimensional Euclidean spaces and (u^α) is a *multi-hypoconvergent sequence* with limit u. The work of Bagh [4] is closer to our approach. He introduces the notion of *variational convergence* for sequences of games and recovers approximation-based existence results of Dasgupta and Maskin [11], Simon

[18], and Gatti [12]. We will provide a more detailed comparison between our work and that in [4] in Section 3 below.

We begin by introducing the notion of *sequential better-reply security of a game* $G = (X_i, u_i)_{i=1}^N$ *with respect to a net* (X^α, u^α), where (X^α) is a net of subsets of X and (u^α) is an approximating net for $(u_1, ..., u_N)$. We relate this condition to the aforementioned work of Lucchetti and Patrone [14], Gürkan and Pang [13], and Bagh [4], and then use it to prove Statement C and some variants. If $X^\alpha = X$ and $u^\alpha = u$, then we say that $G = (X_i, u_i)_{i=1}^N$ satisfies *sequential better-reply security.* Sequential better-reply security is weaker than Barelli and Soza's [5] *generalized better-reply security*, which is, in turn, weaker than the notion of *better-reply security* introduced in Reny's [16] seminal work. Sequential better-reply security is implied by *weak reciprocal upper semicontinuity* (Bagh and Jofre [3]) and *weak payoff security* (Dasgupta and Maskin [11]), and is weaker than *weak better-reply security*, as defined in Carmona [10]. Moreover, Barelli and Soza's [5] *generalized B-security* and McLennan *et al.*'s [15] *B-security* need not imply Statement C.

In Section 4, we provide appropriate analogues for our definitions for the mixed extension of a game, and extend the approximation results to the case of mixed strategies (Theorem 5 and its corollaries).

Section 5 applies the approximation results derived in Sections 3 and 4 to the analysis of perfect equilibrium in discontinuous games. We briefly survey the existing infinite-game extensions of perfection, including Simon and Stinchcombe's [19] *limit-of-finite* formulations. The limit-of-finite approach takes the view that infinite models are merely convenient representations of "true" models, which are large but finite. Simon and Stinchcombe [19] define an ε-perfect equilibrium as a completely mixed ε-equilibrium (where the players choose actions that are in some sense close to their set of best responses to the other players' strategies), and a *limit-of-finite (lof) perfect equilibrium* as the limit of ε-perfect equilibria for successively larger finite approximations of an infinite game. It is shown in [19] that the notion of lof perfection is ill-suited even in continuous games, for it fails a weakening of admissibility, termed *limit admissibility* in [19], which requires that no player choose an action in the interior of the set of weakly dominated actions. The failure of limit admissibility is due in part to the fact that, in some games, whether or not a particular strategy is included in the finite approximations can drastically change the set of perfect equilibria. This leads Simon and Stinchcombe to "anchor" the finite approximations, a process that ensures that "all" pure strategies are represented. While anchored perfect equilibria are immune to the inclusion of any finite set of strategies in the sequence of finite approximations to the infinite strategy space, it is shown in [19] that anchored perfection also fails limit admissibility.

We introduce a new limit-of-finite approach, in the spirit of Simon and Stinchcombe's [19] formulations, that does not suffer from this drawback. This approach relies on finite approximations to Selten perturbations of a game. A Selten perturbation may be viewed as a model of slight mistakes in which any player may tremble and play any one of her actions. Standard notions of perfection, when stated in terms of Selten perturbations, define a perfect equilibrium as the limit of some sequence of (exact) equilibria of neighboring Selten perturbations of a game. Thus, an equilibrium μ is perfect if there exists a sequence of models of (low-probability) mistakes that have at least one equilibrium close to μ, so that μ describes approximate equilibrium behavior when the players interact in the perturbed game. Our "finitistic" approach to perfection defines a limit-of-finite perfect equilibrium as the uniform limit of sequences of (exact) equilibria of neighboring *finite* Selten perturbations that respect the strategic aspects of the original (infinite) game, in the sense that they can be interpreted as "true" representations of certain infinite Selten perturbations. The consistency between the finite and the infinite models is obtained by requiring that an equilibrium be the uniform limit of some set of sequences of equilibria of finite Selten perturbations, where the set of sequences is sufficiently close to some "mirror" sequence of infinite perturbations. An equilibrium with this property is called a *strong limit-of-finite (lof) perfect* equilibrium. Strong lof perfect equilibria are lof perfect in Simon and Stinchcombe's [19] sense (Proposition 4), but the converse is not true.

Subsection 5.1 provides existence results for strong lof perfect equilibrium profiles. We first state and prove a result relating the convergence theorems furnished in Sections 3 and 4 to the existence of strong lof perfect and trembling-hand perfect equilibria in discontinuous games (Theorem 7). We can then show that the members of a class of possibly discontinuous games considered in Carbonell-Nicolau [6] possess strong lof perfect equilibria that are also trembling-hand perfect (Theorems 8 and 9).

Subsection 5.2 studies the relationship between strong lof perfection and limit admissibility. For continuous games, all strong lof perfect equilibria are trembling-hand perfect (Theorem 10). This result can be combined with results from Simon and Stinchcombe [19] to conclude that strong lof perfection satisfies (unlike Simon and Stinchcombe's [19] lof perfection) limit admissibility in continuous games (Theorem 11).

2 Preliminaries

A ***strategic-form game*** is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $u_i \in B(X)$, where $B(X)$ denotes the space of bounded, real-valued functions defined on $X := \times_{i=1}^N X_i$. We view $B(X)$ as a metric space with associated metric defined by

$$\rho(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

Let $X_{-i} := \times_{j \neq i} X_j$ for each i. We will often abuse notation and simply write $G = (X_i, u_i)$ for $G = (X_i, u_i)_{i=1}^N$. Given i and $(x_i, x_{-i}) \in X_i \times X_{-i}$, we employ the standard convention and write $(x_1, ..., x_N)$ in X as (x_i, x_{-i}). If $G = (X_i, u_i)$ is a game and $Y_i \subseteq X_i$ for each i, we will write $(Y_i, u_i|_{Y_i})_{i=1}^N$ simply as $(Y_i, u_i)_{i=1}^N$ or (Y_i, u_i).

Let $U(X)$ denote cartesian product of N copies of $B(X)$. We also view $U(X)$ as a metric space, and denote, by a slight abuse of notation, the associated metric again by ρ, i.e.,

$$\rho((f_1, ..., f_N), (g_1, ..., g_N)) := \max_{i \in \{1,...,N\}} \left[\sup_{x \in X} |f_i(x) - g_i(x)| \right].$$

Consequently, a net (u^α) in $U(X)$ is convergent with limit u if and only if for each i, the net (u_i^α) is uniformly convergent with limit u_i.

If $G = (X_i, u_i)$ is a game then a ***test net for*** G is a net $(G^\nu) = (X_i^\nu, u_i^\nu)$, where, for each i, (X_i^ν) is a net of nonempty subsets of X_i and (u_i^ν) is a net of functions in $B(X)$. Note that we do not (yet) assume that (u_i^ν) is an approximating net for u_i. The *graph* of $G = (X_i, u_i)$, denoted by Γ_G, is defined by

$$\Gamma_G := \{(x, \alpha) \in X \times \mathbb{R}^N : u_i(x) = \alpha_i, \text{ for all } i\}.$$

Definition 1. If $\varepsilon \geq 0$, then a strategy profile $x = (x_i, x_{-i})$ in X is an ε-***Nash equilibrium*** of $G = (X_i, u_i)$ if $u_i(y_i, x_{-i}) \leq u_i(x) + \varepsilon$ for each $y_i \in X_i$ and every i. A 0-Nash equilibrium will be called a ***Nash equilibrium***.

Define a correspondence $\mathcal{N}_X : U(X) \rightrightarrows X$ that assigns to each profile $u = (u_1, ..., u_N) \in U(X)$ the set of Nash equilibria $\mathcal{N}_X(u)$ of (X_i, u_i). Given $\varepsilon \geq 0$, define a correspondence $\mathcal{N}_X^\varepsilon : U(X) \rightrightarrows X$ that assigns to each profile $u = (u_1, .., u_N) \in U(X)$ the set of ε-Nash equilibria $\mathcal{N}_X^\varepsilon(u)$ of (X_i, u_i).

When each X_i is a (nonempty) topological space, $G = (X_i, u_i)_{i=1}^N$ is called a ***topological game***. When each X_i is a nonempty metric space, we say that G is a ***metric game***. If, in addition each X_i is a compact topological (metric) space, then G is called a ***compact topological (metric) game***. If each X_i is a convex subset of a vector space and, for each i and every $x_{-i} \in X_{-i}$, and the function $u_i(\cdot, x_{-i})$ is quasiconcave on X_i, then we say that G is ***quasiconcave***.

If G is a topological game, then $\overline{\Gamma}_G$ will denote the closure of the graph Γ_G with respect to the induced product topology on $X \times \mathbb{R}^N$. If G is a topological game, we are not (yet) assuming that the strategy spaces X_i are first countable (or Hausdorff for that matter). Consequently, our approximation results in this section are formulated in terms of nets. However, virtually all of our results will hold if we replace "net" and "subnet" with "sequence" and "subsequence."[1] If we were to work only in a metric space framework, then we could have used sequences everywhere and in fact, we explicitly use sequences in Conditions 6-7, Theorem 6, and Corollary 4 below. Throughout the paper, we will use $(x^\alpha) = (x^\alpha)_{\alpha \in D}$ to denote a net with indices belonging to a directed set $(D, \prec)$. As we stated in the introduction, one of our main goals is to identify general conditions under which an approximation result like that posed in Statement C will hold. As we show below, results of this type will be useful in the study of refinements of equilibrium.

If (S^α) is a net of subsets of a topological space S, define the (Painleve-Kuratowski) ***topological limit superior of*** (S^α), denoted $\text{Ls}(S^\alpha)$, to be the set of $y \in S$ such that there exist a subnet (S^β) and a net (y^β) satisfying $y^\beta \in S^\beta$ for each β and $y^\beta \to y$.

[1]Some care is required here. For example, the closedness conclusion of Corollary 1 would not hold if we presented our definitions in terms of sequences rather than nets.

3 Approximation results for pure-strategy Nash equilibria of topological games

3.1 Perturbed payoffs and strategy sets

Condition 1. Let $G = (X_i, u_i)$ be a topological game. Suppose that (X_i^ν, u_i^ν) is a test net for G. The game (X_i, u_i) satisfies ***sequential better-reply security with respect to*** (X_i^ν, u_i^ν) if the following condition is satisfied: if (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν), if $(x^\alpha, u^\alpha(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in X^\alpha$ for each α, and if x is not a Nash equilibrium of (X_i, u_i), then there exist an i, an $\eta > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i^\beta$ and $u_i^\beta(y_i^\beta, x_{-i}^\beta) \geq \eta$.

Condition 2. A topological game $G = (X_i, u_i)$ satisfies ***sequential better-reply security (sbrs)*** if and only if G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν), with $(X_i^\nu, u_i^\nu) = (X_i, u_i)$ for all ν. That is, G satisfies sequential better-reply security if the following holds: if $(x^\alpha, u(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ and if x is not a Nash equilibrium of (X_i, u_i), then there exist an i, an $\eta > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i$ and $u_i(y_i^\beta, x_{-i}^\beta) \geq \eta$.

Remark 1. Let $G = (X_i, u_i)$ be a topological and suppose that (u^ν) is a net of functions in $U(X)$. Then $G = (X_i, u_i)$ satisfies ***sequential better-reply security with respect to*** (X_i, u_i^ν) if G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν) with $X_i^\nu = X_i$ for each ν and each i.

Theorem 1. *Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X_i^ν, u_i^ν) is a test net for G. If $G = (X_i, u_i)$ satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν) (Condition 1), then the following condition is satisfied: If (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν) and if $(x^\alpha, u^\alpha(x^\alpha))$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α, where $\varepsilon^\alpha \to 0$, then $x \in \mathcal{N}_X(u)$.*

Proof. Suppose that (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν). Suppose that $(x^\alpha, u^\alpha(x^\alpha))$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α, where $\varepsilon^\alpha \to 0$. If x is not a Nash equilibrium of (X_i, u_i), then there exist an i, an $\eta > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i^\beta$ and $u_i^\beta(y_i^\beta, x_{-i}^\beta) \geq \eta$. Now choose ε and an index β' so that $\eta > \gamma_i + \varepsilon$, $\frac{\varepsilon}{2} > \varepsilon^{\beta'}$, and $u_i^{\beta'}(x_i^{\beta'}, x_{-i}^{\beta'}) < \gamma_i + \frac{\varepsilon}{2}$. Then

$$u_i^{\beta'}(y_i^{\beta'}, x_{-i}^{\beta'}) \geq \eta > \gamma_i + \varepsilon > u_i^{\beta'}(x_i^{\beta'}, x_{-i}^{\beta'}) + \varepsilon^{\beta'},$$

contradicting the assumption that $x^{\beta'} \in \mathcal{N}_{X^{\beta'}}^{\varepsilon^{\beta'}}(u^{\beta'})$. ■

Theorem 1 provides a very general approximation result for Nash equilibria in discontinuous games in which (X_i^ν, u_i^ν) is an approximating net for the game $G = (X_i, u_i)$ with respect to which G satisfies sequential better-reply security. If (x^α) is a convergent sequence of ε^α-equilibria for some subnet (X_i^α, u_i^α) and if $(u^\alpha(x^\alpha))$ is a convergent net, then $x \in \mathcal{N}_X(u)$. This formulation of the result is useful for establishing the nonemptiness of $\mathcal{N}_X(u)$ since one need only find one net of ε^α-equilibria for which $(x^\alpha, u^\alpha(x^\alpha))$ is convergent. In applications, however, it is especially useful to know when the limit of *any* convergent sequence of approximate equilibria of approximating games is an equilibrium of the limit game. That is, it is useful to identify conditions that would ensure that $\varepsilon^\nu \to 0$ implies that

$$\mathrm{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u).$$

A very weak condition that would yield an approximation result of this form is the following:

Definition 2. Suppose that $G = (X_i, u_i)$ is a topological game. A test net (X_i^ν, u_i^ν) for G satisfies ***weak payoff convergence*** if for every subnet (X_i^α, u_i^α) the net $(u^a(x^\alpha))$ contains a convergent subnet whenever (x^α) is convergent, $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α, and $\varepsilon^\alpha \to 0$.

Theorem 2. *Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X_i^ν, u_i^ν) is a test net for G. Furthermore, suppose that (X_i^ν, u_i^ν) satisfies weak payoff convergence. If G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν) (Condition 1) then $\varepsilon^\nu \to 0$ implies that*

$$\mathrm{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u).$$

Proof. Suppose that $\varepsilon^\nu \to 0$ and $x \in \mathrm{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu))$. Then there exists a subnet $(\mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha))$ and a net (x^α) satisfying $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α such that $x^\alpha \to x$. Since (X_i^ν, u_i^ν) satisfies weak payoff convergence, there exists a further subnet (x_i^β) of (x_i^α) such that $(x^\beta, u^\beta(x^\beta))$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$. Since $\varepsilon^\nu \to 0$ implies that $\varepsilon^\beta \to 0$ and since (X_i^β, u_i^β) is a subnet of (X_i^ν, u_i^ν), we can apply Theorem 1 and conclude that $x \in \mathcal{N}_X(u)$. ■

Weak payoff convergence is quite general and simply requires that some subnet of the players' approximate equilibrium payoffs be convergent along a convergent net of approximate equilibria. In particular, weak payoff convergence does not require that $(u^\nu(x^\nu))$ contain a convergent subnet with limit $u(x)$. However, weak payoff convergence will typically not be easy to verify so we are interested in stronger but more tractable sufficient conditions.

