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Bertrand Delegation Games With Implementability In Weakly Undominated SPNE

Junnosuke Shino ^{*†}

Abstract

In this paper we investigate a delegation game in which the underlying situation is represented by a standard Bertrand price competition. In contrast to Fershtman, Judd, and Kalai ([2], hereafter FJK), we employ a weakly undominated SPNE as the equilibrium concept based on Shino [6] and keep the notion of the implementability unchanged. We first point out that optimal price setting behaviors derived by the assumption of “mutually rational players” in FJK are hard to justify from a standard game theory view in that they are assumed to play a weakly dominated strategy even when each of them has the weakly dominant strategy. Next, it is shown that there exists a compensation scheme fully implementing the “most collusive outcome” in which firms jointly set the monopoly price and equally share the profit.

JEL classifications: C72, D04, L13

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1 Introduction

1.1 Introduction

In this paper we investigate a delegation environment in which the underlying situation is represented by a standard Bertrand price competition¹. For a delegation game defined in Fershtman, Judd, and Kalai ([2], hereafter FJK), Shino [6] employs a refined SPNE (weakly undominated SPNE, U-SPNE) as the equilibrium concept, while keeping the notion of the implementability unchanged. By applying U-SPNE, he shows that in certain classes of 2×2 games - including prisoners' dilemma, coordination games, and battle of sexes - every efficient outcome is fully implementable in U-SPNE. This paper first examines the FJK's assumption of "mutually rational players" in the Bertrand delegation game. We argue that optimal price setting behaviors derived by this assumption are hard to justify from a standard game theory view in that they are assumed to play a weakly dominated strategy even when each of them has the weakly dominant strategy. Given this observation, we apply U-SPNE to the Bertrand delegation game and show that there exists a compensation scheme fully implementing the "most collusive outcome" in which firms jointly set the monopoly price and equally share the profit.

1.2 Delegation Game

We consider a situation where two principals face a conflict represented by a strategic form game $G = (P, \{S_i\}_{i=1,2}, \{u_i\}_{i=1,2})$. Let $P = \{p_1, p_2\}$ be the set of players, and p_i is called *principal i* . S_i is the set of strategies of p_i , and u_i is p_i 's utility function, where $u_i : S \equiv S_1 \times S_2 \rightarrow \mathbf{R}$. We call G an *underlying game*. Denote $u \equiv (u_1, u_2) : S \rightarrow \mathbf{R}^2$.

For G , suppose principal i could delegate agent i and consider the following three-stage strategic delegation environment. At the first stage, each principal simultaneously proposes to her agent a compensation scheme. Principal i 's compensation scheme gives a monetary

¹Since Shelling [5], a great deal of attention has been paid to analyzing strategic delegation by using game-theoretic framework. For example, Fershtman and Judd [1] studied a model of Cournot duopoly with delegation and unobservable efforts by agents. Persson and Tabellini [4] analyzed international monetary policy coordination by casting central banks as agents acting on behalf of the public. Persson *et al.* [3] also examined public finance with a delegation setup.

reward to agent i , depending on which payoff in G is realized by agents' interaction in the last stage. Next, in the second stage, each agent simultaneously decides whether to participate the game or not, after observing not only his own contract but also his opponent's. If (at least) one of agents decides not to participate, the game ends at this stage. In this case, agents obtain common reservation wage $\epsilon > 0$, while principals get a constant payoff normalized to zero. If, on the other hand, both of agents decide to participate, the game moves on to the final stage, and agent i chooses an action $s_i \in S_i$ in the underlying game. Then "total payoff" of $u_i(s_1, s_2)$ is realized and depending on this, monetary rewards are paid to agent i based on the compensation scheme. What principal i obtains is the total payoff, $u_i(s_1, s_2)$, minus the reward for agent i .

Formally, for a underlying game G , its *delegation game*, denoted by $D(G)$, is defined as

$$D(G) \equiv (N, \{C_i, L_i\}_{i=1,2}, \{U_i^p, U_i^a\}_{i=1,2}, \epsilon). \quad (1)$$

- $N \equiv P \cup A$ is the set of players where $P = \{p_1, p_2\}$ is the set of *principals* and $A = \{a_1, a_2\}$ is the set of *agents*.
- C_i is the set of strategies of p_i where $C_i = \{c_i \mid c_i : \mathbf{R} \rightarrow \mathbf{R}_+, \text{non-decreasing function}\}$. $c_i \in C_i$ is called a *compensation function* of p_i . Define $C \equiv C_1 \times C_2$.
- L_i is the set of strategies of a_i defined as follows. First, let P and NP be an agent choice of "Participate" and "Not Participate" respectively. Then agent i 's *participate function* is defined as $D_i = \{d_i \mid d_i : C \rightarrow \{P, NP\}\}$. d_i specifies agent i 's choice of participating or not, contingent on his observation about C . Next, we define *response function* R_i where $R_i = \{r_i \mid r_i : C \rightarrow S_i\}$, which specifies agent i 's an action in the last stage. Finally, agent i 's strategy set L_i is defined as $L_i \equiv D_i \times R_i$, that is, i 's strategy set is the set of the profile of the participation function and the response function.

- U_i^p and U_i^a are principal and agent i 's utility functions in $D(G)$ defined as:

$$U_i^p(c_1, c_2, l_1, l_2) = \begin{cases} u_i(r_1(c), r_2(c)) - c_i(u_i(r_1(c), r_2(c))) & \text{if } d_1(c) = d_2(c) = P \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$U_i^a(c_1, c_2, l_1, l_2) = \begin{cases} c_i(u_i(r_1(c), r_2(c))) & \text{if } d_1(c) = d_2(c) = P \\ \epsilon & \text{otherwise.} \end{cases} \quad (3)$$

For the following analysis, we denote a strategy profile of all players (agents and principals) by $(c, l) \equiv (c_1, c_2, l_1, l_2) \equiv (c_1, c_2, (d_1, r_1), (d_2, r_2))$.

For a compensation scheme profile $c \in C$, its subsequent game played at the last stage in the case where both agents participate can be specified. We call it *induced game by c* and denote by $G(c)$. For $c \in C$, we say $(s_1^*, s_2^*) \in S$ is a *Nash equilibrium (NE) in $G(c)$* iff $c_i(u_i(s_1^*, s_2^*)) \geq c_i(u_i(s_i, s_j^*))$ for all $s_j \in S_j$ and for $i \in \{1, 2\}$.

1.3 The Solution Concept

In this subsection, we give a formal definition of the solution concept applied to the Bertrand delegation game. For detailed arguments, see Shino [6].

