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Finite dimensional realizations of forward price term structure models

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Finite dimensional realizations of forward price term structure models

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Abstract

In this paper we study a fairly general Wiener driven model for the term structure of forward prices.

The model, under a fixed martingale measure, Q , consists of two infinite dimensional stochastic differential equations (SDEs). The first system is a standard HJM model for (forward) interest rates, driven by a multidimensional Wiener process W . The second system is an infinite SDE for the term structure of forward prices on some specified underlying asset driven by the same W .

We are primarily interested in the *forward prices*. However, since for any fixed maturity T , the forward price process is a martingale under the T -forward neutral measure, the zero coupon bond volatilities will enter into the drift part of the SDE for these forward prices. The interest rate system is, thus, needed as input into the forward price system.

Given this setup we use the Lie algebra methodology of Björk *et al.* to investigate under what conditions on the volatility structure of the forward prices and/or interest rates, the inherently (doubly) infinite dimensional SDE for forward prices can be realized by a finite dimensional Markovian state space model.

Key words: forward prices, term structures, state space models, Markovian realizations, HJM models.

JEL Classification: E43, G13

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1 Introduction

In this paper we study forward price models and, in particular, we want to understand when the inherently infinite forward price process can be realized by means of a (Markovian) finite state space model.

From a theoretical point of view, forward prices' term structures are a more complex object than interest rates' term structures or futures prices' term structures. The extra complexity results from the fact that forward prices, with maturity T , are martingales under the T -forward measure, Q^T . Under such T -forward measures, zero-coupon bonds (with maturity T) are numeraires which implies that zero-coupon bond prices volatilities will enter into the dynamics of forward prices. Consequently, in general, term structures of forward prices cannot be studied in isolation, they must be studied under some interest rate setting and a needed input to forward price term structures is the term structure of interest rates.

We model the dynamics directly under the risk-neutral measure Q and our forward price model is described by using two infinite dimensional stochastic differential equations (SDEs), one defining the interest rate setting, another defining the forward contract setting. For the interest rate setting we consider a standard HJM model for (forward) interest rates, driven by a multidimensional Wiener process W . For the forward contract setting we use the Q^T -martingale property of forward prices and the bond prices dynamics induced by the interest rate setting to get a second infinite SDE for the term structure of forward prices, on some specified underlying asset, under the risk-neutral measure Q . Without loss of generality we consider that the Wiener process W is the same for both SDEs.

The theoretical literature on forward prices term structures is not big and has mainly focused on understanding under what conditions, on the dynamics of the state space variables (which are *assumed* to be finite), the term structure is of an *a priori given* specific functional form. Included in this traditional approach are the studies on affine and quadratic term structures of forward prices (see [11] for a recent study integrating these two types of term structures and references).

In this paper we choose a fundamentally different approach. We do *not* assume that the state space model is finite, nor that the term structure of forward prices is of a given specific function form. Instead, we try to understand under what conditions, in terms of the volatility of forward prices and interest rates, we can have a finite dimensional realization (FDR) of forward prices term structure models.

This more systematic way of thinking about term structures was proposed by Björk and Christensen [4] and Björk and Svensson [6], and a more geometric way of thinking about FDR of term structures, was then introduced. The main technical tool of these studies is the Frobenius Theorem, and the main result is that there exists a FDR if and only if the Lie algebra generated by the drift and diffusion terms, of the underlying infinite dimensional (Stratonovich) SDE, is finite dimensional. Filipović and Teichman [10] and [9] increased the applicability of the geometric approach by showing how the theory can be extended to much more general settings than initially considered. Finally, Björk and Landén [5] addressed the question of the actual construction of finite-dimensional realizations, making this geometrical analysis interesting also from an application point of view.

The main area of application of these ideas has been (forward) interest rate term structures, which was the object of study in all the above mentioned papers (for a review study on the geometry of interest rate models see also [2]). More recently this geometric machinery has also

been applied to study futures prices term structures (see [3]). As far as our knowledge goes, this techniques have not yet been applied to study forward prices (or any other Q^T -martingales). In the present paper we, thus, take this next natural step.

The main contributions of this paper are as follows.

- We adapt the geometrical analysis of term structures to the case of doubly infinite systems.
- We obtain necessary and sufficient conditions for the existence of a FDR of forward rate term structure models.
- Given that such conditions are satisfied, we derive the dynamics of the underlying finite state space variable.

The organization of the paper is the following.

In Section 2 we present the basic setup, derive the doubly infinite SDE that will be the object of study and present the main questions to be answered. Section 3 briefly reviews the basic geometrical concepts behind the method of analysis. Sections 4, 5 and 6 are devoted to the actual study of forward price models answering the proposed questions. Section 7 resumes our main conclusions and discusses the applicability of the results.

2 Setup

The main goal of this study is the study of *forward prices* in a general stochastic interest rate setting.

We, thus, consider a financial market living on a filtered probability space $\{\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t \geq 0}\}$ carrying an m -dimensional Wiener process W . For reasons that will soon become clear, the main assets we consider are forward contracts (written on some given underlying asset under consideration) and zero-coupon bonds¹.

Let $f_0(t, T)$ denote the forward prices at time t of a forward contract maturing at time T , and $p_0(t, T)$ denote the price at time t of a zero-coupon bond maturing at time T .

Besides the trivial boundary conditions

$$\begin{aligned} f_0(T, T) &= S(T) \\ p_0(t, T) &= 1 \end{aligned}$$

where S is the price process of the underlying asset to the forward contract, arbitrage arguments yields

$$p_0(t, T) = E_t^Q \left[e^{-\int_t^T R(s) ds} \right] \tag{1}$$

$$f_0(t, T) = E_t^T [f_0(T, T)] \tag{2}$$

where R is the short rate of interest and $E_t^Q [\cdot]$, $E_t^T [\cdot]$ denote, respectively, expectations, conditional on \mathcal{F}_t , under the martingale measure Q and under the *forward* martingale measure Q^T .

¹For a textbook discussion of forward contracts and zero-coupon bonds see for instance [1].

It is also well-known that under *deterministic* interest rate settings, or complete orthogonality between the underlying and the interest rate random sources, forward prices are the same as futures prices (a similar but conceptually easier contract).²

In this study we focus on *forward prices* and, in particular, we are interested in analyzing the settings where they are not equivalent to futures prices. Our analysis will, therefore, assume *stochastic interest rates* and that there are at least some *common random sources* driving *both* the interest rates *and* the underlying to the forward contract.

Thus, in our context, a **forward price model** is only fully defined once we have specified both forward prices dynamics and interest rates dynamics under a same measure (which we choose to be Q) and assumed that these two dynamics are, at least partially, driven by common elements of our multidimensional Wiener process.

Before we present in detail our setting, we start by reparameterizing our variables. A more suitable parameterization for our purposes is the so called Musiela parameterization ([7] and [13]). Under the Musiela parameterization, forward prices and bond prices are given in terms of t and x , where x denotes time **to** maturity, in contrast to T which defined time **of** maturity. Therefore, we will use

$$f(t, x) = f_0(t, t + x) \quad p(t, x) = p_0(t, t + x) \quad (3)$$

2.1 The interest rate curve

We consider a standard HJM model for the (forward) interest rates, driven by a multidimensional Wiener process W . Using the Musiela parameterization the dynamics for the interest rates, under Q , are given by³

$$dr(t, x) = \left\{ \frac{\partial}{\partial x} r(t, x) + \sigma(t, x) \int_0^x \sigma^*(t, s) ds \right\} dt + \sigma(t, x) dW_t \quad (4)$$

where $\sigma(t, x)$ is a given adapted process in \mathbb{R}^m and $*$ denotes transpose.

From the relation between (forward) interest rates and bond prices, we can derive the bond price Q -dynamics.

Lemma 2.1 *Assume the (forward) interest rates dynamics in (4). Then the dynamics of the zero-coupon bond prices, using the Musiela parameterization, is given by*

$$dp(t, x) = \{R(t) - r(t, x)\} p(t, x) dt + p(t, x) v(t, x) dW_t$$

where R is the short interest rate⁴ and the bond prices' volatility, v , is obtained from the (forward) interest rate volatilities as

$$v(t, x) = - \int_0^x \sigma(t, s) ds \quad (5)$$

and hence also adapted.

²For a futures price definition, see for instance [1] or [11].

³For a textbook treatment of HJM models and the Musiela parameterization for such models, see [1].

⁴Recall that in the Musiela parametrization the short rate of interest is $R(t) = r(t, 0)$.

Proof. Recall the standard relation between (forward) interest rates and bond prices

$$p(t, x) = e^{-\int_0^x r(t, s) ds}.$$

Let us set $y(t, x) = -\int_0^x r(t, s) ds$. Applying the Itô lemma we get

$$\begin{aligned} dy(t, x) &= -\int_0^x dr(t, s) ds \\ &= -\int_0^x \left[\left(\frac{\partial}{\partial s} r(t, s) + \sigma(t, s) \int_0^s \sigma^*(t, u) du \right) dt + \sigma(t, s) dW_t \right] ds \\ &= -\int_0^x \frac{\partial}{\partial s} r(t, s) ds dt - \int_0^x \sigma(t, s) \int_0^s \sigma^*(t, u) duds dt - \underbrace{\int_0^x \sigma(t, s) ds}_{v(t, x)} dW_t \\ &= \left[\underbrace{r(t, 0)}_{R(t)} - r(t, x) - \int_0^x \sigma(t, s) \int_0^s \sigma^*(t, u) duds \right] dt + v(t, x) dW_t \end{aligned}$$

The result follows from $dp(t, x) = p(t, x) dy(t, x) + \frac{1}{2} p(t, x) [dy(t, x)]^2$ and by noting that

$$\begin{aligned} \sigma(t, x) \int_0^x \sigma^*(t, u) du &= \frac{1}{2} 2\sigma(t, x) \int_0^x \sigma^*(t, u) du \\ \int_0^x \sigma(t, s) \int_0^s \sigma^*(t, u) duds &= \frac{1}{2} \left(\int_0^x \sigma^*(t, u) du \right)^2 \end{aligned}$$

□

2.2 The forward price curve

Since the forward prices are Q^T -martingales (recall (2)), we assume Q^T -dynamics of the form

$$df_0(t, T) = f_0(t, T) \gamma_0(t, T) dW_t^T \quad (6)$$

where we also take γ_0 to be a given adapted process.

Here we use the fact that martingales have zero drift. Note however that, by choosing to model the forward price dynamics as in (6), forward prices with different maturities T are modeled under a different martingale measures Q^T .

Reparameterizing using $f(t, x) = f_0(t, t+x)$ give us

$$df(t, x) = \left\{ \frac{\partial}{\partial x} f(t, x) \right\} dt + f(t, x) \gamma(t, x) dW_t^T.$$

where $T = t+x$ and $\gamma(t, x) = \gamma_0(t, T)$.

It will also simplify matters if we work with the logarithm of forward prices instead of the forward prices themselves. Thus setting

$$q(t, x) = \ln f(t, x) \quad (7)$$

we have

$$dq(t, x) = \left\{ \frac{\partial}{\partial x} q(t, x) - \frac{1}{2} \|\gamma(t, x)\|^2 \right\} dt + \gamma(t, x) dW_t^T. \quad (8)$$

Note that analyzing the logarithm of forward prices is equivalent to analyzing the forward prices themselves, as we can always use (7) to transfer any results on the logarithm of forward prices into results on forward prices.

Finally, to obtain the dynamics of (the logarithm) of forward prices, under the risk-neutral martingale measure, Q , we use the change of numeraire technique (introduced in [12]). Denoting by L the Radon-Nikodym derivative,

$$L(t) = \frac{dQ}{dQ^T} \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T,$$

and recalling the money account B is the numeraire under Q , we have in Musiela parameterization

$$L(t) = p(0, T) \frac{B(t)}{p(t, x)}$$

where $x = T - t$.

Thus, the dynamics of our likelihood process L are given by

$$dL(t) = L(t) \{-v(t, x)\} dW_t^T$$

i.e., the Girsanov kernel, for the transition from Q^T to Q , is the symmetric of the volatility of zero-coupon bond price with maturity T or, equivalently, with time to maturity $x = T - t$.

Using the above Girsanov kernel, we can easily obtain the (logarithm of) forward prices Q -dynamics from (8), that is

$$dq(t, x) = \left\{ \frac{\partial}{\partial x} q(t, x) - \frac{1}{2} \|\gamma(t, x)\|^2 - \gamma(t, x) v^*(t, x) \right\} dt + \gamma(t, x) dW_t \quad (9)$$

where $*$ denotes transpose and where, without loss of generality, we can take W to be multidimensional and the same as in (4).

Taking a geometrically oriented interpretation of equations (4) and (9), we can see each of these equations as infinite dimensional objects. The main infinite dimensional object under study in this paper is (the logarithm) of the **forward price curve**, i.e., the curve $x \rightarrow q(t, x)$. This object, however, for general adapted processes σ and γ may depend on, the **interest rate curve**, i.e., the curve $x \rightarrow r(t, x)$, another infinite dimensional object.

In principle, both adapted processes σ and γ could depend on q and r . It seems, however, unrealistic to assume that a forward price on a specific underlying (be it the price of a stock, or any other asset) should influence the interest rate volatility.

The opposite is true for forward prices. As mentioned before, these prices are only interesting to study in *stochastic* interest rate settings. This tells us that, maybe, it is realistic that the forward price volatility depends on the interest rates' curve.

With this basic intuition in mind we set some more structure on the volatility processes σ and γ .

Assumption 2.1 *The adapted processes $\gamma(t, x)$, and $\sigma(t, x)$ have the following functional form in terms of r and q*

$$\gamma(t, x) = \gamma(q_t, r_t, x) \quad (10)$$

$$\sigma(t, x) = \sigma(r_t, x) \quad (11)$$

where, with a slight abuse of notation, the r.h.s. occurrence of γ and σ denotes deterministic mappings

$$\begin{aligned} \gamma &: \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m \\ \sigma &: \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m. \end{aligned}$$

where \mathcal{H}_q and \mathcal{H}_r are special Hilbert spaces of functions where forward price curves and interest rate curves live, respectively. ⁵

Note that by imposing Assumption 2.1, σ does not depend on q_t (the logarithm of the forward price process) and that we restrict ourselves to the study of time homogenous models. Extensions to non-homogeneous models have been considered in [3] and turn out to be straightforward generalization of the homogeneous results.

From now on we will use the short-hand notation $q_t = q(t, x)$, $r_t = r(t, x)$ where we suppress the x -dependence. This shorter notation will be helpful when the expressions get messy and is more intuitive from a geometrical point of view.

Using the short-hand notation and Assumption 2.1, we can rewrite equations (4)-(9) as

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2} \|\gamma(q_t, r_t)\|^2 - \gamma(q_t, r_t)v^*(r_t) \right\} dt + \gamma(q_t, r_t)dW_t \quad (12)$$

$$dr_t = \left\{ \mathbf{F}r_t - \sigma(r_t)v^*(r_t) \right\} dt + \sigma(r_t)dW_t \quad (13)$$

where $\mathbf{F} = \frac{\partial}{\partial x}$ and we can interpret the entire system as an object $\hat{q} = (q_t, r_t)^* \in \mathcal{H}_q \times \mathcal{H}_r$.

As we can see from equations (12)-(13), the interest rates equation (13) does not depend on the forward prices equation (12), so the interest rates curve r exist and can be studied in isolation. For a survey study on the geometry of interest rate models see [2]. In contrast to this, the (logarithm of the) forward price equation (12), is linked to the interest rate equation (13) through $\gamma(q, r)$ and/or $v(r)$ ⁶. This means, that in general, to study forward prices we will have to study the entire system (12)-(13).

In the following analysis we will refer to **forward price equation** when referring only to (12), to **interest rate equation** when referring only to (13), and to **forward price system** when referring to the entire system (12)-(13).

We can now formulate our main problems.

⁵For details on the construction of the Hilbert spaces \mathcal{H}_q and \mathcal{H}_r we refer to [6], [10] and [9].

⁶Given the definition of the bond price volatility, v , in (5), if $\sigma(t, x) = \sigma(r_t, x)$ then also $v(t, x) = v(r_t, x)$.

2.3 Main Problems

- Problem 1* : Under what conditions we have Markovian forward prices?
- Problem 2* : Is it possible to have a finite realization for the forward prices equation (12) but not for interest rates equation (13)?
- Problem 3* : When can the inherently infinite forward price system (12)-(13) be realized by means of a finite dimensional state space model?
- Problem 4* : In the cases when a finite dimensional realization (FDR) exists, can we determine the finite dimensional state space model?

The next section introduces the method of analysis.

3 Method and Basic Geometric Concepts

In this section we describe the general method we will use to attack the presented problems.

The method relies on geometric results from differential geometry and was firstly applied to finance in [4] and [6]. In this section, we adapt the framework of [4] and [6] to our doubly-infinite system case.

To be able to apply the concepts and intuitions of ordinary differential geometry to (stochastic) Itô calculus, we need to rewrite the analysis in terms of Stratonovich integrals instead of Itô integrals.

Definition 3.1 For given semi martingales X and Y , the **Stratonovich integral** of X with respect to Y , $\int_0^t X_s \circ dY_s$, is defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2} \langle X, Y \rangle_t. \quad (14)$$

where the first term on the r.h.s. is the Itô integral and we can define the quadratic variation process $\langle X, Y \rangle$ can be computed via

$$d\langle X, Y \rangle = dX_t dY_t$$

with the usual multiplication rules: $dW \cdot dt = dt \cdot dt = 0$, $dW \cdot dW = dt$.

The Stratonovich formulation is geometrically more convenient because the Itô formula, in Stratonovich calculus, takes the form of the standard *chain rule* in ordinary calculus.

Lemma 3.1 Assume that a function $F(t, y)$ is smooth. Then we have

$$dF(t, Y_t) = \frac{\partial F}{\partial t}(t, Y_t) dt + \frac{\partial F}{\partial y}(t, Y_t) \circ dY_t. \quad (15)$$

Let us begin by specifying exactly what we mean with a finite dimensional realization for the forward prices generated by volatilities.