Before addressing this issue, we will relate sequential better-reply security to several other approximation results in the literature due to Gürkan and Pang [13], Lucchetti and Patrone [14] and, most recently, Bagh [4]. We begin with a notion of convergence from variational analysis (see, for example, Gürkan and Pang [13] and Lucchetti and Patrone [14]) that is appropriate for the study of approximation problems in game theory.

Definition 3. Let $X_1, ..., X_N$ be topological spaces. A net (u^α) in $U(X)$ is ***multi-hypoconvergent with limit*** u if for each i and every $x \in X$, the following conditions hold:

- If (y_{-i}^α) is a convergent net in X_{-i} with limit x_{-i}, then there exists a convergent net (z_i^α) in X_i with limit x_i such that

$$\underline{\lim}\, u_i^\alpha(z_i^\alpha, y_{-i}^\alpha) \geq u_i(x).$$

- If (y^α) is a convergent net in X with limit x, then

$$\overline{\lim}\, u_i^\alpha(y^\alpha) \leq u_i(x).$$

This definition can be adapted to nets of games (X_i^ν, u_i^ν) in a straightforward way.

Definition 4. Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X_i^ν, u_i^ν) is a test net for G. The net (X^ν, u^ν) is ***multi-hypoconvergent with limit*** (X, u) if for each i and every $x \in X$, the following conditions hold for every subnet (X_i^α, u_i^α) of (X_i^ν, u_i^ν):

- If (y^α_{-i}) is a convergent net in X_{-i} with limit x_{-i} satisfying $y^\alpha_{-i} \in X^\alpha_i$ for each α, then there exists a convergent net (z^α_i) with limit x_i satisfying $z^\alpha_{-i} \in X^\alpha_i$ for each α such that

$$\underline{\lim}\, u^\alpha_i(z^\alpha_i, y^\alpha_{-i}) \geq u_i(x).$$

- If (y^α) is a convergent net with limit x satisfying $y^\alpha \in X^\alpha$ for each α, then

$$\overline{\lim}\, u^\alpha_i(y^\alpha) \leq u_i(x).$$

Multi-hypoconvergence is related to the notion of variational convergence introduced in Bagh [4].

Given a test net of games $(G^\nu) = (X^\nu_i, u^\nu_i)$ for a game G and $(x, \gamma) \in X \times \mathbb{R}^N$, it follows that that $(x, \gamma) \in \mathrm{Ls}(\Gamma_{G^\nu})$ if and only if there exists a subnet (G^α) and a net $(x^\alpha, u^\alpha(x^\alpha)) \to (x, \gamma)$ with $x^\alpha \in X^\alpha$ for each α.

Definition 5. Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X^ν_i, u^ν_i) is a test net for G. The net (X^ν_i, u^ν_i) is ***variationally convergent with limit*** $G = (X_i, u_i)$ if the following holds:

(A) For each i, $\varepsilon > 0$, $z_i \in X_i$, and $x \in \mathrm{Ls}(\mathcal{N}_{X^\nu}(u^\nu))$, the following conditions are satisfied: if $(G^\alpha) = (X^\alpha_i, u^\alpha_i)$ is a subnet of (G^ν) and $x^\alpha_{-i} \to x_{-i}$ with $x^\alpha \in \mathcal{N}_{X^\alpha}(u^\alpha)$ for each α, then for each α there exists $y^\alpha_i \in X^\alpha_i$ and there exists an α' such that

$$u^\alpha_i(y^\alpha_i, x^\alpha_{-i}) > u_i(z_i, x_{-i}) - \varepsilon$$

whenever $\alpha' \prec \alpha$.

(B) For each $(x, \gamma) \in [\mathrm{Ls}(\Gamma_{G^\nu})] \backslash \Gamma_G$, there exist an i and $\xi_i \in X_i$ such that $u_i(\xi_i, x_{-i}) > \gamma_i$.

Definition 6. Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X^ν_i, u^ν_i) is a test net for G. The net (X^ν_i, u^ν_i) is ***strongly variationally convergent with limit*** $G = (X_i, u_i)$ if the following conditions are satisfied:

(A) For each i, $\varepsilon > 0$, $z_i \in X_i$, and $x \in \mathrm{Ls}(X^\nu)$, the following conditions are satisfied: if $(G^\alpha) = (X^\alpha_i, u^\alpha_i)$ is a subnet of (G^ν) and $x^\alpha_{-i} \to x_{-i}$ with $x^\alpha_{-i} \in X^\alpha_{-i}$ for each α, then for each α there exists $y^\alpha_i \in X^\alpha_i$ and there exists an α' such that

$$u^\alpha_i(y^\alpha_i, x^\alpha_{-i}) > u_i(z_i, x_{-i}) - \varepsilon$$

whenever $\alpha' \prec \alpha$.

(B) For each $(x,\gamma) \in [\mathrm{Ls}(\Gamma_{G^\nu})]\backslash\Gamma_G$, there exist an i and $\xi_i \in X_i$ such that $u_i(\xi_i, x_{-i}) > \gamma_i$.

Remark 2. Strong variational convergence differs from variational convergence in that the latter only requires the convergence properties of part (A) to hold along sequences of Nash equilibria. It should be noted that sequential better-reply security with respect to (X_i^ν, u_i^ν) can be weakened in a similar fashion: if (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν), if $(x^\alpha, u^\alpha(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x,\gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in \mathcal{N}_{X^\alpha}(u^\alpha)$ for each α, and if x is not a Nash equilibrium of (X_i, u_i), then there exist an i, an $\eta > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i^\beta$ and $u_i^\beta(y_i^\beta, x_{-i}^\beta) \geq \eta$. From the proofs, it is hopefully clear that all of our results (Theorem 1 in particular) will hold under this weaker assumption.

The next two results show the relationship between multi-hypoconvergence, strong variational convergence, and sequential better-reply security.

Proposition 1. *If the net $(G^\nu) = (X_i^\nu, u_i^\nu)$ is strongly variationally convergent with limit $G = (X_i, u_i)$, then G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν).*

Proof. Suppose that (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν), $(x^\alpha, u^\alpha(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x,\gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in X^\alpha$ for each α and x is not a Nash equilibrium of (X_i, u_i). Since x is not a Nash equilibrium of (X_i, u_i), there exist i and $z_i \in X_i$ such that $u_i(z_i, x_{-i}) > u_i(x)$. If $u(x) = \gamma$, then

$$u_i(z_i, x_{-i}) > \gamma_i.$$

Choose $\varepsilon > 0$ such that $\eta = u_i(z_i, x_{-i}) - \varepsilon > \gamma_i$. Since $x \in \mathrm{Ls}(X^\alpha) \subseteq \mathrm{Ls}(X^\nu)$, Condition (A) in Definition 6 implies that for each α there exist $y_i^\alpha \in X_i^\alpha$ and there exists an α' such that

$$u_i^\alpha(y_i^\alpha, x_{-i}^\alpha) > u_i(z_i, x_{-i}) - \varepsilon = \eta$$

whenever $\alpha' \prec \alpha$. If $u(x) \neq \gamma$, then $(x,\gamma) \in [\mathrm{Ls}(\Gamma_{G^\nu})]\backslash\Gamma_G$. Condition (B) in Definition 6 implies that there exist j and $\xi_j \in X_j$ such that $u_j(\xi_j, x_{-j}) > \gamma_j$. Choose $\varepsilon > 0$ such that $\eta = u_j(\xi_j, x_{-j}) - \varepsilon > \gamma_j$. Condition (A) in Definition 6 implies that for each α there exist $y_j^\alpha \in X_j^\alpha$ and α' such that

$$u_j^\alpha(y_j^\alpha, x_{-j}^\alpha) > u_j(\xi_j, x_{-j}) - \varepsilon = \eta$$

whenever $\alpha' \prec \alpha$. Hence, G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν). ■

Next, we show that multi-hypoconvergence implies strong variational convergence. In fact, hypoconvergence is strong enough to imply weak payoff convergence as well, and we obtain the following generalization of Theorem 1 in Gürkan and Pang [13] (see also Theorem 4.2 in Lucchetti and Patrone [14]).

Theorem 3. *Suppose that $G = (X_i, u_i)$ is a topological game and suppose that $(G^\nu) = (X_i^\nu, u_i^\nu)$ is a test net for G. If the net (X_i^ν, u_i^ν) is multi-hypoconvergent with limit $G = (X_i, u_i)$, then (X_i^ν, u_i^ν) is strongly variationally convergent with limit G . Furthermore, (X_i^ν, u_i^ν) satisfies weak payoff convergence, and $\varepsilon^\nu \to 0$ implies that $\mathrm{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u)$.*

Proof. Suppose that (X_i^ν, u_i^ν) is multi-hypoconvergent with limit $G = (X_i, u_i)$. To show that (A) in Definition 6 is satisfied, choose i, $\varepsilon > 0$, $z_i \in X_i$, and $x \in \mathrm{Ls}(X^\nu)$. Next, suppose that $(G^\alpha) = (X_i^\alpha, u_i^\alpha)$ is a subnet of (G^ν) and $x_{-i}^\alpha \to x_{-i}$ with $x_{-i}^\alpha \in X_{-i}^\alpha$ for each α. From the definition of multi-hypoconvergence, it follows that there exists a convergent net (y_i^α) with limit z_i satisfying $y_i^\alpha \in X_i^\alpha$ for each α such that

$$\underline{\lim}\, u_i^\alpha(y_i^\alpha, x_{-i}^\alpha) \geq u_i(z_i, x_{-i}).$$

Consequently, there exists an α' such that

$$u_i^\alpha(y_i^\alpha, x_{-i}^\alpha) > u_i(z_i, x_{-i}) - \varepsilon$$

whenever $\alpha' \prec \alpha$ and condition (A) in Definition 6 is satisfied.

To show that (B) in Definition 6 is satisfied, suppose that $(x, \gamma) \in [\mathrm{Ls}(\Gamma_{G^\nu})] \backslash \Gamma_G$. Then there exists a subnet $(G^\alpha) = (X_i^\alpha, u_i^\alpha)$ of (G^ν) and $(x^\alpha, u^\alpha(x^\alpha)) \to (x, \gamma)$ with $x^\alpha \in X^\alpha$ for each α and a player i such that $u_i(x) \neq \gamma_i$. From the definition of multi-hypoconvergence, it follows that

$$\gamma_i = \lim\ u_i^\alpha(x^\alpha) = \overline{\lim}\, u_i^\alpha(x^\alpha) \leq u_i(x)$$

Since $u_i(x) \neq \gamma_i$, this implies that $u_i(x) > \gamma_i$.

To show that (X_i^ν, u_i^ν) satisfies weak payoff convergence, suppose that (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν) with $x^\alpha \to x \in X$ satisfying $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α, where $\varepsilon^\alpha \to 0$. The definition of multi-hypoconvergence implies that, for each i, there exists a convergent sequence (z_i^α) with limit x_i such that $z_i^\alpha \in X_i^\alpha$ for each α and

$$\underline{\lim}\, u_i^\alpha(z_i^\alpha, x_{-i}^\alpha) \geq u_i(x).$$

Since $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$ for each α, it follows that $u_i^\alpha(x^\alpha) + \varepsilon^\alpha \geq u_i^\alpha(z_i^\alpha, x_{-i}^\alpha)$ for each α, implying that

$$\underline{\lim}\, [u_i^\alpha(x^\alpha) + \varepsilon^\alpha] \geq \underline{\lim}\, u_i^\alpha(z_i^\alpha, x_{-i}^\alpha) \geq u_i(x).$$

Since $x^\alpha \in X^\alpha$ for each α and $x^\alpha \to x$, it follows that

$$\overline{\lim}\, [u_i^\alpha(x^\alpha) + \varepsilon^\alpha] \leq \overline{\lim}\, u_i^\alpha(x^\alpha) + \overline{\lim}\, \varepsilon^\alpha = \overline{\lim}\, u_i^\alpha(x^\alpha) \leq u_i(x).$$

Consequently,

$$u_i^\alpha(x^\alpha) + \varepsilon^\alpha \to u_i(x)$$

and we conclude that $(u_i^\alpha(x^\alpha))$ is convergent with limit $u_i(x)$.

Applying Theorem 2 and Proposition 1, we conclude that $\varepsilon^\nu \to 0$ implies that $\text{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u)$. ■

If a net (u^ν) in $U(X)$ is uniformly bounded, then weak payoff convergence is clearly satisfied and, in this case, the conclusion of Theorem 3 is necessary and sufficient for sequential better-reply security.

Theorem 4. *Suppose that $G = (X_i, u_i)$ is a topological game and suppose that (X_i^ν, u_i^ν) is a test net for G. Furthermore, suppose that (u^ν) is a uniformly bounded net in $U(X)$. Then G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν) (Condition 1) if and only if $\varepsilon^\nu \to 0$ implies that*

$$\text{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u).$$

Proof. Suppose that G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν). Since (u^ν) is a uniformly bounded net in $U(X)$, it follows that (X_i^ν, u_i^ν) satisfies weak payoff convergence and the conclusion follows from Theorem 2.

Conversely, suppose that $\text{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u)$ whenever $\varepsilon^\nu \to 0$.[2] To show that G satisfies sequential better-reply security with respect to (X_i^ν, u_i^ν), suppose that (X_i^α, u_i^α) is a subnet of (X_i^ν, u_i^ν), $(x^\alpha, u^\alpha(x^\alpha)) \in X \times \mathbb{R}^N$ is a

[2]We thank an anonymous referee for suggesting the argument for the converse.

convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in X^\alpha$ for each α, and suppose that x is not a Nash equilibrium of (X_i, u_i). For each α and i, define

$$v_i^\alpha(x^\alpha) := \sup_{\xi_i \in X_i^\alpha} u_i^\alpha(\xi_i, x_{-i}^\alpha)$$

and let

$$\varepsilon^\alpha := \max_{i \in \{1,\ldots,N\}} \left[v_i^\alpha(x^\alpha) - u_i^\alpha(x^\alpha) \right].$$

Note that $\varepsilon^\alpha \geq 0$ for each α and, since the net (u^ν) is uniformly bounded in $U(X)$, we may assume without loss of generality that (ε^α) is convergent with limit ε. Note that $v_i^\alpha(x^\alpha) - u_i^\alpha(x^\alpha) \leq \varepsilon^\alpha$ for each i implies that $x^\alpha \in \mathcal{N}_{X^\alpha}^{\varepsilon^\alpha}(u^\alpha)$. Therefore, it follows that $x \in \mathrm{Ls}(\mathcal{N}_{X^\nu}^{\varepsilon^\nu}(u^\nu)) \backslash \mathcal{N}_X(u)$, implying that $\varepsilon > 0$. Consequently, there exists α' such that for every $\alpha \succ \alpha'$, the following conditions hold: $\varepsilon^\alpha > \frac{\varepsilon}{2}$, $u_j^\alpha(x^\alpha) > \gamma_j - \frac{\varepsilon}{4}$ for each j, and there exists a player i for whom $v_i^\alpha(x^\alpha) > u_i^\alpha(x^\alpha) + \frac{\varepsilon}{2}$. Since the player set is finite, it follows that there exist an i, a subnet (x^β), and a net (y_i^β) satisfying $y_i^\beta \in X_i^\beta$ for each β and

$$u_i^\beta(y_i^\beta, x_{-i}^\beta) > u_i^\beta(x^\beta) + \tfrac{\varepsilon}{2} > \gamma_j + \tfrac{\varepsilon}{4} := \eta,$$

as desired. ■

If $G = (X_i, u_i)$ is a topological game, then, letting $X^\nu = X$ and $u^\nu = u$ for all ν, we obtain an immediate corollary of Theorem 4.