As we will define, the solution concept, called *undominated subgame perfect Nash equilibrium (U-SPNE)*, is a strategy profile $(c_1^*, c_2^*, l_1^*, l_2^*)$ satisfying the conditions of (6), (7), and (8). As a first step, consider the last stage game played among agents. For a given compensation profile c , denote set of NEs in $G(c)$ by

$$EA(c) = \{(s_1, s_2) \in S \mid (s_1, s_2) \text{ is an NE in } G(c)\} \quad (4)$$

and define $UEA(c)$ as the set of weakly undominated Nash equilibria in $G(c)$:

$$UEA(c) = \{(s_1, s_2) \in S \mid (s_1, s_2) \text{ is a N.E. in } G(c) \text{ and } s_i \in S_i (i = 1, 2) \text{ is not weakly dominated strategy in } G(c)\} \quad (5)$$

where s_i is said to be a weakly dominated strategy in $G(c)$ if there is a strategy $\tilde{s}_i \in S_i$ such that $c_i(u_i(\tilde{s}_i, s_j)) \geq c_i(u_i(s_i, s_j))$ for all $s_j \in S_j$ and $c_i(u_i(\tilde{s}_i, s_j)) > c_i(u_i(s_i, s_j))$ for some $s_j \in S_j$. We employ

(5) as the first condition of our solution concept. Noting that $UEA(c)$ can take empty set for some c , the first condition of U-SPNE is defined as the following:

$$\begin{aligned} r^*(c) \in UEA(c) \quad \forall c \in C \text{ with } UEA(c) \neq \phi \\ r^*(c) \in EA(c) \quad \forall c \in C \text{ with } UEA(c) = \phi \end{aligned} \quad (6)$$

Next, given the action profile played at the last stage, we consider agents' participation choices at the second stage. For a given strategy profile of (c_1, c_2, l_1, l_2) , the payoff matrix at the second stage can be described as follows:

| | | |
|------|--|----------------------|
| G | P | NP |
| P | $c_1(u_1(r_1(c), r_2(c))), c_2(u_2(r_1(c), r_2(c)))$ | ϵ, ϵ |
| NP | ϵ, ϵ | ϵ, ϵ |

Figure 1: Agents' payoff matrix at the second stage by backward induction

Depending on the magnitudes of $c_i(u_i(r_1(c), r_2(c)))$ and ϵ , three cases could happen: (A) P is weakly dominant strategy, (B) NP is weakly dominant strategy, and (C) P and NP are completely indifferent. For case (A) and (B), it is totally natural to assume that the weakly dominant strategy would be played. For simplicity, we make the standard assumption that if the agent is indifferent between participating and not participating, he chooses to participate. Putting it all together, we introduce the following as the second condition of U-SPNE:

$$\text{For every } c \in C, d_i^*(c) = P \text{ iff } c_i(u_i(r_1^*(c), r_2^*(c))) \geq \epsilon. \quad (7)$$

The last step is to consider the first stage in which principals propose compensation schemes. The last condition on U-SPNE is the standard one:

$$U_1^p(c_1^*, c_2^*, l_1^*, l_2^*) \geq U_1^p(c_1, c_2^*, l_1^*, l_2^*) \quad \forall c_1 \in C_1 \text{ and same for principal 2.} \quad (8)$$

As a result, our solution concept applied to the delegation game is expressed as the following definition:

Definition 1.1

$(c_1^*, c_2^*, l_1^*, l_2^*)$ is U-SPNE if it satisfies the conditions of (6), (7), and (8)

Then we define implementation and full implementation.

Definition 1.2

c^* implements $\omega \in \mathbf{R}^2$ in U-SPNE via $l^* \in L$ iff (c^*, l^*) is a U-SPNE of $D(G)$ with (1) $u(r^*(c^*)) = \omega$ and (2) $d_1^*(c^*) = d_2^*(c^*) = P$.

Definition 1.3

c^* fully implements $\omega \in \mathbf{R}^2$ in U-SPNE via $l^* \in L$ iff c^* implements ω via l^* in U-SPNE and the following holds:

$$\text{if } c^* \text{ also implements } \tilde{\omega} \in \mathbf{R}^2 \text{ in U-SPNE via some } \tilde{l}, \text{ then } \tilde{\omega} = \omega. \tag{9}$$

2 Bertrand Delegation Game

This section investigates the delegation game in which the underlying game is a standard Bertrand price competition. After introducing the model and notations in Subsection 2.1, Subsection 2.2 points out that price optimal setting behaviors derived by FJK’s assumption of mutually rational players are hard to justify from a standard game theory view in that they are assumed to play a weakly dominated strategy *even when each of them has the weakly dominant strategy*. Next, in Subsection 2.3, it is shown that there exists a compensation scheme fully implementing the “most collusive outcome” in which firms jointly set the monopoly price and equally share the profit.

2.1 The Model

Consider a standard Bertrand game where two firms produce an identical goods thus face the demand function given by;

$$D(P_i, P_j) = \begin{cases} A - P_i & \text{if } P_i < P_j \\ \frac{1}{2}(A - P_i) & \text{if } P_i = P_j \\ 0 & \text{if } P_i > P_j \end{cases} \tag{10}$$

and both firms have common marginal cost MC with $A > MC$. Thus firm i 's profit π_i is expressed as

$$\pi_i(P_i, P_j) = \begin{cases} (A - P_i)(P_i - MC) & \text{if } P_i < P_j \\ \frac{1}{2}(A - P_i)(P_i - MC) & \text{if } P_i = P_j \\ 0 & \text{if } P_i > P_j. \end{cases} \quad (11)$$

i 's strategy set is $[0, \infty)$. Denote $\pi \equiv (\pi_1, \pi_2) : [0, \infty) \times [0, \infty) \rightarrow \mathbf{R}^2$. Let P^{NE} and π^{NE} be the price and the profit in the unique equilibrium of the one-shot game respectively, that is,

$$(P^{NE}, \pi^{NE}) = (MC, 0). \quad (12)$$

On the other hand, if the "best collusion" is successfully attained, both firms would choose the monopoly price, maximize the sum of their profits, and equally share them. Such collusive price and the profit, denoted by P^c and π^c , are given by

$$(P^c, \pi^c) = \left(\frac{1}{2}(A + MC), \frac{1}{8}(A - MC)^2 \right). \quad (13)$$

Hereafter we call (π^c, π^c) the "most collusive outcome."

2.2 FJK's compensation scheme

In FJK argument, the most collusive outcome (π^c, π^c) can be fully implemented with mutually rational agents when ϵ is small. Particularly, "Folk Theorem" established in their paper proposes the following compensation scheme c_i^{FJK} .

$$c_i^{FJK}(\pi_i) = \begin{cases} \epsilon & \text{if } \pi_i \geq \pi^c \\ 0 & \text{if otherwise.} \end{cases} \quad (14)$$

In the induced game $G(c_1^{FJK}, c_2^{FJK}) \equiv G(c^{FJK})$, (P^c, P^c) is the only action profile which gives ϵ to both agents. Therefore, given the notion of implementability with mutually rational agents, the only outcome which can be implemented by c^{FJK} is (π^c, π^c) , and they show that c^{FJK} indeed fully implements (π^c, π^c) . For their analysis, note that (π^c, π^c) is implemented by making both

agents play P^c in $G(c^{FJK})$, and we first point out some remarks below.

Remark 2.1

In $G(c^{FJK})$, P^c is a weakly dominated strategy.

Proof of Remark 2.1. See Appendix.

