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# JENA ECONOMIC RESEARCH PAPERS



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# New Insights Into Optimal Control of Nonlinear Dynamic Econometric Models: Application of a Heuristic Approach

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## Abstract

Optimal control of dynamic econometric models has a wide variety of applications including economic policy relevant issues. There are several algorithms extending the basic case of a linear-quadratic optimization and taking nonlinearity and stochastics into account, but being still limited in a variety of ways, e.g., symmetry of the objective function and identical data frequencies of control variables. To overcome these problems, an alternative approach based on heuristics is suggested. To this end, we apply a 'classical' algorithm (OPTCON) and a heuristic approach (Differential Evolution) to three different econometric models and compare their performance. In this paper we consider scenarios of symmetric and asymmetric quadratic objective functions. Results provide a strong support for the heuristic approach encouraging its further application to optimum control problems.

**Keywords:** Differential evolution; dynamic programming; nonlinear optimization; optimal control

**JEL Classification:** C54, C61, E27, E61, E62.

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## 1 Introduction

In many areas of science from engineering to economics, determining the optimal way of controlling a system is required in a great number of applications. In economics, one frequently asked question is how a policy maker should choose appropriate values for given controls, such as taxes or public consumption in order to, e.g., increase the growth rate of GDP, decrease unemployment rate or achieve other targets. In this case, calculation of the targeted state variables is restricted by a system of equations representing an econometric model of the country of interest.

Solving such an optimum control problem for nonlinear econometric models is the core of this paper. To this end, two different methods are considered, namely the OPTCON algorithm (Matulka and Neck (1992), Blueschke-Nikolaeva *et al.* (2011)), where classical techniques of linear-quadratic optimization are used, and Differential Evolution (DE, Storn and Price (1997)), which is a population based stochastic optimization method. Among DE's main advantages are the ability to explore complex search spaces with multiple local minima thanks to cooperation and competition of individual solutions in the DE's population, and the application easiness as it needs little parameter tuning (Maringer (2008)). The non-heuristic approach, the OPTCON algorithm, on the other hand, is a more reliable and fast instrument for solving optimum control problems in standard applications.

However, like nearly all 'classical' methods, the OPTCON algorithm has several limitations. One, which is sometimes criticized in literature, is the required symmetry of the objective function. For the problems considered in this paper, the objective function is given in quadratic tracking form and equally penalizes positive and negative deviations from the given target values. In many situations, however, incorporation of different penalizing procedures for positive and negative deviations (in form of additional weighting coefficients) or inclusion of some indifference intervals would be desirable. Whereas it is nearly impossible to allow for this extension in the classic algorithm, it can be achieved by using a heuristic approach.

Before approaching the case of an asymmetric objective function, one has to make sure that DE can deliver a 'good' solution to the basic case of an optimum control problem. To demonstrate this, DE and OPTCON are applied to three macroeconometric models (with a deterministic scenario) and the performance of the two strategies is compared. Both methods are implemented in Matlab 7.11 to simplify their comparison. Due to the stochastic nature of DE and resulting need for several restarts of the strategy, a higher computational time is expected. For this reason, several possibilities to increase DE computational efficiency are also discussed.

Once the applicability of the heuristic approach has been demonstrated for the basic problem, it is extended and applied to solve the optimum control problem with an asymmetric objective function to three macroeconomic models. To this end, certain thresholds around the target values are introduced, inside which the objective function can be handled differently for positive and negative deviations. The resulting changes in the solutions are carefully analyzed and discussed both from the technical and economical perspectives.

The paper proceeds as follows. In Section 2 we define the class of problems to be tackled by the algorithms and describe the limitations, which are present in the OPTCON algorithm and are typical for 'classical' optimization methods. Section 3 briefly reviews the OPTCON algorithm as a classical approach and introduces DE as an alternative heuristic strategy. In Section 4 we analyze simulation results obtained for the two approaches with symmetric objective functions and extend DE to the asymmetric objective function scenario, testing the two strategies based on three econometric models (SLOVNL, SLOPOL4 and SLOPOL8). Section 5 concludes with a summary of the main findings and an outlook to further research.

## 2 Theoretical background

### 2.1 Type of problems

The task is to solve an optimum control problem with a quadratic objective function (a loss function to be minimized) and a nonlinear multivariate discrete-time dynamic system under additive and parameter uncertainties. The intertemporal objective function is formulated in quadratic tracking form, which is often used in applications of optimal control theory to econometric models. It can be written as

$$J = E \left[ \sum_{t=1}^T L_t(x_t, u_t) \right] \quad (1)$$

with

$$L_t(x_t, u_t) = \frac{1}{2} \begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}' W_t \begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}, \quad (2)$$

where  $x_t$  is an  $n$ -dimensional vector of state variables that describes the state of the economic system at any point in time  $t$ ,  $u_t$  is an  $m$ -dimensional vector of control variables,  $\tilde{x}_t \in R^n$  and  $\tilde{u}_t \in R^m$  are given 'ideal' (desired, target) levels of the state and control variables, respectively.  $T$  denotes the terminal

time period of the finite planning horizon.  $W_t$  is an  $((n + m) \times (n + m))$  matrix specifying the relative weights of the state and control variables in the objective function. The  $W_t$  matrix may also include a discount factor  $\alpha$ ,  $W_t = \alpha^{t-1}W$ .  $W_t$  (or  $W$ ) is symmetric.

The dynamic system of nonlinear difference equations has the form

$$x_t = f(x_{t-1}, x_t, u_t, \theta, z_t) + \varepsilon_t, \quad t = 1, \dots, T, \quad (3)$$

where  $\theta$  is a  $p$ -dimensional vector of parameters that is assumed to be constant but unknown to the policy maker (parameter uncertainty),  $z_t$  denotes an  $l$ -dimensional vector of non-controlled exogenous variables, and  $\varepsilon_t$  is an  $n$ -dimensional vector of additive disturbances (system error).  $\theta$  and  $\varepsilon_t$  are assumed to be independent random vectors with expectations,  $\hat{\theta}$  and  $O_n$ , and covariance matrices,  $\Sigma^{\theta\theta}$  and  $\Sigma^{\varepsilon\varepsilon}$ , respectively.  $f$  is a vector-valued function with  $f^i(\dots)$  representing the  $i$ -th component of  $f(\dots)$ ,  $i = 1, \dots, n$ . Solving an optimum control problem means, therefore, to find a certain set of controls  $(u_1^*, u_2^*, \dots, u_T^*)$  which minimizes the objective function  $J$ , i.e. to find  $u^* = \arg \min_u J$  with respect to (2).

For the study presented in this paper the deterministic case is considered only assuming the model parameters and the model equations to be exactly true. It means that parameters in  $\theta$  are given without uncertainty and the error terms are zero. Applying a heuristic approach for stochastic case is a further research question and will be discussed in the future.

## 2.2 Related restrictions

Among limitations for the existing methods, symmetric penalization of the deviations in the objective function is mostly reported.

As some motivation to understand the limitation of tackling symmetric objective functions only (henceforth, the 'symmetry limitation'), let us consider a simple example. Assume an optimum control problem with the government of Austria as decision-maker. Its objective state variable is the growth rate of GDP. Let us assume that the target for this objective is given by 4%. Final values of growth rate of GDP given by 2% and 6% will be penalized in a standard objective function equally. But from the economic point of view a growth rate of 6% is clearly more preferable compared to 2%. In a similar way, Cukierman (2002, p. 23) describes the quadratic (penalization) function as the one being 'chosen mainly for analytical convenience rather than for descriptive realism'.

Moreover, this symmetry limitation is not only restricted to deviations in output. The same applies for several other economic indicators. For example,

Nobay and Peel (2003) show that the ECB target of 2% inflation is explicitly asymmetric and suggest that the Bank of England had an asymmetric target at least in its first few years of formulation.

There are several solution concepts for the symmetry limitation. Among them the Linex form, introduced first by Varian (1974) and Zellner (1986) in the context of Bayesian econometric analysis and proposed by Nobay and Peel (2003) in the optimal monetary policy literature, and the piecewise quadratic objective function introduced by Friedman (1972) are probably the most referred. But the common issue for them is an advanced analytical transformation of the optimization problem which makes these methods only applicable for small-sized linear models. As mentioned above, an alternative solution would be to use a heuristic method. In the case of the asymmetric objective function, our approach is to define thresholds around the given target values. Inside these thresholds or rather in the intervals between the defined thresholds and the given target value the positive and negative deviations can be handled differently. Outside of the threshold intervals the standard penalizing procedure is applied.

