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Abstract

This paper analyses Rawls's celebrated difference principle, and its lexicographic extension, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise Rawlsian egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed by [12]. This is quite surprising, because the Harm principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents.

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Keywords. Difference principle, leximin, weak harm principle, infinite utility streams.

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1 Introduction

Almost four decades after its publication, *A Theory of Justice* ([13]) maintains a prominent role in political philosophy, economics, and social choice. Among the most influential contributions of the book is the *difference principle* contained in Rawls's second principle of justice, according to which inequalities should be allowed only insofar as they benefit the worst-off members of society ([13], p.303). Both the difference principle, formally captured by the well-known *maximin* social welfare relation, and especially its lexicographic extension, the *leximin* social welfare relation, have generated a vast literature across disciplinary borders.

Rawls's difference principle and its extension are usually considered to have a strong egalitarian bias and are taken to represent the main alternative to libertarian and utilitarian approaches (see, e.g., [21] and the discussion in [16]). The classic characterisation of leximin, in fact, is due to Hammond ([9]) and it requires an axiom (the so-called *Hammond Equity axiom*) with a marked egalitarian content: in a welfaristic framework, Hammond Equity asserts that if $x_i < y_i < y_j < x_j$ for two utility profiles x and y , for which $x_h = y_h$ for all agents $h \neq i, j$, then y should be (weakly) socially preferred to x . In a recent contribution, however, Mariotti and Veneziani ([12]) show that the leximin can be characterised using an axiom - the Harm Principle - that incorporates a liberal view of non-interference, without any explicit egalitarian content. This result is surprising and it raises a number of interesting issues for liberal approaches emphasising notions of individual autonomy or freedom in political philosophy and social choice, but it also sheds new light on the normative foundations of standard egalitarian principles.

This paper extends the analysis of the implications of liberal views of noninterference, as expressed in the Harm Principle, and it generalises ([12]) in a number of directions. Formally, it is shown that a weaker version of the Harm Principle, together with standard axioms in social choice, provides a unified axiomatic framework to analyse a set of social welfare relations originating from the difference principle in a welfaristic framework. Theoretically, the analysis provides a novel statement, based on liberal principles, of the ethical intuitions behind a family of normative principles stemming from Rawls's seminal work. On the one hand, the Harm Principle is formally different from standard informational invariance axioms (see, e.g., [8]) and, unlike the latter, it has a clear normative content. On the other hand, unlike the Hammond Equity axiom, the Harm principle does not incorporate an egalitarian intuition. Therefore, quite surprisingly, the ethical foundations of two social welfare relations traditionally considered as rather egalitarian - the difference principle and its lexicographic extension - rest only on the two standard axioms of Anonymity and Pareto efficiency, and on a liberal principle incorporating a noninterfering view. No axiom with a clear

egalitarian content is necessary, and indeed our analysis provides a new meaning to the label ‘liberal egalitarianism’ usually associated with Rawls’s approach. Actually, our analysis sheds new light on the importance of the notion of justice as impartiality incorporated in the Anonymity axiom in egalitarian approaches. This is particularly clear in societies with a finite number of agents: the Harm principle and the Pareto principle are consistent with some of the least egalitarian social welfare orderings (e.g. the lexicographic dictatorships), and the Anonymity axiom plays a pivotal role in determining the egalitarian outcome. Our analysis also raises some interesting issues concerning the actual implications of liberal approaches emphasising a notion of individual autonomy, or freedom: if one endorses some standard axioms - such as Anonymity and the Pareto principle - the adoption of an arguably weak liberal view of noninterference leads straight to welfare egalitarianism. As noted by Mariotti and Veneziani ([12]), liberal noninterference implies equality, an insight that is proved to be quite robust in this paper.

To be specific, first of all, in economies with a finite number of agents, it is shown that a weaker version of the Harm Principle analysed in ([12]) is sufficient to characterise the leximin social welfare ordering. This result is interesting because the weak Harm Principle captures liberal, noninterfering views even more clearly than the original Harm Principle. Further, based on the weak Harm Principle, a new characterisation of the *maximin* social welfare ordering is provided. Second, this paper analyses the maximin and the leximin in the context of societies with an infinite number of agents. This is arguably a crucial task for egalitarians. In fact, despite Rawls’s claims to the contrary, there is no compelling reason to restrict the application of the difference principle to intra-generational justice. In the intergenerational context, a basic concern for impartiality arguably implies that principles of justice be applied to all present and future generations. The extension to the case with an infinite number of generations, however, is problematic for *all* the main approaches, and indeed impossibility results easily obtain, for there exists no social welfare ordering that satisfies the standard axioms of Anonymity and Strong Pareto (see [11]). A number of recent contributions have provided characterisation results for social welfare relations by dropping either completeness (see, among the others, [4], [2], [5], [7], [3]) or transitivity (see, e.g., [18]). But the definition of suitable anonymous and paretian social welfare relations is still an open question in the infinite context (for a thorough discussion, see [1]).

In this tradition, this paper provides various new characterisations of the maximin and the leximin social welfare relations, based on the weak Harm principle in economies with an infinite number of agents. Although various formal frameworks and definitions have been proposed to analyse infinitely-lived societies, it is shown that the weak Harm Principle can be used to derive interesting results in all the main approaches.

The rest of the paper is structured as follows. Section 2 lays out the basic framework of analysis. Section 3 characterises the leximin and the maximin social welfare orderings in economies with a finite number of agents. Section 4 provides a number of characterisation results for leximin and maximin social welfare relations in societies with an infinite number of agents, in various different frameworks. Section 5 briefly concludes.