Corollary 1. *A topological game $G = (X_i, u_i)$ satisfies sequential better-reply security if and only if $\varepsilon^\nu \to 0$ implies that $\mathrm{Ls}(\mathcal{N}_X^{\varepsilon^\nu}(u)) \subseteq \mathcal{N}_X(u)$. In particular, $\mathcal{N}_X(u)$ is a closed set if $G = (X_i, u_i)$ satisfies sequential better-reply security.*

3.2 Perturbed payoffs and fixed strategy sets

For the purposes of this paper, the most important special case of Theorem 1 arises when $X^\alpha = X$ for all α and (u^α) is an approximating sequence for u in the sense that u is the limit of (u^α) according to some notion of convergence. This scenario arises in the study of perfect and essential equilibria. If possible, it is also desirable to identify a condition satisfied by the limit game $G = (X_i, u_i)$ that will yield the approximation result for *any* net (u^α) converging to u according to the specified mode of convergence. Convergence in $(U(X), \rho)$ (*i.e.*, uniform convergence), when combined with sequential better-reply security, yields exactly this type of approximation result and we obtain the following corollary of Theorem 4.

Corollary 2. *Suppose that $G = (X_i, u_i)$ is a topological game satisfying sequential better-reply security. Furthermore, suppose that (u^ν) is convergent in $U(X)$ with limit u. Then (i) G satisfies sequential better-reply security with respect to (X_i, u_i^ν) and (ii) $\varepsilon^\nu \to 0$ implies that* $\mathrm{Ls}(\mathcal{N}_X^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u)$.

Proof. To prove (i), suppose that (u_i^α) is a subnet of (u_i^ν), $(x^\alpha, u^\alpha(x^\alpha))$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ satisfying $x^\alpha \in X$ for each α, and x is not a Nash equilibrium of (X_i, u_i). Since G satisfies sequential better-reply security, there exist an i, an $\eta' > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i^\beta$ and $u_i(y_i^\beta, x_{-i}^\beta) \geq \eta'$.

Next, choose $\varepsilon > 0$ satisfying $0 < \varepsilon < \eta' - \gamma_i$, and index β' so that for each $\beta \succ \beta'$, we have $\rho(u_i(\cdot, x_{-i}^{\beta'}), u_i^{\beta'}(\cdot, x_{-i}^{\beta'})) < \varepsilon$. Then $\beta \succ \beta'$ implies that

$$\begin{aligned} u_i^{\beta'}(y_i^{\beta'}, x_{-i}^{\beta'}) &> u_i(y_i^{\beta'}, x_{-i}^{\beta'}) - \varepsilon \\ &\geq \eta' - \varepsilon \\ &> \gamma_i. \end{aligned}$$

Defining $\eta = \eta' - \varepsilon$, we conclude that there exist an i, an $\eta > \gamma_i$, a subnet (x^β) of (x^α) and a net (y_i^β) such that, for each β, $y_i^\beta \in X_i^\beta$ and $u_i^\beta(y_i^\beta, x_{-i}^\beta) \geq \eta$. This shows that G satisfies sequential better-reply security with respect to (u_i^ν).

If (u^ν) is a uniformly convergent net in $U(X)$ with limit u, then (u^ν) is a uniformly bounded net since u is bounded, and (ii) follows from Theorem 4. ■

We conclude this section by comparing sequential better-reply security to several related concepts in the literature in order to show that the approximation result presented in Corollary 2 is applicable in the presence of these other conditions. In addition, we provide a discussion on the relationship between sequential better-reply security and Barelli and Soza's [5] *generalized B-security.*

Condition 3 (***generalized better-reply security (gbrs)*** (Barelli and Soza [5])**).** Suppose that each X_i is a subset of a vector space. If $(x, \gamma) \in \overline{\Gamma}_{(X_i, u_i)}$ and x is not a Nash equilibrium of (X_i, u_i), then there exist an i, an $\eta > \gamma_i$, an open neighborhood $U_{x_{-i}}$ of x_{-i}, and a convex-valued, compact-valued, nonempty-valued, and upper hemicontinuous correspondence $\varphi_i : U_{x_{-i}} \rightrightarrows X_i$ such that $u_i(y_i, y_{-i}) \geq \eta$ for each (y_{-i}, y_i) in the graph of φ_i.

If each X_i is a subset of a vector space, it follows that every *better-reply secure (brs)* game as defined in Reny [16] is generalized better-reply secure and every generalized better-reply secure game is sequentially better-reply secure.

Definition 7 (Bagh and Jofre [3])**.** A topological game $G = (X_i, u_i)$ satisfies ***weak reciprocal upper semicontinuity (wrusc)*** if for any $(x, \alpha) \in \overline{\Gamma}_G \backslash \Gamma_G$, there exist i and $y_i \in X_i$ such that $u_i(y_i, x_{-i}) > \alpha_i$.

Definition 8 (Dasgupta and Maskin [11])**.** A topological game $G = (X_i, u_i)$ is ***weakly payoff secure (wps)*** if for each i, each $x \in X$, and each $\varepsilon > 0$, there exists an open neighborhood $U_{x_{-i}}$ of x_{-i} such that the following condition is satisfied: for each $y_{-i} \in U_{x_{-i}}$, there exists a $y_i \in X_i$ such that $u_i(y_i, y_{-i}) \geq u_i(x) - \varepsilon$.

Weak reciprocal upper semicontinuity is weaker than *reciprocal upper semicontinuity (rusc)* defined in Simon [18]. Weak payoff security is weaker than *generalized payoff security (gps)* defined in Barelli and Soza [5], and generalized payoff security is weaker than *payoff security (ps)* defined in Reny [16]. Different combinations of these conditions imply different notions of better-reply security and the reader should consult Reny [16], Bagh and Jofre [3], and Barelli and Soza [5] for detailed treatments. In particular, Barelli and Soza [5] showed that a game satisfying weak reciprocal upper semicontinuity and generalized payoff security is generalized better-reply secure.The next result shows that, by further weakening generalized payoff security to weak payoff security, one obtains sequential better-reply security.

Proposition 2. *If a topological game G is weakly reciprocal upper semicontinuous and weakly payoff secure, then G is sequentially better-reply secure.*

Proof. Suppose that $G = (X_i, u_i)$ is weakly reciprocal upper semicontinuous and weakly payoff secure. Furthermore, suppose that $(x^\alpha, u(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ and suppose that x is not a Nash equilibrium of G. Then there exist an i and a $y_i \in X_i$ such that $u_i(y_i, x_{-i}) > u_i(x)$. If $u(x) = \gamma$, then $u_i(y_i, x_{-i}) > \gamma_i$. Choose $\varepsilon > 0$ such that $\eta = u_i(y_i, x_{-i}) - \varepsilon > \gamma_i$. Weak payoff security implies that there exists an α' such that, for each $\alpha' \prec \alpha$, there exists $y_i^\alpha \in X_i$ such that

$$u_i(y_i^\alpha, x_{-i}^\alpha) > u_i(y_i, x_{-i}) - \varepsilon = \eta.$$

If $u(x) \neq \gamma$, then $(x, \gamma) \in \overline{\Gamma}_G \backslash \Gamma_G$ and weak reciprocal upper semicontinuity implies that there exist an i and $y_i \in X_i$ such that $u_i(y_i, x_{-i}) > \gamma_i$. Choose

$\varepsilon > 0$ such that $\eta = u_i(y_i, x_{-i}) - \varepsilon > \gamma_i$. We can now again apply weak payoff security to conclude that there exists an α' such that, for each $\alpha' \prec \alpha$, there exists $y_i^\alpha \in X_i$ such that $u_i(y_i^\alpha, x_{-i}^\alpha) > u_i(y_i, x_{-i}) - \varepsilon = \eta$. ■

Note that, by combining Proposition 2 and Corollary 1, we recover Proposition 4 of Carmona [9]. Carmona [10] provides another variation on the theme of better-reply security. Before stating it formally, we need the following definition.

Definition 9 (Barelli and Soza [5])**.** Suppose that each X_i is a subset of a vector space. A topological game $G = (X_i, u_i)$ is ***generalized payoff secure (gps)*** if for each i, each $x \in X$ and each $\varepsilon > 0$, there exist an open set $U_{x_{-i}}$ with $x_{-i} \in U_{x_{-i}}$ and a convex-valued, compact-valued, nonempty-valued, and upper hemicontinuous correspondence $\varphi_i : U_{x_{-i}} \rightrightarrows X_i$ such that $u_i(y_i, y_{-i}) \geq u_i(x) - \varepsilon$ for each (y_{-i}, y_i) in the graph of φ_i.

Definition 10 (Carmona [10])**.** Suppose that each X_i is a subset of a vector space. A topological game $G = (X_i, u_i)$ is ***weakly better-reply secure (wbrs)*** if there exists a game $\underline{G} = (X_i, \underline{u}_i)$ satisfying:

(i) $\underline{u}_i \leq u_i$ for each i;

(ii) if $(x, \gamma) \in \overline{\Gamma}_G$ and x is not a Nash equilibrium in G, then there exist an i and $y_i \in X_i$ such that $\underline{u}_i(y_i, x_{-i}) > \gamma_i$; and

(iii) $\underline{G}$ is generalized payoff secure.

Sequential better-reply security is weaker than weak better-reply security.

Proposition 3. *If G is weakly better-reply secure, then G is sequentially better-reply secure.*

Proof. Suppose that $G = (X_i, u_i)$ is weakly better-reply secure. Furthermore, suppose that $(x^\alpha, u(x^\alpha)) \in X \times \mathbb{R}^N$ is a convergent net with limit $(x, \gamma) \in X \times \mathbb{R}^N$ and suppose that x is not a Nash equilibrium of G. Since $(x, \gamma) \in \overline{\Gamma}_G$, we conclude from condition (ii) of Definition 10 that there exist an i and a $y_i \in X_i$ such that $\underline{u}_i(y_i, x_{-i}) > \gamma_i$. Choose $\varepsilon > 0$ such that $\underline{u}_i(y_i, x_{-i}) - \varepsilon > \gamma_i$. Generalized payoff security of $\underline{G} = (X_i, \underline{u}_i)$ implies that there exists an α' such that, for each $\alpha' \prec \alpha$, there exists $y_i^\alpha \in X_i$ such that

$$\underline{u}_i(y_i^\alpha, x_{-i}^\alpha) > \underline{u}_i(y_i, x_{-i}) - \varepsilon = \eta.$$

Defining $\underline{u}_i(y_i, x_{-i}) - \varepsilon = \eta$ and using condition (i) of Definition 10, we conclude that for each $\alpha' \prec \alpha$, there exists $y_i^\alpha \in X_i$ such that $u_i(y_i^\alpha, x_{-i}^\alpha) \geq \underline{u}_i(y_i^\alpha, x_{-i}^\alpha) > \eta$. ■

If each X_i is a subset of a vector space, the following table summarizes the relationships between the various notions of upper semicontinuity, payoff security and better-reply security.[3]

$$\begin{array}{ccccccccc}
\text{rusc} & & \text{wrusc} & & \text{wrusc} & & \text{wrusc} & & \\
\text{ps} & \Rightarrow & \text{ps} & \Rightarrow & \text{gps} & \Rightarrow & \text{wps} & & \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & \text{brs} & \Rightarrow & \text{gbrs} & \Rightarrow & \text{sbrs} & \Leftarrow & \text{wbrs}
\end{array}$$

Unlike better-reply security of Reny [16], generalized better-reply security of Barelli and Soza [5], and weak better-reply security of Carmona [10], our notion of sequential better-reply security does not ensure the existence of Nash equilibria in compact, quasiconcave games. In fact, the example of Section 3.3 in Carmona [9] is a compact, quasiconcave game satisfying weak reciprocal upper semicontinuity and weak payoff security (hence sequential better-reply security) for which no Nash equilibria exist.

To complete our discussion of sequential better-reply security, we now relate sequential better-reply security to Barelli and Soza's [5] *generalized B-security*, which weakens McLennan *et al.*'s [15] *B-security*, which, in turn, generalizes the combination of Reny's [16] better-reply security and quasiconcavity.

Definition 11. Suppose that each X_i is a subset of a vector space. A topological game $G = (X_i, u_i)$ is ***generalized B-secure at*** $x \in X$ if there exist $(\alpha_{x1}, ..., \alpha_{xN}) \in \mathbb{R}^N$, an open neighborhood U_x of x, and a convex-valued, compact-valued, nonempty-valued, and upper hemicontinuous correspondence $\varphi_x : U_x \rightrightarrows X$ such that

(a) $\varphi_{xi}(y) \subseteq B_i(y, \alpha_{xi}) := \{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_{xi}\}$ for each i and every $y \in U_x$; and

(b) for each $y \in U_x$ there exists i such that y_i does not belong to the convex hull of $B_i(y, \alpha_{xi})$.

[3] The assumption that each X_i is a subset of a vector space is only made in order to situate generalized better-reply security and very weak better-reply security in this taxonomy.

The game G is ***generalized*** $\boldsymbol{B}$***-secure*** if it is generalized B-secure at x for every x that is not a Nash equilibrium of G.

Generalized B-security and sequential better-reply security are not logically nested. The aforementioned example in Section 3.3 of Carmona [9] satisfies sequential better-reply security but does not satisfy generalized B-security as a result of Corollary 3.5 in Barelli and Soza [5]. In the next example, we construct a generalized B-secure game that does not satisfy sequential better-reply security.

Example 1. Consider the game $G = ([0,1],[0,1],u_1,u_2)$, where

$$u_2(x_1,x_2) := \begin{cases} 1 & \text{if } x_2 = 1 \text{ and } x_1 = 1, \\ 1 - x_1 & \text{if } x_2 = 1 \text{ and } x_1 \in [0,1), \\ -1 & \text{if } x_1 = 1 \text{ and } x_2 \in (0,1), \\ -1 & \text{if } x_2 = 0, \\ -x_2 & \text{otherwise}, \end{cases}$$

and u_1 is identically 0. The set of Nash equilibria is $[0,1] \times \{1\}$. To see that G satisfies generalized B-security, suppose that $x = (x_1,x_2)$ is not a Nash equilibrium of G. Then $x_2 < 1$.