Given the volatility mappings $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ from Assumption 2.1, the forward prices equation will, in general, depend on the interest rate curve. Thus, our main object of study will be a Stratonovich forward price *system* of the following form:

$$d\hat{q}_t = \begin{bmatrix} \mu^q(q_t, r_t) \\ \mu^r(r_t) \end{bmatrix} dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} \circ dW_t, \quad (16)$$

where $\hat{q} = \begin{bmatrix} q \\ r \end{bmatrix} \in \mathcal{H}_q \times \mathcal{H}_r$.

In special cases, the forward price dynamics may be independent of the interest rate curve, then our object of study is the Stratonovich forward price *equation*,

$$dq_t = \mu^q(q_t)dt + \gamma(q_t) \circ dW_t. \quad (17)$$

where $q \in \mathcal{H}_q$, and we say that the (logarithm of) forward prices is **Markovian**.

When referring to the **forward price model** we refer to either (16) or (17), depending on the circumstances.

In this study, we will consider these two possibilities.

Definition 3.2 *We say that the doubly-infinite SDE (16) has a (local) d -dimensional realization at $\hat{q}^0 = (q^0, r^0)^*$, if there exists a point $z_0 \in \mathbb{R}^d$, smooth vector fields $\hat{a}, \hat{b}_1 \cdots, \hat{b}_m$ on some open subset $\hat{\mathcal{Z}}$ of \mathbb{R}^d and a smooth (sub manifold) map $\hat{G} : \hat{\mathcal{Z}} \rightarrow \mathcal{H}_q \times \mathcal{H}_r$, such that $\hat{q} = (q, r)^*$ has a local representation*

$$\hat{q}_t = \hat{G}(Z_t) \quad \text{i.e.,} \quad \begin{bmatrix} q_t \\ r_t \end{bmatrix} = \begin{bmatrix} \hat{G}^q(Z_t) \\ \hat{G}^r(Z_t) \end{bmatrix} \quad \text{a.s.}$$

where Z is the strong solution of the d -dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= \hat{a}(Z_t) + \hat{b}(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases} \quad (18)$$

and where W is the same as in (16).

Likewise, we say that the SDE (17) has a (local) n -dimensional realization at q^0 , if there exist $z_0 \in \mathbb{R}^n$, smooth vector fields $a, b_1 \cdots, b_m$ on some open subset \mathcal{Z} of \mathbb{R}^n and a smooth (sub manifold) map $G : \mathcal{Z} \rightarrow \mathcal{H}_q$, such that q has a local representation

$$q_t = G(Z_t) \quad \text{a.s.}$$

where Z is the strong solution of the d -dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t) + b(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases} \quad (19)$$

where W is the same as in (17).

If the SDE under analysis, (16) or (17), has a finite dimensional realization (FDR), we say that our forward rate model admits a FDR.

The method of studying existence and construction of FDR for forward price models, relies on some basic concepts from infinite dimensional differential geometry, which we now introduce.

3.1 Basic Geometric Concepts

The presentation of the needed geometric concepts follows [6]. These basic concepts will be presented for a general real Hilbert Space Y and we denote by y an element of Y . In practice, the Hilbert space under analysis will be either \mathcal{H}_q when studying Markovian forward prices or $\mathcal{H}_q \times \mathcal{H}_r$ when dealing with the entire forward price system.

Consider a real Hilbert space Y . By an n -dimensional **distribution** we mean a mapping F , which to each $y \in Y$ associates an n -dimensional subspace $F(y) \subseteq Y$. A mapping (vector field) $f : Y \rightarrow Y$, is said to **lie in** F if $f(y) \in F(y)$ for every $y \in Y$. A collection f_1, \dots, f_n of vector fields lying in F **generates** (or spans) F if $\text{span}\{f_1(y), \dots, f_n(y)\} = F(y)$ for every $y \in Y$, where “span” denotes the linear hull over the real field. The distribution is **smooth** if, for every $y \in Y$, there exist smooth vector fields f_1, \dots, f_n spanning F . A vector field is smooth if it belongs to C^∞ . If F and G are distributions and $G(y) \subseteq F(y)$ for all y we say that F **contains** G , and we write $G \subseteq F$. The **dimension** of a distribution F is defined pointwise as $\dim F(y)$.

Let f and g be smooth vector fields on Y . Their **Lie bracket** is the vector field $[f, g]$, defined by

$$[f, g] = f'g - g'f,$$

where f' denotes the Frechet derivative of f at y , and similarly for g' . We will sometimes write $f'[g]$ instead of $f'g$ to emphasize that the Frechet derivative is operating on g . A distribution F is called **involutive** if for all smooth vector fields f and g lying in F on Y , their lie bracket also lies in F , i.e.

$$f, g \in F \quad \Rightarrow \quad [f, g] \in F$$

for all $y \in Y$.

We are now ready to define the concept of a Lie algebra which will play a central role in what follows.

Definition 3.3 *Let F be a smooth distribution on Y . The **Lie algebra** generated by F , denoted by $\{F\}_{LA}$ or by $\mathcal{L}\{F\}$, is defined as the minimal (under inclusion) involutive distribution containing F .*

If, for example, the distribution F is spanned by the vector fields f_1, \dots, f_n then, to construct the Lie algebra $\{f_1, \dots, f_n\}_{LA}$, you simply form all possible brackets, and brackets of brackets, etc. of the fields f_1, \dots, f_n , and adjoin these to the original distribution until the dimension of the distribution is no longer increased.

When one tries to compute a concrete Lie algebra the following observations are often very useful. Taken together, they basically say that, when computing a Lie algebra, you are allowed to perform Gaussian elimination.

Lemma 3.2 *Take the vector fields f_1, \dots, f_k as given. It then holds that the Lie algebra $\{f_1, \dots, f_k\}_{LA}$ remains unchanged under the following operations.*

- The vector field f_i may be replaced by αf_i , where α is any smooth nonzero scalar field.
- The vector field f_i may be replaced by

$$f_i + \sum_{j \neq i} \alpha_j f_j,$$

where $\alpha_1, \dots, \alpha_k$ are any smooth scalar fields.

Let F be a distribution and let $\varphi : Y \rightarrow Y$ be a diffeomorphism on Y . Then we can define a new distribution φ_*F on Y by

$$(\varphi_*F)(\varphi(y)) = \varphi'(y)F(y).$$

For any smooth vector field f on Y the field φ_*f is defined analogously. It is straightforward to verify that

$$\varphi_*[f, g] = [\varphi_*f, \varphi_*g]. \quad (20)$$

We now define an useful operator on our Hilbert space Y .

Definition 3.4 Let f be a smooth vector field on Y , and let y be a fixed point in Y . Consider the ODE

$$\begin{cases} \frac{dy_t}{dt} = f(y_t) \\ y_0 = y \end{cases}$$

We denote the solution y_t as $e^{ft}y$.

Finally, and for future reference, we define a particular type of functions –the quasi-exponential functions – that will turn out useful.

Definition 3.5 A *quasi-exponential* (or *QE*) function is by definition any function of the form

$$f(x) = \sum_u e^{\lambda_u x} + \sum_j e^{\alpha_j x} [p_j(x) \cos(w_j x) + q_j(x) \sin(w_j x)]$$

where λ_u, α_j, w_j are real numbers, whereas p_j and q_j are real polynomials.

Important properties of QE functions are given in Lemma 3.3.

Lemma 3.3 The following holds for quasi-exponential functions

- A function is QE if and only if it is a component of the solution of a vector valued linear ODE with constant coefficients.
- A function is QE if and only if it can be written as $f(x) = ce^{Ax}b$. Where c is a row vector, A is a square matrix and b is a column vector.
- If f is QE, then f' is QE.
- If f is QE, then its primitive function is QE.
- If f and g are QE, then fg is QE.

3.2 Main results from the literature

We can now adapt two important theorems from [6] to our forward price problem. The first theorem gives us the general necessary and sufficient conditions for existence of a FDR.

Theorem 3.4 (Björk and Svensson) *Consider the SDE in (16) and denote $\gamma_1, \dots, \gamma_m$ and $\sigma_1, \dots, \sigma_m$ the elements of γ and σ , respectively. Assume that the dimension of the Lie algebra*

$$\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \dots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA}$$

is constant near the initial point $\hat{q}^0 = (q^0, r^0)^ \in \mathcal{H}_q \times \mathcal{H}_r$.*

Then (16) possesses an FDR if and only if

$$\dim \left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \dots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA} < \infty$$

in a neighborhood of \hat{q}^0 .

Likewise, for Markovian forward prices we consider the SDE (17), and assume that the dimension of the lie algebra of $\{\mu, \gamma_1, \dots, \gamma_m\}_{LA}$ is constant near $q^0 \in \mathcal{H}_q$. Then (17) possesses and FDR if and only if

$$\dim \{\mu, \gamma_1, \dots, \gamma_m\}_{LA} < \infty$$

Remark 3.1 *To shorten notation we will sometimes use $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma \\ \sigma \end{bmatrix} \right\}_{LA}$ instead of $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \sigma_1 \end{bmatrix}, \dots, \begin{bmatrix} \gamma_m \\ \sigma_m \end{bmatrix} \right\}_{LA}$ and $\{\mu, \gamma\}_{LA}$ instead of $\{\mu, \gamma_1, \dots, \gamma_m\}_{LA}$.*

The second theorem gives us a parameterization of the curves produced by the forward price model and is a crucial step to the understanding of the construction algorithm.

Theorem 3.5 (Björk and Svensson) *Assume that the Lie algebra $\left\{ \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix}, \begin{bmatrix} \gamma \\ \sigma \end{bmatrix} \right\}_{LA}$ is spanned*

by the smooth vector fields $\hat{f}_1, \dots, \hat{f}_d$ in $\mathcal{H}_q \times \mathcal{H}_r$.

Then, for the initial point $\hat{q}^0 = (q^0, r^0)^$, all forward price and interest rate curves produced by the model will belong to the manifold $\hat{\mathcal{G}} \in \mathcal{H}_q \times \mathcal{H}_r$, which can be parameterized as $\hat{\mathcal{G}} = \text{Im}[\hat{G}]$, where*

$$\hat{G}(z_1, \dots, z_d) = e^{\hat{f}_d z_d} \dots e^{\hat{f}_1 z_1} \begin{bmatrix} q^0 \\ r^0 \end{bmatrix}, \quad (21)$$

and where the operator $e^{\hat{f}_i z_i}$ is given in Definition 3.4.

Likewise, in the case of Markovian forward prices, and assuming that the Lie algebra $\{\mu, \gamma\}_{LA}$ is spanned by the smooth vector fields f_1, \dots, f_d in \mathcal{H}_q . Then, for the initial point q^0 , all forward price curves produced by the model will belong to the manifold $\mathcal{G} \in \mathcal{H}$, which can be parameterized as $\mathcal{G} = \text{Im}[G]$, where

$$G(z_1, \dots, z_d) = e^{f_d z_d} \dots e^{f_1 z_1} q^0 \quad (22)$$

and where the operator $e^{f_i z_i}$ is given in Definition 3.4.

The manifolds $\hat{\mathcal{G}}$ and \mathcal{G} in the above theorem are obviously invariant under the forward price model dynamics. Therefore, they will be referred to as the **invariant manifolds** in the sequel. \hat{G} and G are, thus, local parameterizations of the invariant manifolds $\hat{\mathcal{G}}$ and \mathcal{G} , respectively.

The construction algorithm (Björk and Landén) introduced in [5] is based on idea that, if we are in the case when the forward price system generated by the volatilities $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ admits a FDR, we have

$$\begin{bmatrix} q \\ r \end{bmatrix} = \hat{G}(Z)$$

and for

$$dZ = \hat{a}(Z)dt + \hat{b}(Z) \circ dW_t,$$

it must hold that

$$\hat{G}_* \hat{a} = \begin{bmatrix} \mu^q \\ \mu^r \end{bmatrix} \quad \hat{G}_* \hat{b} = \begin{bmatrix} \gamma \\ \sigma \end{bmatrix}. \quad (23)$$

Equivalently for Markovian forward prices, if forward price model generated by $\gamma : \mathcal{H}_q \times R_+ \rightarrow R^m$ admits a FDR, then we have

$$q = G(Z)$$

and or

$$dZ = a(Z)dt + b(Z) \circ dW_t,$$

it must hold that,

$$G_* a = \mu \quad G_* b = \gamma. \quad (24)$$

Since we can compute \hat{G} and G from (21) and (22), we can solve the system (23) for \hat{a} and \hat{b} , or the system (24) for a and b . We note that the equations in (23) and (24) do not necessarily have unique solutions, but it is enough to find *one* solution.

Note also that by solving (23) or (24), we obtain the Stratonovich dynamics of our FDR. The Itô dynamics, (which, in general, looks nicer) can be easily obtained using (15).

3.3 Strategy of analysis

In the next few sections we will address our four main problems.

In Section 4, we answer our *problem one* characterizing the settings when forward prices are Markovian. As it turns out, during this analysis, we will also be able to give a partial answer to *problem two*.

In Section 5, we study existence and construction of FDR for Markovian forward prices.

In Section 6, we deal with the cases when forward prices are **not** Markovian, studying existence and construction of FDR for the entire forward price system. Studying the entire system we are able to give a complete answer to *question two*.

In sections 5 and 6, *problems three and four* we will be answered following the scheme.

- Choose a number of vector fields f_0, f_1, \dots that spans the Lie-algebra we are interested on. For that purpose Lemma 3.2 is useful to help simplifying the vector fields.

- Conclude under what conditions our Lie-algebra is of finite dimension in view of Theorem 3.4.
- Assuming that those conditions hold, compute a local parameterization of the invariant manifold using Theorem 3.5.
- Given that parameterization, solve a system of equations of the type (23) or (24) to obtain the finite state variables dynamics.

4 On the Existence of Markovian Forward Prices

Having described the setup and the general method, we now start our analysis.

Recall that our main object of study is the forward price system

$$\begin{cases} dq_t &= \left\{ \mathbf{F}q_t - \frac{1}{2} \|\gamma(q_t, r_t)\|^2 - \gamma(q_t, r_t)v^*(r_t) \right\} dt + \gamma(q_t, r_t)dW_t \\ dr_t &= \{ \mathbf{F}r_t - \sigma(r_t)v^*(r_t) \} dt + \sigma(r_t)dW_t \end{cases} \quad (25)$$

where $\mathbf{F} = \frac{\partial}{\partial x}$ and $v(x, r) = - \int_0^x \sigma(s, r) ds$.

Before we go on, and to exclude pathological cases from the analysis, we need to impose a regularity condition on forward price models.

Assumption 4.1 *If $\gamma_i(q_t, r_t) \neq 0$ and $v_i(r_t) \neq 0$ for some $i \in \{1, \dots, m\}$, then the following regularity condition holds:*

$$\frac{1}{2} \|\gamma(q_t, r_t)\|^2 + \gamma(q_t, r_t)v^*(r_t) \neq 0.$$

Given Assumption 4.1 and by mere inspection of (25), we see that the answer to our *first problem* – on whether forward prices can be studied without considering the interest rate equation – is *yes* if and only if the terms $\gamma(q_t, r_t)$ and $\gamma(q_t, r_t)v^*(r_t)$ do not depend on r_t .

Remark 4.1 *The (logarithm of the) forward price equation is Markovian if and only if the mappings $\gamma : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $\gamma v^* : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ are constant w.r.t. r .*

The first condition – that γ cannot depend on r – is quite straightforward, but let us take a moment to understand what “ γv^* constant w.r.t. to r ” really means.

Given that γ is not dependent on r , does this mean that v must also be independent of r ? The answer to this question is trivially *no*, when we take into consideration the fact that both γ and v are multidimensional. The exercise of explaining this answer, however, helps to establish crucial notation.

Recall that our m - dimensional Wiener process W drives both forward prices and interest rates, and that a multidimensional Wiener process can be seen as a vector of scalar independent Wiener processes. With this in mind, it is possible to understand that, depending on the applications, we may face all the following situations.

- The scalar Wiener processes driving interest rates are orthogonal to the scalar Wiener processes driving the forward prices;
- The scalar Wiener processes driving interest rates and forward prices are the same;
- A part of the scalar Wiener processes driving the interest rates also drives the forward prices (or vice versa);
- Interest rates and forward prices are partially driven by orthogonal scalar Wiener processes and partial driven by the same Wiener processes.

Without loss of generality, we can reorder the scalar Wiener processes inside a multidimensional Wiener process. Assumption 4.2 bellow, give us the reordering we will assume for our multidimensional Wiener process W .

Assumption 4.2 *The Q -Wiener process, W , driving both the forward prices and the interest rates is m -dimensional, and the same for both processes. Furthermore, we suppose that W has been reordered as*

$$W = \begin{bmatrix} W^A \\ W^B \\ W^C \end{bmatrix}$$

where W^A , W^B and W^C are, possibly multidimensional, Wiener process such that

- W^A drives only the forward prices q ,
- W^B drives only the interest rates r ,
- W^C drives both forward prices q and interest rates r .

Finally we establish that $i \in A$ means “ W_i is a element of W^A ”, and similarly for $i \in B$ and $i \in C$.

Assumption 4.2 has obvious implications for the matrices γ and σ which become then of the following form.

$$\begin{aligned} \gamma &= \begin{bmatrix} \gamma_A & 0 & \gamma_C \end{bmatrix} \\ \sigma &= \begin{bmatrix} 0 & \sigma_B & \sigma_C \end{bmatrix} \end{aligned}$$

thus, using $v(x, r) = - \int_0^x \sigma(s, r) ds$, we have

$$v = \begin{bmatrix} 0 & v_B & v_C \end{bmatrix},$$

and

$$\gamma v^* = \begin{bmatrix} \gamma_A & 0 & \gamma_C \end{bmatrix} \begin{bmatrix} 0 \\ v_B^* \\ v_C^* \end{bmatrix} = \gamma_C v_C^*.$$

From this we see that requiring γ and γv^* independent of r , is nothing but requiring that, γ_A , γ_C and v_C do not depend on r .