2 The framework

Let $X \equiv \mathbb{R}^{\mathbb{N}}$ be the set of countably infinite utility streams, where \mathbb{R} is the set of real numbers and \mathbb{N} is the set of natural numbers. An element of X is ${}_1u = (u_1, u_2, \dots)$ and u_t is the utility level of a representative member of generation $t \in \mathbb{N}$. For $T \in \mathbb{N}$, ${}_1u_T = (u_1, \dots, u_T)$ denotes the T -head of ${}_1u$ and ${}_{T+1}u = (u_{T+1}, u_{T+2}, \dots)$ denotes its T -tail, so that ${}_1u = ({}_1u_T, {}_{T+1}u)$; ${}_1\underline{u}_T$ denotes the welfare level of the worst-off generation of the T -head of ${}_1u$, and $\min({}_1u) = \min(\{u_1, u_2, \dots\})$ denotes the welfare level of the worst-off generation of ${}_1u$ whenever it exists. ${}_{con}\epsilon$ denotes the stream of constant level of well-being equal to $\epsilon \in \mathbb{R}$: for the sake of notational simplicity, the T -head of ${}_{con}\epsilon$ is denoted as ${}_1\epsilon_T$. A permutation π is a bijective mapping of \mathbb{N} onto itself. A permutation π of \mathbb{N} is finite if there is $T \in \mathbb{N}$ such that $\pi(t) = t, \forall t > T$, and Π is the set of all finite permutations of \mathbb{N} . For any ${}_1u \in X$ and any $\pi \in \Pi$, let $\pi({}_1u) = (u_{\pi(t)})_{t \in \mathbb{N}}$ be a permutation of ${}_1u$. For any $T \in \mathbb{N}$ and ${}_1u \in X$, ${}_1\bar{u}_T$ is a permutation of ${}_1u_T$ such that the components are ranked in ascending order.

For any ${}_1u, {}_1v \in X$, we write ${}_1u \geq {}_1v$ to mean $u_t \geq v_t, \forall t \in \mathbb{N}$; ${}_1u > {}_1v$ to mean ${}_1u \geq {}_1v$ and ${}_1u \neq {}_1v$; and ${}_1u \gg {}_1v$ to mean $u_t > v_t, \forall t \in \mathbb{N}$.

Let \succsim be a (binary) relation over X . For any ${}_1u, {}_1v \in X$, we write ${}_1u \succsim {}_1v$ for $({}_1u, {}_1v) \in \succsim$ and ${}_1u \not\succeq {}_1v$ for $({}_1u, {}_1v) \notin \succsim$; \succsim stands for “at least as good as”. The asymmetric factor \succ of \succsim is defined by ${}_1u \succ {}_1v$ if and only if ${}_1u \succsim {}_1v$ and ${}_1v \not\succeq {}_1u$, and the symmetric part \sim of \succsim is defined by ${}_1u \sim {}_1v$ if and only if ${}_1u \succsim {}_1v$ and ${}_1v \succsim {}_1u$. They stand, respectively, for “strictly better than” and “indifferent to”. A relation \succsim on X is said to be: *reflexive* if, for any ${}_1u \in X$, ${}_1u \succsim {}_1u$; *complete* if, for any ${}_1u, {}_1v \in X$, ${}_1u \not\succeq {}_1v$ implies ${}_1u \succsim {}_1v$ or ${}_1v \succsim {}_1u$; *transitive* if, for any ${}_1u, {}_1v, {}_1w \in X$, ${}_1u \succsim {}_1v \succsim {}_1w$ implies ${}_1u \succsim {}_1w$. \succsim is a quasi-ordering if it is reflexive and transitive, while \succ is an ordering if it is a complete quasi-ordering. Let \succsim and \succsim' be relations on X : \succsim' is an extension of \succsim if $\succsim \subseteq \succsim'$ and $\succ \subseteq \succ'$.

If there are only a finite set $\{1, \dots, T\} = N \subset \mathbb{N}$ of agents, or generations, X_T denotes the set of utility streams of X truncated at $T = |N|$, where $|N|$ is the cardinality of N . In order to simplify the notation, in economies with a finite number of agents we write u for ${}_1u_T$. With the obvious adaptations, the notation spelled out above is carried over utility streams in X_T .

3 Egalitarian Principles in Societies with a Finite Number of Agents

This section analyses liberal egalitarianism in societies with a finite number of agents. First, the characterisation of the leximin social welfare ordering (henceforth, SWO) derived by Mariotti and Veneziani ([12], Theorem 1, p.126) is strengthened by weakening the main axiom incorporating a liberal view of noninterference, the Harm Principle. Then, based on the weak Harm Principle, a novel characterisation of Rawls's difference principle, as formalised in the maximin SWO, is provided.

3.1 The Leximin

According to the leximin, that society is best which lexicographically maximises the welfare of its worst-off members. Formally, the *leximin relation* $\succsim^{LM} = \succ^{LM} \cup \sim^{LM}$ on X_T is defined as follows. The asymmetric factor \succ^{LM} of \succsim^{LM} is defined by:

$$u \succ^{LM} v \Leftrightarrow \bar{u}_1 > \bar{v}_1 \text{ or } [\exists i \in N \setminus \{1\} : \bar{u}_j = \bar{v}_j (\forall j \in N : j < i) \text{ and } \bar{u}_i > \bar{v}_i].$$

The symmetric factor \sim^{LM} of \succsim^{LM} is defined by:

$$u \sim^{LM} v \Leftrightarrow \bar{u}_i = \bar{v}_i, \forall i \in N.$$

\succsim^{LM} is easily shown to be an ordering. Classic analyses of the leximin SWO typically involve the following three axioms (see [9]).

STRONG PARETO OPTIMALITY, **SPO**: $\forall u, v \in X_T : u > v \Rightarrow u \succ v$.

ANONYMITY, **A**: $\forall u \in X_T : \exists \pi \text{ of } N \Rightarrow u \sim \pi(u)$.

HAMMOND EQUITY, **HE**. $\forall u, v \in X_T : u_i < v_i < v_j < u_j \exists i, j \in N, u_k = v_k \forall k \in N \setminus \{i, j\} \Rightarrow v \succ u$.