In principle, a weakly dominated strategy can be a (or even the unique) Nash strategy (See Fig.2). Indeed, in the standard one-shot Bertrand game which we employ as the underlying game, the unique Nash strategy $P^{NE} = MC$ is in weakly dominated strategies. However, since this Nash equilibrium is unique, it is reasonable to accept this as the “equilibrium,” even if P^{NE} is weakly dominated. However, if both weakly dominated Nash *and* undominated Nash exist, it may be reasonable to assume that the latter would be chosen. The following remark shows this point: in $G(c^{FJK})$, there exists other Nash equilibrium and its Nash strategy is not weakly dominated. Moreover, the remark has even much stronger implication: in $G(c^{FJK})$ there is the weakly *dominant* strategy. In other words, FJK solution concepts assumes that an agent “optimally” plays a weakly dominated strategy even when there is the weakly dominant strategy in a induced game. Such argument looks unusual in a standard game theory view.

| | | |
|----------|---------------------|--------------|
| G | s_{21} | s_{22} |
| s_{11} | 0, <u>2</u> | <u>3</u> , 1 |
| s_{12} | <u>1</u> , 1 | 2, <u>3</u> |
| s_{13} | <u>1</u> , <u>2</u> | 1, 1 |

Figure 2: Weakly Dominated Nash Strategy s_{13}

Remark 2.2

In $G(c^{FJK})$, there exists the weakly dominant strategy $\underline{P}_i^* \equiv \frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC)$.

\underline{P}_i^* is the (smaller) solution of $(A - P_i)(P_i - MC) = \pi^c$ and $\underline{P}_i^* < P^c$ holds. Intuitively, \underline{P}_i^* weakly dominates P^c as the following reason (see also Fig.3 and 4): since $\underline{P}_i^* < P^c$, to play \underline{P}_i^* is less likely to fall into a bad situation where i 's price is higher than j thus i earns zero profit. On the other hand, as long as $P_i < P_j$ satisfies, to play P^c maximize the firm i 's profit to $2\pi^c$, while to play \underline{P}_i^* gives only π^c to i . However, agents i is indifferent between these two profits under c_i^{FJK} because c_i^{FJK} reward ϵ for both of them. Therefore, \underline{P}_i^* is at least “not bad” compared to P^c .

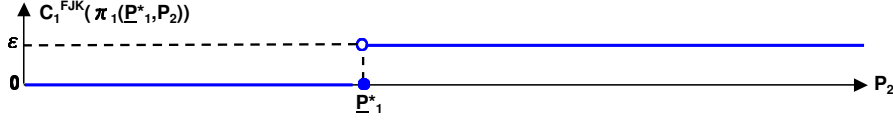


Figure 3: a_1 's compensation under c_1^{FJK} when $P_1 = \underline{P}_1^*$

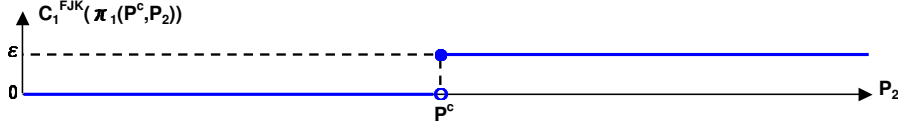


Figure 4: a_1 's compensation under c_1^{FJK} when $P_1 = P^c$

Proof of Remark 2.2. See Appendix.

Remark 2.2 implies that the compensation profile c^{FJK} can not implement (π^c, π^c) in U-SPNE because $(P^c, P^c) \notin UEA(c^{FJK}) = \{(\underline{P}_1^*, \underline{P}_2^*)\}$. It might be natural, therefore, to investigate whether c^{FJK} can implement any outcomes by using our solution concepts. The following remark shows the negative result to this question.

Remark 2.3

$c^{FJK} \equiv (c_1^{FJK}, c_2^{FJK})$ can not implement any outcomes in U-SPNE.

Proof. Suppose that, in negation, c^{FJK} implements ω in U-SPNE via some $l^* = (l_1^*, l_2^*) = ((d_1^*, r_1^*), (d_2^*, r_2^*)) \in L$. Since both agents have the weakly dominant strategy \underline{P}_i^* in $G(c^{FJK})$, $UEA(c^{FJK}) = \{(\underline{P}_1^*, \underline{P}_2^*)\}$. Therefore, $r_i^*(c^{FJK}) = \underline{P}_i^*$ must hold for $i = 1, 2$. However, under c^{FJK} ,

$$c_i^{FJK}(\pi_i(r_1^*(c^{FJK}), r_2^*(c^{FJK}))) = c_i^{FJK}(\pi_i(\underline{P}_1^*, \underline{P}_2^*)) = c_i^{FJK} \left(\frac{1}{16}(A - MC)^2 \right) = c_i^{FJK} \left(\frac{1}{2}\pi^c \right) = 0.$$

From (7), this implies $d_i^*(c^{FJK}) = NP$, contradicting to Definition 1.2 of implementation. ■

Remark 2.3 implies that, under the compensation scheme c^{FJK} , when both agents choose the (unique) weakly dominant strategy \underline{P}_i^* , the reward from the realized profit will be zero. Given this, both of them choose not participating and obtaining the reservation wage ϵ , rather entering the delegation game.

Remark 2.3 also implies that, in order to construct compensation scheme which implements an outcome in U-SPNE, we cannot employ c^{FJK} . Before constructing alternative scheme,

we check if, in our Bertrand delegation game, there exists at least one undominated Nash equilibrium for every induced game. The following remark is obvious without proof.

Remark 2.4

Let $c_i^\alpha(\pi_i(P_1, P_2)) = \alpha(\pi_i(P_1, P_2))$ where $0 < \alpha < 1$. Then the only Nash in $G(c_1^\alpha, c_2^\alpha)$ is weakly dominated, that is, $UEA(c^\alpha) = \phi$ and $EA(c^\alpha) = \{(MC, MC)\}$.

Remark 2.4 implies that $UEA(c) = \phi$ in some $G(c)$. In applying our solution concept, therefore, the second condition of (6) does matter. This condition requires that players could play a weakly dominated Nash if there is no weakly undominated Nash $((s_{13}, s_{21})$ in Fig.2). Again, note that exactly same logic works when we employ Nash equilibrium as a solution concept in the standard Bertrand Nash, and that this additional condition never violates our methodology to use the notion of weak dominance relationship as equilibrium refinement. Furthermore, as we will see, under the compensation scheme c^* which fully implements the most collusive outcome (π^c, π^c) , $UEA(c^*) \neq \phi$. This means that weakly undominated Nash equilibrium is played *on the equilibrium path*.