If $x_1 \in [0,1)$, define

$$(\alpha_{x1}, \alpha_{x2}) := (0, \delta)\,,$$

where $0 < \delta < 1 - y_1$ for every y_1 in some (sufficiently small) open neighborhood of x_1. Choose an open neighborhood U_x of x such that $0 < \delta < 1 - y_1$ and $y_2 < 1$ whenever $(y_1,y_2) \in U_x$. Next, define a correspondence $\varphi_x : U_x \rightrightarrows X$ as follows:

$$\varphi_x(y_1,y_2) := \{(x_1,1)\}, \quad \text{for each } (y_1,y_2) \in U_x. \tag{1}$$

If $(y_1,y_2) \in U_x$, then $(\xi_1,\xi_2) \in \varphi_x(y_1,y_2) = \{(x_1,1)\}$ implies that

$$u_1(x_1,y_2) = 0 = \alpha_{x1} \quad \text{and} \quad u_2(y_1,1) = 1 - y_1 > \delta = \alpha_{x2}.$$

In addition, $(y_1,y_2) \in U_x$ implies that

$$y_2 \notin \{1\} = \{z_2 \in [0,1] : u_2(y_1,z_2) \geq \delta = \alpha_{x2}\},$$

and hence y_2 does not belong to the convex hull of

$$\{z_2 \in [0,1] : u_2(y_1,z_2) \geq \delta\}.$$

If $x_1 = 1$, define

$$(\alpha_{x1}, \alpha_{x2}) := (0, 0) .$$

Choose an open neighborhood U_x of x such that $y_2 < 1$ whenever $(y_1, y_2) \in U_x$. Next, define $\varphi_x : U_x \rightrightarrows X$ as in (1), so that, for each $(y_1, y_2) \in U_x$, $\varphi_x(y_1, y_2) = \{(1, 1)\}$ since $x_1 = 1$. If $(y_1, y_2) \in U_x$, then $(\xi_1, \xi_2) \in \varphi_x(y_1, y_2) = \{(1, 1)\}$ implies that

$$u_1(1, y_2) = 0 = \alpha_{x1} \text{ and } u_2(y_1, 1) > 0 = \alpha_{x2}.$$

In addition, $(y_1, y_2) \in U_x$ implies that

$$y_2 \notin \{1\} = \{z_2 \in [0, 1] : u_2(y_1, z_2) \geq 0 = \alpha_{x2}\}$$

and hence y_2 does not belong to the convex hull of

$$\{z_2 \in [0, 1] : u_2(y_1, z_2) \geq 0\}.$$

It is worth noting that G satisfies not only generalized B-security but also McLennan *et al.*'s [15] B-security.

To show that G does not satisfy sequential better-reply security, let

$$u_2^n(x_1, x_2) := \begin{cases} 1 & \text{if } x_2 = 1 \text{ and } x_1 = 1, \\ 1 - x_1 & \text{if } x_2 = 1 \text{ and } x_1 \in [0, 1), \\ -1 & \text{if } x_1 = 1 \text{ and } x_2 \in (0, 1), \\ -1 & \text{if } x_2 = 0, \\ -x_2 + \frac{1}{n} & \text{otherwise,} \end{cases}$$

so that u_2^n converges to u_2 uniformly, and let

$$y^n = (y_1^n, y_2^n) := \left(1 - \tfrac{1}{2n}, \tfrac{1}{2n}\right), \quad \text{for each } n.$$

Then y^n is a ε^n-Nash equilibrium for the game

$$([0, 1], [0, 1], u_1, u_2^n)$$

with $\varepsilon^n = \frac{1}{2n}$. However, $y^n \to (1, 0)$ and $(1, 0)$ is not a Nash equilibrium of G, for $u_2(1, 0) = -1 < 1 = u_2(1, 1)$. Therefore, Corollary 2 implies that G does not satisfy sequential better-reply security.

Although generalized B-security fails to deliver the approximation results of Corollary 2, it can be shown that generalized B-security does guarantee that the set of Nash equilibria is closed.

4 Approximation results for mixed-strategy Nash equilibria of Borel games

If S is a topological space, let $\mathcal{B}(S)$ denote the class of Borel subsets of S. The cone of nonnegative, countably additive, regular measures on $\mathcal{B}(S)$ is denoted by $M_+(S)$. The subset of $M_+(S)$ consisting of probability measures endowed with the topology of weak convergence is denoted by $\Delta(S)$.

A topological game $G = (X_i, u_i)$ with each u_i a bounded Borel measurable function is a ***Borel game***. A topological game $G = (X_i, u_i)$ with each X_i a separable (compact) metric space and each u_i a bounded Borel measurable function is a ***separable (compact) metric Borel game***.

The ***mixed extension*** of a Borel game G is the strategic-form game

$$\overline{G} := (\Delta(X_i), u_i)_{i=1}^N,$$

where $u_i : \times_{i=1}^N \Delta(X_i) \to \mathbb{R}$ denotes the usual extension defined by

$$u_i(\mu) := \int_X u_i d\mu_1 \cdots d\mu_n.$$

We will abuse notation and define $\Delta(X) := \times_{i=1}^N \Delta(X_i)$. Next, define a correspondence $\mathcal{N}_{\Delta(X)} : U(X) \rightrightarrows \Delta(X)$ that assigns to each profile $u = (u_1, .., u_N) \in U(X)$ the set of Nash equilibria $\mathcal{N}_{\Delta(X)}(u)$ of the mixed extension $\overline{G} = (\Delta(X_i), u_i)$.

We now define analogues of Conditions 1 and 2 in terms of the mixed extension of a strategic-form game.

Condition 4. Suppose that $G = (X_i, u_i)$ is a Borel game with mixed extension $\overline{G} = (\Delta(X_i), u_i)$. Suppose that for each i, (u_i^ν) is a net of bounded, Borel measurable payoff functions and (S_i^ν) is a net of nonempty subsets of $\Delta(X_i)$. Then $\overline{G}$ satisfies ***sequential better-reply security with respect to*** (S_i^ν, u_i^ν) if the following condition is satisfied: if (S_i^α, u_i^α) is a subnet of (S_i^ν, u_i^ν), if $(\mu^\alpha, u^\alpha(\mu^\alpha)) \in \Delta(X) \times \mathbb{R}^N$ is a convergent net with limit $(\mu, \gamma) \in \Delta(X) \times \mathbb{R}^N$ satisfying $\mu^\alpha \in S^\alpha$ for each α, and if μ is not a Nash equilibrium of $\overline{G} = (\Delta(X_i), u_i)$, then there exist i, $\eta > \gamma_i$, a subnet (μ^β) of (μ^α) and a net (p_i^β) such that for each β, $p_i^\beta \in S^\beta$ and $u_i(p_i^\beta, \mu_{-i}^\beta) \geq \eta$.

Condition 5 (***sequential better-reply security***)**.** Suppose that $G = (X_i, u_i)$ is a Borel game with mixed extension $\overline{G} = (\Delta(X_i), u_i)$. Then $\overline{G}$ satisfies ***sequential better-reply security*** if the following condition is satisfied:

if $(\mu^\alpha, u(\mu^\alpha)) \in \Delta(X)\times\mathbb{R}^N$ is a convergent net with limit $(\mu,\gamma) \in \Delta(X)\times\mathbb{R}^N$ and if μ is not a Nash equilibrium of $\overline{G} = (\Delta(X_i), u_i)$, then there exist an i, an $\eta > \gamma_i$, a subnet (μ^β) of (μ^α) and a net (p_i^β) such that for each β, $p_i^\beta \in \Delta(X_i)$ and $u_i(p_i^\beta, \mu_{-i}^\beta) \geq \eta$.

The next result is an adaptation of Theorem 4 to the mixed extension of a game and the proof is the essentially identical.

Theorem 5. *Suppose that $G = (X_i, u_i)$ is a Borel game. Suppose that for each i, (u_i^ν) is a net of bounded, Borel measurable payoff functions and (S_i^ν) is a net of nonempty subsets of $\Delta(X_i)$. Suppose that $\overline{G} = (\Delta(X_i), u_i)$ satisfies sequential better-reply security with respect to (S_i^ν, u_i^ν) (Condition 4). If (S_i^α, u_i^α) is a subnet of (S_i^ν, u_i^ν), (u^α) is convergent in $U(X)$ with limit u, and (μ^α) is a convergent net in $\Delta(X)$ with limit $\mu \in \Delta(X)$ such that $\mu^\alpha \in \mathcal{N}_{S^\alpha}(u^\alpha)$ for each α, then $\mu \in \mathcal{N}_{\Delta(X)}(u)$.*

Corollary 3. *Suppose that (X_i, u_i) is a Borel game whose mixed extension satisfies sequential better-reply security (Condition 5). Suppose that (u^α) is a net of bounded, Borel measurable payoff functions convergent in U(X) with limit u and (μ^α) is a convergent net in $\Delta(X)$ with limit $\mu \in \Delta(X)$ satisfying $\mu^\alpha \in \mathcal{N}_{\Delta(X)}(u^\alpha)$ for each α. Then $\mu \in \mathcal{N}_{\Delta(X)}(u)$.*

Proof. Apply Theorem 5 with $S_i^\alpha = \Delta(X_i)$ for each α and i. ■

We now state analogues of Conditions 4 and 5, Theorem 5, and Corollary 3 in terms of metric games.

Condition 6. Suppose that $G = (X_i, u_i)$ is a Borel, metric game with mixed extension $\overline{G} = (\Delta(X_i), u_i)$. Suppose that for each i, (u_i^n) is a sequence of bounded, Borel measurable payoff functions and (S_i^n) is a sequence of nonempty subsets of $\Delta(X_i)$. Then $\overline{G}$ satisfies ***limit better-reply security with respect to*** (S_i^n, u_i^n) if the following condition is satisfied: if (S_i^m, u_i^m) is a subsequence of (S_i^n, u_i^n), if $(\mu^m, u^m(\mu^m)) \in \Delta(X) \times \mathbb{R}^N$ is a convergent sequence with limit $(\mu,\gamma) \in \Delta(X) \times \mathbb{R}^N$ satisfying $\mu^m \in S^m$ for each m, and if μ is not a Nash equilibrium of $\overline{G} = (\Delta(X_i), u_i)$, then there exist i, $\eta > \gamma_i$, a subsequence (μ^k) of (μ^m) and a sequence (p_i^k) such that for each k, $p_i^k \in S^k$ and $u_i(p_i^k, \mu_{-i}^k) \geq \eta$.

Condition 7 (***limit better-reply security***)**.** Suppose that $G = (X_i, u_i)$ is a Borel, metric game with mixed extension $\overline{G} = (\Delta(X_i), u_i)$. Then $\overline{G}$

satisfies ***limit better-reply security*** if the following condition is satisfied: if $(\mu^n, u(\mu^n)) \in \Delta(X) \times \mathbb{R}^N$ is a convergent sequence with limit $(\mu, \gamma) \in \Delta(X) \times \mathbb{R}^N$ and if μ is not a Nash equilibrium of of $\overline{G} = (\Delta(X_i), u_i)$, then there exist an i, an $\eta > \gamma_i$, a subsequence (μ^{n_k}) of (μ^n) and a sequence (p_i^k) such that for each k, $p_i^k \in \Delta(X_i)$ and $u_i(p_i^k, \mu_{-i}^{n_k}) \geq \eta$.

Theorem 6. *Suppose that $G = (X_i, u_i)$ is a Borel, metric game. Suppose that for each i, (u_i^n) is a sequence of bounded, Borel measurable payoff functions and (S_i^n) is a sequence of nonempty subsets of $\Delta(X_i)$. Suppose that $\overline{G}$ satisfies limit better-reply security with respect to (S_i^n, u_i^n) (Condition 6). If (S_i^m, u_i^m) is a subsequence of (S_i^n, u_i^n), (u^m) is convergent with limit u, and (μ^m) is a convergent sequence in $\Delta(X)$ with limit $\mu \in \Delta(X)$ such that $\mu^m \in \mathcal{N}_{S^m}(u^m)$ for each m, then $\mu \in \mathcal{N}_{\Delta(X)}(u)$.*

Corollary 4. *Suppose that (X_i, u_i) is a Borel metric game whose mixed extension satisfies limit better-reply security (Condition 7). Suppose that (u^n) is a sequence of bounded, Borel measurable payoff functions convergent in $U(X)$ with limit u and (μ^n) is a convergent net in $\Delta(X)$ with limit $\mu \in \Delta(X)$ satisfying $\mu^n \in \mathcal{N}_{\Delta(X)}(u^n)$ for each n. Then $\mu \in \mathcal{N}_{\Delta(X)}(u)$.*

Proof. Apply Theorem 6 with $S_i^n = \Delta(X_i)$ for each n and i. ■

5 Application to perfect equilibrium

Several authors have studied perfect equilibria in games with infinitely many actions (*e.g.*, Simon and Stinchcombe [19], Al-Najjar [2], Carbonell-Nicolau [6, 7, 8]). The approximation results of Sections 3 and 4 can be used to derive new results on the existence of perfect equilibrium in discontinuous games.

Simon and Stinchcombe [19] present several extensions of Selten's [17] notion of perfection to games with infinitely many actions including their concept of *limit-of-finite (lof) perfect equilibrium.* An lof perfect equilibrium is defined as the limit of ϵ-perfect equilibria for successively finer *finite* approximations to an infinite game. It is shown in [19] that limit-of-finite perfection is ill-suited as a general solution concept even in continuous games. Before illustrating this idea, we introduce formal definitions of perfection, lof perfection, and admissibility.

Throughout the sequel, unless otherwise indicated, we assume that $G = (X_i, u_i)$ is a separable metric Borel game. In this case, the topology of weak convergence and the Prokhorov metric topology coincide and, consequently, sequences will be sufficient to define all weak limit concepts. In particular, $\Delta(X_i)$ is sequentially compact if X_i is a compact metric space. If X_i is only assumed to be metric, then $\Delta(X_i)$ is metrizable using the Prokhorov metric but the Prokhorov metric topology will be stronger than the topology of weak convergence and the latter may not be metrizable. Throughout the paper, we will abuse notation and use π to denote the Prokhorov metric on both $\Delta(X_i)$ and the Cartesian product $\Delta(X)$.

A measure $\mu_i \in \Delta(X_i)$ is ***strictly positive*** if $\mu_i(U) > 0$ for every nonempty open set U in X_i. Let $M_{++}(X_i)$ denote the set of all strictly positive measures in $M_+(X_i)$, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive probability measures in $\Delta(X_i)$, and let $\widehat{\Delta}(X) := \times_{i=1}^N \widehat{\Delta}(X_i)$. If $\eta_i \in M_{++}(X_i)$ and $0 < \eta_i(X_i) < 1$, we define the perturbed mixed strategy set of player i as

$$\Delta(X_i, \eta_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq \eta_i\}.$$

Given a profile $\eta = (\eta_1, ..., \eta_N) \in \times_{i=1}^N M_{++}(X_i)$ of perturbations, let $\Delta(X, \eta) := \times_i \Delta(X_i, \eta_i)$. Define the associated ***Selten perturbation*** of G to be the game

$$\overline{G}_\eta = (\Delta(X_i, \eta_i), u_i)_{i=1}^N .$$

Definition 12. Suppose that $G = (X_i, u_i)$ is a separable metric Borel game. A strategy profile $\mu \in \Delta(X)$ is ***trembling-hand perfect (thp)*** in G if there exist a sequence of perturbation profiles (η^n) and a sequence of mixed strategy profiles (μ^n) such that $\eta^n \to 0, \mu^n \to \mu$, and each μ^n is a Nash equilibrium of $\overline{G}_{\eta^n}$.

Thus, μ is a thp profile in G if it is the limit of some sequence of Nash equilibria of neighboring Selten perturbations of G. It is important to note that a thp strategy profile for $G = (X_i, u_i)$ may not be a Nash equilibrium in the mixed extension $\overline{G} = (\Delta(X_i), u_i)$.