The first two axioms are standard in social choice theory and need no further comment. It is important to note, instead, that **HE** expresses a clear concern for equality in welfare distributions, for it asserts that among any two welfare allocations which differ only in two components, society should prefer the more egalitarian one. The classic characterisation by Hammond ([9]) states that a SWO is the leximin ordering if and only if it satisfies **SPO**, **A**, and **HE**.¹

In a recent contribution, Mariotti and Veneziani ([12]) drop **HE** and introduce a new axiom, called the Harm Principle (**HP**), which is meant to capture a liberal view of noninterference whenever

¹See also the related Hammond ([10]) and the generalisation by Tungodden ([20]; [21]).

individual choices have no effect on others. To be precise, starting from two welfare allocations u and v for which u is socially preferred to v , consider two different allocations u' and v' such that agent i is worse off at these than at the corresponding starting allocations, the other agents are equally well off, and agent i prefers u' to v' . The decrease in agent i 's welfare may be due to her negligence or her bad luck, but in any case **HP** states that society's preference over u' and v' should coincide with i 's preferences. In this sense, **HP** requires that having already suffered a welfare loss in both allocations, agent i should not be punished in the SWO by changing social preferences against her. This seems a rather intuitive way of capturing a liberal view of noninterference, and the name of the axiom is meant to echo John Stuart Mill's famous formulation in his essay *On Liberty* (see [22], and the discussion in [12]). Yet, although it has no explicit egalitarian content, quite surprisingly, Mariotti and Veneziani ([12], Theorem 1, p.126) prove that, jointly with **SPO** and **A**, **HP** characterises the leximin SWO.

In this paper, the implications of liberal, noninterfering views in social choice are explored further. As a first step, though, a weaker version of **HP** is presented, which can be formally stated as follows.

WEAK HARM PRINCIPLE, **WHP**: $\forall u, v, u', v' \in X_T : u \succ v$ and u', v' are such that, $\exists i \in N$,

$$\begin{aligned} u'_i &< u_i \\ v'_i &< v_i \\ u'_j &= u_j \quad \forall j \in N \setminus \{i\} \\ v'_j &= v_j \quad \forall j \in N \setminus \{i\} \end{aligned}$$

implies $v' \not\succeq u'$ whenever $u'_i > v'_i$.

WHP weakens the axiom proposed by Mariotti and Veneziani ([12]) in that it does not require that society's preferences over u' and v' be identical with agent i 's, but only that society should *not reverse* the strict preference between ${}_1u$ and ${}_1v$ to a strict preference for ${}_1v'$ over ${}_1u'$ (possibly except when i prefers otherwise). In this sense, the liberal content of **WHP**, and the requirement that agent i should not be punished in the SWO by changing social preferences against her, is even clearer, and **WHP** strongly emphasises the negative prescription of the Harm Principle. The surprising characterisation result provided in ([12]) can then be strengthened.

Theorem 1 *A SWO \succsim on X_T is the leximin ordering if and only if it satisfies Anonymity (**A**), Strong Pareto Optimality (**SPO**), and the Weak Harm Principle (**WHP**).*

Proof. (\Rightarrow) Let \succsim on X_T be the leximin ordering, i.e., $\succsim = \succsim^{LM}$. Since **WHP** is weaker than **HP**, the proof that \succsim^{LM} on X_T meets **SPO**, **A**, and **WHP** follows from the proof of necessity in ([12], Theorem 1, p.126).

(\Leftarrow) Let \succsim on X_T be a SWO satisfying **SPO**, **A**, and **WHP**. We show that \succsim on X_T is the leximin SWO. Thus, we should prove that, $\forall u, v \in X_T$,

$$u \sim^{LM} v \Leftrightarrow u \sim v \quad (3.1)$$

and

$$u \succ^{LM} v \Leftrightarrow u \succ v \quad (3.2)$$

First, we prove the implication \Rightarrow of 3.1. If $u \sim^{LM} v$, then $\bar{u} = \bar{v}$, and so $u \sim v$, by **A**.

Next, we prove the implication \Rightarrow of 3.2. Suppose that $u \succ^{LM} v$, and so, by definition $\exists t \in \{1, \dots, T\}$ such that $\bar{u}_s = \bar{v}_s \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Suppose, by contradiction, that $v \succ u$. Note that since \succsim satisfies **A**, in what follows we can focus, without loss of generality, either on u and v , or on the ranked vectors \bar{u} and \bar{v} . Therefore, suppose $\bar{v} \succ \bar{u}$. As **SPO** holds it must be the case that $\bar{v}_l > \bar{u}_l$ for some $l > t$. Let

$$k = \min\{t < l \leq T \mid \bar{v}_l > \bar{u}_l\}.$$

By **A**, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_{k-g}$, for some $1 \leq g < k$, where $\bar{u}_{k-g} > \bar{v}_{k-g}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors u', v' and the corresponding ranked vectors \bar{u}', \bar{v}' in X formed from \bar{u}, \bar{v} as follows: first, \bar{u}_{k-g} is lowered to $\bar{u}_{k-g} - d_1$ such that $\bar{u}_{k-g} - d_1 > \bar{v}_{k-g}$; next, \bar{v}_k is lowered to $\bar{v}_k - d_2$ such that $\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_{k-g} - d_1$; finally, all other entries of \bar{u} and \bar{v} are unchanged. By construction $\bar{u}'_j \geq \bar{v}'_j$ for all $j \leq k$, with at least two inequalities, $\bar{u}'_{k-g} > \bar{v}'_{k-g}$ and $\bar{u}'_k > \bar{v}'_k$, whereas **WHP**, combined with **A**, implies $\bar{v}' \succ \bar{u}'$. By **SPO**, $d_1, d_2 > 0$ can be chosen so that $\bar{v}' \succ \bar{u}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{u}_k$, but $\bar{u}_l \geq \bar{v}_l$ for all $l > k$. It follows that $\bar{u}' > \bar{v}'$, and so **SPO** implies that $\bar{u}' \succ \bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{u}_l$ for some $l > k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{u}'_l = \bar{u}_l$ for all $l > k$. Then, let

$$k' = \min\{k < l \leq T \mid \bar{v}'_l > \bar{u}'_l\}.$$

where $k' > k$. The above argument can be applied to \bar{u}', \bar{v}' to derive vectors \bar{u}'', \bar{v}'' such that $\bar{u}''_j \geq \bar{v}''_j$ for all $j \leq k'$, whereas **WHP**, combined with **A** and **SPO**, implies $\bar{v}'' \succ \bar{u}''$. And so on. After a finite number of iterations s , two vectors \bar{u}^s, \bar{v}^s can be derived such that, by **WHP**, combined with **A** and **SPO**, we have that $\bar{v}^s \succ \bar{u}^s$, but **SPO** implies $\bar{u}^s \succ \bar{v}^s$, yielding a contradiction.