The important point here is that *no condition* is imposed on σ_B .

We can now restate Remark 4.1, using the notation introduced by Assumption 4.2.

Lemma 4.1 *Suppose that Assumptions 4.1 and 4.2 holds. The (logarithm of) forward prices will be **Markovian** if and only if the volatility mappings γ_A , γ_C and σ_C are constant w.r.t. r . No condition is imposed on σ_B .*

Proof. If γ_A , γ_C and σ_C are constant w.r.t. r , so are $\|\gamma_A\|$, $\|\gamma_C\|$, v_C and $\gamma_C v_C$. The dynamics in of q in (25) does not depend on r and forward prices are, thus, Markovian.

To prove the “only if” part we show that dependence of r by γ_A , γ_C or σ_C suffices, under the regularity conditions of Assumption 4.1, to guarantee non-Markovian forward prices. Suppose, first, that γ_A depends on r . Then $\|\gamma_A\|^2$ also depends on r making the forward prices non-Markovian. Suppose now that γ_C depends on r , then $\|\gamma_C\|^2$ and $\gamma_C v_C^*$ also depend on r . Assumption 4.1 guarantees that there is no full cancelation and the forward prices are non-Markovian. Finally suppose that σ_C depends on r , then v_c depends on r (since the integral is w.r.t. the variable s and we know $\sigma_C \neq 0$). Since v_c depends on r so does $\gamma_C v_C^*$ and the forward prices are non-Markovian. \square

Having established conditions for the forward prices being Markovian, we can go on and try to answer our *second problem* – on whether there exist models which admit a FDR for forward prices but not for interest rates. It turns out that, our unrestricted σ_B for Markovian forward prices, together with general results from the previous literature on interest rates FDR, allows us to give a partial answer already now.

From the previous literature on FDR of **interest rates** we know that only some particular functions $\sigma : \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ will generate interest rate models that admit a FDR. Concretely, it is shown in [10] that **every** component σ_i must be a weighted sum of quasi-exponential deterministic functions of x , weighted by scalar fields in \mathcal{H}_r .

Hence, the fact that σ_B is not driving the forward price equation and can be of *any* form for Markovian forward price models, tell us that, existence of FDR for Markovian forward prices is, in some sense, independent from existence of FDR for interest rates. This leads us to an early answer to our *second question*.

Remark 4.2 *As long as there are FDR for Markovian forward prices, there exist forward price models that allow for a FDR for forward prices but **not** for interest rates.*

5 Markovian Forward Prices

We now focus on the task of studying FDR for the forward price equation, in the special case where we have *Markovian* forward prices. Thus, in this section the following assumption holds (recall Lemma 4.1).

Assumption 5.1 *Consider Assumption 2.1 and 4.2 We, assume that the mappings γ_A , γ_C and σ_C are of the following special forms,*

$$\gamma_A(q, r, x) = \gamma_A(q, x) \quad \gamma_C(q, r, x) = \gamma_C(q, x) \quad \sigma_C(r, x) = \sigma_C(x).$$

Note that the specific functional form of σ_C implies we have deterministic σ_C -volatilities and we can interpret σ_i for $i \in C$ as **constant** vector fields in \mathcal{H}_q .

Given Assumption 5.1, the Q -dynamics of (the logarithm) of forward prices can be written as

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2} \left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2 \right] - \gamma_C(q_t)v_C^* \right\} dt + [\gamma_A(q_t) \quad 0 \quad \gamma_C(q_t)] dW_t. \quad (26)$$

Now, we rewrite equation (26) in Stratonovich form and obtain

$$\begin{aligned} dq_t &= \left\{ \mathbf{F}q_t - \frac{1}{2} \left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2 \right] - \gamma_C(q_t)v_C^* \right\} dt - \frac{1}{2} \sum_{i \in A, C} d\langle \gamma_i(q_t), W_t^i \rangle \\ &\quad + [\gamma_A(q_t), \quad 0, \quad \gamma_C(q_t)] \circ dW_t. \end{aligned}$$

To compute the Stratonovich correction we use the infinite dimensional Itô formula (see [8]) to obtain

$$d\gamma_i(q_t) = (\dots) dt + \gamma'_i(q_t)\gamma_i(q_t)dW_t^i \quad i \in A, C$$

and, thus

$$d\langle \gamma_i(q_t), W_t^i \rangle = \gamma'_i(q_t)\gamma_i(q_t)dt \quad i \in A, C.$$

Given the above computations we can write the Stratonovich dynamics of q as

$$\begin{aligned} dq_t &= \left\{ \mathbf{F}q_t - \frac{1}{2} \left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2 \right] - \gamma_C v_C^* - \frac{1}{2} [\gamma'_A(q_t)\gamma_A(q_t) + \gamma'_C(q_t)\gamma_C(q_t)] \right\} dt \\ &\quad + [\gamma_A(q_t) \quad 0 \quad \gamma_C(q_t)] \circ dW_t \end{aligned} \quad (27)$$

where γ'_A and γ'_B denotes the Frechet derivative. The terms $\gamma'_A(q_t)\gamma_A(q_t)$ and $\gamma'_C(q_t)\gamma_C(q_t)$ should be interpreted as follows,

$$\gamma'_A(q_t)\gamma_A(q_t) = \sum_{i \in A} \gamma'_i(q_t)\gamma_i(q_t) \quad \gamma'_C(q_t)\gamma_C(q_t) = \sum_{i \in C} \gamma'_i(q_t)\gamma_i(q_t).$$

We start by studying the two easier cases:

- (i) the case when γ (i.e, γ_A and γ_C) is also deterministic (σ_C is deterministic by Assumption 5.1), and
- (ii) the case when γ is not deterministic, but has deterministic direction.

5.1 Deterministic Volatility

We first consider the case when the functions γ_A and γ_C do not depend on q , so they have the special form

$$\gamma_i(q, x) = \gamma_i(x) \quad i \in A, C. \quad (28)$$

γ_i for $i \in A, C$ are, thus, **constant** vector fields in \mathcal{H}_q .

Recall from Assumption 5.1 that $\sigma_C(r, x) = \sigma_C(x)$.

In this case, the Stratonovich correction term is zero, and equation (27) becomes

$$dq_t(x) = \mu(q_t, x)dt + \gamma(x) \circ dW_t \quad (29)$$

where

$$\mu(q, x) = \mathbf{F}q - \frac{1}{2} \left[\|\gamma_A(x)\|^2 + \|\gamma_C(x)\|^2 \right] - \gamma_C(x)v_C^*(x) \quad (30)$$

$$\gamma(x) = \begin{bmatrix} \gamma_A(x) & 0 & \gamma_C(x) \end{bmatrix}. \quad (31)$$

Since this is a simple case, we choose to include all computations behind the results in the main text to exemplify the technique. In the next sections, when dealing with more complex cases, most of the computations will instead be presented in the appendix, leaving to the main text the intuition behind the results and their discussion.

5.1.1 Existence of a FDR

From Theorem 3.4 we know that a FDR exists if and only if

$$\dim \{ \mu, \gamma_i; \quad i \in A, C \}_{LA} < \infty.$$

We, thus, need to compute the Lie-algebra, $\{ \mu, \gamma \}_{LA}$. Computing the Lie brackets we have, for each i

$$[\mu, \gamma_i] = \mathbf{F}\gamma_i =: f_{1i}$$

$$[\gamma_i, f_{1i}] = 0$$

$$[\mu, f_{1i}] = \mathbf{F}f_{1i} = \mathbf{F}^2\gamma_i =: f_{2i}$$

⋮

It follows that

$$\{ \mu, \gamma \}_{LA} = \text{span} \{ \mu, \mathbf{F}^k \gamma_i; \quad k = 0, 1 \dots \quad i \in A, C \}. \quad (32)$$

Obviously, if a FDR exists, there must exist an n_i for each i , such that

$$\mathbf{F}^{n_i+1} \gamma_i = \sum_{k=0}^{n_i} c_{i,k} \mathbf{F}^k \gamma_i \quad (33)$$

where the $c_{i,k}$ are real numbers.

Proposition 5.1 tell us under what conditions we will have $\dim \{ \mu, \gamma \}_{LA} < \infty$.

Proposition 5.1 *The (logarithm of the) forward price equation (29) admits a finite dimensional realization (FDR) if and only if each component of γ is quasi-exponential (QE). No functional restriction is imposed on the deterministic function σ_C , so in particular, σ_C does not have to be a QE function, it can be any deterministic function.*

Proof. Recall from Lemma 3.3 that γ_i solves the ODE (33) if and only if it is a QE function. \square

Note that, for Markovian forward prices, the interest rate volatility plays no role in determining existence of FDR. The only restriction on interest rate volatility is that σ_C is deterministic, but that is a result of the Markovian property, not an added requirement imposed to guarantee existence. One other way to see this is to note that only γ shows up in (32). As we will soon see, this is specific to the totally deterministic setting.

Remark 5.1 *In the simple deterministic setting, where $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, Markovian realizations of forward prices are generated only by the volatility of forward prices γ .*

In the next example, and to stress this point, we check existence of FDR in a simple model without even specifying the deterministic function σ_C .

Example 1 *Assume that forward prices are driven by a one-dimensional Wiener-process W which also drives the interest rates⁷. Furthermore, assume that the interest rate volatility associated to W is deterministic and that we have for the forward price volatility,*

$$\gamma(x) = \gamma_C(x) = \alpha e^{-ax}$$

where $\alpha, a \in \mathbb{R}$.

In this case, we have $A \cup B = \emptyset$, $C = \{1\}$ and

$$\mathbf{F}\gamma(x) = -a\alpha e^{-ax} \quad \Rightarrow \quad n_1 = 0 \quad c_{1,0} = -a.$$

Hence,

$$\{\mu, \gamma\}_{LA} = \text{span} \{\mu, \gamma\},$$

and the Lie-algebra $\{\mu, \gamma\}_{LA}$ has dimension two. Consequently, there exist a FDR for forward prices in this case. We will get back to this example in the construction part.

Finally, we want to make a remark on the exact dimension of the lie-algebra.

Remark 5.2 *It follows from (33) that*

$$\dim \{\mu, \gamma\}_{LA} = \dim \{\mu, \mathbf{F}^k \gamma_i ; \quad k = 0, \dots, n_i \quad i \in A, C\} \leq 1 + \sum_{i=1}^m n_i. \quad (34)$$

The “ \leq ” in (34) just reminds us that, given the possibility of Gaussian elimination, there may exist some cancelation effects.

To a better understanding of the above remark, we take the following example.

Example 2 *Suppose that*

$$\gamma_1(x) = e^{-bx} \quad \gamma_2(x) = xe^{-bx}.$$

Thus, $n_1 = 0$, $n_2 = 1$, and $\dim \{\mu, \gamma_1, \gamma_2, \mathbf{F}\gamma_2\} \leq 4$.

However since $\mathbf{F}\gamma_2 = \gamma_1 - b\gamma_2$,

$$\text{span} \{\mu, \gamma_1, \gamma_2, \mathbf{F}\gamma_2\} = \text{span} \{\mu, \gamma_1, \gamma_2\}.$$

Hence, in this case, we actually have $\dim \{\mu, \gamma\}_{LA} = 3$.

⁷This does not exclude the possibility of more Wiener-processes driving only the interest rates.

5.1.2 Construction of FDR

We now go on to the construction of FDR, in the totally deterministic volatility setup.

First, we obtain a parameterization G of the invariant manifold \mathcal{G} . In this case we have that

$$\{\mu, \gamma\}_{LA} = \text{span} \left\{ \mu, \mathbf{F}^k \gamma_i ; \quad k = 0, 1, \dots, n_i \quad i \in A, C \right\},$$

and we recall γ_i solves the ODE (33).

Using Theorem 3.5 we obtain G by computing the operators $\exp \{\mu z_0\}$ and $\exp \{\mathbf{F}^k \gamma_i z_{i,k}\}$. In order to get $\exp \{\mu z_0\} q_0$ we solve

$$\begin{cases} \frac{dy_t}{dt} &= \mu(y_t, x) \\ &= \mathbf{F}y_t + \mathcal{D}(x) \\ y_0 &= q_0 \end{cases}$$

for

$$\mathcal{D}(x) = -\frac{1}{2} \sum_{i \in A, C} \gamma_i^2(x) - \sum_{i \in C} \gamma_i(x) v_i(x). \quad (35)$$

Hence by Definition 3.4, we have ⁸

$$\begin{aligned} e^{\mu t} q_0(x) &= e^{\mathbf{F}t} q_0(x) + \int_0^t e^{\mathbf{F}(t-s)} \mathcal{D}(x) ds \\ &= q_0(x+t) + \int_0^t \mathcal{D}(x+t-s) ds. \end{aligned}$$

To obtain the remaining operators we solve

$$\begin{cases} \frac{dy_t}{dt} &= \mathbf{F}^k \gamma_i \\ y_0 &= y \end{cases}.$$

Because γ does not depend on t , the solution is

$$e^{\mathbf{F}^k \gamma_i(x) t} y = y + \mathbf{F}^k \gamma_i(x) t.$$

It follows that

$$G(z_0, z_{i,k} ; \quad i \in A, C \quad k = 0, 1, \dots, n_i) \quad (36)$$

$$\begin{aligned} &= \prod_{i \in A, C; k=0, \dots, n_i} \left(e^{\mathbf{F}^k \gamma_i(x) z_{i,k}} \right) e^{\mu(q,x) z_0} q_0 \\ &= q_0(x+z_0) + \int_0^{z_0} \mathcal{D}(x+z_0-s) ds + \sum_{i \in A, C} \sum_{k=0}^{n_i} \mathbf{F}^k \gamma_i(x) z_{i,k}. \end{aligned} \quad (37)$$

⁸From the context, it is clear that $e^{\mathbf{F}t} : \mathcal{H}_q \rightarrow \mathcal{H}_q$. From the usual series expansion of the exponential function we have, $e^{\mathbf{F}t} f = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{F}^n f$. In our case, $\mathbf{F}^n = \frac{\partial^n}{\partial x^n}$, so we have $[e^{\mathbf{F}t} f](x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n f}{\partial x^n}(x)$, which is a Taylor expansion of f around the point x , so for analytic f we have $[e^{\mathbf{F}t} f](x) = f(x+t)$.

Note that the volatility of interest rates show up (through the terms v_i for $i \in C$) only in the deterministic term \mathcal{D} defined in (35).

We are now interested in finding a set of factors Z such that

$$q_t = G(Z_t)$$

while Z is given by a strong solution to the SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases}.$$

For that we need to find a solution to

$$G_* a = \mu \quad G_* b^i = \gamma_i \quad i \in A, C.$$

Simple computations yields

$$\begin{aligned} G'(z_0, z_{j,k} \ ; \ j \in A, C \ k = 0, 1, \dots, n_j) & \begin{pmatrix} h_0 \\ h_{1,0} \\ h_{1,1} \\ \vdots \\ h_{m,n_m} \end{pmatrix} (x) \\ &= \left[\frac{\partial q_0}{\partial z_0}(x + z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x + z_0 - s) ds \right] h_0 + \sum_{i \in A, C} \sum_{k=0}^{n_j} \mathbf{F}^k \gamma_i(x) h_{i,k}. \end{aligned}$$

We can now use the fact that $q = G(Z)$ and that γ satisfies the ODE (33) to get

$$\begin{aligned} \mu(q_t, x) &= \mathbf{F} q_t + \mathcal{D}(x) \\ &= \frac{\partial}{\partial x} q_0(x + z_0) + \int_0^{z_0} \frac{\partial}{\partial x} \mathcal{D}(x + z_0 - s) ds + \sum_{j \in A, C} \sum_{k=0}^{n_j} \mathbf{F}^{k+1} \gamma_j(x) z_{j,k} + \mathcal{D}(x) \\ &= \frac{\partial}{\partial x} q_0(x + z_0) + \int_0^{z_0} \frac{\partial}{\partial x} \mathcal{D}(x + z_0 - s) ds + \mathcal{D}(x) + \sum_{j \in A, C} \sum_{k=1}^{n_j} \mathbf{F}^k \gamma_j(x) z_{j,k-1} \\ &\quad + \sum_{j \in A, C} \mathbf{F}^{n_j+1} \gamma_j(x) z_{j,n_j} \\ &= \frac{\partial}{\partial x} q_0(x + z_0) + \int_0^{z_0} \frac{\partial}{\partial x} \mathcal{D}(x + z_0 - s) ds + \mathcal{D}(x) + \sum_{j \in A, C} \sum_{k=1}^{n_j} \mathbf{F}^k \gamma_j(x) z_{j,k-1} \\ &\quad + \sum_{j \in A, C} \sum_{k=0}^{n_j} c_{j,k} \mathbf{F}^k \gamma_j(x) z_{j,n_j}. \end{aligned}$$

Thus, from $G_\star a = \mu$, we get

$$\begin{aligned}
& \left[\frac{\partial q_0}{\partial z_0}(x + z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x + z_0 - s) ds \right] a_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j} \mathbf{F}^k \gamma_j(x) a_{j,k} \\
&= \frac{\partial}{\partial x} q_0(x + z_0) + \int_0^{z_0} \frac{\partial}{\partial x} \mathcal{D}(x + z_0 - s) ds + \mathcal{D}(x) + \sum_{j \in A, C} \sum_{k=1}^{n_j} \mathbf{F}^k \gamma_j(x) z_{j,k-1} \\
&+ \sum_{j \in A, C} \sum_{k=0}^{n_j} c_{j,k} \mathbf{F}^k \gamma_j(x) z_{j,n_j}
\end{aligned}$$

and, since the above expression should hold for all x , while a is not allowed to depend on x , it is possible to identify the following expressions for a

$$\begin{aligned}
a_0 &= 1 \\
a_{j,0} &= c_{j,0} z_{j,n_j}, & j \in A, C \\
a_{j,k} &= c_{j,k} z_{j,n_j} + z_{j,k-1} & j \in A, C \quad k = 1, \dots, n_j.
\end{aligned}$$

Likewise, from $G_\star b^i = \gamma_i$ for each $i \in A, C$ we get

$$\left[\frac{\partial q_0}{\partial z_0}(x + z_0) + \mathcal{D}(x) + \int_0^{z_0} \frac{\partial \mathcal{D}}{\partial z_0}(x + z_0 - s) ds \right] b_0 + \sum_{i \in A, C} \sum_{k=0}^{n_j} \mathbf{F}^k \gamma_j(x) b_{j,k}^i = \gamma_i(x)$$

and by simple identification of terms

$$\begin{aligned}
b_0^i &= 0 & i \in A, C \\
b_{j,0}^i &= 1 & i = j \quad i, j \in A, C \\
b_{j,k}^i &= 0 & \text{for all other } j \text{ and } k.
\end{aligned}$$

In this totally deterministic setting, the Stratonovich and the Itô dynamics are equivalent, so we have proved the following result.