The reader is referred to Carbonell-Nicolau [7] for alternative, equivalent definitions of trembling-hand perfection.

Simon and Stinchcombe's [19] limit-of-finite perfect equilibrium is defined as follows. Let $G = (X_i, u_i)$ be a separable metric Borel game. For Y_i a nonempty Borel subset of X_i and for $\mu \in \Delta(X)$, let $Br_i(Y_i, \mu)$ denote player i's (possibly empty) set of best responses in $\Delta(Y_i)$ to the profile μ:

$$Br_i(Y_i, \mu) := \left\{ \sigma_i \in \Delta(Y_i) : u_i(\sigma_i, \mu_{-i}) = \sup_{p_i \in \Delta(Y_i)} u_i(p_i, \mu_{-i}) \right\}.$$

Definition 13 (Simon and Stinchcombe [19]). For each i and $\delta > 0$, let X_i^δ denote a finite subset of X_i within Hausdorff distance δ of X_i. A profile $\mu^{(\epsilon,\delta)} \in \times_i \widehat{\Delta}(X_i^\delta)$ is (ϵ, δ)***-perfect with respect to*** $\times_i X_i^\delta$ if for all i,

$$d_i^{X_i^\delta}(\mu_i^{(\epsilon,\delta)}, Br_i(X_i^\delta, \mu^{(\epsilon,\delta)})) < \epsilon,$$

where

$$d_i^{X_i^\delta}(\mu_i, \nu_i) := \sum_{x_i \in X_i^\delta} |\mu_i(\{x_i\}) - \nu_i(\{x_i\})| \,.$$

A strategy profile $\mu \in \Delta(X)$ in $G = (X_i, u_i)$ is ***limit-of-finite (lof) perfect in*** G if it is the weak limit as $(\epsilon^n, \delta^n) \to 0$ of (ϵ^n, δ^n)-perfect profiles with respect to some sequence (X^{δ^n}).

Thus, μ is lof perfect strategy profile if it is the limit of (ϵ, δ)-perfect profiles for successively finer finite approximations of G.

Definition 14. A strategy $x_i \in X_i$ is ***weakly dominated for*** i if there exists a strategy $\mu_i \in \Delta(X_i)$ such that $u_i(\mu_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, with strict inequality for some x_{-i}.

Definition 15. A strategy profile $\mu \in \Delta(X)$ is ***limit admissible*** if $\mu_i(W_i) = 0$ for all i, where W_i denotes the interior of the set of strategies weakly dominated for i.

Simon and Stinchcombe [19] provide an example of a continuous game in which each player has a single pure strategy that weakly dominates all other strategies. In this example, whether or not the dominant strategy is included in the finite approximations can drastically change the character of the game. For finite approximations excluding the dominant strategy, lof perfect profiles may involve play of weakly dominated strategies, thereby violating limit admissibility. This leads Simon and Stinchcombe to stregnthen the notion of lof perfection to *anchored perfection.* However, Example 2.5 in Simon and Stinchombe [19] illustrates that even anchored perfect profiles may fail limit admissibility in continuous games.

Our limit-of-finite notion of perfection, which is also a strengthening of Simon and Stinchcombe's [19] lof perfection, does satisfy limit admissibility within the class of continuous games (Subsection 5.2).

Definition 16. Let $G = (X_i, u_i)$ be a separable metric Borel game and let η be a profile of perturbations. A ***finite ε-approximation*** of the Selten perturbation $\overline{G}_\eta$ of G is a strategic-form game

$$\overline{G}_{(Y,\xi)} = (\Delta(Y_i, \xi_i), u_i)\,,$$

where

- Y_i is a finite subset of X_i within Hausdorff distance ε of X_i for each i.
- ξ_i is a measure with finite support Y_i such that $0 < \xi_i(Y_i) < 1$, $|\xi_i(Y_i) - \eta_i(X_i)| < \varepsilon$, and the Prokhorov distance between the probability measures $\frac{\xi_i}{\xi_i(Y_i)}$ and $\frac{\eta_i}{\eta_i(X_i)}$ is less than ε.

Definition 17. Given sequences (ε^n) and (η^n), and a corresponding sequence $(\overline{G}_{\eta^n})$ of Selten perturbations of G, the sequence $(\overline{G}_{(Y^n,\xi^n)})$ is a ***finite*** (ε^n)-***approximation*** of $(\overline{G}_{\eta^n})$ if $\overline{G}_{(Y^n,\xi^n)}$ is an ε^n-approximation of $\overline{G}_{\eta^n}$ for each n.

Observe that the Selten perturbation $\overline{G}_\eta$, where $0 < \eta_i(X_i) < 1$ for each i, can be interpreted as a game in which each player i is constrained to choose the mixed strategy $\frac{\eta_i}{\eta_i(X_i)}$ with probability $\eta_i(X_i)$, while the player is free to choose any mixed strategy in $\Delta(X_i)$ with probability $1 - \eta_i(X_i)$. A similar interpretation applies to an approximation $\overline{G}_{(Y,\xi)}$: each player i can choose any mixed strategy in $\Delta(Y_i)$ with probability $1 - \xi_i(Y_i)$ but is forced to play the mixed strategy $\frac{\xi_i}{\xi(Y_i)}$ with probability $\xi_i(Y_i)$.

Thus, when ε is small, a finite ε-approximation $\overline{G}_{(Y,\xi)}$ of $\overline{G}_\eta$ is "close" to $\overline{G}_\eta$, in the sense that the mistakes the players make in $\overline{G}_\eta$ are "similar" to the mistakes they make in $\overline{G}_{(Y,\xi)}$, since the set of choices available in $\overline{G}_{(Y,\xi)}$ is "close" to the set of actions available in $\overline{G}_\eta$.

Definition 18. Given a double sequence $(\mu_i^{m,n})$ in $\Delta(X_i)$ and $\mu_i \in \Delta(X_i)$, we write

$$\mu_i^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu_i \tag{2}$$

if the following condition is satisfied: for each $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that

$$\pi(\mu_i^{m,n}, \mu_i) < \varepsilon, \quad \text{for each } m \text{ and each } n \geq n^*.$$

If (2) holds for each i, we write

$$\mu^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu.$$

Definition 19. A strategy profile μ in $\Delta(X)$ is ***strongly limit-of-finite (lof) perfect*** in $G = (X_i, u_i)$ if there exist a sequence (η^n) with $\eta^n \to 0$ and a double sequence $(\varepsilon^{m,n})$ with $\varepsilon^{m,n} > 0$ for each m and n,

$$\varepsilon^{m,n} \xrightarrow[n\to\infty]{} 0, \quad \text{for each } m, \tag{3}$$

and

$$\varepsilon^{m,n} \xrightarrow[m\to\infty]{} 0, \quad \text{for each } n, \tag{4}$$

such that for every m, there exists a finite $(\varepsilon^{m,n})$-approximation $(\overline{G}_{(Y^{m,n},\xi^{m,n})})$ of $(\overline{G}_{\eta^n})$ such that each $\overline{G}_{(Y^{m,n},\xi^{m,n})}$ possesses a Nash equilibrium $\mu^{m,n}$, and $\mu^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu$.

Thus, a strong lof perfect strategy profile is the limit of sequences of (exact) equilibria of neighboring *finite* Selten perturbations that respect the strategic aspects of the original (infinite) game, in the sense that they can be interpreted as "true" approximations of certain infinite Selten perturbations: If μ cannot be obtained as the limit of a sequence of equilibria extracted from some finite (ε^n)-approximation sufficiently close to some sequence $(\overline{G}_{\eta^n})$ of (infinite) Selten perturbations of G, then the ability to approximate μ by a sequence of equilibria in finite "models of slight mistakes" critically relies on the sequence of finite perturbations being "far" from the infinite variants in the sequence $(\overline{G}_{\eta^n})$: either the trembles in the finite approximation are "far" from those in $(\overline{G}_{\eta^n})$ or the approximation's finite action spaces place indispensable constraints on how the players can optimize their responses to the others' strategies and trembles (relative to their performance in $(\overline{G}_{\eta^n})$). In either case, there is an essential "gap" between the sequence $(\overline{G}_{\eta^n})$ and the finite (ε^n)-approximation used to approach μ. We view this as an undesirable property of the approximating sequence because it is based on a fundamental inconsistency between the modeling of the original game and that of its Selten perturbations: the refinement specification is subject to "manipulation" via arbitrary omission (in the finite approximating sequence) of certain strategies otherwise available within the original game.

The role of the requirement that convergence to μ be uniform across finite approximations of the sequence $(\overline{G}_{\eta^n})$ is somewhat subtle. This requirement, together with the other conditions in Definition 19, ensures that for continuous games, strong lof perfect equilibria are also trembling-hand perfect. This plays a role in the proof of Theorem 10.

The following proposition establishes the relationship between strong lof perfection and Simon and Stinchcombe's [19] lof perfection.

Proposition 4. *If $G = (X_i, u_i)$ is a separable metric Borel game, then every strong limit-of-finite perfect profile is a limit-of-finite perfect profile.*

Proof. Suppose that μ is a strong lof perfect profile in $G = (X_i, u_i)$. Then there exist a sequence (η^n) with $\eta^n \to 0$ and a double sequence $(\varepsilon^{m,n})$ with $\varepsilon^{m,n} > 0$ for each m and n,

$$\varepsilon^{m,n} \xrightarrow[n\to\infty]{} 0, \quad \text{for each } m,$$

and

$$\varepsilon^{m,n} \xrightarrow[m\to\infty]{} 0, \quad \text{for each } n,$$

such that for every m, there exists a finite $(\varepsilon^{m,n})$-approximation $(\overline{G}_{(Y^{m,n},\xi^{m,n})})$ of $(\overline{G}_{\eta^n})$ such that each $\overline{G}_{(Y^{m,n},\xi^{m,n})}$ possesses a Nash equilibrium $\mu^{m,n}$, and $\mu^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu$. We claim that μ is an lof perfect strategy profile.

Fix m. To lighten notation we omit the index m: $Y^n := Y^{m,n}$, $\xi^n := \xi^{m,n}$, and $\mu^n := \mu^{m,n}$. Each μ^n is a Nash equilibrium of $\overline{G}_{(Y^n,\xi^n)}$, and $Y_i^n \to X_i$ for each i. To complete the proof, we must show that for each i the distance (as defined in Definition 13) between

$$\mu_i^n \in \underset{p\in\Delta(Y_i^n,\xi_i^n)}{\arg\max}\; u_i(p, \mu_{-i}^n)$$

and

$$Br_i(Y_i^n, \mu^n) = \underset{p\in\Delta(Y_i^n)}{\arg\max}\, u_i(p, \mu_{-i}^n)$$

converges to 0. This will be true if each μ_i^n can be expressed as

$$\mu_i^n := (1-\delta^n)\rho_i^n + \delta^n p_i^n,$$

for some sequence (δ^n) with $[0,1] \ni \delta^n \to 0$, some $\rho_i^n \in Br_i(Y_i^n, \mu^n)$, and some $p_i^n \in \Delta(Y_i^n)$, since, in that case,

$$\begin{aligned} d_i^{Y_i^n}(\mu_i^n, \rho_i^n) &= \sum_{x_i \in Y_i^n} |\mu_i^n(\{x_i\}) - \rho_i^n(\{x_i\})| \\ &= \delta^n \sum_{x_i \in Y_i^n} |\rho_i^n(\{x_i\}) - p_i^n(\{x_i\})| \\ &\le 2\delta^n. \end{aligned}$$

To see this, write $\mu_i^n = \hat{\mu}_i^n + \xi_i^n$, where $\hat{\mu}_i^n := \mu_i^n - \xi_i^n$. Since $1 > \xi_i^n(Y_i^n) > 0$, $\hat{\mu}_i^n(Y_i^n) = \mu_i^n(Y_i^n) - \xi_i^n(Y_i^n) = 1 - \xi_i^n(Y_i^n) > 0$. It follows that

$$\mu_i^n = \hat{\mu}_i^n(Y_i^n)\tfrac{\hat{\mu}_i^n}{\hat{\mu}_i^n(Y_i^n)} + \xi_i^n(Y_i^n)\tfrac{\xi_i^n}{\xi_i^n(Y_i^n)}. \tag{5}$$

Since $\hat{\mu}_i^n(Y_i^n) + \xi_i^n(Y_i^n) = \mu_i^n(Y_i^n) = 1$ and $\xi_i^n(Y_i^n) \to 0$, it only remains to show that $\frac{\hat{\mu}_i^n}{\hat{\mu}_i^n(Y_i^n)} \in Br_i(Y_i^n, \mu^n)$. If $\frac{\hat{\mu}_i^n}{\hat{\mu}_i^n(Y_i^n)} \notin Br_i(Y_i^n, \mu^n)$, then (using (5)) there exists a $\rho_i \in Br_i(Y_i^n, \mu^n)$ such that

$$u_i(\rho_i, , \mu_{-i}^n) > u_i\left(\tfrac{\hat{\mu}_i^n}{\hat{\mu}_i^n(Y_i^n)}, \mu_{-i}^n\right).$$

Using linearity, we can rearrange this inequality to obtain

$$u_i\left(\hat{\mu}_i^n(Y_i^n)\rho_i + \xi_i^n(Y_i^n)\tfrac{\xi_i^n}{\xi_i^n(Y_i^n)}, \mu_{-i}^n\right) > u_i(\mu_i^n, \mu_{-i}^n),$$

contradicting the assumption that

$$\mu_i^n \in \underset{p \in \Delta(Y_i^n, \xi_i^n)}{\arg\max}\; u_i(p, \mu_{-i}^n).$$

This completes the proof. ■

Remark 3. Simon and Stinchcombe [19] define lof perfection in terms of ϵ-perfect equilibria, while strong lof perfection is defined in terms of finite approximations of Selten perturbations. There is, however, an alternative formulation of lof perfection in terms of finite Selten perturbations. In fact, it can be shown that Definition 13 is equivalent to the following: A strategy profile $\mu \in \Delta(X)$ is lof perfect in $G = (X_i, u_i)$ if there are sequences $(X^n) = (X_1^n, ..., X_N^n)$, (ξ^n), and (μ^n) such that each ξ_i^n has support X_i^n, $\xi^n \to 0$, each $X_i^n \subseteq X_i$ is finite, $X_i^n \to X_i$ for each n and i, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of $\overline{G}_{(X^n, \xi^n)}$.

This alternative formulation of lof perfection is more easily compared with strong lof perfection (Definition 19).

An lof perfect strategy profile in $G = (X_i, u_i)$ need not be a mixed-strategy equilibrium, even if an lof perfect strategy profile exists for $G = (X_i, u_i)$. We first address the question of existence of strong lof perfect equilibrium profiles. In Subsection 5.2 we discuss some properties of strong lof perfection.

5.1 Existence of strong limit-of-finite perfect equilibrium

Theorem 7 below establishes the relationship between the approximation results of Section 4 and the existence of strong lof perfect equilibria, while Theorem 8 is an existence result in terms of the data of the original game, $G = (X_i, u_i)$. The proof of Theorem 7 relies on the following auxiliary results.