We have proved that if $u \succ^{LM} v$ then $u \succ v$. Suppose now, by contradiction, that $v \sim u$, or equivalently $\bar{v} \sim \bar{u}$. Since, by our supposition, $\bar{v}_t < \bar{u}_t$, there exists $\epsilon > 0$ such that $\bar{v}_t < \bar{u}_t - \epsilon < \bar{u}_t$. Let ${}_1\bar{u}^\epsilon \in X$ be a vector such that $\bar{u}_t^\epsilon = \bar{u}_t - \epsilon$ and $\bar{u}_j^\epsilon = \bar{u}_j$ for all $j \neq t$. It follows that $\bar{u}^\epsilon \succ^{LM} \bar{v}$ but $\bar{v} \succ \bar{u}^\epsilon$ by **SPO** and the transitivity of \succ . Hence, the above argument can be applied to \bar{v} and \bar{u}^ϵ , yielding the desired contradiction. ■

The properties in Theorem 1 are clearly independent.

Theorem 1 has a number of interesting theoretical implications. First of all, Theorem 1 implies that **HE** and **WHP** are equivalent in the presence of **A** and **SPO**, even though they are completely independent. Actually, it can be proved that if $N = \{1, 2\}$, then in the presence of **SPO**, **HE** implies **WHP**, but the converse is not true. This implies that the above characterisation is far from trivial, given that, at least in some cases, and if **SPO** is assumed, **HE** is actually stronger than **WHP**. Secondly, and perhaps more interestingly, Theorem 1 puts the normative foundations of leximin under a rather different light. For, unlike in standard results, the egalitarian SWO is characterised without appealing to any axioms with a clear egalitarian content. Actually, it is easily shown that **SPO** and **WHP** alone are compatible with some of the least egalitarian SWOs, namely the lexicographic dictatorships, which forcefully shows that **WHP** imposes no significant egalitarian restriction. As a result, Theorem 1 interestingly shows the normative strength of the Anonymity axiom in determining the egalitarian outcome, an important insight which is not obvious in standard characterisations based on **HE**.

The main implication of Theorem 1, however, is that it proves that the core intuition of Mariotti and Veneziani ([12], Theorem 1, p.126) concerning the implications of liberal noninterfering views is robust: a strongly egalitarian SWO can be characterised with an even weaker axiom that only incorporates a liberal view of non-interference. In the next sections, this intuition is extended further and it is shown that the counterintuitive implications of liberal noninterfering principles in terms of egalitarian orderings are quite general and robust. Analogous characterisations of a whole *family* of principles inspired by Rawls's theory are obtained in societies with both finite and infinite populations, based on the weak Harm Principle.

3.2 The Difference Principle

The *maximin relation* $\succ^M = \succ^M \cup \sim^M$ on X_T is defined as follows. The asymmetric factor \succ^M of \succ^M is defined by:

$$u \succ^M v \Leftrightarrow \bar{u}_1 > \bar{v}_1.$$

The symmetric factor \sim^M of \succsim^M is defined by:

$$u \sim^M v \Leftrightarrow \bar{u}_1 = \bar{v}_1.$$

\succsim^M is easily shown to be an ordering. The maximin SWO formalises Rawls's difference principle. As is well-known, the maximin does not satisfy **SPO**, and therefore the following, weaker axiom is imposed on the SWO.

WEAK PARETO OPTIMALITY, WPO: $\forall u, v \in X_T : u \gg v \Rightarrow u \succ v$.

Second, a continuity axiom is imposed, which represents a standard interprofile condition requiring the SWO to vary continuously with variations in utility streams. This axiom is common in characterisations of the maximin SWO (see, e.g., [6]).

CONTINUITY, C: $\forall u \in X_T, \{v \in X_T | v \succ u\}$ is closed and $\{v \in X_T | u \succ v\}$ is closed.

The next Theorem shows that the combination of Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and the Weak Harm Principle (**WHP**) characterises the maximin SWO.

Theorem 2 *A SWO \succsim on X_T is the maximin ordering if and only if it satisfies Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and the Weak Harm Principle (**WHP**).*

Proof. (\Rightarrow) Let \succsim on X_T be the maximin ordering, i.e., $\succsim = \succsim^M$. It can be easily verified that \succsim^M on X_T satisfies **WPO**, **A**, **C**, and **WHP**.

(\Leftarrow) Let \succsim on X_T be a SWO satisfying **A**, **WPO**, **WHP**, and **C**. We show that \succsim is the maximin SWO. We shall prove that, $\forall u, v \in X_T$,

$$u \succ^M v \Leftrightarrow u \succ v \tag{3.3}$$

and

$$u \sim^M v \Leftrightarrow u \sim v. \tag{3.4}$$

Note that as \succsim on X_T satisfies **A**, in what follows we can focus either on u and v , or on the ranked vectors \bar{u} and \bar{v} , without loss of generality.

First, we show that the implication (\Rightarrow) of (3.3) is satisfied. Take any $u, v \in X_T$. Suppose that $u \succ^M v \Leftrightarrow \bar{u}_1 > \bar{v}_1$ and assume, by contradiction, that $v \succ u$, or equivalently, $\bar{v} \succ \bar{u}$. As **WPO** holds, $\bar{v}_j \geq \bar{u}_j$ for some $j \in N$, otherwise a contradiction immediately obtains. We proceed according to the following steps.