Proposition 5.2 *Given the initial forward price curve q_0 , the forward prices system generated by γ_A , γ_C and σ_C has a finite dimensional realization given by*

$$q_t = G(Z_t)$$

where G is defined as in (37) and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0 &= dt \\ dZ_{j,0} &= c_{j,0} Z_{j,n_j} dt + dW_t^j & j \in A, C \\ dZ_{j,k} &= (c_{j,k} Z_{j,n_j} + Z_{j,k-1}) dt & j \in A, C \quad k = 1, \dots, n_j \end{cases}$$

We first take the easiest example: the one-dimensional deterministic constant volatility.

Example 3 *Assume that forward prices q are driven by a one-dimensional Wiener process that also drive the interest rates r ($C = \{1\}$). Furthermore, assume that the forward price volatility γ is of the following form*

$$\gamma(x) = \gamma_C(x) = \alpha$$

where $\alpha \in \mathbb{R}$.

We leave the (scalar) function $\sigma_C(x)$ (thus $v_C(x)$) unspecified to stress the point that it plays no role determining the dimension of the Lie-algebra or constructing the realization.

Then we know $\mathbf{F}\gamma = 0$, thus $n = 1$, $c_{1,0} = 0$ and the dimension of the $\{\mu, \gamma\}_{LA}$ is two.

The invariant manifold is given by

$$G(z_0, z_{1,0}) = q_0(x + z_0) - \frac{1}{2}\alpha^2 z_0 - \alpha \int_0^{z_0} v_C(x + z_0 - s) ds + \alpha z_{1,0}$$

for some deterministic function v_C .

Using Proposition 5.2, we have $q_t = G(Z_t)$ for the state variable $Z = \begin{bmatrix} Z_0 \\ Z_{1,0} \end{bmatrix}$ with dynamics given by

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0} &= dW_t \end{cases}$$

We now recover again Example 1.

Example 1 (Cont.) Recall that we assumed

$$\gamma(x) = \gamma_C(x) = \alpha e^{-ax}$$

which implies $n_1 = 0$, $c_{1,0} = -a$.

Once again we leave σ_C (thus v_C) as an unspecified deterministic function.

In the previous comments it was explained that the Lie-algebra is of dimension 2, so the invariant manifold can be obtained from (37),

$$G(z_0, z_{1,0}) = q_0(x + z_0) + \int_0^{z_0} \mathcal{D}(x + z_0 - s) ds + \gamma(x) z_{1,0}.$$

In this case

$$\begin{aligned} \mathcal{D}(x) &= -\frac{1}{2}\gamma_C^2(x) - \gamma_C(x)v_C(x) \\ &= -\frac{1}{2}\alpha^2 e^{-2ax} - \alpha e^{-ax}v_C(x). \end{aligned}$$

Thus,

$$\begin{aligned} G(z_0, z_{1,0}) &= q_0(x + z_0) - \int_0^{z_0} \left(\frac{1}{2}\alpha^2 e^{-2a(x+z_0-s)} + \alpha e^{-a(x+z_0-s)} v_C(x + z_0 - s) \right) ds + \alpha e^{-ax} z_{1,0} \\ &= q_0(x + z_0) + \frac{1}{2} \frac{\alpha^2}{2a} \left[e^{-2ax} - e^{-2a(x+z_0)} \right] - \alpha \int_0^{z_0} \left(e^{-a(x+z_0-s)} v_C(x + z_0 - s) \right) ds \\ &\quad + \alpha e^{-ax} z_{1,0} \end{aligned}$$

and from Proposition 5.2 it follows that the FDR is given by

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0} &= -aZ_{1,0}dt + dW_t \end{cases}.$$

5.2 Deterministic Direction Volatility

We now deal with the second simplest case, that of having *deterministic direction* forward prices volatilities.

Then, we have the following special functional forms for γ_A and γ_C in Assumption 5.1.

$$\gamma_i(q, x) = \lambda_i(x)\varphi_i(q) \quad i \in A, C$$

where λ_i is a deterministic function of x (constant vector field in \mathcal{H}_q) and φ_i is a scalar vector field in \mathcal{H}_q (i.e., it does not depend on x and depends only on the current forward price curve).

Omitting the x -dependence,

$$\gamma_i(q) = \lambda_i\varphi_i(q) \quad i \in A, C \quad (38)$$

and for future reference we also define

$$\lambda = [\lambda_A, \quad 0, \quad \lambda_C]. \quad (39)$$

On what concerns interest rate volatilities, we maintain the requirement that σ_C is deterministic (since we are still dealing with Markovian forward prices).

In this particular case, the forward price equation can be rewritten as

$$dq_t = \mu(q_t)dt + \gamma(q_t) \circ dW_t \quad (40)$$

for

$$\mu(q_t) = \mathbf{F}q_t - \frac{1}{2} \left[\|\gamma_A(q_t)\|^2 + \|\gamma_C(q_t)\|^2 \right] - \gamma_C(q_t)v_C^* - \frac{1}{2} [\gamma'_A(q_t)\gamma_A(q_t) + \gamma'_C(q_t)\gamma_C(q_t)] \quad (41)$$

$$\gamma(q_t) = [\gamma_A(q_t) \quad 0 \quad \gamma_C(q_t)],$$

and given the functional form for γ_A and γ_C in (38) we have the following Frechet derivatives

$$\gamma'_A(q_t)\gamma_A(q_t) = \sum_{i \in A} \lambda_i \varphi'_i(q_t) [\lambda_i \varphi_i(q_t)] \quad \gamma'_C(q_t, x)\gamma_C(q_t, x) = \sum_{i \in C} \lambda_i(x) \varphi'_i(q_t) [\lambda_i \varphi_i(q_t)].$$

We see that μ in (41) is much more complex than the one previously studied (compare to (30)), so, the task of actually computing the Lie algebra $\mathcal{L} = \{\mu, \gamma\}_{LA}$ will not be as straightforward as before.

Using the specific functional forms of γ_A and γ_C in (38) we have

$$\begin{aligned} \mu(q) &= \mathbf{F}q - \frac{1}{2} \sum_{i \in A, C} \underbrace{\lambda_i^2}_{D_i} \underbrace{\varphi_i^2(q)}_{\phi_i(q)} - \sum_{i \in C} \overbrace{\varphi_i(q)}^{\text{scalar field}} \underbrace{\lambda_i v_i}_{V_i} - \frac{1}{2} \sum_{i \in A, C} \overbrace{\varphi'_i(q) [\lambda_i \varphi_i(q)]}_{\text{scalar field}} \lambda_i \\ \gamma_i(q) &= \lambda_i \overbrace{\varphi_i(q)}^{\text{scalar field}} \end{aligned}$$

and, given the possibility of Gaussian elimination (Lemma 3.2), we see that the Lie algebra is in fact generated by the simpler system of vector fields,

$$\begin{aligned} f_0(q) &= \mathbf{F}q - \frac{1}{2} \sum_{k \in A, C} D_k \phi_k(q) - \sum_{k \in C} \varphi_k(q) V_k \\ f_{1_i}(q) &= \lambda_i \quad i \in A, C. \end{aligned}$$

where

$$D_k = \lambda_k^2 \quad V_k = \lambda_k v_k.$$

We now start computing Lie brackets and simplifying. For all $i \in A, C$ we have

$$\begin{aligned} [f_0, f_{1i}] &= f_0' f_{1i} - f_{1i}' f_0 \\ &= \mathbf{F} \lambda_i - \frac{1}{2} \sum_{k \in A, C} D_k \underbrace{\phi_k'[\lambda_i]}_{\text{scalar field}} - \sum_{k \in C} V_k \underbrace{\varphi_k'[\lambda_i]}_{\text{scalar field}} \\ &= f_{2i} \end{aligned}$$

and using this new field we have for all $i, j \in A, C$

$$\begin{aligned} [f_{1i}, f_{2j}] &= f_{1i}' f_{2j} - f_{2j}' f_{1i} \\ &= +\frac{1}{2} \sum_{k \in A, C} D_k \underbrace{\phi_k''[\lambda_j; \lambda_i]}_{\text{scalar field}} + \sum_{k \in C} V_k \underbrace{\varphi_k''[\lambda_j; \lambda_i]}_{\text{scalar field}} \\ &= f_{3ij}. \end{aligned}$$

We know realize that the our lie algebra is hard to handle (even in the one-dimensional Wiener process case⁹). At this point it seems a good idea to note the following.

Remark 5.3 *The Lie-algebra*

$$\mathcal{L} = \{\mu, \gamma\}_{LA} \quad (42)$$

is included in the larger Lie-algebra

$$\bar{\mathcal{L}} = \{\mathbf{F}, \lambda_i, D_i, V_j; \quad i \in A, C \quad j \in C\}_{LA}. \quad (43)$$

That is

$$\{\mu, \gamma\}_{LA} = \{f_0, f_{1i}; \quad i \in A, C\}_{LA} \subseteq \{\mathbf{F}, \lambda_i, D_i, V_j; \quad i \in A, C \quad j \in C\}_{LA}.$$

There are three important points to make here.

- The fields in the larger Lie-algebra, $\bar{\mathcal{L}}$, are simpler than those in \mathcal{L} . That is, none of the field contains sums.
- From the inclusion $\mathcal{L} \subseteq \bar{\mathcal{L}}$ it is obvious that if $\bar{\mathcal{L}}$ has finite dimension also \mathcal{L} does. So, studying the conditions that guarantee $\bar{\mathcal{L}}$ to have finite dimension, give us, at least, *sufficient* conditions for \mathcal{L} to have also finite dimension.
- We conjecture that conditions that guarantee $\bar{\mathcal{L}}$ to have finite dimension are also *necessary* conditions for \mathcal{L} to have also finite dimension. The intuition is that since the fields in $\bar{\mathcal{L}}$ are all contained in the fields of \mathcal{L} (as parcels of various sums), if they are “nasty” enough to make the dimension of $\bar{\mathcal{L}}$ infinite, they should make the fields that contain them in \mathcal{L} even “nastier”. We will formalize this intuition below.

⁹We note once again, that if we consider the Wiener process to be one-dimensional, the only interesting case to consider is when that Wiener process drives both forward prices and interest rates, i.e. it belongs to the W^C set. Otherwise, we fall into the futures case already studied in [3]. So, even in that case, we cannot avoid having two parcels not easy to simplify.

- Finally, even if the analysis of $\bar{\mathcal{L}}$ is, in the above sense, equivalent to the analysis of \mathcal{L} , in the construction sense, studying $\bar{\mathcal{L}}$ will, in principle, generate finite realizations with state variables of higher dimension. This is obviously the price one has to pay for dealing with easier fields. We call these realizations *non-minimal realizations*. An advantage of *non-minimal* realizations is that they are always possible to obtain (as long as the dimension of $\bar{\mathcal{L}}$ is finite).

The following conjecture formally states the idea behind our third point above (and the sketch of the proof, for the one-dimensional case, can be found in the appendix).

Conjecture 5.1 *Consider \mathcal{L} in (42) and $\bar{\mathcal{L}}$ in (43). Then the following holds*

$$\dim(\mathcal{L}) < \infty \quad \Leftrightarrow \quad \dim(\bar{\mathcal{L}}) < \infty .$$

These ideas can be applied in a more complex setting. They will be used extensively in Section 6, when dealing with the entire forward price system.

We now continue our analysis studying the larger Lie-algebra $\bar{\mathcal{L}}$.

5.2.1 Existence of FDR

As mentioned before, in the current deterministic direction setting, the larger Lie-algebra, $\bar{\mathcal{L}}$, is given by

$$\bar{\mathcal{L}} = \{\mathbf{F}, \lambda_i, D_i, V_j ; \quad i \in A, C \quad j \in C\}_{LA} .$$

thus, the basic fields of the enlarged Lie-algebra are

$$\begin{aligned} g_0(q) &:= \mathbf{F}q \\ g_{1i}(q) &:= \lambda_i && i \in A, C \\ g_{2i}(q) &:= D_i && i \in A, C = 1, \dots, m \\ g_{3j}(q) &:= V_j && j \in C \end{aligned}$$

Computing the Lie-brackets we have, in the first step,

$$\begin{aligned} [g_0, g_{1i}] &= \mathbf{F}\lambda_i =: g_{4i} \\ [g_0, g_{2i}] &= \mathbf{F}D_i =: g_{5i} \\ [g_0, g_{3j}] &= \mathbf{F}V_j =: g_{6j} \end{aligned}$$

all remaining combinations of lie-brackets from the fields in (43) are zero. Using the new vector fields, we easily see that

$$\begin{aligned} [g_0, g_{4i}] &= \mathbf{F}^2\lambda_i \\ [g_0, g_{5i}] &= \mathbf{F}^2D_i \\ [g_0, g_{6j}] &= \mathbf{F}^2V_j \end{aligned}$$

and again all others lie-brackets are zero.

Continuing with similar iterations, it is easy to check that

$$\bar{\mathcal{L}} = \text{span} \{ \mathbf{F}, \mathbf{F}^k \lambda_i, \mathbf{F}^k D_i, \mathbf{F}^k V_j ; \quad i \in A, C \quad j \in C \quad k = 0, 1, \dots \}.$$

Thus, for $\dim(\bar{\mathcal{L}}) < \infty$ there must exist orders n_j^1 , n_j^2 and n_j^3 for which

$$\begin{aligned} \mathbf{F}^{n_j^1+1} \lambda_j &= \sum_{k=0}^{n_j^1} c_{j,k}^1 \mathbf{F}^k \lambda_j & \mathbf{F}^{n_j^2+1} D_j &= \sum_{k=0}^{n_j^2} c_{j,k}^2 \mathbf{F}^k D_j & j \in A, C \\ \mathbf{F}^{n_j^3+1} V_j &= \sum_{k=0}^{n_j^3} c_{j,k}^3 \mathbf{F}^k V_j & & & j \in C \end{aligned}$$

where $c_{j,k}^I$ are real constants, $k = 0, \dots, n_j^I$, $j \in A, C$, and $I = 1, 2, 3$.

Indeed, if (44) holds,

$$\dim(\bar{\mathcal{L}}) \leq 1 + \sum_{j \in A, C} (n_j^1 + 1) + \sum_{j \in C} (n_j^2 + 1) + \sum_{j \in C} (n_j^3 + 1) < \infty. \quad (44)$$

We now start a brief side discussion on the exact dimension of $\bar{\mathcal{L}}$. As before, we note that the sign “ \leq ” above results from the possibility of Gaussian elimination across the various derivatives of various different fields. This is particularly relevant for the concrete fields of $\bar{\mathcal{L}}$ since $V_j = \lambda_j v_j$ and thus derivatives of λ_j may help to Gaussian simplify the derivatives of V_j . This point will be made clear in Example 4 below.

Unfortunately, these simplifications are instance dependent and, thus, in an abstract way it is impossible to be more exact about the dimension of $\bar{\mathcal{L}}$ than in (44). The consequence is that when doing the abstract construction of realizations we cannot take into account case-specific Gaussian eliminations. Therefore, when applying the abstract results to concrete models, we may get unnecessarily large realizations. However, since we are considering non-minimal realizations anyway (because we analyze $\bar{\mathcal{L}}$ instead of \mathcal{L}), this does not seem a major disadvantage¹⁰. With this we conclude the side discussion and go on with the abstract analysis.

Proposition 5.3 give us the necessary and sufficient conditions that guarantee $\dim(\bar{\mathcal{L}}) < \infty$.

Proposition 5.3 *The dimension of the Lie-algebra $\bar{\mathcal{L}}$ in (43) is finite if and only if each component of λ and σ_C is QE.*

Proof. It follows from Lemma 3.3 that (44) holds if and only if λ_i , D_i for $i \in A, C$ and V_j for $j \in C$ are QE functions. It also follows from the properties of QE functions that, λ_i for $i \in A, C$ and σ_j for $j \in C$ QE, suffices to guarantee this requirement. To check the last statement note that λ_i QE $\Rightarrow D_i = \lambda_i^2$ QE; σ_j QE, $\Rightarrow v_j(x) = -\int_0^x \sigma_j(s) ds$ QE; and finally λ_j and v_j QE $\Rightarrow V_j = \lambda_j v_j$ QE. \square

Taking together Propositions 5.3 and Conjecture 5.1 we have the following general result for forward prices with deterministic direction volatilities.

¹⁰The unsatisfied reader can always, when faced with a concrete situation, use the techniques presented and, whenever possible, derive a smaller realization instead of using the abstract results. In Example 4 below, we use both approaches to exemplify the kind of difference one can expect.

Proposition 5.4 *Assume that Conjecture 5.1 holds. The (logarithm of) forward price equation (40) admits a finite dimensional realization if and only if each component of λ and σ_C are quasi-exponential.*

We note that, in contrast to the deterministic forward price volatilities case, in our present setting of deterministic direction forward price volatilities, existence of FDR imposes requirements on the concrete functional form of the deterministic function σ_C : it must be a QE function. This was to be expected, since this time, σ_C actually drives the forward price equation indirectly through the fields $V_j = \lambda_j v_j$ for $j \in C$.