Given $(\delta, \mu) \in [0,1)^N \times \Delta(X)$, let

$$G_{(\delta,\mu)} := (X_i, u_i^{(\delta,\mu)})_{i=1}^N$$

be the strategic-form game where the payoff function $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is defined by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, ..., (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Here, $(1-\delta_i)x_i + \delta_i\mu_i$ denotes the measure $\sigma_i \in \Delta(X_i)$ defined by $\sigma_i(B) := (1-\delta_i)\delta_{x_i}(B) + \delta_i\mu_i(B)$ where $\delta_{x_i} \in \Delta(X_i)$ is the Dirac measure with support $\{x_i\}$.

Lemma 1. *Suppose that (X_i, u_i) is a compact, metric, Borel game. Suppose that $Q_i \subseteq X_i$ is countable for each i. If $(\delta, \mu) \in [0,1)^N \times \Delta(X)$ and $[0,1)^N \ni \delta^n \to \delta$, then there exists a sequence (μ^n) with $\Delta(X) \ni \mu^n \to \mu$ such that each μ_i^n has finite support X_i^n, $X_i^n \subseteq X_i^{n+1}$ for each n and i, $\bigcup_{n=1}^\infty X_i^n \supseteq Q_i$ for each i, and $u^{(\delta^n,\mu^n)} \to u^{(\delta,\mu)}$ in $U(X)$.*

The proof of Lemma 1 is relegated to Section 6.

Lemma 2. *Let X be a compact metric space and suppose that (μ^n) is a sequence in $\Delta(X)$ weakly converging to $\mu \in \Delta(X)$. Then there exists a subsequence which we also denote by (μ^n) and a set $S \subseteq X$ such that* $\text{supp}(\mu) \subseteq S$ *and* $(\text{supp}(\mu^n))$ *is convergent in the Hausdorff metric topology with limit S.*

Proof. Since X is compact, the hyperspace of nonempty compact subsets of X is a compact metric space with respect to the Hausdorff metric. The set $\text{supp}(\mu^n)$ is closed, hence compact in X. Consequently, there exists a subsequence which we also denote by (μ^n) and a compact set $S \subseteq X$ such that $\text{supp}(\mu^n) \to S$. To see that $\text{supp}(\mu) \subseteq S$, suppose that there exists $x \in \text{supp}(\mu) \setminus S$. Then since S is closed there exist a neighborhood V_x of x and an open set U containing S such that $V_x \cap U = \emptyset$. Note that $\mu(V_x) > 0$

since $x \in \text{supp}(\mu)$. Since $V_x \cap U = \emptyset$ and $\text{supp}(\mu^n) \to S$ it follows that $\mu^n(V_x) = 0$ for any large enough n. On the other hand since $\mu^n \to \mu$ and $\mu(V_x) > 0$ we have $0 = \underline{\lim}\, \mu^n(V_x) \geq \mu(V_x)$, an impossibility. ■

Theorem 7. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game. Suppose further that*

- (i) *$\overline{G}$ satisfies limit better-reply security (Condition 7), and*
- (ii) *there exist sequences (ν^n) and (δ^n) with $\nu^n \in \widehat{\Delta}(X)$ and $(0,1)^N \ni \delta^n \to 0$ and, for each i, a countable subset Q_i of X_i such that for each n, the following condition holds: $(\Delta(X_i), u_i^{(\delta^n,\nu^n)})$ satisfies limit better-reply security with respect to any sequence*

$$\left(\Delta(X_i^n(m)), u_i^{(\delta^n,\sigma^n(m))}\right)_{m\geq 1}$$

(Condition 6), where for each i, $(X_i^n(m))_{m\geq 1}$ is an increasing sequence of finite subsets of X_i, $X_i^n(m)$ is the finite support of $\sigma_i^n(m)$,

$$\bigcup_{m=1}^{\infty} X_i^n(m) \supseteq Q_i,$$

$$X_i^n(m) \xrightarrow[m\to\infty]{} X_i,\ u_i^{(\delta^n,\sigma^n(m))} \xrightarrow[m\to\infty]{} u_i^{(\delta^n,\nu^n)} \text{ in } B(X), \text{ and } \sigma^n(m) \xrightarrow[m\to\infty]{} \nu^n.$$

Then G possesses a strong limit-of-finite perfect equilibrium, which is also trembling-hand perfect.

Proof. Suppose that (X_i, u_i) is a compact, metric, Borel game. Assume (i) and (ii) above.

For each n, we can apply Lemma 1 and deduce the existence of a sequence of games

$$\left(\Delta(X_i^n(m)), u_i^{(\delta^n,\sigma^n(m))}\right)_{m\geq 1} \tag{6}$$

with the following properties:

- for each i, $X_i^n(m) \subseteq X_i$ is the finite support of $\sigma_i^n(m)$;
- $X_i^n(m) \subseteq X_i^n(m+1)$ for each m;
- $\bigcup_{m=1}^{\infty} X_i^n(m) \supseteq Q_i$;

- $u_i^{(\delta^n,\sigma^n(m))} \xrightarrow[m\to\infty]{} u_i^{(\delta^n,\nu^n)}$ in $B(X)$; and
- $\sigma^n(m) \xrightarrow[m\to\infty]{} \nu^n$.

Since $\nu^n \in \widehat{\Delta}(X)$ implies that $\mathrm{supp}(\nu_i^n) = X_i$, we can apply Lemma 2 and assume (passing to a subsequence if necessary) that $X_i^n(m) \xrightarrow[m\to\infty]{} X_i$.

Applying (ii), it follows that the game $\overline{G} = (\Delta(X_i), u_i^{(\delta^n,\nu^n)})$ satisfies limit better-reply security (*i.e.*, Condition 6) with respect to the sequence in (6).

For each m, the game $(\Delta(X_i^n(m)), u_i^{(\delta^n,\sigma^n(m))})$ has a Nash equilibrium $\mu^n(m)$, and (because $\Delta(X)$ is sequentially compact) we have (passing to a subsequence if necessary) $\mu^n(m) \xrightarrow[m\to\infty]{} \mu^n$ for some $\mu^n \in \Delta(X)$. Since $u^{(\delta^n,\sigma^n(m))} \xrightarrow[m\to\infty]{} u^{(\delta^n,\nu^n)}$ in $U(X)$, Theorem 6 implies that $\mu^n \in \mathcal{N}_{\Delta(X)}(u^{(\delta^n,\nu^n)})$. Since $\Delta(X_i)$ is sequentially compact, we conclude (extracting a subsequence if necessary) that there exists a $\mu \in \Delta(X)$ such that $\mu^n \to \mu$. To show that μ is a thp equilibrium, first observe that $u^{(\delta^n,\nu^n)} \to u$ in $U(X)$ since $\delta^n \to 0$. Applying Corollary 4, it follows that $\mu \in N_{\Delta(X)}(u)$. Next, define $q_i^n := (1-\delta_i^n)\mu_i^n + \delta_i^n\nu_i^n$ for each i and note that $q^n = (q_1^n, .., q_N^n)$ is a Nash equilibrium in the Selten perturbation $(\Delta(X_i, \delta_i^n\nu_i^n), u_i)$. Since $q^n \to \mu$, we conclude that μ is a thp profile and a Nash equilibrium.

Since μ is a Nash equilibrium, the proof will be complete if we show that μ is a strong lof perfect strategy profile. To begin, note that $(\Delta(X_i, \delta_i^n\nu_i^n), u_i)$ is a Selten perturbation of G and that $\delta_i^n\nu_i^n \to 0$ for each i. The sequence $(\delta_i^n\nu_i^n)$ will play the role of the sequence (η_i^n) in Definition 19.

Define $\varepsilon^{k,n} := \frac{1}{kn}$ for $(k,n) \in \mathbb{N}^2$. Clearly, $\varepsilon^{k,n} > 0$ for each k and n, and

$$\varepsilon^{k,n} \xrightarrow[n\to\infty]{} 0, \quad \text{for each } k,$$

and

$$\varepsilon^{k,n} \xrightarrow[k\to\infty]{} 0, \quad \text{for each } n.$$

The double sequence $(\varepsilon^{k,n})$ will play the role of $(\varepsilon^{m,n})$ in Definition 19.

Define

$$p_i^n(m) := (1-\delta_i^n)\mu_i^n(m) + \delta_i^n\sigma_i^n(m).$$

Then $p^n(m) := (p_1^n(m), .., p_N^n(m))$ is a Nash equilibrium of the (finite) Selten perturbation $(\Delta(X_i^n(m), \delta_i^n\sigma_i^n(m)), u_i)$. For each k and each n, choose $m_{kn} \in$

$\mathbb{N}$ so that for each i,

$$\begin{aligned} haus(X_i^n(m_{kn}), X_i) &< \varepsilon^{k,n}, \\ \pi(\sigma_i^n(m_{kn}), \nu_i^n) &< \varepsilon^{k,n}, \\ \text{and } \pi(\mu_i^n(m_{kn}), \mu_i^n) &< \varepsilon^{k,n}. \end{aligned}$$

Here, $haus(\cdot, \cdot)$ denotes the Hausdorff distance induced by the metric on X_i. Since $haus(X_i^n(m_{kn}), X_i) < \varepsilon^{k,n}$ and $\pi(\sigma_i^n(m_{kn}), \nu_i^n) < \varepsilon^{k,n}$, it follows that for each k the sequence

$$(\Delta(X_i^n(m_{kn}), \delta_i^n \sigma_i^n(m_{kn})), u_i)_{n \geq 1}$$

is a finite $(\varepsilon^{k,n})$-approximation of $(\Delta(X_i, \delta_i^n \nu_i^n), u_i)$. Furthermore,

$$p^n(m_{kn}) := (p_1^n(m_{kn}), ..., p_N^n(m_{kn}))$$

is a Nash equilibrium of

$$(\Delta(X_i^n(m_{kn}), \delta_i^n \sigma_i^n(m_{kn})), u_i).$$

It only remains to show that $p^n(m_{kn}) \xrightarrow[n\to\infty]{\text{unif}} \mu$. Fix i. We need to show that $p_i^n(m_{kn}) \xrightarrow[n\to\infty]{\text{unif}} \mu_i$, *i.e.*, for each $\varepsilon > 0$, there exists n^* such that $\pi(p_i^n(m_{kn}), \mu_i) < \varepsilon$ for each k and each $n \geq n^*$.

Pick $\varepsilon > 0$. There exists n' such that for each k and each $n \geq n'$,

$$\pi\left((1-\delta_i^n)\mu_i^n(m_{kn}) + \delta_i^n \sigma_i^n(m_{kn}), \mu_i^n(m_{kn})\right) < \tfrac{\varepsilon}{3}. \tag{7}$$

To see this, choose n' so that $\delta_i^n < \frac{\varepsilon}{6}$ for each $n \geq n'$ and choose a Borel set $B_i \subseteq X_i$. Given k and $n \geq n'$, we have

$$\begin{aligned} (1-\delta_i^n)\mu_i^n(m_{kn})(B_i) + \delta_i^n \sigma_i^n(m_{kn})(B_i) &\leq (1-\delta_i^n)\mu_i^n(m_{kn})(B_i) + \delta_i^n \\ &= \mu_i^n(m_{kn})(B_i) + \delta_i^n(1 - \mu_i^n(m_{kn})(B_i)) \\ &\leq \mu_i^n(m_{kn})(N_{\frac{\varepsilon}{6}}(B_i)) + \delta_i^n(1 - \mu_i^n(m_{kn})(B_i)) \\ &\leq \mu_i^n(m_{kn})(N_{\frac{\varepsilon}{6}}(B_i)) + \tfrac{\varepsilon}{6} \end{aligned}$$

and

$$\begin{aligned} \mu_i^n(m_{kn})(B_i) &\leq (1-\delta_i^n)\mu_i^n(m_{kn})(B_i) + \delta_i^n \\ &\leq (1-\delta_i^n)\mu_i^n(m_{kn})(N_{\frac{\varepsilon}{6}}(B_i)) + \delta_i^n \\ &\leq (1-\delta_i^n)\mu_i^n(m_{kn})(N_{\frac{\varepsilon}{6}}(B_i)) + \delta_i^n \sigma_i^n(m_{kn})(N_{\frac{\varepsilon}{6}}(B_i)) + \tfrac{\varepsilon}{6}, \end{aligned}$$

implying that (7) holds for each k and each $n \geq n'$. Next, choose n'' such that, for each $n \geq n''$,

$$\pi(\mu_i^n, \mu_i) < \tfrac{\varepsilon}{3} \quad \text{and} \quad \tfrac{1}{kn} < \tfrac{\varepsilon}{3} \text{ for each } k,$$

and define

$$n^* := \max\{n', n''\}.$$

For k and $n \geq n^*$ we have

$$\begin{aligned}
\pi(p_i^n(m_{kn}), \mu_i) &= \pi\left((1-\delta_i^n)\mu_i^n(m_{kn}) + \delta_i^n \sigma_i^n(m_{kn}), \mu_i\right) \\
&\leq \pi\left((1-\delta_i^n)\mu_i^n(m_{kn}) + \delta_i^n \sigma_i^n(m_{kn}), \mu_i^n(m_{kn})\right) + \pi\left(\mu_i^n(m_{kn}), \mu_i\right) \\
&< \tfrac{\varepsilon}{3} + \pi\left(\mu_i^n(m_{kn}), \mu_i\right) \\
&\leq \tfrac{\varepsilon}{3} + \pi\left(\mu_i^n(m_{kn}), \mu_i^n\right) + \pi\left(\mu_i^n, \mu_i\right) \\
&\leq \tfrac{\varepsilon}{3} + \tfrac{1}{kn} + \tfrac{\varepsilon}{3} \\
&< \varepsilon,
\end{aligned}$$

as desired. ■

We now seek conditions on the payoff functions of the original game (X_i, u_i), rather than perturbations of the form $G_{(\delta,\mu)}$, that imply the hypotheses of Theorem 7.

Condition (A*). There exists $(\nu_1, ..., \nu_N) \in \widehat{\Delta}(X)$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f_i^\varepsilon : X_i \to X_i$ with countable range such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f_i^\varepsilon(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for every $z_{-i} \in O_{y_{-i}}$.

(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\nu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_i^\varepsilon(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

Lemma 3. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game satisfying Condition* (A*). *Then, there exist $\nu \in \widehat{\Delta}(X)$, and, for each i, a countable set $Q_i \subseteq X_i$ such that the following is satisfied: given $\delta \in [0,1)$, $n \in \mathbb{N}$, i, and $\sigma = (\sigma_1, ..., \sigma_N) \in \Delta(X, \delta\nu)$, there exists $\varrho_i \in \Delta(X_i)$ with $\varrho_i(Q_i) = 1$ and a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that*

$$u_i((1-\delta)\varrho_i + \delta\nu_i, p_{-i}) > u_i(\sigma) - \tfrac{1}{n}, \quad \textit{for all } p_{-i} \in O_{\sigma_{-i}}. \tag{8}$$

Proof. Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game satisfying Condition (A*). By Condition (A*), there exists $\nu \in \widehat{\Delta}(X)$ such that for each i and every $n \in \mathbb{N}$ there is a Borel measurable map $f_{in} : X_i \to X_i$ with countable range such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f_{in}(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \frac{1}{n}$ for every $z_{-i} \in O_{y_{-i}}$.

(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\nu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_{in}(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \frac{1}{n}$ for all $z_{-i} \in V_{y_{-i}}$.