## 3 Optimization algorithms

### 3.1 OPTCON

The OPTCON algorithm determines approximate solutions to optimum control problems with a quadratic objective function and a nonlinear multivariate dynamic system under additive and parameter uncertainties. It combines elements of previous algorithms developed by Chow (1975) and Chow (1981), which incorporate nonlinear systems but no multiplicative uncertainty, and by Kendrick (1981), who deals with linear systems and all kinds of uncertainty. In our experiments we use the last version of the OPTCON algorithm, which is called OPTCON2. In this Section only its basic idea for open-loop solutions is presented, for more details see Blueschke-Nikolaeva *et al.* (2011).

It is well known in stochastic control theory that a general analytical solution to dynamic stochastic optimization problems cannot be achieved even for very simple control problems. The main reason is the so-called dual effect of control under uncertainty, meaning that controls not only contribute directly to achieving the stated objective, but also affect future uncertainty and, hence, the possibility of indirectly improving the system performance by providing better information about the system (see, e.g., Aoki (1989) and Neck (1984)). Therefore, only approximations to the true optimum for such problems are feasible, with various schemes existing to deal with the problem

of information acquisition.

The problem with the nonlinear system is tackled iteratively, starting with a tentative path of state and control variables. The tentative path of the control variables is given for the first iteration. In order to find the corresponding tentative path for the state variables, the nonlinear system is solved numerically using the Newton-Raphson method. Alternatively, the Gauss-Seidel method or perturbation methods (see Chen and Zadrozny (2009)) may be used for this purpose.

After the tentative path is found, the iterative approximation of the optimal solution starts. The solution is sought from one time path to another until the algorithm converges or the maximal number of iterations is reached. During this search the system is linearized around the previous iteration's result as a tentative path and the problem is solved for the resulting time-varying linearized system. The criterion for convergence demands that the difference between the values of current and previous iterations be smaller than a pre-specified number. The approximate optimal solution of the problem for the linearized system is found under the above-mentioned simplifying assumptions about the information pattern. Then this solution is used as the tentative path for the next iteration, starting off the procedure all over again.

Every iteration, i.e. every solution of the problem for the linearized system, has the following structure: the objective function is minimized using Bellman's principle of optimality to obtain the parameters of the feedback control rule. This uses known results for the stochastic control of LQG problems (optimization of linear systems with Gaussian noise under a quadratic objective function). A backward recursion over time starts in order to calculate the controls for the first period. Next, the optimal values of the state and the control variables are calculated by applying forward recursion, i.e. beginning with  $u_1$  and  $x_1$  at period 1 and finishing with  $u_T$  and  $x_T$  at the terminal period  $T$ . If the convergence criterion is fulfilled, the solution of the last iteration is taken as the approximately optimal solution to the problem and the algorithm stops. Finally, the value of the objective function is calculated for this solution. For more details, see Matulka and Neck (1992) and Blueschke-Nikolaeva *et al.* (2011). Figure 1 summarizes the open-loop solution of the OPTCON2 algorithm.



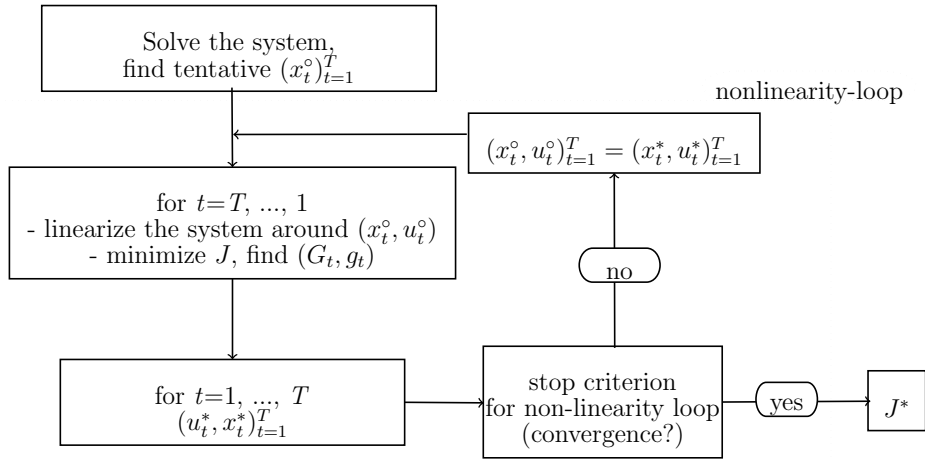


Figure 1: Flow chart of OPTCON2, open-loop solution

## 3.2 Heuristic optimization

Thanks to the recent advances in computing technology, new nature-inspired optimization methods called heuristics have become available. These methods are designed to provide ways of tackling complex combinatorial optimization problems and detect global optima of various objective functions (eligible for certain constraints). For an overview of these optimization techniques see Winker (2001) and Gilli and Winker (2009).

### 3.2.1 Differential Evolution

Differential Evolution (DE), proposed by Storn and Price (1997), is a population based optimization technique for continuous objective functions. For applications of DE in finance and risk management see Lyra *et al.* (2010) and Winker *et al.* (2011), respectively. In short, starting with an initial population of solutions, DE updates this population by linear combination and crossover of four different solution vectors into one, and selects the fittest solutions among the original and the updated population. This continues until some stopping criterion is met. Algorithm 1 provides a pseudocode of the DE implementation.

More specifically, the algorithm starts with a randomly initialized set of candidate solutions  $P_{j,t,i}^{(1)}$  ( $j = 1, \dots, m; t = 1, \dots, T, i = 1, \dots, p$ ) of the  $m \times T \times p$

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**Algorithm 1** Pseudocode for Differential Evolution