2.3 Full implementability in U-SPNE

Now we look for a compensation scheme which could implement an outcome in U-SPNE. In the last part of this section, it will be shown that such compensation scheme exists, and the scheme fully implements the most collusive outcome (π^c, π^c) . First of all, consider the following compensation scheme:

$$c_i^*(\pi_i) = \begin{cases} 2\alpha\pi^c & \text{if } \pi_i = 2\pi^c \left(\equiv \frac{1}{4}(A - MC)^2 \right) \\ \epsilon & \text{if } \pi^c \leq \pi_i < 2\pi^c \\ 0 & \text{if } \pi_i < \pi^c \end{cases} \quad (15)$$

where α satisfies $2\alpha\pi^c > \epsilon$. Note that $2\pi^c$ is the solution for

$$\max_{P_i} (A - P_i)(P_i - MC), \quad (16)$$

and $\arg \max_{P_i} (A - P_i)(P_i - MC) = P^c$. Namely, $2\pi^c$ is the maximum profit among those which can be attained by choosing P^c in the underlying game.

Recall that, under c^{FJK} , $\pi(P^c, P^c) = (\pi^c, \pi^c)$ cannot be implemented in U-SPNE because P^c is weakly dominated by \underline{P}_i^* . Intuitively, this weak dominance relation means that \underline{P}_i^* is at least “not bad” compared to P^c for every P_j from the viewpoint of agent i who faces c_i^{FJK} . On the other hand, under our compensation scheme c_i^* , to play P^c can be a better choice than \underline{P}_i^* (and even any other choices) for some P_j . This is because, different from c_i^{FJK} , c_i^* gives the largest amount of reward $2\alpha\pi^c$ to agent i only when P^c is played and the maximum profit $2\pi^c$ is realized (see Fig.5 and 6).

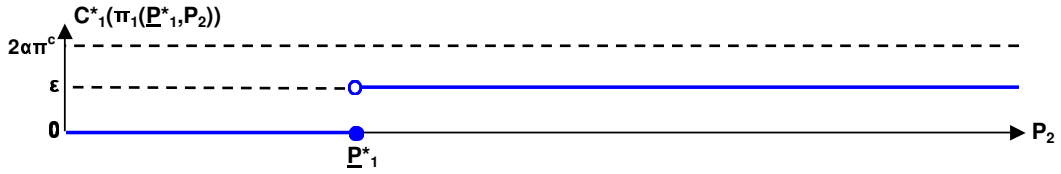


Figure 5: a_1 's compensation under c_1^* when $P_1 = \underline{P}_1^*$

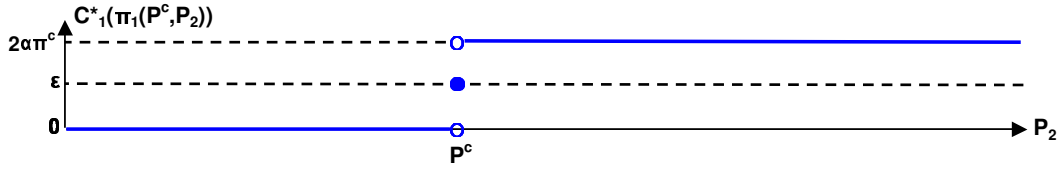


Figure 6: a_1 's compensation under c_1^* when $P_1 = P^c$

First we establish the following two remark about weakly dominated strategies in $G(c_1^*, c_2^*)$.

Remark 2.5

In $G(c_1^*, c_2^*)$, agent i 's strategy P^c weakly dominates P_i such that (1) $P_i > P^c$ and (2) $P_i < \frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC)$.

Proof. First, note that (see also Fig.6)

$$c_1^*(\pi_1(P^c, P_2)) = \begin{cases} 0 & \text{if } P_2 < P^c \\ \epsilon & \text{if } P_2 = P^c \\ 2\alpha\pi^c & \text{if } P_2 > P^c. \end{cases} \quad (17)$$

If $P_1 > P^c$, it is obvious that P^c weakly dominates P_1 . Now suppose $P_1 < P^c$. Recall that

$$\pi_1(P_1, P_2) = \begin{cases} 0 & \text{if } P_1 > P_2 \\ \frac{1}{2}(A - P_1)(P_1 - MC) & \text{if } P_1 = P_2 \\ (A - P_1)(P_1 - MC) & \text{if } P_1 < P_2. \end{cases} \quad (18)$$

Therefore, from (15), P^c weakly dominates P_1 if

$$\frac{1}{2}(A - P_1)(P_1 - MC) < \pi^c \text{ and } (A - P_1)(P_1 - MC) < \pi^c. \quad (19)$$

Obviously the latter implies the former in (19), and the latter is identical to

$$-P_1^2 + (A + MC)P_1 - \frac{1}{8}(A^2 + 6A \cdot MC + MC^2) < 0. \quad (20)$$

The condition on P_1 to satisfy (20) under the assumption of $P_1 < P^c$ is given by $P_1 < \frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC)$. ■

The results of Remark 2.5 is depicted in Fig.7.

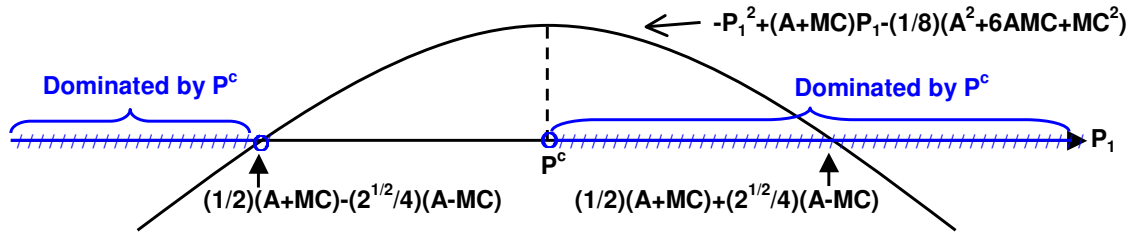


Figure 7: Weakly Dominated Region by P_c under c_1^*

Remark 2.6

Let $\underline{P}_i^* \equiv \frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC)$. In $G(c_1^*, c_2^*)$, \underline{P}_i^* weakly dominates P_i such that $\underline{P}_i^* < P_i < P^c$.

Proof. First, note that (see also Fig.5)

$$c_1^*(\pi_1(\underline{P}_1^*, P_2)) = \begin{cases} 0 & \text{if } P_2 \leq \underline{P}_1^* \\ \epsilon & \text{if } P_2 > \underline{P}_1^*. \end{cases} \quad (21)$$

Pick P_1 with $\underline{P}_1^* < P_1 < P^c$. Note that (see also Fig.14 in Appendix):

$$\pi_1(P_1, P_2) = \begin{cases} 0 & \text{if } P_2 < P_1 \\ \frac{1}{2}(A - P_1)(P_1 - MC) (< \pi^c) & \text{if } P_2 = P_1 \\ (\pi^c <) (A - P_1)(P_1 - MC) (< 2\pi^c) & \text{if } P_2 > P_1. \end{cases} \quad (22)$$

Therefore, for such P_i ,

$$c_1^*(\pi_1(P_1, P_2)) = \begin{cases} 0 & \text{if } P_2 \leq P_1 \\ \epsilon & \text{if } P_2 > P_1. \end{cases} \quad (23)$$

By comparing (23) with (21), we conclude that \underline{P}_1^* weakly dominates P_1 ■

The results of Remark 2.6 is depicted in Fig.8.