Step 1. Let

$$k = \min \{l \in N | \bar{v}_l \geq \bar{u}_l\}.$$

By **A**, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_1$. Then, consider two real numbers $d_1, d_2 > 0$, and two vectors u', v^* - together with the corresponding ranked vectors $\bar{u}', \bar{v}^* \in X_T$ - formed from \bar{u}, \bar{v} as follows: \bar{u}_1 is lowered to $\bar{u}_1 - d_1 > \bar{v}_1$; \bar{v}_k is lowered to $\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_1 - d_1$; and all other entries of \bar{u} and \bar{v} are unchanged. By construction $\bar{u}'_j > \bar{v}^*_j$ for all $j \leq k$, whereas by **WHP** and **A**, we have $\bar{v}^* \succ \bar{u}'$.

Step 2. Let

$$0 < \epsilon < \min\{\bar{u}'_j - \bar{v}^*_j \mid \forall j \leq k\}$$

and define $\bar{v}' = \bar{v}^* + \epsilon$. By construction, $\bar{v}^* \ll \bar{v}'$, and $\bar{v}'_j < \bar{u}'_j$ for all $j \leq k$. **WPO** implies $\bar{v}' \succ \bar{v}^*$. As $\bar{v}^* \succ \bar{u}'$, by *step 1*, the transitivity of \succ implies $\bar{v}' \succ \bar{u}'$.

If $\bar{u}'_j > \bar{v}'_j$ for all $j \in N$, **WPO** implies $\bar{u}' \succ \bar{v}'$, a contradiction. Otherwise, let $\bar{v}'_l \geq \bar{u}'_l$ for some $l > k$. Then, let

$$k' = \min\{l \in N \mid \bar{v}'_l \geq \bar{u}'_l\}$$

where $k' > k$.

The above steps 1-2 can be applied to \bar{u}', \bar{v}' to derive vectors \bar{u}'', \bar{v}'' such that $\bar{u}''_j > \bar{v}''_j$ for all $j \leq k'$, whereas $\bar{v}'' \succ \bar{u}''$. By **WPO**, a contradiction is obtained whenever $\bar{u}''_j > \bar{v}''_j$ for all $j \in N$. Otherwise, let $\bar{v}''_l \geq \bar{u}''_l$ for some $l > k$. And so on. After a finite number s of iterations, two vectors \bar{u}^s, \bar{v}^s can be derived such that $\bar{v}^s \succ \bar{u}^s$, by steps 1-2, but $\bar{u}^s \succ \bar{v}^s$, by **WPO**, a contradiction.

Therefore, it must be $\bar{u} \succ \bar{v}$ whenever $\bar{u} \succ^M \bar{v}$. We have to rule out the possibility that $\bar{u} \sim \bar{v}$. We proceed by contradiction. Suppose that $\bar{u} \sim \bar{v}$. Since $\bar{v}_1 < \bar{u}_1$, there exists $\epsilon > 0$ such that $\bar{v}^\epsilon = \bar{v} + \epsilon$ and $\bar{v}_1^\epsilon < \bar{u}_1$ so that $\bar{u} \succ^M \bar{v}^\epsilon$. However, by **WPO** and transitivity of \succ it follows that $\bar{v}^\epsilon \succ \bar{u}$. Then the above reasoning can be applied to vectors \bar{v}^ϵ and \bar{u} to obtain the desired contradiction.

Now, we show that the implication (\Rightarrow) of (3.4) is met as well. Suppose that $\bar{u} \sim^M \bar{v} \Leftrightarrow \bar{u}_1 = \bar{v}_1$. Assume, to the contrary, that $\bar{u} \not\sim \bar{v}$. Without loss of generality, let $\bar{u} \succ \bar{v}$. By **A**, it must be $\bar{u} \neq \bar{v}$. As $\bar{u} \succ \bar{v}$, it follows from **C** that there exists neighborhoods $S(\bar{u})$ and $S(\bar{v})$ of \bar{u} and \bar{v} such that $u' \succ v'$ for all $u' \in S(\bar{u})$ and for all $v' \in S(\bar{v})$. Then, there exists $v' \in S(\bar{v})$ such that $v' \gg \bar{v}$ and $\bar{u} \succ v' \sim \bar{v}'$, so that $\bar{u} \succ \bar{v}'$ but $\bar{v}' \succ^M \bar{u}$. By the implication (3.3) proved above, it follows that $\bar{v}' \succ \bar{u}$, a contradiction. ■

The properties in Theorem 2 are clearly independent.²

²It is worth noting that a stronger characterisation result of the maximin SWO can be provided by replacing the standard continuity property **C** with the following weaker property, where ϵ denotes the stream of constant level of well-being equal to $\epsilon \in \mathbb{R}$:

WEAK CONTINUITY, **WC**: $\forall u, v \in X, u \succ v \Rightarrow \exists \epsilon > 0 : u \succ v + \epsilon, \exists \epsilon' > 0 : {}_1u - \epsilon' \succ {}_1v$.

The theoretical relevance of the latter result can be appreciated if Theorem 2 is compared with alternative characterisations. On the one hand, unlike axioms on informational comparability often used in the literature (see, e.g., [14], [15]), the weak Harm Principle has a clear ethical foundation, but, as noted above it has no obvious egalitarian implication. In a recent contribution, Bosmans and Ooghe ([6]) characterise the maximin SWO using Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and Hammond Equity (**HE**). Instead, as in the case of the leximin ordering analysed above, Theorem 2 characterises an egalitarian SWO such as the maximin without appealing to an axiom like **HE**, which arguably has a marked egalitarian content, and using instead **WHP**, which only incorporates a liberal, noninterfering view of society.