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```

1: Initialize parameters  $m, T, p, F$  and  $CR$ 
2: Randomly initialize  $P_{j,t,i}^{(1)}$ ,  $j = 1, \dots, m$ ;  $t = 1, \dots, T$ ;  $i = 1, \dots, p$ 
3: while the stopping criterion is not met do
4:    $P^{(0)} = P^{(1)}$ 
5:   for  $i = 1$  to  $p$  do
6:     Generate  $r_1, r_2, r_3 \in 1, \dots, p$ ,  $r_1 \neq r_2 \neq r_3 \neq i$ 
7:     Compute  $P_{\dots,i}^{(v)} = P_{\dots,r_1}^{(0)} + F \times (P_{\dots,r_2}^{(0)} - P_{\dots,r_3}^{(0)})$ 
8:     for  $j = 1$  to  $m$  and  $t = 1$  to  $T$  do
9:       if  $u < CR$  then  $P_{j,t,i}^{(n)} = P_{j,t,i}^{(v)}$  else  $P_{j,t,i}^{(n)} = P_{j,t,i}^{(0)}$ 
10:    end for
11:    if  $J(P_{\dots,i}^{(n)}) < J(P_{\dots,i}^{(0)})$  then  $P_{\dots,i}^{(1)} = P_{\dots,i}^{(n)}$  else  $P_{\dots,i}^{(1)} = P_{\dots,i}^{(0)}$ 
12:  end for
13: end while

```

---

size (step 2), where  $m \times T$  is the dimension of a single candidate solution and  $p$  is the population size. At this point it is important to explain how a DE candidate solution in the case of an optimum control problem looks like and how we choose an initial population.

We propose to use a candidate solution containing all control variables for all time periods. Thus, each candidate  $i = 1, \dots, p$  represents an alternative complete solution path for the whole optimum control problem, and is given as an  $(m \times T)$ -matrix  $P_{\dots,i}^{(1)} = (P_{j,t,i}^{(1)})_{\substack{j=1,\dots,m \\ t=1,\dots,T}} = (u_1^{(1),i}, u_2^{(1),i}, \dots, u_T^{(1),i})$ , where  $u_t^{(1),i}$  is an  $m$ -dimensional vector of controls. As a result, the dimension of the problem for each candidate solution is given by  $d = m \times T$ , with  $m$  being the number of control variables and  $T$  – the size of the planning horizon.

It is important to mention that each candidate solution is also described by the time paths of corresponding state variables, which result from the dynamic system  $f$  and the selected controls, i.e.  $(x_{t=1,\dots,T}^{(1),i} = f(\dots, u_{t=1,\dots,T}^{(1),i}, \dots))$ . For each candidate solution (for each set of control variables) there is a unique set of state variables. These state variables are not directly included in a candidate solution but they contribute to the objective function which is to minimize. In order to calculate these state variables an appropriate nonlinear system solver like Newton-Raphson or Gauss-Seidel is used. The objective function as given by equations (1) and (2) summarizes the weighted quadratic deviation of the  $n$  state variables and the  $m$  control variables for all time periods and has the dimension  $(m + n) \times T$ . The value  $J$  of this objective function is used as the fitness of each candidate solution.

One candidate solution  $P_{\dots,1}^{(1)}$  as described above is available at the beginning of the optimization procedure and is given by the tentative path

of control variables. The remaining  $p - 1$  candidate solutions of the initial population ( $P_{\dots,i}^{(1)}$ ,  $i = 2, \dots, p$ ) are constructed from this given path  $P_{\dots,1}^{(1)}$  by adding uniformly distributed error terms. The key aspect for creating this distribution is the assumed variance of the control variable  $j$ . In order to calculate these individual variances, the volatility of the corresponding time series and/or the differences between the tentative path and the OPTCON2 solution can be used.<sup>1</sup>

Then, in each generation the algorithm constructs a new candidate solution  $P_{\dots,i}^{(v)}$  (containing information on all control variables for all time periods) for member  $i$  from three different members of the current population (steps 6-7). For this reason, the scale factor  $F$  determines the shrinkage rate in exploring the search space. After that, the elements of the two solutions,  $P_{\dots,i}^{(v)}$  and  $P_{\dots,i}^{(0)}$ , are shuffled in an updated solution  $P_{\dots,i}^{(n)}$  according to the crossover rate  $CR$  and the uniform random variable  $u \sim U(0, 1)$  (steps 8-10). Finally, the fitness of the new candidate solution is compared with the one of the original population (step 11). If the new solution is better, the new candidate replaces the old one. The above process is repeated until the population of solutions has converged to a single vector, or until the predefined maximal number of generations  $g$  is reached.

### 3.2.2 DE calibration

Some guidelines for DE calibration can be found in Price *et al.* (2005).<sup>2</sup> Although DE performs well for many problems with  $F = 0.8$ ,  $CR = 0.8$  and  $p = 10d$ , tuning of the parameters is a problem specific issue. For this reason, we conduct a series of simulation experiments calibrating the DE parameters.<sup>3</sup> As it was done in Winker *et al.* (2011), initially we fix  $F$  and  $CR$  to be both equal to 0.55 (average value) and test different population sizes (between  $5d$  and  $30d$ ) increasing  $g$  until DE results in the same outcome for several replications. This is achieved with the population of  $10d$  size and 750 generations. Whereas  $5d$  does not allow for a successive identification of the same outcome, experiments with  $30d$  provide identical results in each restart. Since a larger population reduces the convergence speed,  $p = 10d$  and  $g = 750$  for a standard symmetric optimization problem are selected.

The difficulty in applying DE to an optimum control problem comes from the repeated computation of the state variables for each new calculated candidate solution, i.e.  $x_t^{(l),i} = f(\dots, u_t^{(l),i}, \dots)$ , for all time periods

<sup>1</sup>The latter is the case when the observed variance equals zero.

<sup>2</sup>A practical advice for this is also given on [www.icsi.berkeley.edu/storn/code](http://www.icsi.berkeley.edu/storn/code).

<sup>3</sup>In the following we describe the procedure for the SLOVNL model. Similar findings are made for the other two models, SLOPOL4 and SLOPOL8.

$t = 1, \dots, T$ , for all members of the population  $i = 1, \dots, p$  and for all generations  $l = 1, \dots, g$ . Comparing the quality of two available system solving algorithms, Newton-Raphson (line-search extension) and Gauss-Seidel, the latter one demonstrates a slightly better performance<sup>4</sup> and is employed for our computations in the following.

Furthermore, we examine the impact of the convergence criterion inside the nonlinear system solver on the DE performance.<sup>5</sup> Analyzing DE progress over the search process (see right panel of Figure 2 for the SLOVNL model), we notice that high precision (set equal to  $1 \times 10^{-5}$ ) of the system solving algorithm does not play an important role throughout the full search process, but only at its end. This precision, however, constitutes a significant computational challenge. For this reason, we decrease the accuracy up to 1 for the first 85% of generations leaving the full precision only for the final part of the search period. For all three models this alternation in precision does not affect our findings on the population size and gives an approximately 30% time reduction in comparison to the default (full precision) calculation.<sup>6</sup>