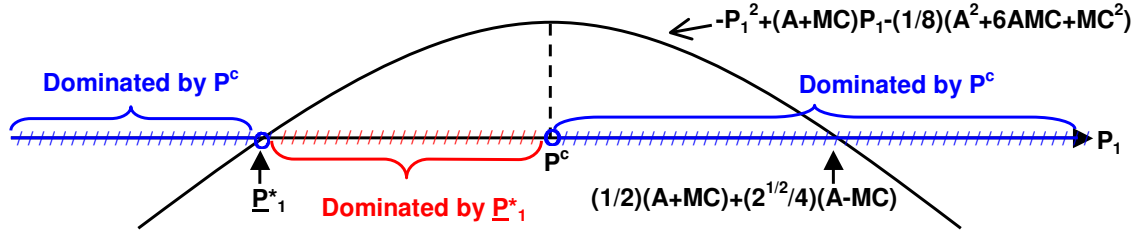


Figure 8: Weakly Dominated Region by P^c and \underline{P}_1^* under c_1^*

Remark 2.5 and 2.6 implies that $UEA(c_1^*, c_2^*) \subseteq \{P^c, \underline{P}_1^*\} \times \{P^c, \underline{P}_2^*\}$. First, we show that $(P^c, P^c) \in UEA(c_1^*, c_2^*)$ thus $UEA(c_1^*, c_2^*)$ is a non-empty set.

Remark 2.7

In $G(c_1^*, c_2^*)$, (P^c, P^c) is an weakly undominated Nash equilibrium.

Proof. (P^c, P^c) is obviously a Nash equilibrium in $G(c_1^*, c_2^*)$ because for $P_2 = P^c$, $c_1^*(\pi_1(P_1, P^c))$ is ϵ if $P_1 = P^c$ and never exceeds ϵ (possibly takes 0 or ϵ) for every $P_1 \neq P^c$. Next, we show that P^c is a weakly undominated strategy in $G(c_1^*, c_2^*)$. From Remark 2.5, it suffices to show that P^c is not weakly dominated by P_1 with $\frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC) \leq P_1 < P^c$. Now suppose $P_1 \in \left[\frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC), P^c\right)$. A necessary condition for P_1 to weakly dominates P^c in $G(c_1^*, c_2^*)$ is that (see also Fig.6):

$$\pi_1(P_1, P_2) = 2\pi^c \quad \forall P_2 \text{ with } P_2 > P^c. \quad (24)$$

However, this is impossible because

$$\max_{P_1 \in \left[\frac{1}{2}(A+MC) - \frac{\sqrt{2}}{4}(A-MC), P^c \right)} (A - P_1)(P_1 - MC) < 2\pi^c. \quad \blacksquare \quad (25)$$

Remark 2.7 implies that, if $c^* = (c_1^*, c_2^*)$ implements $\omega \in \mathbf{R}^2$ in U-SPNE, then ω has to be attained by a weakly undominated Nash equilibrium in $G(c^*)$. Here suppose that agent j chooses P^c . Then, recalling $\underline{P}_i^* < P^c$,

$$\begin{aligned} \pi_i(\underline{P}_i^*, P^c) &= (A - \underline{P}_i^*)(\underline{P}_i^* - MC) \\ &= \frac{1}{8}(A - MC)^2 = \pi^c \\ \pi_i(P^c, P^c) &= \frac{1}{8}(A - MC)^2 = \pi^c \end{aligned}$$

Next, suppose that agent j chooses \underline{P}_j^* . Then

$$\begin{aligned} \pi_i(\underline{P}_i^*, \underline{P}_j^*) &= \frac{1}{2}(A - \underline{P}_i^*)(\underline{P}_i^* - MC) \\ &= \frac{1}{16}(A - MC)^2 = \frac{1}{2}\pi^c \\ \pi_i(P^c, \underline{P}_j^*) &= 0. \end{aligned}$$

In terms of compensation value, these can be summarized in Fig.9 below:

$$\begin{array}{c|c|c} \pi(P_1, P_2) & \underline{P}_2^* & P^c \\ \hline \underline{P}_1^* & \frac{1}{2}\pi^c, \frac{1}{2}\pi^c & \pi^c, 0 \\ \hline P^c & 0, \pi^c & \pi^c, \pi^c \end{array} \implies \begin{array}{c|c|c} c^*(\pi(P_1, P_2)) & \underline{P}_2^* & P^c \\ \hline \underline{P}_1^* & 0, 0 & \epsilon, 0 \\ \hline P^c & 0, \epsilon & \epsilon, \epsilon \end{array}$$

Figure 9: $\pi(P_1, P_2)$ and $c^*(\pi(P_1, P_2))$ in $G(c^*)$

Given this, we next show the following remark.

Remark 2.8

If c^* implements $\omega \in \mathbf{R}^2$ in U-SPNE, then $\omega = (\pi^c, \pi^c)$.

Proof. Suppose, in negation, that c^* implements $\omega \in \mathbf{R}^2$ in U-SPNE via $\tilde{l} = ((\tilde{d}_1, \tilde{r}_1), (\tilde{d}_1, \tilde{r}_1))$ such that $\omega \neq (\pi^c, \pi^c)$. Since $UEA(c^*) \neq \phi$ from Remark 2.7, $\tilde{r}(c^*) \in \{P^c, \underline{P}_1^*\} \times \{P^c, \underline{P}_2^*\}$ must hold

(see Fig.8). Moreover, since $\omega \neq (\pi^c, \pi^c)$, $\tilde{r}(c^*) \in \{(P^c, P_2^*), (P_1^*, P^c), (P_1^*, P_2^*)\} \equiv \Phi$ must hold (see the left of Fig.9). However, for every $\tilde{r}(c^*) \in \Phi$, there exists $i \in \{1, 2\}$ such that (see the right of Fig.9) $c_i^*(\pi_i(\tilde{r}(c^*))) = 0 < \epsilon$. From (7), this implies $\tilde{d}_i(c^*) = NP$, which contradicts to the second condition of Definition 1.2. ■

Now we show the following main theorem.

Theorem 2.1

(c_1^*, c_2^*) defined in (15) fully implements (π^c, π^c) in U-SPNE.

Proof. From Remark 2.8, it suffices to show that (c_1^*, c_2^*) implements (π^c, π^c) in U-SPNE (Step 4 is not needed).

Step 1. Consider the strategy profile $(c_1^*, c_2^*, l_1^*, l_2^*) \equiv (c_1^*, c_2^*, (d_1^*, r_1^*), (d_2^*, r_2^*))$ such that:

$$c_i^* \text{ satisfying (15),} \tag{26}$$

$$d_i^* \text{ satisfying (7),} \tag{27}$$

and

$$(r_1^*(c_1, c_2), r_2^*(c_1, c_2)) = \begin{cases} (P^c, P^c) & \text{if } c = (c_1^*, c_2^*) \\ (P_1^*, P_2^*) \in UEA(c) & \text{if } UEA(c) \neq \phi \text{ and } c \neq c^* \\ (MC, MC) & \text{if } UEA(c) = \phi. \end{cases} \tag{28}$$

Step 2. $\pi(r^*(c^*)) = \pi(P^c, P^c) = (\pi^c, \pi^c)$ is obviously satisfied. From (26), $c_i^*(\pi_i(r_1^*(c^*), r_2^*(c^*))) = c_i^*(\pi^c) = \epsilon$. Therefore, from (27), $d_i^*(c^*) = P$.