The following example gives us one, very simple instance where we have a finite dimensional realization for forward prices, considering deterministic direction volatilities.

Example 4 *Suppose that forward prices are driven by a one-dimensional Wiener process ($i = 1$) that also drives interest rates ($i \in C$) and that*

$$\gamma(q, x) = \gamma_C(q, x) = \underbrace{\alpha e^{-bx}}_{\lambda(x)} \underbrace{q}_{\varphi(q)} \quad \sigma_C(x) = \delta e^{-ax}.$$

for $\alpha, \beta, a, b \in \mathbb{R}$.

Then we have

$$\begin{aligned} v(x) &= -\delta \int_0^x e^{-as} ds = \frac{\delta}{a} (e^{-ax} - 1) \\ \lambda(x) &= \alpha e^{-bx} \\ D(x) &= \lambda^2(x) = \alpha^2 e^{-2bx} \\ V(x) &= \lambda(x)v(x) = \alpha e^{-bx} \frac{\delta}{a} (e^{-ax} - 1) = \frac{\alpha\delta}{a} [e^{-(a+b)x} - e^{-bx}] \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}\lambda(x) &= -b\alpha e^{-bx} = -b\lambda(x) & \Rightarrow & n_1^1 = 0 & c_{1,0}^1 &= -b \\ \mathbf{F}D(x) &= -2b\alpha^2 e^{-2bx} = -2bD(x) & \Rightarrow & n_1^2 = 0 & c_{1,0}^2 &= -2b \\ \mathbf{F}V(x) &= \frac{\alpha\delta}{a} [-(a+b)e^{-(a+b)x} - (-b)e^{-bx}] \\ &= -bV(x) - \alpha\delta e^{-(a+b)x} \\ \mathbf{F}^2V(x) &= -b\mathbf{F}V(x) + (a+b)\alpha\delta e^{-(a+b)x} \\ &= -b\mathbf{F}V(x) + (a+b)\alpha\delta [-bV(x) - \mathbf{F}V(x)] \\ &= -(a+2b)\mathbf{F}V(x) - (a+b)bV(x) & \Rightarrow & n_1^3 = 1 & c_{1,0}^3 &= -(a+b)b \\ & & & & c_{1,1}^3 &= -(a+2b) \end{aligned}$$

so we have

$$\{\mu, \gamma\}_{LA} \subseteq \{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA}.$$

Thus, $\dim \{\mu, \gamma\}_{LA} \leq \dim \{\mathbf{F}, \lambda, D, V\}_{LA} \leq 5$.

Alternatively, we may note that

$$V(x) = \frac{\alpha\delta}{a} [e^{-(a+b)x} - e^{-bx}] = \frac{\alpha\delta}{a} e^{-(a+b)x} - \frac{\delta}{a} \underbrace{\alpha e^{-bx}}_{\lambda(x)}$$

thus, using Gaussian elimination we can substitute V by $\tilde{V}(x) = e^{-(a+b)x}$. And since

$$\mathbf{F}\tilde{V}(x) = -(a+b)e^{-(a+b)x} = -(a+b)\tilde{V}(x) \quad \Rightarrow \quad \tilde{n}_1^3 = 0 \quad \tilde{c}_{1,0}^3 = -(a+b)$$

and we can compute the exact dimension of $\{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA}$,

$$\dim \{\mathbf{F}, \lambda, D, V, \mathbf{F}V\}_{LA} = \dim \{\mathbf{F}, \lambda, D, \tilde{V}\}_{LA} = 4.$$

Obviously, in either way we are able to conclude that the forward prices admit a FDR.

5.2.2 Construction of FDR

We would now like to construct a FDR for forward prices, whenever we know that it exists. Since we are using the larger Lie-algebra $\tilde{\mathcal{L}}$ and we cannot use case-specific Gaussian elimination in the general case, we are aiming to get *non-minimal* finite realizations.

As before we would like to derive a parameterization \bar{G} of the invariant manifold $\bar{\mathcal{G}}$ and infer, from the functional form of that parameterization, the dynamics of the state variables.

Given the simplicity of the fields spanning $\tilde{\mathcal{L}}$, it is straightforward to compute the operators:

$$\begin{aligned} e^{\mathbf{F}z_0}q_0 &= q_0(x + z_0), \\ e^{\mathbf{F}^k \lambda_j z_{j,k}^1}q &= q + \mathbf{F}^k \lambda_j z_{j,k}^1, & e^{\mathbf{F}^k D_j z_{j,k}^2}q &= q + \mathbf{F}^k D_j z_{j,k}^2 & j \in A, C; k = 0, 1, \dots, n_j \\ e^{\mathbf{F}^k V_j z_{j,k}^3}q &= q + \mathbf{F}^k V_j z_{j,k}^3, & & & j \in C; k = 0, 1, \dots, n_j. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{G}(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3) &= \prod_{j \in A, C} e^{\mathbf{F}^k \lambda_j(x) z_{j,k}^1} \cdot \prod_{j \in A, C} e^{\mathbf{F}^k D_j(x) z_{j,k}^2} \cdot \prod_{j \in C} e^{\mathbf{F}^k V_j(x) z_{j,k}^3} \\ &= q_0(x + z_0) + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j z_{j,k}^2 \\ &\quad + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j z_{j,k}^3. \end{aligned} \tag{45}$$

Hence, in order to find the dynamics of the state variable Z such that $q = \bar{G}(Z)$, and as in the previous section, we take the Stratonovich Q -dynamics to be given by

$$\begin{cases} dZ_t &= \bar{a}(Z_t)dt + \bar{b}(Z_t) \circ dW_t \\ Z_0 &= z_0 \end{cases}, \tag{46}$$

and we solve

$$\bar{G}_* \bar{a} = \mu \quad \bar{G}_* \bar{b} = \gamma \tag{47}$$

with μ and γ given in (38)-(41), and \bar{G} from (45), to get a strong solution to the SDE (46).

The steps are then the usual ones, but with much more messy computations. Using the functional form of \bar{G} in (45), it is possible to compute the Frechet derivatives. Then, from the expression for $q = \bar{G}(Z)$ and equations (44), we can find an concrete expression to the term $\mathbf{F}q$

in our μ (equation (41)). Identification of term in equations (47) allows us to determine the Stratonovich drift and diffusion terms. And finally, to obtain the Itô dynamics we calculate the Stratonovich correction term. Following these steps gives us the result in Proposition 5.5 (the actual computations of can be found in the appendix).

Proposition 5.5 *Given the initial forward price curve q_0 , the forward prices system generated by γ_A, γ_C as in (38) and σ_C deterministic, has a finite dimensional realization given by*

$$q_t = \bar{G}(Z_t)$$

where \bar{G} is defined as in (45) and the dynamics of the state space variables Z are given by

$$\left\{ \begin{array}{ll} dZ_0 & = dt \\ dZ_{j,0}^1 & = c_{j,0}^1 Z_{j,n_j^1}^1 dt + \varphi(\bar{G}(Z)) dW_t^j & j \in A, C \\ dZ_{j,k}^1 & = \left(Z_{j,k-1}^1 + c_{j,k}^1 Z_{j,n_j^1}^1 \right) dt & j \in A, C \quad k = 1, \dots, n_j^1 \\ dZ_{j,0}^2 & = \left(c_{j,0}^2 Z_{j,n_j^2}^2 - \frac{1}{2} \varphi_j^2(\bar{G}(Z)) \right) dt & j \in A, C \\ dZ_{j,k}^2 & = \left(Z_{j,k-1}^2 + c_{j,k}^2 Z_{j,n_j^2}^2 \right) dt & j \in A, C \quad k = 1, \dots, n_j^2 \\ dZ_{j,0}^3 & = \left(c_{j,0}^3 Z_{j,n_j^3}^3 + \varphi(\bar{G}(zZ)) \right) dt & j \in C \\ dZ_{j,k}^3 & = \left(Z_{j,k-1}^3 + c_{j,k}^3 Z_{j,n_j^3}^3 \right) dt & j \in C \quad k = 1, \dots, n_j^3 \end{array} \right.$$

Example 4 (cont.) *Recall that we studied an one-dimensional model where $A \cap B = \emptyset$, $C = \{1\}$ and*

$$\gamma(q, x) = \gamma_C(q, x) = \underbrace{\alpha e^{-bx}}_{\lambda(x)} \underbrace{q}_{\varphi(q)} \quad \sigma_C(x) = \delta e^{-ax}.$$

As in the first part of the example, we will first directly apply the abstract results (in the construction part, that is Proposition 5.5). Then we derive a smaller realization that can be obtained from the case-specific simpler fields.

We start by directly applying Proposition 5.5. Recall from previous computations that we had $n_1^1 = 0$, $c_{1,0}^1 = -b$, $n_1^2 = 0$, $c_{1,0}^2 = -2b$, $n_1^3 = 1$, $c_{1,0}^3 = -(a+b)b$ and $c_{1,1}^3 = -(a+2b)$ and

$$V(x) = \frac{\alpha\delta}{a} \left[e^{-(a+b)x} - e^{-bx} \right] \quad \mathbf{FV}(x) = \frac{\alpha\delta}{a} \left[-(a+b)e^{-(a+b)x} - (-b)^k e^{-bx} \right].$$

Using this we get from (45) the parameterization of the realization to be

$$\begin{aligned} \bar{G}(z_0, z_{1,0}^1, z_{1,0}^2, z_{1,0}^3, z_{1,1}^3) &= q_0(x + z_0) + \lambda(x) z_{1,0}^1 + D(x) z_{1,0}^2 + V(x) z_{1,0}^3 + \mathbf{FV}(x) z_{1,1}^3 \\ &= q_0(x + z_0) + \alpha e^{-bx} z_{1,0}^1 + \alpha^2 e^{-2bx} z_{1,0}^2 + \frac{\alpha\delta}{a} \left[e^{-(a+b)x} - e^{-bx} \right] z_{1,0}^3 \\ &\quad + \frac{\alpha\delta}{a} \left[-(a+b)e^{-(a+b)x} - (-b)^k e^{-bx} \right] z_{1,1}^3. \end{aligned}$$

We now note that in our case $\varphi(q) = q$ and by Proposition 5.5 it follows that the realization is

$$\begin{cases} dZ_0 &= dt \\ dZ_{1,0}^1 &= -bZ_{1,0}^1 dt + \bar{G}(Z) dW_t \\ dZ_{1,0}^2 &= \left(-2bZ_{1,0}^2 - \frac{1}{2}(\bar{G}(Z))^2\right) dt \\ dZ_{1,0}^3 &= \left(-b(a+b)Z_{1,1}^3 + \bar{G}(Z)\right) dt \\ dZ_{1,1}^3 &= \left(Z_{1,0}^3 - (a+2b)Z_{1,1}^3\right) dt \end{cases}$$

Now recall that for this particular model we have $\bar{\mathcal{L}} = \{\mathbf{F}, \lambda, D, \tilde{V}\}_{LA}$ with $\tilde{V}(x) = e^{-(a+b)x}$.

Thus another (smaller) parameterization, \tilde{G} is given by

$$\begin{aligned} \tilde{G}(\tilde{z}_0, \tilde{z}_{1,0}^1, \tilde{z}_{1,0}^2, \tilde{z}_{1,0}^3) &= q_0(x + \tilde{z}_0) + \lambda(x)\tilde{z}_{1,0}^1 + D(x)\tilde{z}_{1,0}^2 + \tilde{V}(x)\tilde{z}_{1,0}^3 \\ &= q_0(x + \tilde{z}_0) + \alpha e^{-bx}\tilde{z}_{1,0}^1 + \alpha^2 e^{-2bx}\tilde{z}_{1,0}^2 + e^{-(a+b)x}\tilde{z}_{1,0}^3. \end{aligned}$$

Proposition 5.5 cannot be used directly, but we can compare the two parameterizations \bar{G} and \tilde{G} (of the same invariant manifold) above, to get

$$\tilde{z}_0 = z_0 \quad \tilde{z}_{1,0}^1 = z_{1,0}^1 - \frac{\delta}{a}z_{1,0}^3 + b\frac{\delta}{a}z_{1,1}^3 \quad \tilde{z}_{1,0}^2 = z_{1,0}^2 \quad \tilde{z}_{1,0}^3 = \frac{\alpha\delta}{a}z_{1,0}^3 - \frac{\alpha\delta}{a}(a+b)z_{1,1}^3.$$

Finally using Itô and simplifying we have

$$\begin{cases} d\tilde{Z}_0 &= dt \\ d\tilde{Z}_{1,0}^1 &= \left(-b\tilde{Z}_{1,0}^1 - \frac{\delta}{a}\tilde{G}(Z)\right) dt + \tilde{G}(Z) dW_t \\ d\tilde{Z}_{1,0}^2 &= \left(-2b\tilde{Z}_{1,0}^2 - \frac{1}{2}(\tilde{G}(Z))^2\right) dt \\ d\tilde{Z}_{1,0}^3 &= \left(-(a+b)\tilde{Z}_{1,0}^3 + \frac{\alpha\delta}{a}\tilde{G}(Z)\right) dt \end{cases}$$

The two realizations are equivalent.

5.3 The General Case

In the sections 5.1 and 5.2, we analyzed existence and construction of FDR of Markovian forward prices, under the specific setting of *deterministic* and *deterministic direction* forward price volatility.

A natural question at this point is: *what is the most general functional form, for the volatilities γ and σ , consistent with FDR of Markovian forward prices?* The answer follows from previous results in [10] and from . In the following proposition we adapt it to the Markovian forward prices case. We state it in the form of a proposition.

Proposition 5.6 *Suppose Assumption 2.1 holds. There exist a FDR of Markovian forward*

prices if and only if

$$\begin{aligned}\gamma_j(q, r, x) &= \sum_{k=0}^{N_\gamma^j} \varphi_k^j(q) \lambda_k^j(x) & j \in A, C \\ \sigma_j(r, x) &= \begin{cases} \beta_j(x) & \text{if } \gamma_j(q, r, x) = \gamma_j(x) \\ \delta_j(x) & \text{if } \gamma_j(q, r, x) = \gamma_j(q, x) \end{cases} & j \in C\end{aligned}$$

where β_j are unrestricted deterministic functions, δ_j, λ_k^j are QE deterministic functions, and φ_k^j are scalar vector fields in \mathcal{H}_q .

From Proposition 5.6, we see the most general situation can be attained by extending *deterministic direction* forward price volatilities to finite sums of deterministic direction parcels. This represents, of course, a relevant extension in terms of model flexibility, but not in terms of complexity of analysis.

The results from section 5.2 extend naturally to this most general case, the computations are exactly the same and, in concrete applications, easy to derive. In abstract terms, however, computation get much messier given the additional indices one must keep track of¹¹.

We now will consider the case when forward prices are **not** Markovian.

6 Non-Markovian Forward Prices

Recall that under Assumption 2.1 – our basic assumption on the volatility processes for forward prices and interest rates γ and σ , respectively – the (logarithm of the) forward price curve q cannot, in general, be studied without incorporating in the analysis the interest rate curve r (recall (12)-(13)).

In this section, we want to study the circumstances which were not covered by Section 5. In that case our forward price model is a doubly infinite system and we set

$$\hat{q} = \begin{bmatrix} q \\ r \end{bmatrix}.$$

and \hat{q} belongs to $\mathcal{H}_q \times \mathcal{H}_r$.

The Itô dynamics of \hat{q} can, thus, also be written in block matrix notation as

$$d \begin{bmatrix} q_t \\ r_t \end{bmatrix} = \left\{ \mathbf{F} \begin{bmatrix} q_t \\ r_t \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \|\gamma(q_t, r_t)\|^2 \\ 0 \end{bmatrix} - \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} v^*(r_t) \right\} dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} dW_t.$$

where, as before, we take W to be an m -dimensional Wiener process, and

$$\mathbf{F} = \frac{\partial}{\partial x} \quad v(r, x) = - \int_0^x \sigma(r, s) ds.$$

¹¹We do not present the abstract results and derivations, as we believe the reader would spend more time understanding the notation, than extending the results of section 5.2 to concrete, slightly more general, applications.

In this case the Stratonovich correction term is given by

$$-\frac{1}{2}d\left\langle \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix}, W_t \right\rangle = -\frac{1}{2} \sum_{i=1}^m d\left\langle \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix}, W_t^i \right\rangle$$

Since we have (from the infinite Itô formula)

$$d \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} = (\dots)dt + \begin{bmatrix} \gamma'_{iq}(q, r) & \gamma'_{ir}(q, r) \\ 0 & \sigma'_{ir}(r) \end{bmatrix} \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} dW_t$$

with γ'_q , γ'_r and σ'_r the partial Frechet derivatives.