Some additional reduction in computational time can be achieved by analyzing and using the structure of the  $W_t$  matrix of the weights, which has the dimension  $((n + m) \times (n + m))$ . Usually, not all of the available (state) variables are considered in the objective function. Especially for large models this can result in the fact that the statement  $rank(W) \leq n + m$  is in reality a strong inequality  $rank(W) < n + m$ . In order to reduce the CPU time, we prevent the program from evaluating the objective function for non-objective variables. We use the structure of the  $W_t$  matrix and evaluate only the state variables which correspond to non-zero elements of the vector of weights. Although the computational costs for the procedure are low for one evaluation, the total gain can be significant due to the very large frequency of the objective function evaluation for DE (e.g., for SLOVNL one restart results in  $10 \times 3 \times 12 \times 750 = 270000$  evaluations). Depending on the model considered this performance improvement leads to a time reduction of around 1-10%, where the largest improvement is identified for the SLOPOL8 model.

Next, we run DE for different  $CR$  and  $F$  ranging between 0.1 and 1 and construct a *phase portrait* (see Price *et al.* (2005)) that pictures combinations of parameter values with the lowest average number of generations required

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<sup>4</sup>Lower computational time and nearly no difference in the quality of the final solutions.

<sup>5</sup>More precisely, we examine the impact of the convergence criterion's precision, which requires the relative deviations between the values of objective variables in the current and the previous solution loop of the system solver to be less than a certain value.

<sup>6</sup>This finding only holds for symmetric optimization problems. In contrast, for asymmetric case the precision is found to be more sensitive throughout the entire search path.

to achieve the value-to-reach ( $VTR$ ).<sup>7</sup> The combination is highlighted if the minimum objective value obtained becomes less than or equal to  $VTR$  in less than or equal to 750 generations. This process is illustrated in Algorithm 2.

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**Algorithm 2** Calibration of tuning parameters

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- 1: Initialize parameters  $p, g$
  - 2: Initialize population  $P_{j,i}^{(1)}, j = 1, \dots, d, i = 1, \dots, p$
  - 3: **for**  $F = 0.1$  to  $1$  **do**
  - 4:     **for**  $CR = 0.1$  to  $1$  **do**
  - 5:         **for**  $k = 1$  to  $g$  **do**
  - 6:             Repeat statements 4-12 from Algorithm 1
  - 7:             **if**  $J_{DE} \leq 1.0001J_{OPTCON2}$  **then** mark  $F$  and  $CR$
  - 8:             **end for**
  - 9:     **end for**
  - 10: **end for**
- 

Figure 2 (left panel) demonstrates the resulting phase portrait for the SLOVNL model. Whereas  $CR$  favourable values vary predominantly between 0.1 and 0.4,  $F$  is concentrated in  $[0.4, 0.6]$ . Choosing the combination with the highest fitness, we set  $F = 0.4$  and  $CR = 0.1$ .<sup>8</sup> In addition, on the right panel of Figure 2 the DE progress plot over the evaluation time is demonstrated.

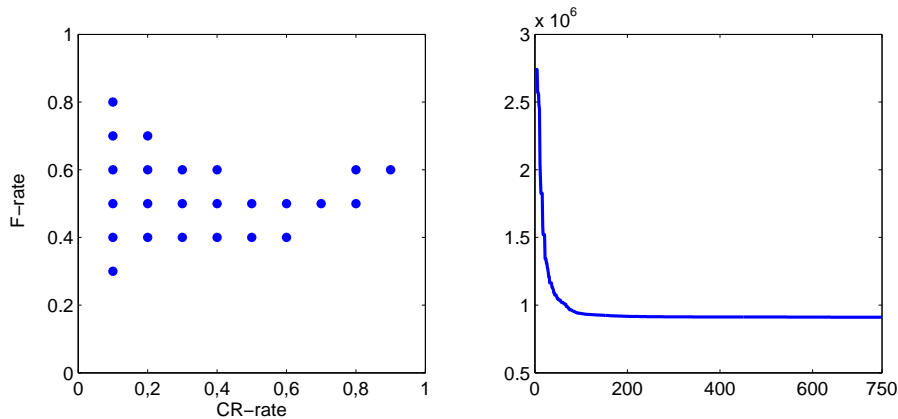


Figure 2: Phase portrait and progress plot for SLOVNL

To illustrate convergence of the resulting objective function values, we apply DE with 100 restarts for different  $g$  (number of generations). In the

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<sup>7</sup> $VTR$  is set to 100.0001% of the objective value achieved by OPTCON2. Thus, the deviation of .0001% (e.g., less than 100 for SLOVNL) is acceptable for illustrative reasons.

<sup>8</sup>In contrast, employing high precision in the fitness evaluation over the full search process would result in the  $F = 0.4, CR = 0.5$  combination.

upper left plot of Figure 3 the cumulative distribution function  $F(J)$  for different  $g$  is given, whereas the other plots are histograms of objective function values identified. Increasing  $g$  the distribution shifts left and becomes less dispersed (see also Savin and Winker (forthcoming)).

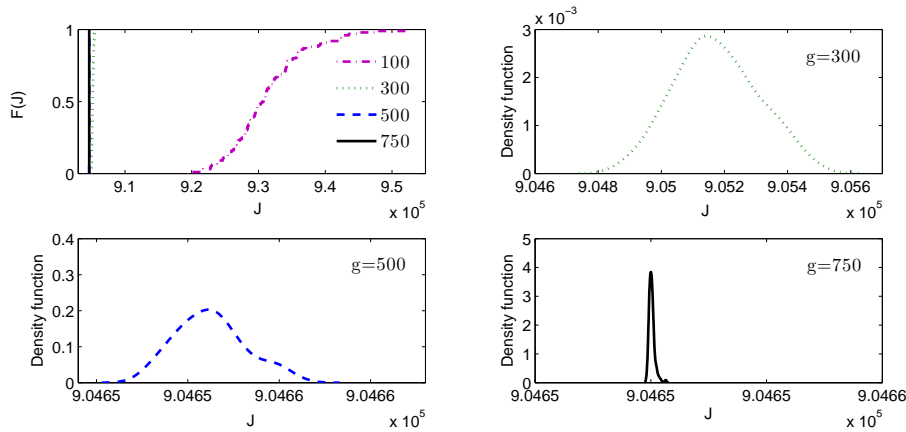


Figure 3: Empirical distribution of objective function values for different  $g$

Since DE is a stochastic method, the algorithm is restarted ten times and the solution with the best objective value is selected.

The corresponding computational time for the three models tested in this study varies depending on the complexity of a particular problem. For SLOVNL 750 generations are sufficient to obtain a solution, and require less than one minute using Matlab 7.11 and Pentium IV 3.3 GHz.

## 4 Simulation results

### 4.1 Comparison of OPTCON and DE

Before analyzing the impacts of introducing asymmetric objective functions, one has to consider the standard objective values for three macroeconomic models (SLOVNL, SLOPOL4 and SLOPOL8).

We start with the results for the SLOVNL model, a small nonlinear econometric model of the Slovenian economy, which consists of 8 equations and includes 8 state variables, 4 exogenous non-controlled variables, 3 control variables and 16 unknown (estimated) parameters (see Appendix 6.1 and Blueschke-Nikolaeva *et al.* (2011) for more detail). Here only the information about the values of the objective function is presented.

The objective function value achieved by OPTCON2 for SLOVNL model equals 2759744 in the uncontrolled simulation and 904649.68 for the optimal solution. Heuristic solution with symmetric objective function approximates the optimal solution fairly well. As DE gives by each restart slightly different values, we also report its standard deviations (in parentheses) to account for the variance in results. With 10 restarts and 750 generations the best value identified is 904649.98. All objective value results for symmetric and asymmetric solutions for three models considered can be seen in Table 1.