Step 3. Now we check (6) to (8). But from the construction of r_i^* , (6) is directly satisfied. As a result, it suffices to show that principal i (w.o.l.g, we consider p_1) has no strictly preferable deviation from $(c_1^*, c_2^*, l_1^*, l_2^*)$ to $(\tilde{c}_1, c_2^*, l_1^*, l_2^*)$.

First of all, note that $U_1^p(c_1^*, c_2^*, l_1^*, l_2^*) = \pi^c - \epsilon > 0$. If $UEA(\tilde{c}_1, c_2^*) = \phi$, (MC, MC) is played thus such deviation is never profitable to p_1 . Now suppose $UEA(\tilde{c}_1, c_2^*) \neq \phi$. Noting that the compensation scheme proposed to a_2 is not changed in $G(\tilde{c}_1, c_2^*)$, we derive a_2 's best response function for every a_1 's action in $G(\tilde{c}_1, c_2^*)$.

If $0 \leq P_1 \leq \underline{P}_1^*$, then

$$c_2^*(\pi_2(P_1, P_2)) = 0 \quad \forall P_2 \quad (29)$$

If $\underline{P}_1^* < P_1 < P_c$, then

$$c_2^*(\pi_2(P_1, P_2)) = \begin{cases} 0 & \text{if } P_2 \geq P_1 \\ \epsilon & \text{if } \underline{P}_2^* \leq P_2 < P_1 \\ 0 & \text{if } P_2 < \underline{P}_2^* \end{cases} \quad (30)$$

If $P_1 = P^c$, then

$$c_2^*(\pi_2(P_1, P_2)) = \begin{cases} 0 & \text{if } P_2 > P_1 \\ \epsilon & \text{if } P_2 = P^c \\ \epsilon & \text{if } \underline{P}_2^* < P_2 < P^c \\ 0 & \text{if } P_2 < \underline{P}_2^* \end{cases} \quad (31)$$

If $P_1 > P^c$, then

$$c_2^*(\pi_2(P_1, P_2)) \begin{cases} = 2\alpha\pi^c & \text{if } P_2 = P^c \\ < 2\alpha\pi^c & \text{if otherwise} \end{cases} \quad (32)$$

From (29) to (32), a_2 's best response under $G(\cdot, c_2^*)$ can be summarized as follows:

$$\begin{cases} \text{Any} & \text{if } 0 \leq P_1 \leq \underline{P}_1^* \\ \underline{P}_2^* \leq P_2 < P^c & \text{if } \underline{P}_1^* < P_1 < P_c \\ \underline{P}_2^* \leq P_2 \leq P^c & \text{if } P_1 = P^c \\ P^c & \text{if } P_1 > P^c \end{cases} \quad (33)$$

See also the left graph of Fig.10. The shaded area means a_2 's best responses for every P_1 .

Furthermore, since $UEA(\tilde{c}_1, c_2^*) \neq \phi$, a_2 must choose a weakly undominated strategy, that is (See also the right of Fig.10),

$$r_2^*(\tilde{c}_1, c_2^*) \in \begin{cases} \{\underline{P}_2^*, P^c\} & \text{if } 0 \leq r_1^*(\tilde{c}_1, c_2^*) \leq \underline{P}_1^* \\ \{\underline{P}_2^*\} & \text{if } \underline{P}_1^* < r_1^*(\tilde{c}_1, c_2^*) < P_c \\ \{\underline{P}_2^*, P^c\} & \text{if } r_1^*(\tilde{c}_1, c_2^*) = P^c \\ P^c & \text{if } r_1^*(\tilde{c}_1, c_2^*) > P^c. \end{cases} \quad (34)$$

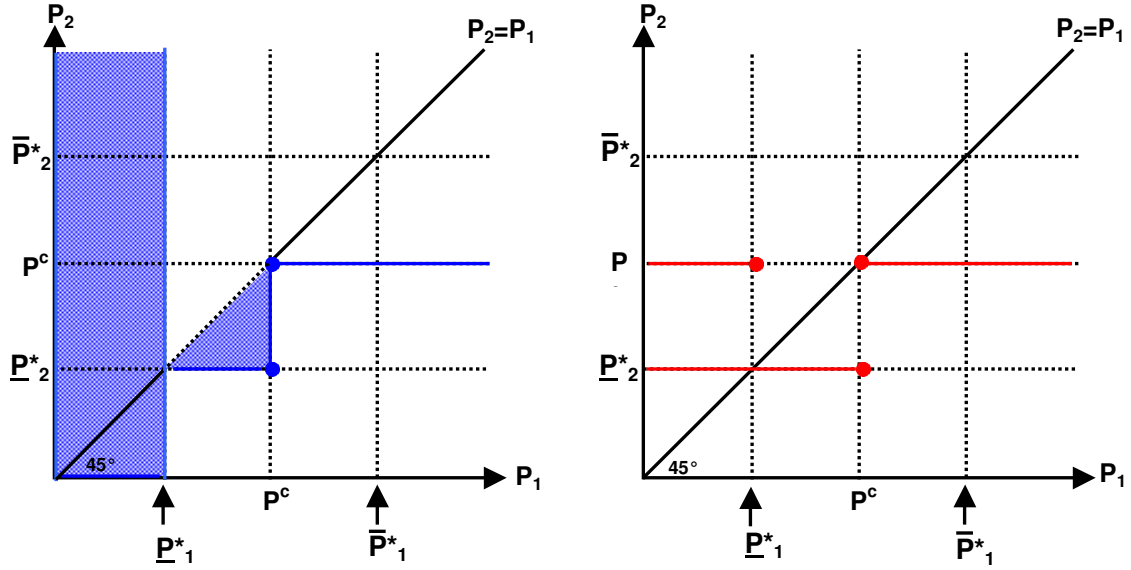


Figure 10: a_2 's Best responses (left) and weakly undominated best responses (right) in $G(\cdot, c_2^*)$

Given this a_2 's "weakly undominated best responses" expressed as (34), p_1 has to construct \tilde{c}_1 such that $U_1^p(\tilde{c}_1, c_2^*, l_1^*, l_2^*) > U_1^p(c_1^*, c_2^*, l_1^*, l_2^*) = \pi^c - \epsilon$.

First, suppose that $0 \leq r_1^*(\tilde{c}_1, c_2^*) \leq \underline{P}_1^*$. In this case, possibly maximum $\pi_1(r^*(\tilde{c}_1, c_2^*))$ can be attained when $r_1^*(\tilde{c}_1, c_2^*) = \underline{P}_1^*$ and $r_2^*(\tilde{c}_1, c_2^*) = P^c$ and $\pi_1(r^*(\tilde{c}_1, c_2^*)) = \pi^c$. For the deviation to \tilde{c}_1 to be a strictly profitable deviation, the only possibility is that $\tilde{c}_1(\pi^c) = \delta$ with $\delta < \epsilon$. However, this implies $d_1(\tilde{c}_1, c_2^*) = NP$, which results in zero payoff to p_1 . Therefore, such \tilde{c}_1 cannot be a strictly preferable deviation.