For each i, define

$$Q_i := \bigcup_{n=1}^{\infty} f_{in}(X_i).$$

Since each f_{in} has a countable range, Q_i is countable. Consequently, the proof will be complete if we show that given $\delta \in [0, 1)$, $n \in \mathbb{N}$, i, and $\sigma = (\sigma_1, ..., \sigma_N) \in \Delta(X, \delta\nu)$, there exists $\varrho_i \in \Delta(X_i)$ with $\varrho_i(Q_i) = 1$ and a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that (8) holds. Fix $\delta \in [0, 1)$, $n \in \mathbb{N}$, i, and $\sigma = (\sigma_1, ..., \sigma_N) \in \Delta(X, \delta\nu)$. From the proof of Lemma 2 in Carbonell-Nicolau [6], it follows that there exist $\varrho_{in} \in \Delta(X_i)$ with $\varrho_{in}(f_{in}(X_i)) = 1$ and a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$u_i((1-\delta)\varrho_{in} + \delta\nu_i, p_{-i}) > u_i(\sigma) - \tfrac{1}{n}, \quad \text{for all } p_{-i} \in O_{\sigma_{-i}}.$$

Since $1 = \varrho_{in}(f_{in}(X_i)) \leq \varrho_{in}(Q_i) \leq 1$, the proof is complete. ■

Theorem 8. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game satisfying Condition* (A*)*. Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G possesses a strong limit-of-finite perfect equilibrium, which is also trembling-hand perfect.*

Proof. Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game satisfying Condition (A*). Suppose further that $\sum_i u_i$ is upper semicontinuous. It suffices to show that the hypotheses of Theorem 7 are satisfied.

That $\overline{G}$ satisfies Condition 7 (even better-reply security) follows from Condition (A*) and upper semicontinuity of $\sum_i u_i$. Indeed, it is clear that (A*) is stronger than payoff security, while upper semicontinuity of $\sum_i u_i$ implies reciprocal upper semicontinuity of G, and it is well-known that payoff security and reciprocal upper semicontinuity imply better-reply security (and hence Condition 7).

Next, choose a sequence (δ^n) with $\delta^n \in (0,1)$ and $\delta^n \to 0$. Since G satisfies Condition (A*), we can choose $\nu \in \widehat{\Delta}(X)$ and, for each i, a countable set $Q_i \subseteq X_i$ satisfying the conclusion in Lemma 3. We will show that part (ii) of Theorem 7 is satisfied for the sequence (δ^n), the countable sets $Q_1, ..., Q_N$, and the constant sequence (ν^n) with $\nu^n = \nu$ for all n. In the remainder of the proof, we will use δ^n to represent the number $\delta^n \in (0,1)$ and the vector $(\delta_1^n, ..., \delta_N^n)$ with $\delta_i^n = \delta^n$ for each i.

To begin, fix n and let

$$\left(\Delta(X_i(m)), u_i^{(\delta^n, \sigma(m))}\right)_{m \geq 1} \tag{9}$$

be a sequence where for each i, $(X_i(m))_{m\geq 1}$ is an increasing sequence of finite subsets of X_i, $X_i^n(m)$ is the finite support of $\sigma_i^n(m)$,

$$\bigcup_{m=1}^{\infty} X_i^n(m) \supseteq Q_i,$$

$X_i^n(m) \xrightarrow[m\to\infty]{} X_i$, $u_i^{(\delta^n, \sigma^n(m))} \xrightarrow[m\to\infty]{} u_i^{(\delta^n, \nu^n)}$ in $B(X)$, and $\sigma^n(m) \xrightarrow[m\to\infty]{} \nu^n$. It suffices to show that $(\Delta(X_i), u_i^{(\delta^n,\nu)})$ satisfies Condition 6 with respect to the sequence in (9).

To that end, suppose that $(\Delta(X_i(k)), u_i^{(\delta^n,\sigma(k))})_{k\geq 1}$ is a subsequence of $(\Delta(X_i(m)), u_i^{(\delta^n,\sigma(m))})_{m\geq 1}$ and that $(\mu^k, u^{(\delta^n,\sigma(k))}(\mu^k)) \in \Delta(X) \times \mathbb{R}^N$ is a convergent sequence with limit $(\mu, \gamma) \in \Delta(X) \times \mathbb{R}^N$ satisfying $\mu_i^k \in \Delta(X_i(k))$ for each k and each i. Suppose that μ is not a Nash equilibrium of $(\Delta(X_i), u_i^{(\delta^n,\nu)})$. We need to show that there exist i, $\eta > \gamma_i$, a subsequence (μ^l) of (μ^k) and a sequence (p_i^l) such that, for each l, $p_i^l \in \Delta(X_i(l))$ and $u_i^{(\delta^n,\nu)}(p_i^l, \mu_{-i}^l) \geq \eta$.

If $u_j^{(\delta^n,\nu)}(\mu) = \gamma_j$ for each player j, then, since μ is not a Nash equilibrium of $(\Delta(X_i), u_i^{(\delta^n,\nu)})$, there exist i and $p_i \in \Delta(X_i)$ such that $u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) > u_i^{(\delta^n,\nu)}(\mu)$. Choose a positive integer M and a real number η so that

$$u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) - \tfrac{1}{M} \geq \eta > u_i^{(\delta^n,\nu)}(\mu) = \gamma_i \tag{10}$$

Define

$$\sigma = (\sigma_1, ..., \sigma_N) := ((1-\delta^n)p_i + \delta^n \nu_i, (1-\delta^n)\mu_{-i} + \delta^n \nu_{-i})$$

and note that $\sigma_i \in \Delta(X_i, \delta^n \nu_i)$ for each i and that

$$u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) = u_i((1-\delta^n)p_i + \delta^n \nu_i, (1-\delta^n)\mu_{-i} + \delta^n \nu_{-i}) = u_i(\sigma).$$

Applying Lemma 3, there exist a neighborhood $O_{\sigma_{-i}}$ of $\sigma_{-i} = (1-\delta^n)\mu_{-i} + \delta^n \nu_{-i}$ and $\varrho_i \in \Delta(X_i)$ with $\varrho_i(Q_i) = 1$ such that

$$u_i((1-\delta^n)\varrho_i + \delta^n \nu_i, p_{-i}) > u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) - \tfrac{1}{M}, \quad \text{for all } p_{-i} \in O_{\sigma_{-i}}.$$

Therefore, there exists a neighborhood $O_{\mu_{-i}}$ of μ_{-i} such that

$$\begin{aligned} u_i^{(\delta^n,\nu)}(\varrho_i, q_{-i}) &= u_i((1-\delta^n)\varrho_i + \delta^n \nu_i, (1-\delta^n)q_i + \delta^n \nu_i) \\ &> u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) - \tfrac{1}{M}, \quad \text{for all } q_{-i} \in O_{\mu_{-i}}. \end{aligned} \tag{11}$$

Now let $\Delta(Q_i)$ and $\Delta^f(Q_i)$ denote, respectively, the set of $\theta_i \in \Delta(X_i)$ with $\theta_i(Q_i) = 1$ and the set of $\theta_i \in \Delta(Q_i)$ with finite support. Since $u_i^{(\delta^n,\nu)}$ is bounded, there exists for every $\epsilon > 0$ a measure $\lambda_i \in \Delta^f(Q_i)$ such that

$$|u_i^{(\delta^n,\nu)}(\lambda_i, q_{-i}) - u_i^{(\delta^n,\nu)}(\varrho_i, q_{-i})| < \epsilon, \quad \text{for all } q_{-i} \in O_{\mu_{-i}}.$$

From this observation, (10), and (11), we see that there exists a $\lambda_i \in \Delta^f(Q_i)$ and neighborhood $O_{\mu_{-i}}$ of μ_{-i} such that

$$u_i^{(\delta^n,\nu)}(\lambda_i, q_{-i}) > u_i^{(\delta^n,\nu)}(p_i, \mu_{-i}) - \tfrac{1}{M} \geq \eta > \gamma_i, \quad \text{for all } q_{-i} \in O_{\mu_{-i}}.$$

Recall that $(X_i(k))$ is an increasing sequence of finite subsets of X_i with $\bigcup_k X_i(k) \supseteq Q_i$. Since $\lambda_i \in \Delta^f(Q_i)$ and $\mu^k \to \mu$, it follows that there exists a positive integer L such that $\lambda_i \in \Delta(X_i(k))$ and $\mu^k_{-i} \in O_{\mu_{-i}}$ for all $k \geq L$. Summarizing, we conclude that $\lambda_i \in \Delta(X_i(k))$ and $u_i^{(\delta^n,\nu)}(\lambda_i, \mu^k_{-i}) \geq \eta > \gamma_i$ for each $k \geq L$.

To complete the argument, suppose that $u_j^{(\delta^n,\nu)}(\mu) \neq \gamma_j$ for some j. Since $u^{(\delta^n,\sigma(k))} \xrightarrow[k\to\infty]{} u^{(\delta^n,\nu)}$ in $U(X)$ and $u^{(\delta^n,\sigma(k))}(\mu^k) \xrightarrow[k\to\infty]{} \gamma$, we have $u^{(\delta^n,\nu)}(\mu^k) \xrightarrow[k\to\infty]{} \gamma$. Consequently, since $\mu^k \to \mu$ and $u_j^{(\delta^n,\nu)}(\mu) \neq \gamma_j$, upper semicontinuity of $\sum_i u_i$ (which implies upper semicontinuity of $\sum_i u_i^{(\delta^n,\nu)}$) implies that $u_i^{(\delta^n,\nu)}(\mu) > \gamma_i$ for some i. Choose a positive integer M and a real number η so that

$$u_i^{(\delta^n,\nu)}(\mu_i, \mu_{-i}) - \tfrac{1}{M} \geq \eta > \gamma_i$$

and duplicate the previous argument with μ_i replacing p_i. ■

Condition (A*) is implied by the combination of two independent conditions, *generic entire payoff security* and *generic local equi-upper semicontinuity*, introduced in Carbonell-Nicolau [6, 8]. From the proof of Lemma 4 in [8], it follows that, if G is a compact, metric, Borel game satisfying generic entire payoff security and generic local equi-upper semicontinuity, then G satisfies Condition (A*).[4]

This observation, together with Theorem 8, yields the following result.

Theorem 9. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game satisfying generic entire payoff security and generic local equi-upper semicontinuity. Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G possesses a strong limit-of-finite perfect equilibrium, which is also trembling-hand perfect.*

Remark 4. Theorem 9 generalizes Corollary 1 in Carbonell-Nicolau [6].

Remark 5. Generic entire payoff security and generic local equi-upper semicontinuity are met in a variety of economic games, as illustrated in Section 3 of Carbonell-Nicolau [6].

5.2 On strong limit-of-finite perfect equilibrium and limit admissibility

We now turn to the relationship between strong lof perfection and limit admissibility. Unlike Simon and Stinchcombe's [19] lof perfection and anchored perfection, strong lof perfection satisfies limit admissibility in continuous games. In fact, for continuous games, the statement of Theorem 8 can be strengthened as follows:

Theorem 10. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game. Suppose further that each u_i is continuous. Then G possesses a strong limit-of-finite perfect equilibrium, and all strong limit-of-finite perfect equilibria are trembling-hand perfect.*

[4] Lemma 4 in Carbonell-Nicolau [8] states that a compact, metric, Borel game satisfying generic entire payoff security and generic local equi-upper semicontinuity satisfies a condition weaker than Condition (A*), but the proof actually shows that the two conditions imply Condition (A*).

Proof. That G possesses a strong lof perfect equilibrium follows immediately from Theorem 8. To see that all strong lof perfect equilibria are thp, take a strong lof perfect equilibrium μ. Then there exist a sequence (η^n) with $\eta^n \to 0$ and a double sequence $(\varepsilon^{m,n})$ with $\varepsilon^{m,n} > 0$ for each m and n,

$$\varepsilon^{m,n} \xrightarrow[n\to\infty]{} 0, \quad \text{for each } m,$$

and

$$\varepsilon^{m,n} \xrightarrow[m\to\infty]{} 0, \quad \text{for each } n,$$

such that for every m, there exists a finite $(\varepsilon^{m,n})$-approximation $(\overline{G}_{(Y^{m,n},\xi^{m,n})})$ of $(\overline{G}_{\eta^n})$ such that each $\overline{G}_{(Y^{m,n},\xi^{m,n})}$ possesses a Nash equilibrium $\mu^{m,n}$, and $\mu^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu$.

Observe that for fixed n, and for each i,

$$\begin{aligned}
&Y_i^{m,n} \xrightarrow[m\to\infty]{} X_i, \\
&\xi_i^{m,n}(Y_i^{m,n}) \xrightarrow[m\to\infty]{} \eta_i^n(X_i), \text{ and} \\
&\tfrac{\xi_i^{m,n}}{\xi_i^{m,n}(Y_i^{m,n})} \xrightarrow[m\to\infty]{} \tfrac{\eta_i^n}{\eta_i^n(X_i)}.
\end{aligned}$$

Fix n. To lighten notation we drop the index n:

$$\begin{aligned}
\mu^m &:= \mu^{m,n}, \\
\xi^m &:= \xi^{m,n}, \\
Y^m &:= Y^{m,n}.
\end{aligned}$$

Thus, the sequences (μ_i^m), (ξ_i^m), and (Y_i^m) are such that ξ_i^m is a measure with finite support $Y_i^m \subseteq X_i$, μ^m is a Nash equilibrium of $\overline{G}_{(Y^m,\xi^m)}$ for each m, and, for each i, $Y_i^m \to X_i$, $\xi_i^m(Y_i^m) \to \eta_i^n(X_i)$, and $\frac{\xi_i^m}{\xi_i^m(Y_i^m)} \to \frac{\eta_i^n}{\eta_i^n(X_i)}$.