Similar findings are calculated for SLOPOL4 and SLOPOL8. Thus, the values of the objective function as calculated by OPTCON2 in the uncontrolled simulation are 1000690.19 and 1257518562.81, respectively, while for the optimal solution OPTCON2 achieves 375452.64 and 876621276.32. Heuristic solution with standard objective function approximates the optimal OPTCON2 solutions well, by achieving values even slightly below the OPTCON2 results. In particular, 375403.81 and 876597577.33 are obtained. The latter fact explains a higher variance in DE results for the SLOPOL4 model: since DE slightly outperforms OPTCON2, the corresponding VTR is systematically reached even with the higher variance in DE results.<sup>9</sup> Thus, we find evidence that DE can even beat standard optimum control strategies for complex econometric models compensating by this the higher computational cost required to obtain a solution.

Table 1: Results for both optimization algorithms with different settings

		OPTCON2		Differential Evolution		Benefit	
		uncontrolled	optimal	symmetric	asymmetric	absolute	relative
SLOVNL	min	2759744.00	904649.68	904649.98	479025.65	174946.80	19.3%
	std	(0.00)	(0.00)	(0.62)	(4.66)	(19.00)	
	cpu	.001s	.5s	58s	1378s		
SLOPOL4	min	1000690.19	375452.64	375403.81	351066.92	6373.18	1.7%
	std	(0.00)	(0.00)	(1.78)	(80.99)	(67.58)	
	cpu	.001s	5s	15047s	31005s		
SLOPOL8	min	1257518562.81	876621276.32	876597577.33	788018915.13	1155946.19	0.1%
	std	(0.00)	(0.00)	(679.29)	(124.96)	(1262.33)	
	cpu	.002s	33s	7512s	35610s		

<sup>a</sup>For explanation see Section 4.1.

<sup>9</sup>For SLOPOL8 the standard deviation lies within the 0.0001% (VTR) deviation.

## 4.2 Application to asymmetric objective functions

In order to solve the symmetry limitation of the objective function we define certain thresholds around the given target values. In particular, we concentrate on the intervals between the defined thresholds and the given target values. Let us consider again a simplified example with the growth rate of GDP as the only one objective variable. Let us assume further that the target value is given by 4%. In order to reduce penalty on the positive deviations, we define a threshold larger than the target value of 4% and given, for example, by 8% for the growth rate of GDP. Possible optimization results inside the interval [4%, 8%] can be then handled differently: i.e. penalized less strong compared to negative deviations. To this end, we introduce an additional factor  $\beta \leq 1$  which indicates the level of the asymmetry.<sup>10</sup> Outside of the thresholds standard penalizing procedure is applied (see 4-5), so that an overheating of the economy is punished in the same way as an underperformance.

Our first results use 'one-sided' thresholds which are defined as relative deviations from the given targets. 'One-sided' means that we deal either with positive or negative intervals for one variable in asymmetric way. Two-sided thresholds which allow for simultaneous consideration of positive and negative intervals for each variable are implemented as well. Such experiments could be especially interesting for the problems where controls are allowed to be very flexible but only inside a certain interval. An example could be the nominal prime rate set by a central bank which can vary at low cost inside the interval  $[0, x]$  with  $x = 5\%$  or even higher, but can not be negative, which implies a high penalty weight outside of the defined interval.

The advantage of using relative deviations instead of fixed values arises for the objective variables with targets given as changing time paths. Using fixed values as thresholds would require to define a time path for these thresholds as well. In contrast, defining relative deviations for thresholds allows to calculate the corresponding values using the given target value at the time point  $t$ .<sup>11</sup> The targets and the thresholds (given as relative deviations and denoted by  $\tilde{d}$ ) for all three models under consideration are presented in Table 2.<sup>12</sup> Since some variables in different models have a very similar economic

<sup>10</sup>For the special case  $\beta = 1$  the symmetric case is included as well.

<sup>11</sup>Notice, that in case of variables with target values given as '0' relative thresholds do not work and should be replaced by absolute ones.

<sup>12</sup>There is no unique way in defining thresholds. For the present study some moderate values are chosen using very general economic theory considerations. Allowing for larger intervals would increase the impact of the thresholds on the final solutions. Moreover, some differences to catch model specifications are allowed: for example, in SLOPOL8 model describing an economy in crisis some differences in thresholds chosen (e.g., unemployment



meaning but a different notation (e.g., *GRCPI* and *INFL*), they are grouped in one line to simplify understanding and comparison.

Table 2: Target values  $\tilde{x}_{t=1}$ ,  $\tilde{u}_{t=1}$  and thresholds  $\tilde{d}$

	SLOVNL			SLOPOL4			SLOPOL8		
		$\tilde{x}, \tilde{u}$	$\tilde{d}$		$\tilde{x}, \tilde{u}$	$\tilde{d}$		$\tilde{x}, \tilde{u}$	$\tilde{d}$
States in $t=1$	<i>CR</i>	1920	5%	<i>CR</i>	370	5%			
	<i>INVR</i>	956.9	10%	<i>INVR</i>	173	10%			
	<i>IMPR</i>	2299	0	<i>IMPR</i>	437	0			
	<i>STIRLN</i>	9.78	-20%						
	<i>GDPR</i>	3478	10%	<i>GDPR</i>	679	10%	<i>GDPR</i>	5898	10%
	<i>VR</i>	5783	3%						
	<i>PV</i>	172.5	-1.5%						
				<i>GRCPI</i>	6	-50%	<i>INFL</i>	2.27	-50%
				<i>UR</i>	10	-5%	<i>UR</i>	9.79	-50%
				<i>EXR</i>	437	10%			
				<i>GR</i>	144	0			
Controls in $t=1$	<i>TaxRate</i>	25.2	0						
	<i>GR</i>	629.3	-1%						
	<i>M3N</i>	16050	0%						
				<i>TaxRateLabour</i>	37.45	-10%			
				<i>GN</i>	240	0	<i>GN</i>	1548	0
				<i>TRANSFERSN</i>	197	-3%	<i>TRANSFERSN</i>	1303	-3%
				<i>STIRLN</i>	12	-50%			
							<i>DEBTGDP</i>	23.95	-10%

The value of 5% for *CR*, for example, means that possible final solutions are penalized differently in the interval  $[1920, 1920 + 0.05 \cdot 1920]$ . Thus, some private excess demand is 'tolerated', i.e. penalized less strongly compared with the standard objective function. Similarly, -20% for the *STIRLN* defines the asymmetric interval for negative deviations  $[9.78 - 0.2 \cdot 9.78, 9.78]$ . As mentioned before, this asymmetric penalization procedure is achieved by adding an additional weighting coefficient  $\beta$ ,  $0 \leq \beta \leq 1$ . A lower value of  $\beta$  implies a lower penalty. In the case if  $\beta = 0$  we create an indifference interval, where no penalty for deviations between the optimization results and given target values is applied. The case of  $\beta = 1$  describes the standard penalizing procedure and is used for testing of implementation only.

Thus, in a general form the asymmetric objective function described above rate (*UR*) and level of debt in relation to GDP (*DEBTGDP*) can be observed.

and denoted by  $L_t^{asym}(x_t, u_t)$  penalizes the deviations

$$(x_t - \tilde{x}_t) \text{ for } x_t \notin [\min(\tilde{x}_t, \tilde{x}_t(1 + \tilde{d})), \max(\tilde{x}_t, \tilde{x}_t(1 + \tilde{d}))] \quad (4)$$