Next, suppose that $\underline{P}_1^* < r_1^*(\tilde{c}_1, c_2^*) < P^c$. Then $r_2^*(\tilde{c}_1, c_2^*) = \underline{P}_2^* < r_1^*(\tilde{c}_1, c_2^*)$. Therefore, $U_1^p(\tilde{c}_1, c_2^*, l_1^*, l_2^*) = 0 - \tilde{c}_1(0) < \pi^c - \epsilon = U_1^p(c_1^*, c_2^*, l_1^*, l_2^*)$.

Third, suppose that $r_1^*(\tilde{c}_1, c_2^*) = P^c$. If $r_2^*(\tilde{c}_1, c_2^*) = \underline{P}_2^*$, then $U_1^p(\tilde{c}_1, c_2^*, l_1^*, l_2^*) = 0 - \tilde{c}_1(0) < \pi^c - \epsilon = U_1^p(c_1^*, c_2^*, l_1^*, l_2^*)$. On the other hand, if $r_2^*(\tilde{c}_1, c_2^*) = P^c$, that is, $r^*(\tilde{c}_1, c_2^*) = (P^c, P^c)$, then $\tilde{c}_1(\pi^c) = \delta$ with $\delta < \epsilon$ has to be satisfied. However, this implies $d_1(\tilde{c}_1, c_2^*) = NP$, which results in zero payoff to p_1 . Therefore, such \tilde{c}_1 cannot be a strictly preferable deviation.

Finally, suppose that $r_1^*(\tilde{c}_1, c_2^*) > P^c$. Then $r_2^*(\tilde{c}_1, c_2^*) = P^c < r_1^*(\tilde{c}_1, c_2^*)$. Therefore, $U_1^p(\tilde{c}_1, c_2^*, l_1^*, l_2^*) = 0 - \tilde{c}_1(0) < \pi^c - \epsilon = U_1^p(c_1^*, c_2^*, l_1^*, l_2^*)$.

Therefore, p_1 cannot construct any strictly preferable deviation from $(c_1^*, c_2^*, l_1^*, l_2^*)$ ■

3 Conclusion

In this paper we investigate a delegation game in which the underlying situation is represented by a standard Bertrand price competition. In contrast to Fershtman, Judd, and Kalai ([2], hereafter FJK), we employ a weakly undominated SPNE as the equilibrium concept based on Shino [6] and keep the notion of the implementability unchanged. We first point out that optimal price setting behaviors derived by the assumption of “mutually rational players” in FJK are hard to justify from a standard game theory view in that they are assumed to play a weakly dominated strategy even when each of them has the weakly dominant strategy. Next, it is shown that there exists a compensation scheme fully implementing the “most collusive outcome” in which firms jointly set the monopoly price and equally share the profit.

Our delegation framework can obviously be extended to other IO models which can be represented by a strategic form game. For example, a delegation game where its underlying game is Cournot competition could be intriguing.

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Appendices

Proof of Remark 2.1.

We show that agent 1's action P^c in $G(c^{FJK})$ is weakly dominated by some \tilde{P}_1 . For $P_1 = P^c$, the compensation scheme proposed by principal 1 in $G(c^{FJK})$ is (see also Fig 11):

$$c_1^{FJK}(\pi_1(P^c, P_2)) = \begin{cases} \epsilon & \text{if } P_2 \geq P^c \\ 0 & \text{if } P_2 < P^c. \end{cases} \quad (35)$$

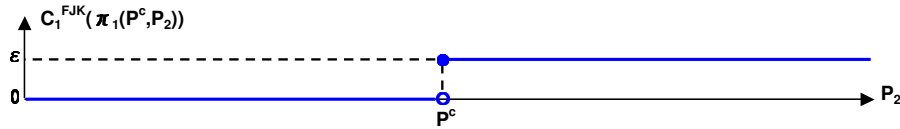


Figure 11: a_1 's compensation under c_1^{FJK} when $P_1 = P^c$

Now pick \tilde{P}_1 such that (note: we can pick such price level)

$$\frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC) \leq \tilde{P}_1 < P^c, \quad (36)$$

then \tilde{P}_1 satisfies

$$(A - \tilde{P}_1)(\tilde{P}_1 - MC) \geq \pi^c. \quad (37)$$

For any \hat{P} with $\tilde{P}_1 < \hat{P}$,

$$\pi_1(\tilde{P}_1, \hat{P}) = (A - \tilde{P}_1)(\tilde{P}_1 - MC) \geq \pi^c. \quad (38)$$

Thus $c^{FJK}(\pi_1(\tilde{P}_1, \hat{P})) = \epsilon$. Furthermore, since $\tilde{P}_1 < P^c$, there exists such \hat{P} satisfying $\tilde{P}_1 < \hat{P} < P^c$ (see Fig 12).

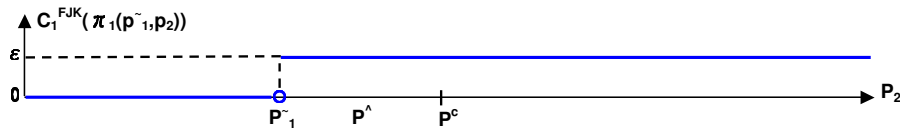


Figure 12: a_1 's compensation under c_1^{FJK} when $P_1 = \tilde{P}_1$

Consequently, \tilde{P}_1 weakly dominates P^c ■

Proof of Remark 2.2.

Let \underline{P}_1^* and \bar{P}_1^* be solutions for $(A - P_1)(P_1 - MC) = \frac{1}{8}(A - MC)^2$, that is,

$$\underline{P}_1^* = \frac{1}{2}(A + MC) - \frac{\sqrt{2}}{4}(A - MC) \quad (39)$$

$$\bar{P}_1^* = \frac{1}{2}(A + MC) + \frac{\sqrt{2}}{4}(A - MC), \quad (40)$$

then we will show that \underline{P}_1^* weakly dominates for all \tilde{P}_1 s with $\tilde{P}_1 \neq \underline{P}_1^*$. First of all,

$$\pi_1(\underline{P}_1^*, P_2) = \begin{cases} 0 & \text{if } P_2 < \underline{P}_1^* \\ \frac{1}{2}(A - \underline{P}_1^*)(\underline{P}_1^* - MC) & \text{if } P_2 = \underline{P}_1^* \\ (A - \underline{P}_1^*)(\underline{P}_1^* - MC) & \text{if } P_2 > \underline{P}_1^*. \end{cases} \quad (41)$$