4 Egalitarian Principles in the Infinitely-Lived Society

In this section, the axiomatic analysis of the difference principle and of its lexicographic refinement is extended to infinitely-lived economies, focusing on the role of liberal views of noninterference as formulated in the weak Harm Principle. As is well-known, the case with an infinite number of agents raises a number of issues concerning the existence and the characterisation of SWOs (see, e.g., [1]) and different definitions of the leximin social welfare relation (henceforth, SWR) can be provided in order to compare (countably) infinite utility streams. In this section, first, the framework proposed by Asheim and Tungodden ([2]) is adopted, and an alternative characterisation of their leximin SWRs is provided. Then, a new characterisation of an infinite-horizon ordering extension of a leximin SWR recently proposed by Bossert, Sprumont and Suzumura ([7]) is provided. Finally, the framework of Asheim and Tungodden ([2]) is extended to analyse the maximin SWR, and a new characterisation of the difference principle is proposed in the context of infinitely-lived economies.

4.1 The Leximin SWR

Following Asheim and Tungodden, there are two different ways of formally defining the leximin SWR. The first one is the so-called ‘weak leximin’, or W-Leximin, and can be formalised as follows.

Definition 1 (*Definition 2, [2], p. 224*) For all ${}_1u, {}_1v \in X$, ${}_1u \sim^{LM*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: ${}_1\bar{u}_T = {}_1\bar{v}_T$, and ${}_1u \succ^{LM*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T} \exists t \in \{1, \dots, T\} \bar{u}_s = \bar{v}_s \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$.

The characterisation results below are based on a number of standard axioms. The first three axioms are similar to those used in the finite case, and need no further comment, except possibly

noting that in this context, **WHP** is weaker than the version in Section 2 above, since it only holds for vectors with the same tail.

FINITE ANONYMITY, **FA**: $\forall_1 u \in X$ and $\forall \pi \in \Pi$, $\pi(1u) \sim_1 u$.

STRONG PARETO OPTIMALITY, **SPO**: $\forall_1 u, v \in X : 1u >_1 v \Rightarrow 1u \succ_1 v$.

WEAK HARM PRINCIPLE, **WHP***: $\forall_1 u, v, u', v' \in X : \exists T \geq 1$ $1u = (1u_{T, T+1} v) \succ_1 v$, and $1u', v'$ are such that, $\exists i \leq T$,

$$\begin{aligned} u'_i &< u_i \\ v'_i &< v_i \\ u'_j &= u_j \quad \forall j \neq i \\ v'_j &= v_j \quad \forall j \neq i \end{aligned}$$

implies $1v' \not\succeq_1 u'$ whenever $u'_i > v'_i$.

Next, following Asheim and Tungodden ([2], p. 223), an axiom is imposed, which represents a mainly technical requirement to deal with infinite dimensional vectors.

WEAK PREFERENCE CONTINUITY, **WPC**: $\forall_1 u, v \in X : \exists \tilde{T} \geq 1$ such that $(1u_{\tilde{T}, \tilde{T}+1} v) \succ_1 v \quad \forall T \geq \tilde{T} \Rightarrow 1u \succ_1 v$.

WPC (and the same is true for the stronger **SPC** discussed below) provides a condition that establishes “a link to the standard finite setting of distributive justice, by transforming the comparison of any two infinite utility paths to an infinite number of comparisons of utility paths each containing a finite number of generations” ([2], p. 223). In the same vein, the next axiom states that the SWR should at least be able to compare (infinite-dimensional) vectors with the same tail. This seems an obviously desirable property which imposes a minimum requirement of completeness.

MINIMAL COMPLETENESS, **MC**: $\forall_1 u, v \in X$, $\exists T \geq 1$ $(1u_{T, T+1} v) \neq_1 v \Rightarrow (1u_{T, T+1} v) \succcurlyeq_1 v$ or $v \succcurlyeq_1 (1u_{T, T+1} v)$.

The next Theorem proves that the combination of Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Weak Preference Continuity (**WPC**), and Minimal Completeness (**MC**), characterises the leximin SWR.

Theorem 3 \succcurlyeq is an extension of \succcurlyeq^{LM*} if and only if \succcurlyeq satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Weak Preference Continuity (**WPC**), and Minimal Completeness (**MC**).

Proof. (\Rightarrow) Let $\succsim^{LM*} \subseteq \succsim$. It is easy to see that \succsim meets **FA** and **SPO**. By observing that \succsim^{LM*} is complete for comparisons between utility streams having the same tail it is also easy to see that \succsim satisfies **WPC** and **MC**. We show that \succsim meets **WHP***. Take any ${}_1u, {}_1v, {}_1u', {}_1v' \in X$ such that $\exists T \geq 1$ ${}_1u = ({}_1u_{T,T+1} v) \succ {}_1v$, and ${}_1u', {}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \forall j \neq i$, $v'_j = v_j \forall j \neq i$. We show that ${}_1u' \succ {}_1v'$ whenever $u'_i > v'_i$. As \succsim^{LM*} is complete for comparisons between utility streams having the same tail, it must be true that ${}_1u \succ^{LM*} {}_1v$. Therefore, by definition, $\exists \tilde{T} \geq 1$ such that $\forall T' \geq \tilde{T} \exists t \in \{1, \dots, T'\} \bar{u}_s = \bar{v}_s \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Take any $T' \geq \tilde{T}$. As $T' < \infty$ it follows from Theorem 1 in [[12], p. 126] that there exists $t^* \leq t \leq T'$ such that $\bar{u}'_s = \bar{v}'_s \forall 1 \leq s < t^*$ and $\bar{v}'_{t^*} < \bar{u}'_{t^*}$. As it holds true for any $T' \geq \tilde{T}$ it follows that ${}_1u' \succ {}_1v'$ as $\succsim^{LM*} \subseteq \succsim$.

(\Leftarrow) Suppose that \succsim satisfies **FA**, **SPO**, **WHP***, **WPC**, and **MC**. We show that $\sim^{LM*} \subseteq \sim$ and $\succ^{LM*} \subseteq \succ$. Take any ${}_1u, {}_1v \in X$.