Then for $i = 1, \dots, m$ we have

$$d\left\langle \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix}, W_t^i \right\rangle = \begin{bmatrix} \gamma'_{iq}(q_t, r_t) & \gamma'_{ir}(q_t, r_t) \\ 0 & \sigma'_{ir}(r_t) \end{bmatrix} \begin{bmatrix} \gamma_i(q_t, r_t) \\ \sigma_i(r_t) \end{bmatrix} dt$$

and the Stratonovich dynamics for \hat{q} ,

$$d\hat{q}_t = \mu(q_t, r_t)dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} \circ dW_t$$

$$\mu(q, r) = \mathbf{F} \begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \|\gamma(q, r)\|^2 \\ 0 \end{bmatrix} - \begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} v^*(r) - \frac{1}{2} \begin{bmatrix} \gamma'_q(q, r) & \gamma'_r(q, r) \\ 0 & \sigma'_r(r) \end{bmatrix} \begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix}.$$

Given that

$$\gamma(q, r) = [\gamma_A(q, r) \quad 0 \quad \gamma_C(q, r)] \quad \sigma(r) = [0 \quad \sigma_B(r) \quad \sigma_C(r)]$$

we note that

$$\begin{aligned} \begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} v(r)^* &= \begin{bmatrix} \gamma_A(q, r) & 0 & \gamma_C(q, r) \\ 0 & \sigma_B(r) & \sigma_C(r) \end{bmatrix} \begin{bmatrix} 0 \\ v_B(r)^* \\ v_C(r)^* \end{bmatrix} \\ &= \begin{bmatrix} \gamma_C(q, r)v_C^*(r) \\ \sigma_B(r)v_C^*(r) + \sigma_C(r)v_C^*(r) \end{bmatrix} \\ \begin{bmatrix} \gamma'_q(q, r) & \gamma'_r(q, r) \\ 0 & \sigma'_r(r) \end{bmatrix} \begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} &= \begin{bmatrix} \gamma'_{Aq}(q, r) & \gamma'_{Ar}(q, r) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_A(q, r) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma'_{Br}(r) \end{bmatrix} \begin{bmatrix} 0 \\ \sigma_B(r) \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma'_{Cq}(q, r) & \gamma'_{Cr}(q, r) \\ 0 & \sigma'_{Cr}(r) \end{bmatrix} \begin{bmatrix} \gamma_C(q, r) \\ \sigma_C(r) \end{bmatrix} \\ &= \begin{bmatrix} \gamma'_{Aq}(q, r)\gamma_A(q, r) + \gamma'_{Cq}(q, r)\gamma_C(q, r) + \gamma'_{Cr}(q, r)\sigma_C(r) \\ \sigma'_{Br}(r)\sigma_B(r) + \sigma'_{Cr}(r)\sigma_C(r) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \gamma'_{Aq}(q, r)\gamma_A(q, r) &= \sum_{i \in A} \gamma'_{Aiq}(q, r)\gamma_{Ai}(q, r), \\ \gamma'_{Cr}(q, r)\sigma_C(r) &= \sum_{i \in C} \gamma'_{Cir}(q, r)\sigma_{Ci}(r), \quad \gamma'_{Cq}(q, r)\gamma_C(q, r) = \sum_{i \in C} \gamma'_{Ciq}(q, r)\gamma_{Ci}(q, r), \\ \sigma'_{Br}(r)\sigma_B(r) &= \sum_{i \in B} \sigma'_{Bir}(r)\sigma_{Bi}(r), \quad \sigma'_{Cr}(r)\sigma_C(r) = \sum_{i \in C} \sigma'_{Cir}(r)\sigma_{Ci}(r). \end{aligned}$$

We can, finally, identify our main object of study as the following doubly infinite Stratonovich SDE

$$d\hat{q}_t = \mu(q_t, r_t)dt + \begin{bmatrix} \gamma(q_t, r_t) \\ \sigma(r_t) \end{bmatrix} \circ dW_t$$

where

$$\begin{aligned} \mu(q, r) &= \mathbf{F} \begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \|\gamma_A(q, r)\|^2 + \|\gamma_C(q, r)\|^2 \\ 0 \end{bmatrix} - \begin{bmatrix} \gamma_C(q, r)v_C^*(r) \\ \sigma_B(r)v_B^*(r) + \sigma_C(r)v_C^*(r) \end{bmatrix} \\ &\quad - \frac{1}{2} \begin{bmatrix} \gamma_{A_q}'(q, r)\gamma_A(q, r) + \gamma_{C_q}'(q, r)\gamma_C(q, r) + \gamma_{C_r}'(q, r)\sigma_C(r) \\ \sigma_{B_r}'(r)\sigma_B(r) + \sigma_{C_r}'(r)\sigma_C(r) \end{bmatrix} \end{aligned} \quad (48)$$

$$\begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} = \begin{bmatrix} \gamma_A(q, r) & 0 & \gamma_C(q, r) \\ 0 & \sigma_B(r) & \sigma_C(r) \end{bmatrix}$$

Given the general functional forms of γ_A , γ_C , σ_B and σ_C the study of all possible special cases¹² would be exhausting.

In this section, we take, therefore a more aggressive strategy: we consider immediately the scenario where each element in γ and σ have **deterministic direction volatility**.

As before, the situation of deterministic direction volatilities can be extended to the case where each element of γ and σ is a finite sum of deterministic direction parcels, and that is the most general possible scenario consistent with existence of FDR¹³. We omit the analysis of this most general scenario because the results can be easily derived from the ones on deterministic direction volatilities, and, in abstract terms, notation becomes almost untractable.

The deterministic direction setting we will work with is formally stated by the next assumption.

Assumption 6.1 *The mappings $\gamma_i : \mathcal{H}_q \times \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $\sigma_i : \mathcal{H}_r \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ are of the following functional form.*

$$\begin{aligned} \gamma_i(x, q, r) &= \lambda_i(x)\varphi_i(q, r) & i \in A, C \\ \sigma_i(x, r) &= \beta_i(x)\phi_i(r) & i \in B, C \end{aligned} \quad (49)$$

where λ_i , β_i are deterministic functions of x and φ_i , ϕ_i are scalar vector fields in $\mathcal{H}_q \times \mathcal{H}_r$ (i.e., they do not depend on x and depend only on the current forward price and interest rate curves).

We note that under Assumption 6.1

$$v_i(x, r) = - \int_0^x \sigma_i(r, s)ds = - \int_0^x \beta_i(s)\phi_i(r)ds = -\phi_i(r) \int_0^x \beta_i(s)ds \quad i \in B, C.$$

Defining

$$B_i(x) = \int_0^x \beta_i(s)ds \quad (50)$$

¹²One particular special case would be to take, say, γ_A and σ_B to have deterministic direction and γ_C , σ_C to be deterministic.

¹³For details on why this is the most general scenario we refer to [10].

we, thus, have

$$v_i(x) = -\phi_i(r)B_i(x) \quad i \in B, C.$$

To check if our forward prices model admits a finite dimensional realization, we need to see if

$$\dim \left\{ \mu, \begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} \right\}_{LA} < \infty.$$

Considering (48), under Assumption 6.1, our basic vector fields can be written as

$$\begin{aligned} \mu(q, r) &= \mathbf{F} \begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2} \sum_{i \in A, C} \varphi_i^2(q, r) \begin{bmatrix} \lambda_i^2 \\ 0 \end{bmatrix} + \sum_{i \in C} \varphi_i(q, r) \phi_i(r) \begin{bmatrix} \lambda_i B_i \\ 0 \end{bmatrix} + \sum_{i \in B, C} \phi_i^2(r) \begin{bmatrix} 0 \\ \beta_i B_i \end{bmatrix} \\ &\quad - \frac{1}{2} \left\{ \sum_{i \in A, C} \varphi'_{iq}(q, r) [\lambda_i] \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} \right. \\ &\quad \left. + \sum_{i \in C} \varphi'_{ir}(q, r) [\beta_i] \phi_i(r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \sum_{i \in B, C} \phi'_{ir}(r) [\beta_i] \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} \right\}, \\ \begin{bmatrix} \gamma_i(q, r) \\ \sigma_i(r) \end{bmatrix} &= \begin{cases} \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} & i \in A \\ \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in B \\ \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in C. \end{cases} \end{aligned}$$

Following the strategy described in Section 3.3, we would now start computing Lie brackets of all possible combinations of these fields and, through Gaussian elimination, hopefully, get to a simple set of generators of our Lie-algebra $\mathcal{L} = \{\mu, \delta\}_{LA}$. Based on properties of these generators we would, also hopefully, be able to understand which γ and σ would guarantee a FDR for forward prices.

The particular complex expression for μ above, and the almost impossibility of Gaussian elimination that results from having to handle two infinite SDE at the same time¹⁴, leads to the conclusion that our best hope is again to study a larger Lie-algebra, $\bar{\mathcal{L}}$, and to choose such a Lie-algebra so that the basic fields would be simple.

The following Lemma give us the desired (simple enough) Lie-algebra $\bar{\mathcal{L}}$.

¹⁴To the usual complexity of dealing with multidimensional cases, there is an additional complexity specific of forward price models that results from the fact that $\varphi_i(q, r) \neq \phi_i(r)$. However even under the unrealistic assumption (since the forward price volatility could not depend on the forward prices) where we would assume $\varphi_i(r) = \phi_i(r)$, the complexity of the Stratonovich correction term would not allow us to obtain simple generators for \mathcal{L} .

Lemma 6.1 Consider the following set of fields in \mathcal{H} .

$$\begin{aligned}
f^0 &= \begin{bmatrix} \mathbf{F}q \\ \mathbf{F}r \end{bmatrix}, \\
f_i^1 &= \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \quad f_i^2 = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \quad i \in A, C \\
f_i^3 &= \begin{bmatrix} V_i \\ 0 \end{bmatrix}, \quad i \in C \\
f_i^4 &= \begin{bmatrix} 0 \\ \beta_i \end{bmatrix}, \quad f_i^5 = \begin{bmatrix} 0 \\ H_i \end{bmatrix} \quad i \in B, C
\end{aligned} \tag{51}$$

where λ_i , β_i and B_i are deterministic functions of x and defined as in (49) and (50) and we further define

$$D_i(x) = \lambda_i^2(x) \quad V_i(x) = \lambda_i(x)B_i(x) \quad H_i(x) = \beta_i(x)B_i(x). \tag{52}$$

Then the following holds

$$\mathcal{L} = \{\mu, \delta\}_{LA} \subseteq \bar{\mathcal{L}} = \left\{ \mathbf{F}, \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \begin{bmatrix} V_j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta_n \end{bmatrix}, \begin{bmatrix} 0 \\ H_n \end{bmatrix}; \quad i \in A, C, j \in C, n \in B, C \right\}_{LA}$$

Proof. First, note that

$$\begin{aligned}
\varphi_i(q, r), \quad \varphi_i^2(q, r), \quad \varphi'_{iq}(q, r) [\lambda_i] \varphi_i(q, r) & \quad i \in A, B \\
\phi_i(r), \quad \phi_i^2(r), \quad \phi'_{ir}(r) [\beta_i] \phi_i(r) & \quad i \in B, C \\
\varphi_i(q, r) \phi_i(r) & \quad i \in C
\end{aligned}$$

are scalar fields in $\mathcal{H}_q \times \mathcal{H}_r$. The conclusion now follows using Gaussian Elimination (Lemma 3.2). \square

6.1 Existence of FDR

Computing Lie-brackets on the basic fields of $\bar{\mathcal{L}}$ is not hard, and the conclusion on the existence of a FDR becomes a straightforward generalization of the easier setups studied in previous sections. Proposition 6.2 give us the needed conditions.

Proposition 6.2 The lie-algebra $\bar{\mathcal{L}}$ is spanned by

$$\text{span} \left\{ \mathbf{F}, \mathbf{F}^k \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} V_j \\ 0 \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} 0 \\ \beta_n \end{bmatrix}, \mathbf{F}^k \begin{bmatrix} 0 \\ H_n \end{bmatrix}; i \in A, C; j \in C; n \in B, C; k = 0, 1, \dots \right\}$$

and will have a finite dimension if and only if each component of λ and β is QE.

Moreover, under those conditions also each component of D , V , W in (52) are QE and

$$\dim \{\bar{\mathcal{L}}\} \leq 1 + \sum_{j \in A, C} (n_j^1 + n_j^2) + \sum_{j \in C} n_j^3 + \sum_{j \in B, C} (n_j^4 + n_j^5) \tag{53}$$

for $n_j^i \in \mathbb{N}$, such that,

$$\begin{aligned}
\mathbf{F}^{n_j^1+1} \lambda_j &= \sum_{k=1}^{n_j^1} c_{j,k}^1 \mathbf{F}^k \lambda_j & \mathbf{F}^{n_j^2+1} D_j &= \sum_{k=1}^{n_j^2} c_{j,k}^2 \mathbf{F}^k D_j & j \in A, C \\
\mathbf{F}^{n_j^3+1} V_j &= \sum_{k=1}^{n_j^3} c_{j,k}^3 \mathbf{F}^k V_j & & & j \in C \\
\mathbf{F}^{n_j^4+1} \beta_j &= \sum_{k=1}^{n_j^4} c_{j,k}^4 \mathbf{F}^k \beta_j & \mathbf{F}^{n_j^5+1} H_j &= \sum_{k=1}^{n_j^5} c_{j,k}^5 \mathbf{F}^k H_j & j \in B, C
\end{aligned} \tag{54}$$

hold for some $c_{j,k}^I \in \mathbb{R}$ with $k = 0, 1, \dots, n_j^I$, $j = 1, \dots, m$, and $I = 1, \dots, 5$.

Proof. Given the fields of $\bar{\mathcal{L}}$, we have a FDR if and only if (54) hold and it follows that (53) holds (the “ \leq ” in (53) accounts for possible case-specific Gaussian elimination across terms). Finally, (54) can be interpreted as ODEs whose solution are QE functions, thus λ_i , for $i \in A, C$; D_j, V_j , for $j \in C$ and H_k for $k \in B, C$ solve (54) if and only if they are QE. It remains to show that requiring λ_i and β_j for $i \in A, C$, $j \in B, C$ is sufficient to guarantee that. Given that

$$D_i(x) = \lambda_i^2(x) \quad V_j(x) = \lambda_j(x) \int_0^b \beta_j(s) ds \quad H_j(x) = \beta_j(x) \int_0^b \beta_j(s) ds,$$

the result follows from Lemma 3.3. \square

6.2 Construction of FDR

Knowing the conditions for existence of a FDR for forward prices, we can now construct the finite dimensional realization. Proposition 6.3 gives us a *non-minimal* (since it is based on $\bar{\mathcal{L}}$ and cannot take into account case-specific Gaussian elimination) parameterization \hat{G} of the invariant manifold $\hat{\mathcal{G}}$.

Note that our parameterization, $\hat{q} = \hat{G}(Z)$, will be of the following block matrix form

$$\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \hat{G}^q(Z) \\ \hat{G}^r(Z) \end{bmatrix}.$$

Furthermore, by close inspection of (51) we realize that the operator generated by f^0 ,

$$e^{\mathbf{F}z_0} \begin{bmatrix} q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \tilde{q}_0 \\ \tilde{r}_0 \end{bmatrix},$$

is the only that will affect both \hat{G}^q and \hat{G}^r . The remaining operators will only affect one component at the time.

The operators generated by f_i^1, f_i^2 for $i \in A, C$ and f_j^3 for $j \in C$, will only affect \hat{G}^q . So,

$$\begin{aligned}
e^{\mathbf{F}^k \lambda_j z_{j,k}^1} q &= q + \lambda_j z_{j,k}^1 q & e^{\mathbf{F}^k D_j z_{j,k}^2} q &= q + D_j z_{j,k}^2 q & j \in A, C \\
e^{\mathbf{F}^k V_j z_{j,k}^3} q &= q + V_j z_{j,k}^3 q & & & j \in C.
\end{aligned}$$

On the other hand, \hat{G}^r will be affected by the operators generated by f_j^4, f_j^5 for $j \in B, C$ and we have

$$e^{\mathbf{F}^k \beta_j z_{j,k}^4} r = r + \beta_j z_{j,k}^4 r \quad e^{\mathbf{F}^k H_j z_{j,k}^5} r = r + H_j z_{j,k}^5 r \quad j \in B, C$$

Once the parameterization has been derived, we can infer the dynamics of the finite dimensional realization, exactly as before. The actual construction of the realization, though cumbersome (and thus presented in the appendix), follow the same ideas of the constructions in previous sections.

Proposition 6.3 *Suppose Assumption 6.1 holds. Given the initial forward price curve q_0 and the initial interest rate curve r_0 , the system generated by forward price and interest rate volatilities defined as in (49) has a finite dimensional realization, given by*

$$\begin{bmatrix} q_t \\ r_t \end{bmatrix} = \hat{G}(Z)$$

where \hat{G} is defined by

$$\hat{G}(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3, z_{j,k}^4, z_{j,k}^5) = \begin{bmatrix} \tilde{q}_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j z_{j,k}^2 + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j z_{j,k}^3 \\ \tilde{r}_0 + \sum_{j \in B, C} \sum_{k=0}^{n_j^4} \mathbf{F}^k \beta_j z_{j,k}^4 + \sum_{j \in B, C} \sum_{k=0}^{n_j^5} \mathbf{F}^k H_j z_{j,k}^5 \end{bmatrix} \quad (55)$$

for $\tilde{q}_0(x) = q_0(x + z_0)$ and $\tilde{r}_0(x) = r_0(x + z_0)$.

Moreover, the dynamics of the state space variables Z are given by

$$\left\{ \begin{array}{ll} dZ_0 & = dt \\ dZ_{j,0}^1 & = c_{j,0}^1 Z_{j,n_j^1}^1 dt + \varphi_j(\hat{G}^q(Z), \hat{G}^r(Z)) dW_t^j & j \in A, C \\ dZ_{j,k}^1 & = \left\{ Z_{j,k-1}^1 + c_{j,k}^1 Z_{j,n_j^1}^1 \right\} dt & j \in A, C \quad k = 1, \dots, n_j^1 \\ dZ_{j,0}^2 & = \left\{ c_{j,0}^2 Z^2 j, n_j^2 - \frac{1}{2} \varphi_j^2(\hat{G}^q(Z), \hat{G}^r(Z)) \right\} dt & j \in A, C \\ dZ_{j,k}^2 & = \left\{ Z_{j,k-1}^2 + c_{j,k}^2 Z_{j,n_j^2}^2 \right\} dt & j \in A, C \quad k = 1, \dots, n_j^2 \\ dZ_{j,0}^3 & = \left\{ c_{j,0}^3 z_{j,n_j^3}^3 + \varphi(\hat{G}^q(Z), \hat{G}^r(Z)) \phi(\hat{G}^r(Z)) \right\} dt & j \in C \\ dZ_{j,k}^3 & = \left\{ Z_{j,k-1}^3 + c_{j,k}^3 Z_{j,n_j^3}^3 \right\} dt & j \in C \quad k = 1, \dots, n_j^3 \\ dZ_{j,0}^4 & = c_{j,0}^4 Z_{j,n_j^4}^4 dt + \phi_j(\hat{G}^r(Z)) dW^j & j \in B, C \\ dZ_{j,k}^4 & = \left\{ c_{j,0}^4 Z_{j,n_j^4}^4 \right\} dt & j \in B, C \quad k = 1, \dots, n_j^4 \\ dZ_{j,0}^5 & = \left\{ c_{j,0}^5 Z_{j,n_j^5}^5 + \phi_j^2(\hat{G}^r(Z)) \right\} dt & j \in B, C \\ dZ_{j,k}^5 & = \left\{ Z_{j,k-1}^5 + c_{j,k}^5 Z_{j,n_j^5}^5 \right\} dt & j \in B, C \quad k = 1, \dots, n_j^5 \end{array} \right.$$

The next example may help to understand Proposition 6.3.