Since $(A - \underline{P}_1^*)(\underline{P}_1^* - MC) = \pi^c$ and $\frac{1}{2}(A - \underline{P}_1^*)(\underline{P}_1^* - MC) < \pi^c$ (see also Fig13),

$$c_1^{FJK}(\pi_1(\underline{P}_1^*, P_2)) = \begin{cases} 0 & \text{if } P_2 \leq \underline{P}_1^* \\ \epsilon & \text{if } P_2 > \underline{P}_1^*. \end{cases} \quad (42)$$

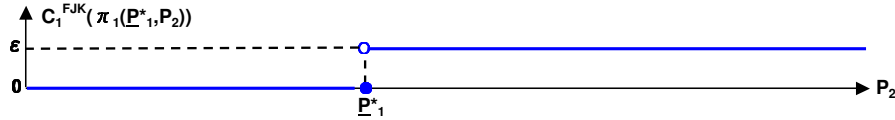


Figure 13: a_1 's compensation under c_1^{FJK} when $P_1 = \underline{P}_1^*$

Now pick \tilde{P}_1 with $\tilde{P}_1 < \underline{P}_1^*$ first. Since $(A - P_1)(P_1 - MC)$ is concave in P_1 and $\tilde{P}_1 < \underline{P}_1^*$, it follows that $(A - \tilde{P}_1)(\tilde{P}_1 - MC) < \pi^c$. Thus

$$c_1^{FJK}(\pi_1(\tilde{P}_1, P_2)) = 0 \quad \forall P_2 \quad (43)$$

by comparing (42) and (43), it turns out that \underline{P}_1^* weakly dominates \tilde{P}_1 .

Next, pick \tilde{P}_1 with $\tilde{P}_1 > \underline{P}_1^*$. Note that (see also Fig 14)

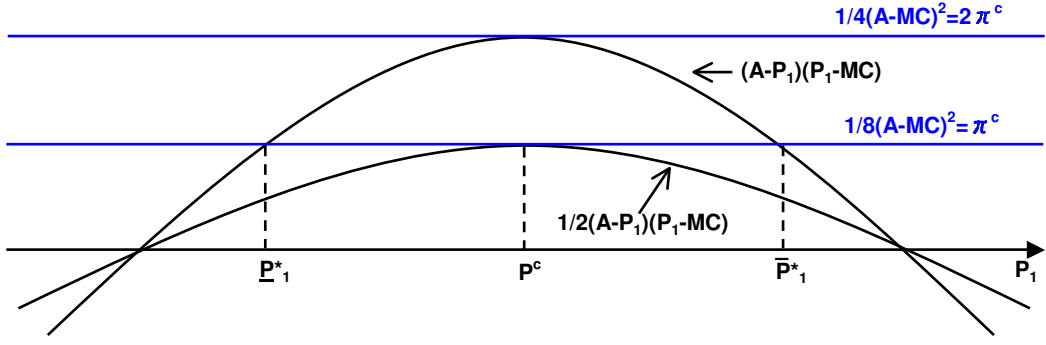


Figure 14:

$$\frac{1}{2}(A - P_1)(P_1 - MC) \begin{cases} = \frac{1}{8}(A - MC)^2 & \text{if } P_1 = P^c \\ < \frac{1}{8}(A - MC)^2 & \text{if } P_1 \neq P^c. \end{cases} \quad (44)$$

(1) If $P_1^* < \tilde{P}_1 < P^c$, then $\frac{1}{2}(A - \tilde{P}_1)(\tilde{P}_1 - MC) < \pi^c$ and $(A - \tilde{P}_1)(\tilde{P}_1 - MC) > \pi^c$. Therefore,

$$c_1^{FJK}(\pi_1(\tilde{P}_1, P_2)) = \begin{cases} 0 & \text{if } P_2 \leq \tilde{P}_1 \\ \epsilon & \text{if } P_2 > \tilde{P}_1. \end{cases} \quad (45)$$

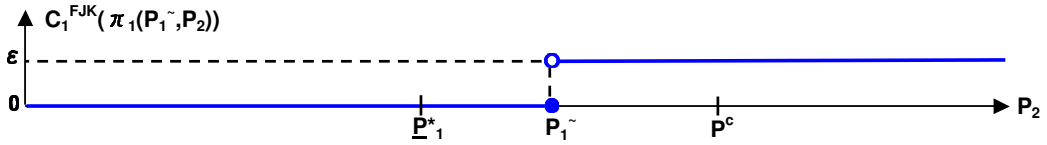


Figure 15: Compensation scheme for $P_1 = \tilde{P}_1$ with $P_1^* < \tilde{P}_1 < P^c$

(2) If $\tilde{P}_1 = P^c$, then from (44),

$$c_1^{FJK}(\pi_1(\tilde{P}_1, P_2)) = \begin{cases} 0 & \text{if } P_2 < \tilde{P}_1 \\ \epsilon & \text{if } P_2 \geq \tilde{P}_1. \end{cases} \quad (46)$$

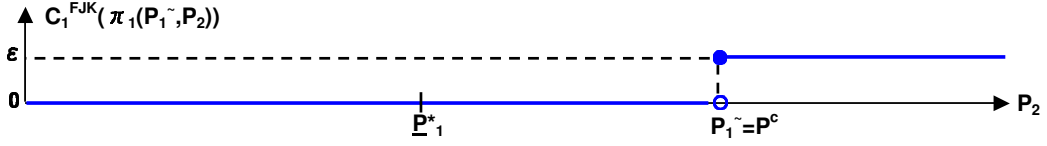


Figure 16: Compensation scheme for $P_1 = \tilde{P}_1$ with $\tilde{P}_1 = P^c$

(3) If $P^c < \tilde{P}_1 \leq \bar{P}_1^*$, then $\frac{1}{2}(A - \tilde{P}_1)(\tilde{P}_1 - MC) < \pi^c$ and $(A - \tilde{P}_1)(\tilde{P}_1 - MC) \geq \pi^c$. Therefore,

$$c_1^{FJK}(\pi_1(\tilde{P}_1, P_2)) = \begin{cases} 0 & \text{if } P_2 \leq \tilde{P}_1 \\ \epsilon & \text{if } P_2 > \tilde{P}_1. \end{cases} \quad (47)$$

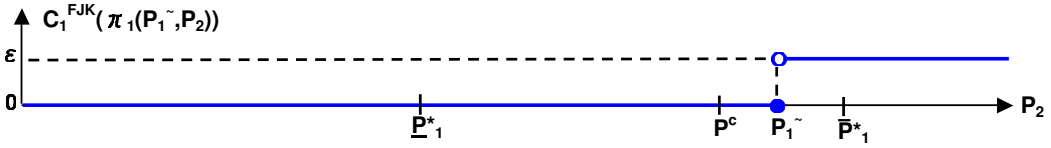


Figure 17: Compensation scheme for $P_1 = \tilde{P}_1$ with $P^c < \tilde{P}_1 \leq \bar{P}_1^*$

(4) If $\bar{P}_1^* < \tilde{P}_1$, then

$$c_1^{FJK}(\pi_1(\tilde{P}_1, P_2)) = 0 \quad \forall P_2. \quad (48)$$

In both cases of (1)~(4), P_1^* weakly dominates \tilde{P}_1 . ■