Assume that ${}_1u \sim^{LM*} {}_1v$. By definition, $\exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T} {}_1\bar{u}_T = {}_1\bar{v}_T$, and so ${}_{T+1}u = {}_{T+1}v$, for any $T \geq \tilde{T}$. It follows that ${}_1u \sim {}_1v$, by **FA**.

Next, suppose that ${}_1u \succ^{LM*} {}_1v$, and so, by definition, $\exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T} \exists t \in \{1, \dots, T\}$ such that $\bar{u}_s = \bar{v}_s \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Take any such T and consider the vector ${}_1w \equiv ({}_1u_{T,T+1} v)$. We want to show that ${}_1w \succ {}_1v$. By **FA** and transitivity, we can consider ${}_1\bar{w} \equiv ({}_1\bar{u}_{T,T+1} v)$ and ${}_1\bar{v} \equiv ({}_1\bar{v}_{T,T+1} v)$. Suppose that ${}_1\bar{v} \succsim {}_1\bar{w}$. We distinguish two cases.

Case 1. ${}_1\bar{v} \succ {}_1\bar{w}$

As **SPO** holds it must be the case that $\bar{v}_l > \bar{w}_l$ for some $l > t$. Let

$$k = \min\{t < l \leq T | \bar{v}_l > \bar{w}_l\}.$$

Let $v_i = \bar{v}_k$ and let $w_i = \bar{w}_{k-g}$, for some $1 \leq g < k$, where $\bar{w}_{k-g} > \bar{v}_{k-g}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors ${}_1w', {}_1v'$ formed from ${}_1\bar{w}, {}_1\bar{v}$ as follows: \bar{w}_{k-g} is lowered to $\bar{w}_{k-g} - d_1$ such that $\bar{w}_{k-g} - d_1 > \bar{v}_{k-g}$; \bar{v}_k is lowered to $\bar{v}_k - d_2$ such that $\bar{w}_k > \bar{v}_k - d_2 > \bar{w}_{k-g} - d_1$; and all other entries of ${}_1\bar{w}$ and ${}_1\bar{v}$ are unchanged. By **FA**, consider ${}_1\bar{w}' = ({}_1\bar{w}'_{T,T+1} v)$ and ${}_1\bar{v}' = ({}_1\bar{v}'_{T,T+1} v)$. By construction $\bar{w}'_j \geq \bar{v}'_j$ for all $j \leq k$, with at least two inequalities, $\bar{w}'_{k-g} > \bar{v}'_{k-g}$ and $\bar{w}'_k > \bar{v}'_k$, whereas **WHP***, combined with **MC** and **FA**, implies $\bar{v}' \succ \bar{w}'$. Furthermore, by **SPO**, it is possible to choose $d_1, d_2 > 0$, such that $\bar{v}' \succ \bar{w}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{w}_k$, but $\bar{w}_l \geq \bar{v}_l$ for all $l > k$. It follows that ${}_1\bar{w}' > {}_1\bar{v}'$, and so **SPO** implies that ${}_1\bar{w}' \succ {}_1\bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{w}_l$ for some $l > k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{w}'_l = \bar{w}_l$ for all $l > k$. Then, let

$$k' = \min\{k < l \leq T | \bar{v}'_l > \bar{w}'_l\}.$$

where $k' > k$. The above argument can be applied to ${}_1\bar{w}', {}_1\bar{v}'$ to derive vectors ${}_1\bar{w}'', {}_1\bar{v}''$ such that $\bar{w}_j'' \geq \bar{v}_j''$ for all $j \leq k'$, whereas **WHP***, combined with **MC**, **FA**, and **SPO**, implies ${}_1\bar{v}'' \succ {}_1\bar{w}''$. And so on. After a finite number of iterations s , two vectors ${}_1\bar{w}^s, {}_1\bar{v}^s$ can be derived such that, by **WHP***, combined with **MC**, **FA**, and **SPO**, we have that ${}_1\bar{v}^s \succ {}_1\bar{w}^s$, but **SPO** implies ${}_1\bar{w}^s \succ {}_1\bar{v}^s$, yielding a contradiction.

Case 2. $\bar{v} \sim \bar{w}$

Since, by our supposition, $\bar{v}_t < \bar{u}_t \equiv \bar{w}_t$, there exists $\epsilon > 0$ such that $\bar{v}_t < \bar{w}_t - \epsilon < \bar{w}_t$. Let ${}_1\bar{w}^\epsilon \in X$ be a vector such that $\bar{w}_t^\epsilon = \bar{w}_t - \epsilon$ and $\bar{w}_j^\epsilon = \bar{w}_j$ for all $j \neq t$. It follows that ${}_1\bar{w}^\epsilon \succ^{LM*} {}_1\bar{v}$ but ${}_1\bar{v} \succ {}_1\bar{w}^\epsilon$ by **SPO** and the transitivity of \succ . Hence, the argument of *Case 1* above can be applied to ${}_1\bar{v}$ and ${}_1\bar{w}^\epsilon$, yielding the desired contradiction.

It follows from **MC** that ${}_1\bar{w} \succ {}_1\bar{v}$. **FA**, combined with the transitivity of \succ , implies that $({}_1u_T, {}_{T+1}v) \succ {}_1v$. Since it holds true for any $T \geq \tilde{T}$, **WPC** implies ${}_1u \succ {}_1v$, as desired. ■

The properties in Theorem 3 are easily shown to be independent (see Appendix).

It is worth stressing again that in societies with an infinite number of agents, or generations, there is no obvious, and unanimously accepted, definition of the leximin SWR. Asheim and Tungodden ([2], p. 224), for example, provide an alternative, stronger definition of the leximin - the S-Leximin - that can be formalised as follows.

Definition 2 (*Definition 1, [2], p. 224*) For all ${}_1u, {}_1v \in X$, ${}_1u \succ_S^{LM*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: either ${}_1\bar{u}_T = {}_1\bar{v}_T$ or $\exists t \in \{1, \dots, T\}$: $\bar{u}_s = \bar{v}_s \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$.