$$(u_t - \tilde{u}_t) \text{ for } u_t \notin [\min(\tilde{u}_t, \tilde{u}_t(1 + \tilde{d})), \max(\tilde{u}_t, \tilde{u}_t(1 + \tilde{d}))]$$

similar to  $L_t^{sym}(x_t, u_t)$ , which denotes the standard penalizing procedure as described in (2), but

$$\sqrt{\beta}(x_t - \tilde{x}_t) \text{ for } x_t \in [\min(\tilde{x}_t, \tilde{x}_t(1 + \tilde{d})), \max(\tilde{x}_t, \tilde{x}_t(1 + \tilde{d}))] \quad (5)$$

$$\sqrt{\beta}(u_t - \tilde{u}_t) \text{ for } u_t \in [\min(\tilde{u}_t, \tilde{u}_t(1 + \tilde{d})), \max(\tilde{u}_t, \tilde{u}_t(1 + \tilde{d}))].$$

$L_t^{asym}(x_t, u_t)$  allows to 'smooth' the parabola of the quadratic objective function inside the defined intervals by factor  $\beta$ .<sup>13</sup>

We apply DE using the advanced ('asymmetric') penalizing procedure<sup>14</sup> starting with an additional weighting coefficient  $\beta = 0.1$ . Comparing the results for both, symmetric and asymmetric, scenarios calculated via DE one can see a very substantial decrease of the objective value by around 47%.

In order to analyze the relevance of this change two additional comparisons are performed next. First, we compare the objective values not for the final results, but for the tentative paths. The objective value in the standard case is given by 2759744. The objective value of the uncontrolled solution in asymmetric case is 2451838. We see that the difference between the standard calculation and the asymmetric calculation using the thresholds as given in Table 2 reduces the uncontrolled objective value by around 12%. This is a considerable reduction, but significantly below the 47% reduction of optimized results. Hence, the 47% difference in the final results is influenced by the penalizing procedure. Furthermore, this indicates a good quality of the target values chosen. Thus, if the optimal values can at 'low cost' take values on the one side (positive or negative) from the target values, then it is sometimes reasonable to adjust target values in this direction. As a result one obtains better target values and better results in the 'low cost' area.

Second, to emphasize the impact of the penalizing procedure we use an additional index which calculates the 'benefit' from asymmetric DE search

<sup>13</sup>As a result it creates a discontinuous point at the threshold position.

<sup>14</sup>The asymmetric penalization scheme naturally affects the corresponding search space of solutions making it more complex and 'unfriendly'. This necessitates a larger population of solutions ( $p = 50d$ ) to screen the search space in more directions simultaneously and, hence, a larger number of generations and CPU time required.

path. To this end, we calculate first the asymmetric objective value for the symmetric DE result, i.e. we obtain first the DE results using standard weighting procedure. Then we take the optimal states and controls from this optimization  $(x_t^{sym}, u_t^{sym})$  and calculate the objective value for the advanced, asymmetric weighting procedure. The resulting difference between the latter objective value and the one obtained via DE and the asymmetric penalized procedure is referred as *Benefit* (last two columns in Table 1):

$$Benefit = E \left[ \sum_{t=1}^T L_t^{asym}(x_t^{sym}, u_t^{sym}) \right] - E \left[ \sum_{t=1}^T L_t^{asym}(x_t^{asym}, u_t^{asym}) \right]. \quad (6)$$

Thus, we apply asymmetric penalization  $L_t^{asym}(\cdot)$  (i.e. the more 'meaningful' penalization from economical point of view) on both, the symmetric and asymmetric, results for control and state variables measuring an improvement in the objective value  $J$ .

To ease the comparison between the benefits for the three different models we report both, the absolute value of the benefit and its share in relation to the symmetric DE result (*relative benefit*) in Table 1. In addition, standard deviations over ten restarts are indicated. Thus, for SLOVNL the benefit from using the asymmetric penalization throughout the search process implies a decrease in the final objective value of slightly more than 19%.

Next, the asymmetric penalizing procedure is applied on the other two models. We use the same additional weighting coefficient  $\beta = 0.1$  and look first at the resulting objective values which are 351066.92 and 788018915.13 for SLOPOL4 and SLOPOL8, respectively. Comparing the asymmetric DE result with the objective value calculated using standard DE one can see a substantial decrease of the objective value by around 6-10%.<sup>15</sup> Similar to SLOVNL, comparison of the objective values for the tentative paths indicates a good quality of the target values chosen: for SLOPOL4 these are 1000690 (symmetric) and 986193 (asymmetric), while for SLOPOL8 1340648701 and 1257518562, respectively. Thus, in contrast to the decrease in the final objective values, the difference amounts only to 1.5-6%. Finally, the benefit obtained from asymmetric penalization throughout the (stochastic) search process accounts for 1.7-0.1%. It is evident that the benefit for these two models is much lower than for SLOVNL, which can be explained by different normalization procedures for the weighting matrix  $W$ .<sup>16</sup>

<sup>15</sup>The lower difference between symmetric and asymmetric objective function values for this models can be explained by logarithmization of some variables' values used in the latter two models in comparison to 'level' values used in SLOVNL.

<sup>16</sup>In order to prevent that the unit of measurement and other time series characteris-

Our paper focuses on the technical application of a heuristic optimization method (DE) to optimum control problems including certain advanced restrictions and comparison of performance between DE and OPTCON. We do not aim (at this stage of research) on giving any policy recommendations. Nevertheless, in the following we give a brief insight into economic interpretation of the DE results for the asymmetric scenario.

In order to understand constituent elements of the resulting differences in the objective function values for the different methods applied (Table 1), let us consider relative deviations in the state and control variables' values obtained from the given targets. To this end, we calculate percentage differences  $dDE$  (i.e. difference between the values obtained via DE with symmetric penalization and the targets) and  $dDE_{asym}$  (i.e. difference between the values obtained via DE with asymmetric penalization and the targets) taken in relation to the respective subtrahend (target).<sup>17</sup> While plots on the deviation in controls are given below, plots for states are presented in Appendix.

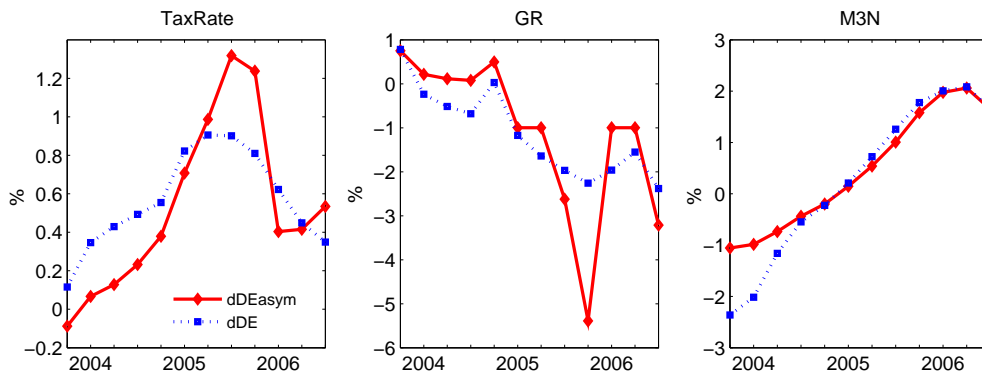


Figure 4: Relative deviations in controls for SLOVNL