Because each μ^m is a Nash equilibrium of $\overline{G}_{(Y^m,\xi^m)}$, we may write $\mu_i^m = \hat{\mu}_i^m + \xi_i^m$ for each i, where $\hat{\mu}_i^m := \mu_i^m - \xi_i^m$ is a Borel measure on X_i. Hence, for each i,

$$\begin{aligned}
\mu_i^m &= \hat{\mu}_i^m(Y_i^m)\tfrac{\hat{\mu}_i^m}{\hat{\mu}_i^m(Y_i^m)} + \xi_i^m(Y_i^m)\tfrac{\xi_i^m}{\xi_i^m(Y_i^m)} \\
&= (1 - \xi_i^m(Y_i^m))\tfrac{\hat{\mu}_i^m}{\hat{\mu}_i^m(Y_i^m)} + \xi_i^m(Y_i^m)\tfrac{\xi_i^m}{\xi_i^m(Y_i^m)}.
\end{aligned}$$

For each i the sequences $\left(\frac{\hat{\mu}_i^m}{\hat{\mu}_i^m(Y_i^m)}\right)$ and $\left(\frac{\xi_i^m}{\xi_i^m(Y_i^m)}\right)$ lie in $\Delta(X_i)$, so we may write (passing to a subsequence if necessary)

$$\mu_i^m = (1-\xi_i^m(Y_i^m))\tfrac{\hat{\mu}_i^m}{\hat{\mu}_i^m(Y_i^m)} + \xi_i^m(Y_i^m)\tfrac{\xi_i^m}{\xi_i^m(Y_i^m)} \to (1-\delta^n)\rho_i^n + \delta^n\sigma_i^n =: p_i^n,$$

for some $\delta^n \in [0,1]$ and some $(\rho_i^n, \sigma_i^n) \in \Delta(X_i) \times \Delta(X_i)$, and since $\frac{\xi_i^m}{\xi_i^m(Y_i^m)} \to \frac{\eta_i^n}{\eta_i^n(X_i)}$ and $\xi_i^m(Y_i^m) \to \eta_i^n(X_i)$, we have $\sigma_i^n = \frac{\eta_i^n}{\eta_i^n(X_i)}$ and $\delta^n = \eta_i^n(X_i)$, so that

$$\begin{aligned} p_i^n &= (1-\eta_i^n(X_i))\rho_i^n + \eta_i^n(X_i)\tfrac{\eta_i^n}{\eta_i^n(X_i)} \\ &= (1-\eta_i^n(X_i))\rho_i^n + \eta_i^n. \end{aligned}$$

Choose n. We claim that $p^n = (p_1^n, ..., p_N^n)$ is a Nash equilibrium of $\overline{G}_{\eta^n}$. To see this, note first that for each i, $p_i^n \in \Delta(X_i)$ and $p_i^n \geq \eta_i^n$, implying that $p_i^n \in \Delta(X_i, \eta_i^n)$. Suppose that p^n is not a Nash equilibrium of $\overline{G}_{\eta^n}$. Then, since $p_i^n \in \Delta(X_i, \eta_i^n)$ for each i, there exist i and $q_i^n \in \Delta(X_i, \eta_i^n)$ such that $u_i(q_i^n, p_{-i}^n) > u_i(p^n)$. Since $q_i^n \in \Delta(X_i, \eta_i^n)$, we can express $q_i^n = \theta_i^n + \eta_i^n$, where $\theta_i^n := q_i^n - \eta_i^n$ is a Borel measure implying that

$$\begin{aligned} q_i^n &= \theta_i^n(X_i)\tfrac{\theta_i^n}{\theta_i^n(X_i)} + \eta_i^n(X_i)\tfrac{\eta_i^n}{\eta_i^n(X_i)} \\ &= (1-\eta_i^n(X_i))\tfrac{\theta_i^n}{\theta_i^n(X_i)} + \eta_i^n(X_i)\tfrac{\eta_i^n}{\eta_i^n(X_i)}. \end{aligned}$$

Because u_i is continuous on $\Delta(X)$, there exist $\epsilon > 0$ and a finitely supported measure υ_i in $\Delta(X_i)$ such that $1 - \eta_i^n(X_i) - \epsilon > 0$ and

$$u_i(q_i^\epsilon, p_{-i}^n) > u_i(p^n),$$

where

$$q_i^\epsilon := (1-\eta_i^n(X_i)-\epsilon)\upsilon_i + (\eta_i^n(X_i)+\epsilon)\tfrac{\eta_i^n}{\eta_i^n(X_i)}.$$

Consequently, because u_i is continuous on $\Delta(X)$ and $\mu^m \to p^n$, we have $u_i(q_i, \mu_{-i}^m) > u_i(\mu^m)$ for each q_i in some neighborhood $V_{q_i^\epsilon}$ of q_i^ϵ and for every sufficiently large m.

Since υ_i is finitely supported in $\Delta(X_i)$, and since $Y_i^m \to X_i$, there is a sequence (υ_i^m) of measures in $\Delta(X_i)$ such that $\mathrm{supp}(\upsilon_i^m) = Y_i^m$ for each m and $\upsilon_i^m \to \upsilon_i$. In fact, let $\{x_{1i}, ..., x_{\kappa i}\}$ be the finite support of υ_i. Since $Y_i^m \to X_i$, for each $k \in \{1, ..., \kappa\}$ one can choose a sequence (y_{ki}^m) such that $y_{ki}^m \in Y_i^m$ for each m and $y_{ki}^m \xrightarrow[m\to\infty]{} x_{ki}$. For each m, define $\widehat{\upsilon}_i^m$ in $\Delta(X_i)$ by

$$\widehat{\upsilon}_i^m(\{y_{ki}^m\}) := \upsilon_i(\{x_{ki}\}), \quad \text{for each } k \in \{1, ..., \kappa\}.$$

Let $\widetilde{\upsilon}_i^m \in \Delta(X_i)$ be defined by $\widetilde{\upsilon}_i^m(y_i^m) := \frac{1}{\#Y_i^m}$ for each $y_i^m \in Y_i^m$, where $\#Y_i^m$ denotes the cardinality of the finite set Y_i^m. Define

$$\upsilon_i^m := \left(1 - \tfrac{1}{m}\right) \widehat{\upsilon}_i^m + \tfrac{1}{m}\widetilde{\upsilon}_i^m.$$

Clearly, $\mathrm{supp}(\upsilon_i^m) = Y_i^m$ for each m. To see that $\upsilon_i^m \to \upsilon_i$, let O_i be an open set in X_i and define $I := \{k : x_{ki} \in O_i\}$. Then there exists an $\widehat{m}$ such that, $y_{ki}^m \in O_i$ for each $m > \widehat{m}$ and each $k \in I$. Therefore, $m > \widehat{m}$ implies that

$$\upsilon_i(O_i \cap \{x_{1i}, ..., x_{\kappa i}\}) = \sum_{k \in I} \upsilon_i(\{x_{ki}\}) = \sum_{k \in I} \widehat{\upsilon}_i^m(\{y_{ki}^m\}) \leq \widehat{\upsilon}_i^m(O_i \cap \{y_{1i}^m, ..., y_{\kappa i}^m\}),$$

so for $m > \widehat{m}$, we have

$$\begin{aligned}
\upsilon_i^m(O_i) &= \left(1 - \tfrac{1}{m}\right) \widehat{\upsilon}_i^m(O_i) + \tfrac{1}{m}\widetilde{\upsilon}_i^m(O_i) \\
&= \widehat{\upsilon}_i^m(O_i) + \tfrac{1}{m}\left(\widetilde{\upsilon}_i^m(O_i) - \widehat{\upsilon}_i^m(O_i)\right) \\
&\geq \widehat{\upsilon}_i^m(O_i) - \tfrac{1}{m} \\
&= \widehat{\upsilon}_i^m(O_i \cap \{y_{1i}^m, ..., y_{\kappa i}^m\}) - \tfrac{1}{m} \\
&\geq \upsilon_i(O_i \cap \{x_{1i}, ..., x_{\kappa i}\}) - \tfrac{1}{m} \\
&= \upsilon_i(O_i) - \tfrac{1}{m},
\end{aligned}$$

from which it follows that $\underline{\lim}\, \upsilon_i^m(O_i) \geq \upsilon_i(O_i)$.

Now define, for each m, $q_i^m \in \Delta(X_i)$ as follows:

$$q_i^m := (1 - \eta_i^n(X_i) - \epsilon)\upsilon_i^m + (\eta_i^n(X_i) + \epsilon)\tfrac{\xi_i^m}{\xi_i^m(Y_i^m)}.$$

Because $\xi_i^m(Y_i^m) \to \eta_i^n(X_i)$, for any large enough m we have $\frac{\eta_i^n(X_i)+\epsilon}{\xi_i^m(Y_i^m)} \geq 1$, and therefore

$$\begin{aligned}
q_i^m &= (1 - \eta_i^n(X_i) - \epsilon)\upsilon_i^m + (\eta_i^n(X_i) + \epsilon)\tfrac{\xi_i^m}{\xi_i^m(Y_i^m)} \\
&\geq \xi_i^m,
\end{aligned}$$

and since q_i^m has support Y_i^m for each m, we have $q_i^m \in \Delta(Y_i^m, \xi_i^m)$ for each m. On the other hand, we have $q_i^m \to q_i^\epsilon$, since $\upsilon_i^m \to \upsilon_i$ and $\frac{\xi_i^m}{\xi_i^m(Y_i^m)} \to \frac{\eta_i^n}{\eta_i^n(X_i)}$.

Since $u_i(q_i, \mu_{-i}^m) > u_i(\mu^m)$ for each q_i in some neighborhood $V_{q_i^\epsilon}$ of q_i^ϵ and for every sufficiently large m, and since $q_i^m \to q_i^\epsilon$, for large enough m we have $u_i(q_i^m, \mu_{-i}^m) > u_i(\mu^m)$, and because $q_i^m \in \Delta(Y_i^m, \xi_i^m)$, this contradicts the assumption that μ^m is a Nash equilibrium of $\overline{G}_{(Y^m,\xi^m)}$.

We conclude that p^n is a Nash equilibrium of $\overline{G}_{\eta^n}$. Since p^n is a Nash equilibrium of $\overline{G}_{\eta^n}$ for each n and $p^n \to \mu$, the proof will be complete if we show that $p^n \to \mu$.

We have seen that for fixed n and for each i,

$$\mu_i^{m,n} \xrightarrow[m\to\infty]{} p_i^n, \quad \text{for each } n. \tag{12}$$

On the other hand, for each i,

$$\mu_i^{m,n} \xrightarrow[n\to\infty]{\text{unif}} \mu_i. \tag{13}$$

This, combined with (12), yields $p_i^n \to \mu_i$ for each i. ■

It is shown in Simon and Stinchcombe [19] that any trembling-hand perfect equilibrium of a compact, metric, Borel game $G = (X_i, u_i)$ with each u_i continuous is limit admissible. From this observation and Theorem 10, we obtain the following result.

Theorem 11. *Suppose that $G = (X_i, u_i)$ is a compact, metric, Borel game. Suppose further that each u_i is continuous. Then any strong limit-of-finite perfect equilibrium of G is limit admissible.*

6 Additional proofs

6.1 Proof of Lemma 1

The proof of Lemma 1 relies on the following lemma.

Lemma 4. *For each $i \in \{1, ..., N\}$, let X_i be a compact metric space. Suppose that $f : X \to \mathbb{R}$ is bounded and Borel measurable, where $X := \times_{i=1}^N X_i$. Suppose that $Q_i \subseteq X_i$ is countable for each i. For each $\mu = (\mu_1, ..., \mu_N) \in \Delta(X)$, there exists a sequence (μ^n) with $\Delta(X) \ni \mu^n \to \mu$ such that each μ_i^n has finite support X_i^n, $X_i^n \subseteq X_i^{n+1}$ for each i and n, $\bigcup_{n=1}^\infty X_i^n \supseteq Q_i$ for each i, and $\int_X f d\mu^n \to \int_X f d\mu$.*

We omit the proof of Lemma 4, which is similar to that of Lemma 6 in [8].

Lemma 1. *Suppose that* (X_i, u_i) *is a compact, metric, Borel game. Suppose that* $Q_i \subseteq X_i$ *is countable for each* i. *If* $(\delta, \mu) \in [0,1)^N \times \Delta(X)$ *and* $[0,1)^N \ni \delta^n \to \delta$, *then there exists a sequence* (μ^n) *with* $\Delta(X) \ni \mu^n \to \mu$ *such that each* μ_i^n *has finite support* X_i^n, $X_i^n \subseteq X_i^{n+1}$ *for each* n *and* i, $\bigcup_{n=1}^{\infty} X_i^n \supseteq Q_i$ *for each* i, *and* $u^{(\delta^n,\mu^n)} \to u^{(\delta,\mu)}$ *in* $U(X)$.

Proof. Suppose that each X_i is a compact subset of a metric space and that each u_i is bounded and Borel measurable. Let $(\delta, \mu) \in [0,1)^N \times \Delta(X)$ and take a sequence (δ^n) with $[0,1)^N \ni \delta^n \to \delta$. Using the convention that $y = (y_1, ..., y_N)$ is the profile of Dirac measures $(\delta_{y_1}, ..., \delta_{y_N})$, we write

$$u_i^{(\delta,y)}(x) := u_i((1-\delta_1)x_1 + \delta_1 y_1, ..., (1-\delta_N)x_N + \delta_N y_N).$$

By Lemma 4, there exists a sequence (μ^n) with $\Delta(X) \ni \mu^n \to \mu$ such that each μ_i^n has finite support X_i^n, $X_i^n \subseteq X_i^{n+1}$ for each i and n, $\bigcup_{n=1}^{\infty} X_i^n \supseteq Q_i$ for each i, and

$$\lim_{n\to\infty} \rho(u_i^{(\delta,\mu^n)}, u_i^{(\delta,\mu)}) = 0.$$

Since

$$u_i^{(\delta,\mu^n)}(x) = \sum_{S \subseteq \{1,..,N\}} \left[\prod_{i \in S} (1-\delta_i) \prod_{j \notin S} \delta_j \right] u_i((x_i)_{i \in S}, (\mu_j^n)_{j \notin S})$$

and u_i is bounded, it follows that

$$\lim_{n\to\infty} \rho(u_i^{(\delta^n,\mu^n)}, u_i^{(\delta,\mu^n)}) = 0,$$

implying that $u_i^{(\delta^n,\mu^n)} \to u_i^{(\delta,\mu)}$ as desired. ■

References

[1] Al-Najjar, N. (1995), "Strategically stable equilibria in games with infinitely many pure strategies," *Mathematical Social Sciences*, 29, 151-164.

[2] Bagh, A., and A. Jofre (2006), "Weak reciprocal upper semi-continuity and Better reply secure games," *Econometrica*, 74, 1715-1721.

[3] Bagh, A. (2010), "Variational convergence: approximation and existence of equilibria in discontinuous games," *Journal of Economic Theory*, 145, 1244-1268.

[4] Barelli, P., and I. Soza (2010), "On the existence of Nash equilibria in discontinuous and qualitative games," mimeo.

[5] Carbonell-Nicolau, O. (2011), "The existence of perfect equilibrium in discontinuous games," *Games*, 2, 235-256.

[6] Carbonell-Nicolau, O., "Perfect and limit admissible perfect equilibria in discontinuous games," *Journal of Mathematical Economics*, forthcoming.

[7] Carbonell-Nicolau, O. (2011), "On the existence of pure-strategy perfect equilibrium in discontinuous games," *Games and Economic Behavior*, 71, 23-48.

[8] Carmona, G. (2009), "An existence result for discontinuous games," *Journal of Economic Theory*, 144, 1333-1340.

[9] Carmona, G. (2010), "Understanding some recent existence results for discontinuous games," *Economic Theory*, forthcoming.

[10] Dasgupta, P., and E. Maskin (1986), "The existence of equilibrium in discontinuous economic games, I: Theory," *Review of Economic Studies* 53, 1–26.

[11] Gatti, R. J. (2005), "A note on the existence of Nash equilibrium in games with discontinuous payoffs," mimeo.

[12] Gürkan, G., and J. S. Pang (2009), "Approximizations of Nash equilibria," *Mathematical Programming B*, 117(1-2), 223-253.

[13] Lucchetti, R, and F. Patrone (1986), "Closure and upper semicontinuity results in mathematical programming, Nash and economic equilibria," *Optimization*, 17, 619-628.

[14] McLennan, A., Monteiro, P. K., and R. Tourky, "Games with discontinuous payoffs: a strengthening of Reny's existence theorem," *Econometrica*, forthcoming.

[15] Reny, P. J. (1999), "On the existence of pure and mixed strategy Nash equilibria in discontinuous games," *Econometrica*, 67, 1029-1056.

[16] Selten, R. (1975), "Reexamination of the perfectness concept for equilibrium points in extensive games," *International Journal of Game Theory*, 4, 25-55.

[17] Simon, L. K., "Games with discontinuous payoffs," *Review of Economic Studies*, 54, 569-597.

[18] Simon, L. K., and M. B. Stinchcombe (1995), "Equilibrium refinement for infinite normal-form games," *Econometrica*, 63, 1421-1443.