6.3 Example

Consider a model with the following volatility matrix

$$\begin{bmatrix} \gamma(q, r) \\ \sigma(r) \end{bmatrix} = \begin{bmatrix} \alpha e^{-bx} q & \rho \\ 0 & \delta e^{-ax} \sqrt{r} \end{bmatrix}.$$

Using the usual notation we have two Wiener processes, one of them of type W^A and another of type W^C . So $A = \{1\}$ and $C = \{2\}$. Since there is only one element of each type, we set $j = A, C$ (instead of $j=1,2$) all over.

We also have

$$\begin{aligned} \gamma_A(q, r, x) = \lambda_A(x) \varphi_A(q, r) &\Rightarrow \lambda_A(x) = \alpha e^{-bx} & \varphi_A(q, r) &= q \\ \gamma_C(q, r, x) = \lambda_C(x) \varphi_C(q, r) &\Rightarrow \lambda_C(x) = \rho & \varphi_C(q, r) &= 0 \\ \sigma_C(r, x) = \beta_C(x) \phi_C(x) &\Rightarrow \beta_C(x) = \delta e^{-ax} & \phi_C(q) &= \sqrt{r} \\ &\Rightarrow v_C(r, x) = -B_C(x) \phi(r) & B_C(x) &= -\frac{1}{a} [e^{-ax} - 1] \end{aligned}$$

Moreover, we have

$$\begin{aligned} D_A(x) &= \lambda_A^2(x) = \alpha^2 e^{-2bx} \\ V_C(x) &= \lambda_C(x) B_C(x) = -\frac{\rho}{a} [e^{-ax} - 1] \\ H_C(x) &= \beta_C(x) B_C(x) = -\frac{\delta}{a} [e^{-2ax} - e^{-ax}]. \end{aligned}$$

Taking all this into account we easily get

$$\begin{aligned} \mathbf{F}\lambda_A &= -b\lambda_A & \Rightarrow n_A^1 &= 0 & c_{A,0}^1 &= -b \\ \mathbf{F}\lambda_C &= 0 & \Rightarrow n_C^1 &= 0 & c_{C,0}^1 &= 0 \\ \mathbf{F}D_A &= -2bD_A & \Rightarrow n_A^2 &= 0 & c_{A,0}^2 &= -2b \\ \mathbf{F}D_C &= 0 & \Rightarrow n_C^2 &= 0 & c_{C,0}^2 &= 0 \\ \mathbf{F}V_C &= \rho e^{-ax} & \mathbf{F}^2 V_C &= -a\mathbf{F}V_C & \Rightarrow n_C^3 &= 1 & c_{C,0}^3 &= 0 \\ & & & & & & c_{C,1}^3 &= -a \\ \mathbf{F}\beta_C &= -a\beta_C & \Rightarrow n_C^4 &= 0 & c_{C,0}^4 &= -a \\ \mathbf{F}H_C &= -aH_C + \delta e^{-2ax} & \mathbf{F}^2 H_C &= -3a\mathbf{F}H_C - 2a^2 H_C & \Rightarrow n_C^5 &= 2 & c_{C,0}^5 &= -2a^2 \\ & & & & & & c_{C,1}^5 &= -3a. \end{aligned}$$

Given this computations, we see the following fields span $\bar{\mathcal{L}}$

$$\left\{ \mathbf{F}, \begin{bmatrix} \lambda_A \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_C \\ 0 \end{bmatrix}, \begin{bmatrix} D_A \\ 0 \end{bmatrix}, \begin{bmatrix} D_C \\ 0 \end{bmatrix}, \begin{bmatrix} V_C \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{F}V_C \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta_C \end{bmatrix}, \begin{bmatrix} 0 \\ H_C \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{F}H_C \end{bmatrix} \right\} \quad (56)$$

thus we know that our forward price model admits a FDR since

$$\dim(\bar{\mathcal{L}}) \leq 10.$$

From (55) we get the parameterization

$$\begin{aligned} &\begin{bmatrix} \hat{G}^q(x, z_0, z_{A,0}^1, z_{C,0}^1, z_{A,0}^2, z_{C,0}^2, z_{C,0}^3, z_{C,1}^3) \\ \hat{G}^r(x, z_0, z_{C,0}^4, z_{C,0}^5, z_{C,1}^5) \end{bmatrix} \\ &= \begin{bmatrix} q_0(x + z_0) + \alpha e^{-bx} z_{A,0}^1 + \rho z_{C,0}^1 + \alpha^2 e^{-2bx} z_{A,0}^2 + \rho^2 z_{C,0}^2 - \frac{\rho}{a} [e^{-ax} - 1] z_{C,0}^3 + \rho e^{-ax} z_{C,1}^3 \\ r_0(x + z_0) + \delta e^{-ax} z_{C,0}^4 - \frac{\delta}{a} [e^{-2ax} - e^{-ax}] z_{C,0}^5 - \frac{\delta}{a} [-2ae^{-2ax} + ae^{-ax}] z_{C,1}^5 \end{bmatrix} \end{aligned} \quad (57)$$

and from Proposition 6.3 we get the Z dynamics

$$\left\{ \begin{array}{ll} dZ_0 = dt & \\ dZ_{A,0}^1 = -bZ_{A,0}^1 dt + \hat{G}^q(Z) dW_t^j & dZ_{C,0}^1 = \hat{G}^q(Z) dW_t^j \\ dZ_{A,0}^2 = \left(-2bZ_{A,1}^2 - \frac{1}{2} \left(\hat{G}^q(Z) \right)^2 \right) dt & dZ_{C,0}^2 = -\frac{1}{2} \left(\hat{G}^q(Z) \right)^2 dt \\ dZ_{C,0}^3 = \hat{G}^q(z) \sqrt{\hat{G}^r(Z)} dt & dZ_{C,1}^3 = (Z_{C,0}^3 - aZ_{C,1}^3) dt \\ dZ_{C,0}^4 = -aZ_{C,0}^4 dt + \sqrt{\hat{G}^r(Z)} dW^j & \\ dZ_{C,0}^5 = \left(-2a^2 Z_{C,1}^5 + \hat{G}^r(Z) \right) dt & dZ_{C,1}^5 = (Z_{C,0}^5 - 3aZ_{C,1}^5) dt \end{array} \right. \quad (58)$$

where \hat{G}^q and \hat{G}^r are as in (57).

It is obvious however, from both (57) and (58), that this realization is unnecessarily larger. Using the following change of variables we can find a realization of dimension 7 (which is the minimal possible with the fields in (56)),

$$\begin{aligned} \tilde{Z}_0 &= Z_0 \\ \tilde{Z}_{A,0}^1 &= Z_{A,0}^1 \\ \tilde{Z}_{A,0}^2 &= Z_{A,0}^2 \\ \tilde{Z}_{A,0}^3 &= -\frac{\rho}{a} Z_{C,0}^3 + \rho Z_{C,1}^3 \\ \tilde{Z}_{C,0}^1 &= \rho Z_{C,0}^1 + \rho^2 Z_{C,0}^2 + \frac{\rho}{a} Z_{C,0}^3 \\ \tilde{Z}_{C,0}^4 &= \delta Z_{C,0}^4 + \frac{\delta}{a} Z_{C,0}^5 - \delta Z_{C,1}^5 \\ \tilde{Z}_{C,0}^5 &= -\frac{\delta}{a} Z_{C,0}^5 + 2\delta Z_{C,1}^5. \end{aligned}$$

We can then use Itô and (58) to derive the dynamics of the new state variables.

6.4 FDR of forward prices versus FDR of interest rates

We finish this study giving a complete answer to our *problem four* – on whether there it is possible to have forward price model which allows for a FDR for forward prices but not for interest rates¹⁵.

We recall a forward price term structure model consists of the following two infinite SDEs.

$$dq_t = \left\{ \mathbf{F}q_t - \frac{1}{2} \|\gamma(q_t, r_t)\|^2 - \gamma(q_t, r_t) v^*(r_t) \right\} dt + \gamma(q_t, r_t) dW_t \quad (59)$$

$$dr_t = \{ \mathbf{F}r_t - \sigma(r_t) v^*(r_t) \} dt + \sigma(r_t) dW_t \quad (60)$$

Proposition 6.4 *In forward price term structure models, inexistence of a FDR for the interest rate equation (59) and existence of a FDR for the forward price equation (60), is possible only*

¹⁵Recall the partial answer given in Remark 4.2.

if forward prices are Markovian **and** the conditions of Proposition 5.6 hold i.e.,

$$\begin{aligned}
\gamma_j(q, r, x) &= \sum_{k=0}^{N_\gamma^j} \varphi_k^j(q) \lambda_k^j(x) & j \in A, C \\
\sigma_j(r, x) &= \beta_j(x) & j \in B \\
\sigma_j(r, x) &= \begin{cases} \beta_j(x) & \text{if } \gamma_j(q, r, x) = \gamma_j(x) \\ \sum_{k=0}^{N_\sigma^j} \omega_k^j \delta_k^j(x) & \text{if } \gamma_j(q, r, x) = \gamma_j(q, x) \end{cases} & j \in C
\end{aligned} \tag{61}$$

where ω_k^j are deterministic constants, β_j are unrestricted deterministic functions, δ_k^j, λ_k^j are QE deterministic functions, and φ_k^j are scalar vector fields in \mathcal{H}_q .

Proof. If forward prices are *not* Markovian, then either γ or γv^* depend on r . If that is the case, we know from Proposition 6.3 that the dynamics of the state variables of type Z^1, Z^2, Z^3 (the ones showing up directly on the parameterization \hat{G}^q) depend on \hat{G}^r . That is, the forward price realization depend on the interest rate realization, indirectly, through the dynamics of the factors showing up in \hat{G}^q . Thus, if $r = \hat{G}^r(Z)$ only holds for infinite Z , the forward prices will also be a function on an infinite state variable and, by definition, do not admit a FDR.

If forward prices are Markovian, we know, from Proposition 5.6, that forward prices admit a FDR if and only if (61) hold. On the other hand, we see (61) imposes no restriction on σ_B , so we can choose σ_B *not to be* a weighted finite sum of quasi-exponential functions, weighted by scalar fields in \mathcal{H}_r , making existence of FDR for interest rates impossible (for further details on this result from the previous literature see [10]). \square

7 Conclusions and Applicability

Forward prices are only interesting objects of study in settings where the forward measures Q^T differ from the risk-neutral measure Q . In these settings, the study of forward prices depends on zero-coupon bond price volatilities. Using the Lie-algebraic approach of Björk *et al.*, we have shown that forward prices term structure models consist of a system of two infinite dimensional SDEs, one describing the dynamics of the forward prices themselves and another characterizing the interest rate setting and where the interest rate equation is an input to the forward price equation.

Despite the apparent non-Markovian nature of forward prices, we were able to show that there exist models for which forward prices are, actually, Markovian and identified necessary and sufficient conditions for this Markovian property to hold. Studying Markovian forward prices we concluded that existence of finite dimensional realization (FDR) for Markovian forward prices is, in some sense, independent of existence of FDR for interest rates.

We studied existence and construction of FDR for Markovian forward prices and derived general conditions for existence of FDR. We considered with special detail the pure deterministic and the deterministic direction volatility special cases. From this analysis, we concluded that some results from previous literature can be extended to the forward price term structure case, but also that forward price term structure models are particularly complex.

The dynamics of forward prices has a specially complex drift under the risk-neutral measure Q . A direct consequence of this complexity is that, as soon as we leave the pure deterministic volatility setting, the best we can hope for is to study *non-minimal* Lie-algebras and to find *non-minimal* FDR. Existence of non-minimal realizations is, of course, sufficient to prove existence of FDR, but is in general not necessary. We showed, however, that given the specificity of the forward price equation drift, and for a specific enlargement of the Lie-algebra, existence of FDR for non-minimal realizations is also necessary for existence of FDR, at least for the one-dimensional case. Then we conjectured that this hold for the higher dimension case.

Even if *non-minimal* Lie-algebras are, in the above sense, satisfactory for existence results, they are not as satisfactory for construction results, since we are bound to find realizations with too many variables. Despite this fact, we exemplified how, given a concrete application, we can use the abstract results to obtain a smaller realization (sometimes the minimal one) simply by using a smart change of variables and Itô's lemma.

For non-Markovian forward prices, we showed that whenever there exist FDR for the forward price equation, the dynamics of the state variable depend on the interest rates. Consequently, term structures of forward prices will always (indirectly) depend on interest rates, and existence of a FDR for the interest rate equation is *necessary* for existence of a FDR for the forward price equation. In order to study non-Markovian forward price term structure models, we handle a system of two infinite dimensional SDEs, thus, computations get quite cumbersome. Still, most results are the expected ones, given the previous literature on FDR of interest rates and the study of the forward price equation in the easier Markovian setting.

In terms of the applicability of the results presented here, it is, first of all, important to stress that the characterization of the conditions that guarantee existence of a FDR for forward price term structure models is crucial, in distinguishing the “good” forward price models from the “bad” forward price models. After all, a term structure model that do not allow for a FDR realization cannot be useful for any practical application. For instance, it is impossible to estimate the dynamics of an infinite state variable.

In addition to this *selection applicability*, perhaps, the most important application results from the actual local parameterization of the term structure. This parameterization can help in understanding what are the needed conditions, on the driving volatility vector fields, that will produce term structures consistent with case-specific realities, helping to design good models. In the present study this *design applicability* of the Lie-algebraic approach was left untouched because it is case dependent, and we have focused on general results.

Finally, let alone forward prices, the results derived here are applicable to study term structures of any Q^T -martingale. Examples financial instruments, with strong connections to Q^T -martingales, are swap rates and credit spreads.

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Appendix: Technical details and Proofs

Conjecture 5.1 Proof. (sketch) The implication \Leftarrow follows immediately from $\mathcal{L} \subseteq \bar{\mathcal{L}}$.

The implication \Rightarrow is much harder to prove. Here, as an illustration, we consider the one-dimensional Wiener process case. We will show the equivalent result that if $\dim(\bar{\mathcal{L}}) = \infty$ then we must also have $\dim(\mathcal{L}) = \infty$.

In this case

$$\begin{aligned}\mathcal{L} &= \{\mu, \gamma\}_{LA} = \{f_0, f_1\}_{LA} \\ \bar{\mathcal{L}} &= \{\mathbf{F}, \lambda, D, V\}.\end{aligned}$$

for

$$\begin{aligned}f_0(q) &= \mathbf{F}q - \frac{1}{2}D\phi(q) + \varphi(q)V \\ f_1(q) &= \lambda \\ D &= \lambda^2 \\ V &= \lambda v\end{aligned}$$

and as usual we take $v(x) = -\int_0^x \sigma(s)ds$.

From Lemma 5.3 we know that $\dim(\bar{\mathcal{L}}) = \infty$ if and only if at least one of the functions γ and σ is not QE. So, assume that γ and σ are *not* QE functions. Then, also D and V are not QE functions.

Let us now, have a look at the original (smaller) Lie-algebra, \mathcal{L} . We will try to see if

$$\mathcal{L} = \{\mu, \gamma\}_{LA} = \{f_0, f_1\}_{LA} < \infty.$$

Computing lie-brackets and simplifying

$$\begin{aligned}[f_0, f_1](q) &= f_0'(q)f_1(q) - f_1'(q)f_0(q) \\ &= \mathbf{F}\lambda - \frac{1}{2}D \underbrace{\phi'(q)[\lambda]}_{\text{scalar field}} - V \underbrace{\varphi'(q)[\lambda]}_{\text{scalar field}} \\ &= f_2(q).\end{aligned}$$

Continuing this way we need to compute $[f_0, f_2] = f_0'(q)f_2 - f_2'f_0(q)$. The first parcel, however give us

$$\begin{aligned}f_0'(q)f_2(q) &= \mathbf{F}^2\lambda - \frac{1}{2}(\mathbf{F}D)\phi'(q)[\lambda] + (\mathbf{F}V)\varphi'(q)[\lambda] \\ &= \mathbf{F}^2\lambda + \mathbf{F}D.(\text{scalar field}) + \mathbf{F}V.(\text{scalar field})\end{aligned}$$

and the second parcel (though involving more messy computations) is of the form

$$f_2'(q)f_0(q) = D.(\text{scalar field}) + V.(\text{scalar field}).$$

So,

$$\begin{aligned}
[f_0, f_2] &= \mathbf{F}^2\lambda + \mathbf{F}D(\text{scalar field}) + \mathbf{F}V(\text{scalar field}) + D(\text{scalar field}) + V(\text{scalar field}) =: f_3 \\
[f_0, f_3] &= \mathbf{F}^3\lambda + \mathbf{F}^2D(\text{scalar field}) + \mathbf{F}^2V(\text{scalar field}) + D(\text{scalar field}) + V(\text{scalar field}) \\
&\vdots
\end{aligned}$$

It is now easy to see why the dimension of our \mathcal{L} will also be infinite, in this case. It would only be finite if λ , D and V were QE functions, but this hypothesis is excluded by assumption. \square

Proposition 5.5 Proof. Using \bar{G} from (45) we have

$$\begin{aligned}
\bar{G}'(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3) &\begin{bmatrix} h_0 \\ h_{1,0}^1 \\ \vdots \\ h_{1,0}^2 \\ \vdots \\ h_{1,0}^3 \\ \vdots \\ h_{m,n_j}^3 \end{bmatrix} = \frac{\partial}{\partial z_0} q(x+z_0)h_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j h_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j h_{j,k}^2 \\
&+ \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j h_{j,k}^3.
\end{aligned}$$

From $q = \bar{G}(Z)$ and once again the functional form of \bar{G} in (45) we get

$$\begin{aligned}
\mathbf{F}q &= \mathbf{F}\tilde{q}_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^{k+1} \lambda_j z_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^{k+1} D_j z_{j,k}^2 + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^{k+1} V_j z_{j,k}^3 \\
&= \frac{\partial}{\partial x} \tilde{q}_0 + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k-1}^1 + \mathbf{F}^{n_j^1+1} \lambda_{n_j} z_{j,n_j}^1 \right] \\
&+ \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^2} \mathbf{F}^k D_j z_{j,k-1}^2 + \mathbf{F}^{n_j^2+1} D_{n_j} z_{j,n_j}^2 \right] + \sum_{j \in C} \left[\sum_{k=1}^{n_j^3} \mathbf{F}^k V_j z_{j,k-1}^3 + \mathbf{F}^{n_j^3+1} V_{n_j} z_{j,n_j}^3 \right] \\
&= \frac{\partial}{\partial x} \tilde{q}_0 + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1 \lambda_k z_{j,n_j}^1 \right] \\
&+ \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^2} \mathbf{F}^k D_j z_{j,k-1}^2 + \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j}^2 \right] + \sum_{j \in C} \left[\sum_{k=1}^{n_j^3} \mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j}^3 \right]
\end{aligned}$$

where we omitted the x -dependence and used $\tilde{q}_0(x) = q_0(x+z_0)$.