Comparing results for the SLOVNL model in control (Figure 4) and state (Figure 7) variables, one realizes that the lower objective value obtained via asymmetric penalization is due to a different fiscal and monetary policy applied. In particular, while in the period 2004-2005 the state introduces a lower

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tics can distort the optimization results, usually the weights in the matrix  $W$  should be normalized. For this purpose an appropriate normalization procedure is applied. For more information about different normalization procedures see Blueschke (2011).

<sup>17</sup>We also have considered differences between the obtained values via OPTCON2 and the targets. However, since DE (with symmetric objective function) approximates the OPTCON2's optimal solution very well, the latter difference is very close to  $dDE$ . For this reason, we do not report those differences in the paper for the sake of brevity. However, the results can be obtained from authors on request.

tax rate (*TaxRate*) and a higher public consumption (*GR*) and, at the same time, shrinking less the money stock (*M3N*), it raises *TaxRate* and reduces *GR* and *M3N* more radically than in the symmetric penalization scenario at the end 2005 - start of 2006. As a result of this (less restrictive) monetary and fiscal policy in 2004-2005, a lower short term interest rate (*STIRLN*) and marginally higher growth in the gross domestic product (*GDPR*), aggregate demand (*VR*), imports of goods and services (*IMPR*), and investments (*INVR*) in the economy within this period is achieved. Furthermore, as one could expect, a somewhat lower growth in *GDPR*, *VR*, *IMPR* and *INVR* is obtained in the first quarter of 2006. However, in total these differences account for the 19% benefit in favour of the asymmetric DE penalization obtained in the objective value.<sup>18</sup> Note that the larger negative deviations from the targets in *GR* (within 1%) and *STIRLN* (within 20%) and also larger positive deviation in *VR* (within 3%), *GDPR* and *INVR* (both within 10%) are 'stimulated' if applying the thresholds given in Table 2.

Next, considering SLOPOL4 (see Figures 5 and 8) one can notice that the larger decreases in short term interest rate (*STIRLN*) and wage tax rate (*TAXRATELABOUR*) - both within the tolerance intervals defined by thresholds in Table 2 - together with a lower nominal public spending (*GN*) and higher transfer payments (*TRANSFERSN*) allow to obtain a slightly larger real GDP (*GDPR*), private consumption (*CR*) and imports (*IMPR*), particularly for the period 2005-2007. This, in its turn, results in a higher growth rates of GDP (*GRGDPR*) in the respective period. Furthermore, the measures described lead to lower unemployment rates (*UR*) and a larger gross fixed capital formation (*INVR*).<sup>19</sup> All this allows to reduce the budget (*DEF%*) and the current account (*CAN%*) disbalances<sup>20</sup> of the economy and produce an almost 2% benefit in terms of the objective function value.

Finally, analyzing the deviations obtained for SLOPOL8 (and illustrated in Figures 6 and 9) one observes a larger public consumption (*GN*) values within the period of 2009-2013, which are accompanied by lower transfer payments to households (*TRANSFERSN*). This, however, leads to only minor differences in the state values considered (unemployment rate (*UR*),

<sup>18</sup>Such a substantial benefit in comparison to relatively small differences in the Figures 4 and 7 can be explained by several reasons: large 'level' values of the variables used in computation, quadratic penalization of the model, different weights of particular variables in the final objective value.

<sup>19</sup>Note again that the lower unemployment rates (within 5%) and higher growth rates in fixed capital formation (within 10%) are tolerated by our thresholds.

<sup>20</sup>Note that the deviations for the latter two variables, which are measured in percentages, are not 'normalized' with respect to the targets and therefore are denoted in Figure 8 with 'abs' (standing for absolute deviations) on the end.

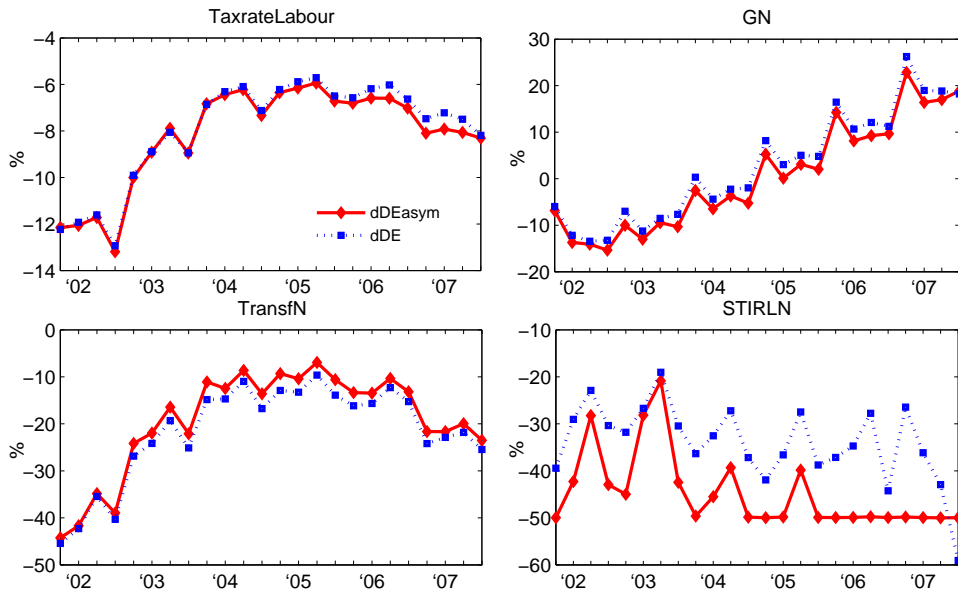


Figure 5: Relative deviations in controls for SLOPOL4

real gross domestic product ( $GDPR$ ), inflation ( $INFL$ ) and debt level in relation to GDP ( $DEBTGDP$ ). Since the deviations in the states are very moderate, we also calculate individual constituent elements of the objective function value for all six variables under consideration (see Figure 10). Thus, while for  $DEBTGDP$ ,  $GN$  and  $TRANSFERSN$  the deviations obtained in the asymmetric scenario are found to be slightly higher (within 3-6%), results for  $UR$ ,  $GDPR$  and particularly  $INFL$  are in contrast in favour of the asymmetric penalization (although they are within 0.2-2%). Thus, obviously the redistribution of funds from  $TRANSFERSN$  in favour of  $GN$  allows to marginally improve the final objective value by 0.1%.

## 5 Conclusions and Outlook

In this paper we apply a heuristic approach (Differential Evolution) to solve nonlinear optimum control problems. The main reason to do that is the DE's flexibility allowing to deliver solutions in the specific situations, where the classical methods fail. To test the quality and performance of DE we compare its results with the ones obtained by the OPTCON algorithm, which uses the classical techniques of linear-quadratic optimization.

Our work can be divided into two parts. First, we demonstrate that DE

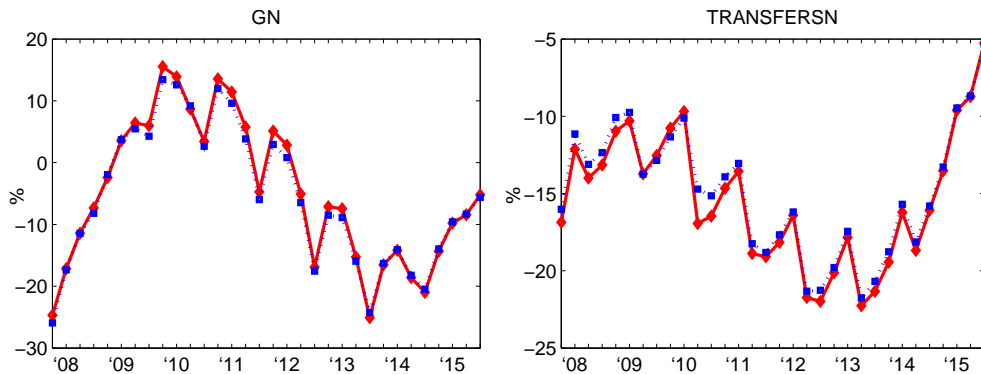


Figure 6: Relative deviations in controls for SLOPOL8

approximates the solutions obtained by OPTCON fairly well based on three econometric models (SLOVNL, SLOPOL4 and SLOPOL8). Moreover, we obtain evidence that the heuristic approach can even beat standard optimum control strategies compensating by this the higher computational cost required. Furthermore, different tuning schemes for performance improvement of DE are analyzed.

Second, we apply DE to situations where classical methods fail. In particular, we focus on the symmetry limitation of the quadratic objective function and relax this condition by introducing certain thresholds (intervals) around given target values for objective variables. Inside these intervals the negative and positive deviations between final results and target values are handled differently, i.e. penalized asymmetrically. We find that the asymmetric application of DE requires even more computational time compared to the symmetric scenario, but does not require any advanced analytical adjustments. Intensive computational experiments with different thresholds and models demonstrate proper robustness of the calculated results, indicate their superiority compared to the results with symmetric penalization and encourage DE's application to asymmetric optimum control problems.

Although our paper focuses on the technical implementation of the two strategies, we provide a brief insight into the economical interpretation of the asymmetric results obtained (which are not meant for any kind of policy recommendations). Thus, it is clear that these results are highly problem specific and correlate to many model internal issues including the choice of targets, weights and thresholds. We observe substantial differences for symmetric and asymmetric scenarios in the optimal policy in nearly all control variables considered, which supports the usefulness of the heuristic methods and the necessity of further research in this area.

In the future, it is highly interesting to consider other limitations of the classical methods (the problem of different data frequencies), to compare the two strategies based on stochastic problems and further elaborate the issue of performance improvement (e.g., consider other heuristic methods).

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## 6 Appendix

### 6.1 The SLOVNL model

The small nonlinear macroeconomic model of the Slovenian economy (SLOVNL) consists of 8 equations, 4 of them behavioral and 4 identities. The model includes 8 state variables, 4 exogenous non-controlled variables, 3 control variables and 16 unknown (estimated) parameters. The quarterly data for the time periods 1995:1 to 2006:4 yield 48 observations and admit a full-information maximum likelihood (FIML) estimation of the expected values and the covariance matrices for the parameters and system errors. The start period for the optimization is 2004:1 and the end period is 2006:4 (12 periods).

*Endogenous (state) variables:*

$x[1]$	<i>CR</i>	real private consumption
$x[2]$	<i>INVR</i>	real investment
$x[3]$	<i>IMPR</i>	real imports of goods and services
$x[4]$	<i>STIRLN</i>	short term interest rate
$x[5]$	<i>GDPR</i>	real gross domestic product
$x[6]$	<i>VR</i>	real total aggregate demand
$x[7]$	<i>PV</i>	general price level
$x[8]$	<i>Pi4</i>	rate of inflation

*Control variables:*

$u[1]$	<i>TaxRate</i>	net tax rate
$u[2]$	<i>GR</i>	real public consumption
$u[3]$	<i>M3N</i>	money stock, nominal

Table 3: Weights of the variables in the SLOVNL model

a: 'raw' weights		b: 'correct' weights	
variable	weight	variable	weight
<i>CR</i>	1	<i>CR</i>	3.457677
<i>INVR</i>	1	<i>INVR</i>	12.16323
<i>IMPR</i>	1	<i>IMPR</i>	1.869532
<i>STIRLN</i>	1	<i>STIRLN</i>	216403.9
<i>GDPR</i>	2	<i>GDPR</i>	2
<i>VR</i>	1	<i>VR</i>	0.333598
<i>PV</i>	1	<i>PV</i>	423.9907
<i>Pi4</i>	0	<i>Pi4</i>	0
<i>TaxRate</i>	2	<i>TaxRate</i>	37770.76
<i>GR</i>	2	<i>GR</i>	63.77052
<i>M3N</i>	2	<i>M3N</i>	0.090549

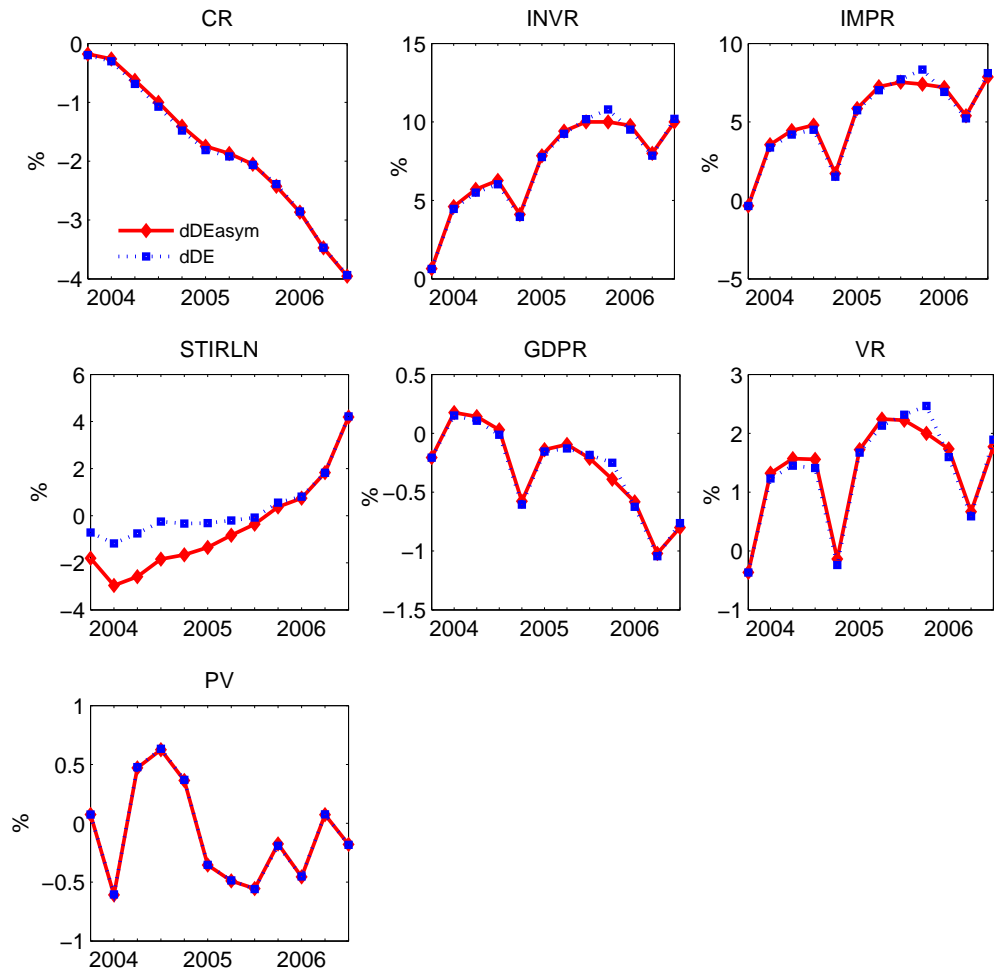


Figure 7: Relative deviations in states for SLOVNL

## 6.2 The SLOPOL4 model

The medium-sized nonlinear macroeconometric model of the Slovenian economy (SLOPOL4) consists of 15 behavioral and 30 identity equations (the total number of equations as programmed for optimum control optimization including several auxiliary equations is 71). The model includes 45 state variables, 4 controls, 11 exogenous non-controlled variables and 59 unknown (estimated) parameters. The exogenous variables include variables outside the influence of Slovenian policy-makers (oil price, world trade, euro area interest rates, population), some policy instruments (public consumption and investment, transfer payments to private households, tax rates and social security contribution rates) and some dummy variables.

The behavioral equations are estimated by ordinary least squares (OLS), using quarterly data for the period 1995:1 until 2005:4. The start period for the optimization is 2002:1 and the end period is 2007:4 (24 periods). For more information see (Neck *et al.* 2004).

*Endogenous (objective) variables:*

$x[1]$ :	<i>GDPR</i>	real gross domestic product
$x[2]$ :	<i>UR</i>	unemployment rate
$x[3]$ :	<i>CR</i>	private consumption, real
$x[4]$ :	<i>EXR</i>	exports, real
$x[5]$ :	<i>IMPR</i>	imports, real
$x[6]$ :	<i>INVR</i>	gross fixed capital formation, real
$x[7]$ :	<i>GR</i>	government consumption, real
$x[8]$ :	<i>CAN%</i>	current account balance in percent of GDP
$x[9]$ :	<i>GRGDPR</i>	growth rate of real gross domestic product
$x[10]$ :	<i>GRCPI</i>	growth rate of consumer price index
$x[11]$ :	<i>DEF%</i>	budget balance in relation to GDP

*Control variables:*

$u[1]$	<i>TAXRATELABOUR</i>	wage tax rate
$u[2]$	<i>GN</i>	government consumption at current prices
$u[3]$	<i>TRANSFERSN</i>	transfer payments to households at current prices
$u[4]$	<i>STIRLN</i>	short term interest rate

Table 4: Weights of the variables in the SLOPOL4 model

variable	state variables		control variables		
	weight	variable	weight	variable	
<i>GDPR</i>	1	<i>IMPR</i>	1	<i>TAXRATELABOUR</i>	10
<i>UR</i>	1000	<i>INVR</i>	1	<i>GN</i>	1
<i>CR</i>	1	<i>GR</i>	1	<i>TRANSFERSN</i>	1
<i>EXR</i>	1	<i>CAN%</i>	1000	<i>STIRLN</i>	100
<i>GRGDPR</i>	1000	<i>GRCPI</i>	1000		
<i>DEF%</i>	1000				

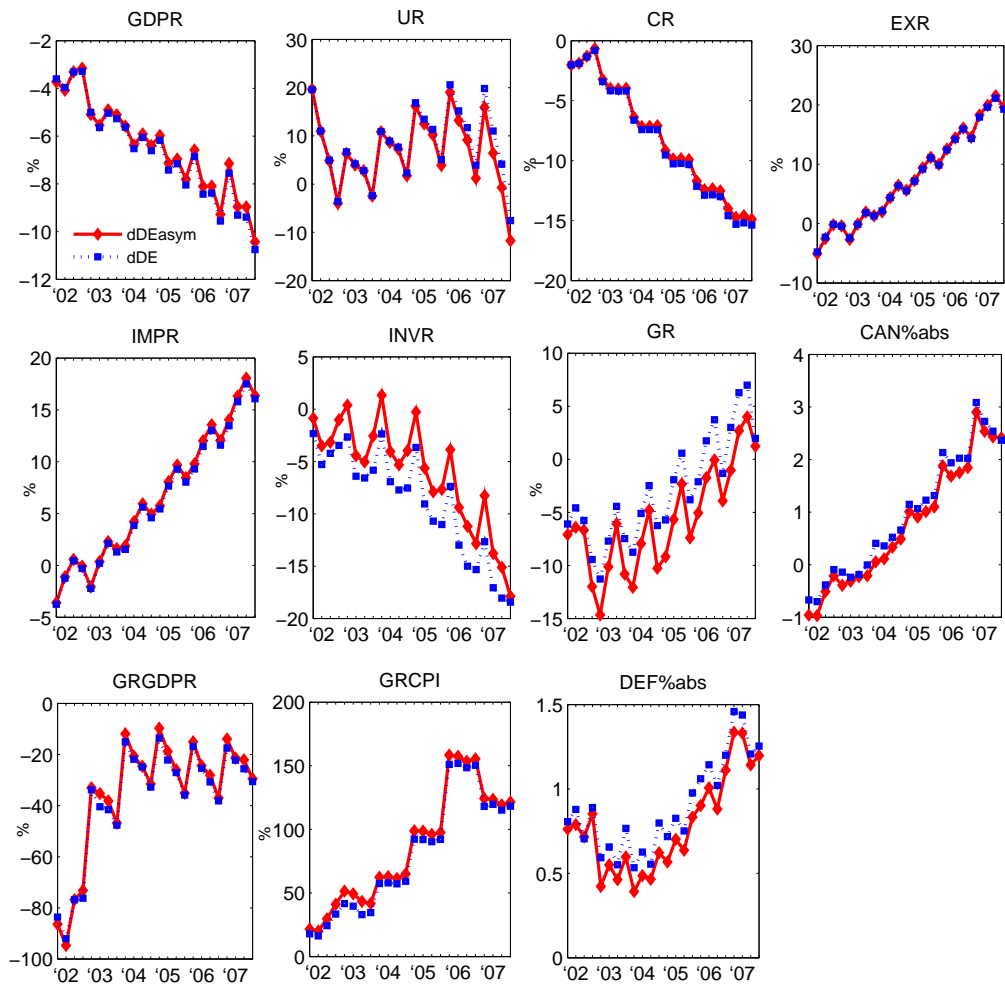


Figure 8: Relative deviations in states for SLOPOL4

### 6.3 The SLOPOL8 model

The medium-sized nonlinear macroeconometric model of the Slovenian economy (SLOPOL8) consists of 24 behavioral and 37 identity equations (the total number of equations as programmed for optimum control optimization including several auxiliary equations is 154). The model includes 61 state variables, 2 controls, 15 exogenous non-controlled variables and 148 unknown (estimated) parameters. Similar to SLOVNL, the exogenous variables also include variables outside the influence of Slovenian policy-makers (oil price, world trade, euro area interest rates, population), some policy instruments (public consumption and investment, transfer payments to private households, tax rates and social security contribution rates) and some dummy variables.

The behavioral equations are estimated by ordinary least squares (OLS), using quarterly data for the period 1995:1 until 2008:4. The start period for the optimization is 2008:1 and the end period is 2015:4 (32 periods). For more information see (Neck *et al.* 2011).

*Endogenous (objective) variables:*

$x[1]$	<i>UR</i>	unemployment rate
$x[2]$	<i>INFL</i>	inflation rate
$x[3]$	<i>GDP</i>	real gross domestic product
$x[4]$	<i>DEBTGDP</i>	debt level in relation to GDP

*Control variables:*

$u[1]$	<i>GN</i>	government consumption at current prices
$u[2]$	<i>TRANSFERSN</i>	transfer payments to households at current prices

Table 5: Weights of the variables in the SLOPOL8 model

state variables		control variables	
variable	weight	variable	weight
<i>UR</i>	1259698.847	<i>GN</i>	28.6
<i>INFL</i>	6763629.434	<i>TRANSFERSN</i>	39.9
<i>GDP</i>	2		
<i>DEBTGDP</i>	148218.3721		



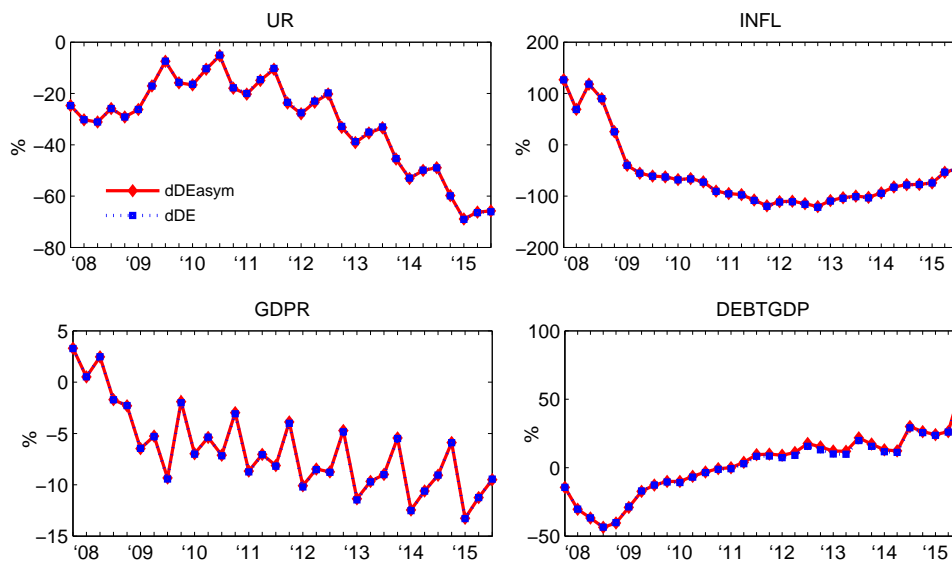


Figure 9: Relative deviations in states for SLOPOL8

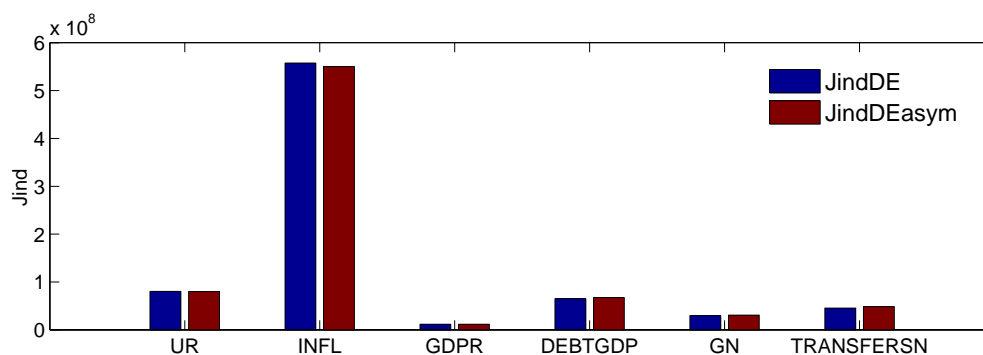


Figure 10: Constituent elements of the objective function value for SLOPOL8