We can now use the above expression to substitute into our μ (recall (41)). Thus, $\bar{G}_\star \bar{a} = \mu$ becomes

$$\begin{aligned}
& \frac{\partial}{\partial z_0} \tilde{q}_0 \bar{a}_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j \bar{a}_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j \bar{a}_{j,k}^2 + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j \bar{a}_{j,k}^3 \\
&= \frac{\partial}{\partial x} \tilde{q}_0 + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1 \lambda_k z_{j,n_j}^1 \right] \\
&+ \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^2} \mathbf{F}^k D_j z_{j,k-1}^2 + \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j}^2 \right] + \sum_{j \in C} \left[\sum_{k=1}^{n_j^3} \mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j}^3 \right] \\
&- \frac{1}{2} \sum_{j \in A, C} D_j \varphi_j^2(q) - \frac{1}{2} \sum_{j \in A, C} \varphi_j'(q) [\lambda_j] \varphi_j(q) \lambda_j
\end{aligned}$$

and $\bar{G}_\star \bar{b}^i = \gamma_i$ for all $i \in A, C$

$$\begin{aligned}
& \frac{\partial}{\partial z_0} \tilde{q}_0 \bar{b}_0^i + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j \bar{b}_{j,k}^{1i} + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j \bar{b}_{j,k}^{2i} + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j \bar{b}_{j,k}^{3i} \\
&= \varphi_i(q) \lambda_i
\end{aligned}$$

Identification of terms, and again use of $q = \bar{G}(Z)$ yields

$$\begin{aligned}
\bar{a}_0 &= 1 \\
\bar{a}_{j,0}^1 &= c_{j,0}^1 z_{j,n_j}^1 - \frac{1}{2} \varphi_j'(\bar{G}(z)) [\lambda_j] \varphi_j(\bar{G}(z)) & j \in A, C \\
\bar{a}_{j,k}^1 &= z_{j,k-1}^1 + c_{j,k}^1 z_{j,n_j}^1 & j \in A, C \quad k = 1, 2, \dots, n_j^1 \\
\bar{a}_{j,0}^2 &= c_{j,0}^2 z^2 j, n_j^2 - \frac{1}{2} \varphi_j^2(\bar{G}(z)) & j \in A, C \\
\bar{a}_{j,k}^2 &= z_{j,k-1}^2 + c_{j,k}^2 z_{j,n_j}^2 & j \in A, C \quad k = 1, 2, \dots, n_j^2 \\
\bar{a}_{j,0}^3 &= c_{j,0}^3 z^3 j, n_j^3 + \varphi(\bar{G}(z)) & j \in C \\
\bar{a}_{j,k}^3 &= z_{j,k-1}^3 + c_{j,k}^3 z_{j,n_j}^3 & j \in C \quad k = 1, 2, \dots, n_j^3 \\
\bar{b}_0^i &= 0 \\
\bar{b}_{j,k}^{1i} &= \begin{cases} \varphi_j(\bar{G}(z)) & \text{if } i = j \text{ and } k = 0 \\ 0 & \text{if } i \neq j \text{ or } k = 1, 2, \dots, n_j^1 \end{cases} & j \in A, C \\
\bar{b}_{j,k}^{2i} &= 0 & j \in A, C \quad k = 1, 2, \dots, n_j^2 \\
\bar{b}_{j,k}^{3i} &= 0 & j \in A, C \quad k = 1, 2, \dots, n_j^3.
\end{aligned}$$

Note that the factors $z_{j,0}^1$ are driven by the scalar wiener process W^j and that all remaining factors have diffusion terms equal to zero. Thus only for $z_{j,0}^1$ the Itô dynamics differ from the Stratonovich dynamics. The above $\bar{a}_{j,0}^1$ is the Stratonovich drift. Given the form of $\bar{a}_{j,0}^1$, the diffusion for $z_{j,0}^1$. We easily get the Itô drift to be simply $\bar{a}_{j,0}^1 \text{Itô} = c_{j,0}^1 z_{j,n_j}^1$. \square

Proposition 6.3 Proof. We first compute the parameterization \hat{G} . From Theorem 3.5 and the special shape of the basic fields in (51), we have

$$\begin{aligned} \begin{bmatrix} \hat{G}^q(Z) \\ \hat{G}^r(Z) \end{bmatrix} &= \begin{bmatrix} \prod_{j \in A, C} \left(e^{\mathbf{F}^k \lambda_j z_{j,k}^1} e^{\mathbf{F}^k D_j z_{j,k}^2} \right) \prod_{j \in C} \left(e^{\mathbf{F}^k V_j z_{j,k}^3} \right) e^{\mathbf{F} z_0} q_0 \\ \prod_{j \in B, C} \left(e^{\mathbf{F}^k \beta_j z_{j,k}^4} e^{\mathbf{F}^k H_j z_{j,k}^5} \right) e^{\mathbf{F} z_0} r_0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{q}_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j z_{j,k}^2 + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j z_{j,k}^3 \\ \tilde{r}_0 + \sum_{j \in B, C} \sum_{k=0}^{n_j^4} \mathbf{F}^k \beta_j z_{j,k}^4 + \sum_{j \in B, C} \sum_{k=0}^{n_j^5} \mathbf{F}^k H_j z_{j,k}^5 \end{bmatrix} \end{aligned} \quad (62)$$

Using \hat{G} from (62) we have

$$\begin{aligned} \hat{G}^q(z_0, z_{j,k}^1, z_{j,k}^2, z_{j,k}^3) \begin{bmatrix} h_0 \\ h_{1,0}^1 \\ \vdots \\ h_{1,0}^2 \\ \vdots \\ h_{m,n_j^3}^3 \end{bmatrix} &= \frac{\partial}{\partial z_0} \tilde{q}_0 h_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^k \lambda_j h_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^k D_j h_{j,k}^2 \\ &\quad + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^k V_j h_{j,k}^3. \end{aligned}$$

and

$$\hat{G}^r(z_0, z_{j,k}^4, z_{j,k}^5) \begin{bmatrix} h_0 \\ h_{1,0}^4 \\ \vdots \\ h_{m,n_j^3}^5 \end{bmatrix} = \frac{\partial}{\partial z_0} \tilde{r}_0 h_0 + \sum_{j \in B, C} \sum_{k=0}^{n_j^4} \mathbf{F}^k \beta_j h_{j,k}^4 + \sum_{j \in A, C} \sum_{k=0}^{n_j^5} \mathbf{F}^k H_j h_{j,k}^5$$

where we omitted the x -dependence and used $\tilde{q}_0(x) = q_0(x + z_0)$, $\tilde{r}_0(x) = r_0(x + z_0)$.

From $q = \hat{G}^q(Z)$, $r = \hat{G}^r(Z)$ and once again the functional form of \hat{G} in (62) we get

$$\begin{aligned}
\mathbf{F}q &= \mathbf{F}\tilde{q}_0 + \sum_{j \in A, C} \sum_{k=0}^{n_j^1} \mathbf{F}^{k+1} \lambda_j z_{j,k}^1 + \sum_{j \in A, C} \sum_{k=0}^{n_j^2} \mathbf{F}^{k+1} D_j z_{j,k}^2 + \sum_{j \in C} \sum_{k=0}^{n_j^3} \mathbf{F}^{k+1} V_j z_{j,k}^3 \\
&= \frac{\partial}{\partial x} \tilde{q}_0 + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k-1}^1 + \mathbf{F}^{n_j^1+1} \lambda_{n_j^1} z_{j,n_j^1}^1 \right] + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^2} \mathbf{F}^k D_j z_{j,k-1}^2 + \mathbf{F}^{n_j^2+1} D_{n_j^2} z_{j,n_j^2}^2 \right] \\
&\quad + \sum_{j \in C} \left[\sum_{k=1}^{n_j^3} \mathbf{F}^k V_j z_{j,k-1}^3 + \mathbf{F}^{n_j^3+1} V_{n_j^3} z_{j,n_j^3}^3 \right] \\
&= \frac{\partial}{\partial x} \tilde{q}_0 + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^1} \mathbf{F}^k \lambda_j z_{j,k-1}^1 + \sum_{k=0}^{n_j^1} c_{j,k}^1 \lambda_k z_{j,n_j^1}^1 \right] + \sum_{j \in A, C} \left[\sum_{k=1}^{n_j^2} \mathbf{F}^k D_j z_{j,k-1}^2 + \sum_{k=0}^{n_j^2} c_{j,k}^2 D_k z_{j,n_j^2}^2 \right] \\
&\quad + \sum_{j \in C} \left[\sum_{k=1}^{n_j^3} \mathbf{F}^k V_j z_{j,k-1}^3 + \sum_{k=0}^{n_j^3} c_{j,k}^3 V_k z_{j,n_j^3}^3 \right] \\
\mathbf{F}r &= \mathbf{F}\tilde{r}_0 + \sum_{j \in b, C} \sum_{k=0}^{n_j^4} \mathbf{F}^{k+1} \beta_j z_{j,k}^4 + \sum_{j \in B, C} \sum_{k=0}^{n_j^5} \mathbf{F}^{k+1} H_j z_{j,k}^5 \\
&= \frac{\partial}{\partial x} \tilde{r}_0 + \sum_{j \in B, C} \left[\sum_{k=1}^{n_j^4} \mathbf{F}^k \beta_j z_{j,k-1}^4 + \mathbf{F}^{n_j^4+1} \beta_{n_j^4} z_{j,n_j^4}^4 \right] + \sum_{j \in B, C} \left[\sum_{k=1}^{n_j^5} \mathbf{F}^k H_j z_{j,k-1}^5 + \mathbf{F}^{n_j^5+1} H_{n_j^5} z_{j,n_j^5}^5 \right] \\
&= \frac{\partial}{\partial x} \tilde{r}_0 + \sum_{j \in B, C} \left[\sum_{k=1}^{n_j^4} \mathbf{F}^k \beta_j z_{j,k-1}^4 + \sum_{k=0}^{n_j^4} c_{j,k}^4 \beta_k z_{j,n_j^4}^4 \right] + \sum_{j \in B, C} \left[\sum_{k=1}^{n_j^5} \mathbf{F}^k H_j z_{j,k-1}^5 + \sum_{k=0}^{n_j^5} c_{j,k}^5 H_k z_{j,n_j^5}^5 \right]
\end{aligned}$$

We can now use the above expressions to substitute into our μ , recall we have

$$\begin{aligned}
\mu(q, r) &= \mathbf{F} \begin{bmatrix} q \\ r \end{bmatrix} - \frac{1}{2} \sum_{i \in A, C} \varphi_i^2(q, r) \begin{bmatrix} \lambda_i^2 \\ 0 \end{bmatrix} + \sum_{i \in C} \varphi_i(q, r) \phi_i(r) \begin{bmatrix} \lambda_i B_i \\ 0 \end{bmatrix} + \sum_{i \in B, C} \phi_i^2(r) \begin{bmatrix} 0 \\ \beta_i B_i \end{bmatrix} \\
&\quad - \frac{1}{2} \left\{ \sum_{i \in A, C} \varphi'_{iq}(q, r) [\lambda_i] \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} \right. \\
&\quad \left. + \sum_{i \in C} \varphi'_{ir}(q, r) [\beta_i] \phi_i(r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \sum_{i \in B, C} \phi'_{ir}(r) [\beta_i] \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} \right\}, \\
\begin{bmatrix} \gamma_i(q, r) \\ \sigma_i(r) \end{bmatrix} &= \begin{cases} \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} & i \in A \\ \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in B \\ \varphi_i(q, r) \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix} + \phi_i(r) \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} & i \in C. \end{cases}
\end{aligned}$$

Defining

$$\mu(q, r) = \begin{bmatrix} \mu^q(q, r) \\ \mu^r(r) \end{bmatrix}$$

identification of terms in $\hat{G}_*^q \hat{a}^q = \mu^q$, $\hat{G}_*^r \hat{a}^r = \mu^r$, gives us

$$\begin{aligned} \hat{a}_0 &= 1 \\ \hat{a}_{j,0}^1 &= \begin{cases} c_{j,0}^1 z_{j,n_j}^1 - \frac{1}{2} \varphi_j'(\hat{G}^q(z), \hat{G}^r(z)) [\lambda_j] \varphi_j(\hat{G}^q(z), \hat{G}^r(z)) & j \in A \\ c_{j,0}^1 z_{j,n_j}^1 - \frac{1}{2} \varphi_j'(\hat{G}^q(z), \hat{G}^r(z)) [\lambda_j] \varphi_j(\hat{G}^q(z), \hat{G}^r(z)) \\ \quad - \frac{1}{2} \varphi_j'(\hat{G}^q(z), \hat{G}^r(z)) [\beta_j] \phi_j(\hat{G}^r(z)) & j \in C \end{cases} \\ \hat{a}_{j,k}^1 &= z_{j,k-1}^1 + c_{j,k}^1 z_{j,n_j}^1 & j \in A, C \quad k = 1, 2, \dots, n_j^1 \\ \hat{a}_{j,0}^2 &= c_{j,0}^2 z_{j,n_j}^2 - \frac{1}{2} \varphi_j^2(\hat{G}^q(z), \hat{G}^r(z)) & j \in A, C \\ \hat{a}_{j,k}^2 &= z_{j,k-1}^2 + c_{j,k}^2 z_{j,n_j}^2 & j \in A, C \quad k = 1, 2, \dots, n_j^2 \\ \hat{a}_{j,0}^3 &= c_{j,0}^3 z_{j,n_j}^3 + \varphi(\hat{G}^q(z), \hat{G}^r(z)) \phi(\hat{G}^r(z)) & j \in C \\ \hat{a}_{j,k}^3 &= z_{j,k-1}^3 + c_{j,k}^3 z_{j,n_j}^3 & j \in C \quad k = 1, 2, \dots, n_j^3 \\ \hat{a}_{j,0}^4 &= c_{j,0}^4 z_{j,n_j}^4 - \frac{1}{2} \phi_j'(\hat{G}^r(z)) [\beta_j] \phi_j(\hat{G}^r(z)) & j \in B, C \\ \hat{a}_{j,k}^4 &= z_{j,k-1}^4 + c_{j,k}^4 z_{j,n_j}^4 & j \in B, C \quad k = 1, 2, \dots, n_j^4 \\ \hat{a}_{j,0}^5 &= c_{j,0}^5 z_{j,n_j}^5 + \phi^2(\hat{G}^r(z)) & j \in B, C \\ \hat{a}_{j,k}^5 &= z_{j,k-1}^5 + c_{j,k}^5 z_{j,n_j}^5 & j \in B, C \quad k = 1, 2, \dots, n_j^5 \end{aligned}$$

To get the diffusion terms we identify terms in $\hat{G}_*^q \hat{b}^{qi} = \gamma_i$ for $i \in A, C$ and $\hat{G}_*^r \hat{b}^{ri} = \sigma_i$ for $i \in B, C$.

$$\begin{aligned} \hat{b}_0^i &= 0 \\ \hat{b}_{j,k}^{1i} &= \begin{cases} \varphi_j(\hat{G}^q(z), \hat{G}^r(z)) & \text{if } i = j, \text{ and } k = 0 \\ 0 & \text{otherwise} \end{cases} & j \in A, C \\ \hat{b}_{j,k}^{2i} &= 0 & j \in A, C \quad k = 1, 2, \dots, n_j^2 \\ \hat{b}_{j,k}^{3i} &= 0 & j \in A, C \quad k = 1, 2, \dots, n_j^3 \\ \hat{b}_{j,k}^{4i} &= \begin{cases} \phi_j(\hat{G}^r(z)) & \text{if } i = j \text{ and } k = 0 \\ 0 & \text{if } i \neq j \text{ or } k = 1, 2, \dots, n_j^4 \end{cases} & j \in B, C \\ \hat{b}_{j,k}^{5i} &= 0 & j \in B, C \quad k = 1, 2, \dots, n_j^5 \end{aligned}$$

Note that this implies that the factors $Z_{j,0}^1, Z_{j,0}^4$ are driven by the scalar wiener process W^j so, in particular for $j \in C$ the same Wiener process drives the two variables. All remaining factors have diffusion terms equal to zero. Thus only for $z_{j,0}^1$ and $z_{j,0}^4$, the Itô dynamics differ from the Stratonovich dynamics. We easily get the Itô drifts to be simply

$$\begin{aligned}\hat{a}_{j,0}^1 \text{ Itô} &= c_{j,0}^1 z_{j,n_j}^1 \\ \hat{a}_{j,0}^4 \text{ Itô} &= c_{j,0}^4 z_{j,n_j}^4\end{aligned}$$

□