The above analysis has focused on the W-Leximin, because the continuity axiom **WPC** is arguably more appealing than the Strong Preference Continuity property adopted by Asheim and Tungodden ([2], p. 223) to characterise the S-leximin, which seems a rather strong requirement (as forcefully argued, for example, by Basu and Mitra [5], p. 358). Strong Preference Continuity can be formalised as follows.

STRONG PREFERENCE CONTINUITY, SPC: $\forall {}_1u, {}_1v \in X$: $\exists \tilde{T} \geq 1$ such that $({}_1u_T, {}_{T+1}v) \succ {}_1v \forall T \geq \tilde{T}$, and $\forall \tilde{T} \geq 1 \exists T \geq \tilde{T}$ such that $({}_1u_T, {}_{T+1}v) \succ {}_1v \Rightarrow {}_1u \succ {}_1v$.

A result analogous to Theorem 3 can be established for the stronger definition 2 by replacing **WPC** with **SPC**. It can be easily obtained through a trivial modification of the parts of the proof of Theorem 3 that involve **WPC**, and by observing that the necessity of **WHP*** can be easily established along the same lines as in Theorem 3.

Theorem 4 \succsim is an extension of $\succsim_S^{LM^*}$ if and only if \succsim satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Strong Preference Continuity (**SPC**), and Minimal Completeness (**MC**).

The properties in Theorem 4 are easily shown to be independent (see Appendix).

Theorems 3 and 4 identify the relevant class of leximin SWRs by postulating a continuity property on the quasi-ordering (respectively, **WPC** and **SPC**), which represents a mainly technical requirement in ranking infinite utility streams. As axioms such as **SPO** and **FA** may be considered ethically more defensible than continuity axioms, Bossert, Sprumont and Suzumura ([7]) have not postulated any continuity property on the quasi-ordering and have provided a characterisation of a subclass of the class of orderings satisfying **SPO**, **FA**, and an infinite version of **HE**. Formally, the relationship between the characterisation of the leximin by Bossert et al. ([7]) and that by Asheim and Tungodden ([2]) is analogous to the relationship between the characterisation of the utilitarian SWR by Basu and Mitra ([5]) and the characterisations of the more restrictive utilitarian SWR induced by the overtaking criterion (see the discussion in [7], p.580). This relationship is explored below by extending the analysis of **WHP*** to the framework developed by Bossert et al. ([7]).

For each $T \in \mathbb{N}$, let the leximin ordering on X_T be denoted as \succsim_T^{LM} . The definition of the leximin SWR proposed by Bossert et al. ([7]) can be formulated as follows. Define a relation $\succsim_T^L \subseteq X \times X$ by letting, for all ${}_1u, {}_1v \in X$,

$${}_1u \succsim_T^L {}_1v \Leftrightarrow {}_1u_T \succsim_T^{LM} {}_1v_T \text{ and } {}_{T+1}u \geq {}_{T+1}v. \quad (4.1)$$

The relation \succsim_T^L can be shown to be reflexive and transitive for all $T \in \mathbb{N}$. Then the leximin SWR is $\succsim^L = \bigcup_{T \in \mathbb{N}} \succsim_T^L$ ([7], p. 586): it is reflexive and transitive, but not necessarily complete. Moreover, Bossert et al. ([7]) show that \succsim^L satisfies the following property ([7], p. 586, equation (14)):

$$\forall {}_1u, {}_1v \in X : \exists T \in \mathbb{N} \text{ such that } {}_1u \succ_T^L {}_1v \Leftrightarrow {}_1u \succ^L {}_1v. \quad (4.2)$$

The set of ordering extensions of \succsim^L characterised by the next theorem, based on Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), and the weak Harm Principle (**WHP***), is shown to be non-empty.

Theorem 5 \succsim is an extension of \succsim^L if and only if \succsim satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), and Weak Harm Principle (**WHP***).

Proof. (\Rightarrow) The proof that any ordering extension of \succsim^L satisfies **FA** and **SPO** is as in ([7], Theorem 2, p. 586). We only need to prove that any ordering extension \succ of \succsim^L satisfies **WHP***. Consider any ${}_1u, {}_1v, {}_1u', {}_1v' \in X$ such that $\exists T \geq 1$ ${}_1u = ({}_1u_{T,T+1} v) \succ {}_1v$, and ${}_1u', {}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \forall j \neq i$, $v'_j = v_j \forall j \neq i$. We show that ${}_1u' \succ {}_1v'$ whenever $u'_i > v'_i$. Since \succsim_T^{LM} is complete and ${}_{T+1}v = {}_{T+1}u$ it cannot be ${}_1v_T \succsim_T^{LM} {}_1u_T$, otherwise $({}_1v, {}_1u) \in \succsim^L \subseteq \succ$ which contradicts ${}_1u \succ {}_1v$. Thus, we have that ${}_1u_T \succsim_T^{LM} {}_1v_T$, ${}_1v_T \not\succeq_T^{LM} {}_1u_T$, and ${}_{T+1}v = {}_{T+1}u$, so that $({}_1u, {}_1v) \in \succ_T^L$ by (4.1). It follows from (4.2) that $({}_1u, {}_1v) \in \succ^L$. As ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \forall j \neq i$, $v'_j = v_j \forall j \neq i$, it can easily be shown, as in ([12]), that ${}_1u'_T \succ_T^{LM} {}_1v'_T$ whenever $u'_i > v'_i$. As ${}_{T+1}v' = {}_{T+1}u'$ and ${}_1u'_T \succ_T^{LM} {}_1v'_T$ it follows from (4.1) that ${}_1u' \succ_T^L {}_1v'$, and therefore ${}_1u' \succ^L {}_1v'$ by (4.2). But since \succ is an ordering extension of \succsim^L it follows that ${}_1u' \succ {}_1v'$.

(\Leftarrow) The proof is identical to ([7], Theorem 2, p. 587) using the characterisation of the T -person leximin in Theorem (1). ■

Theorem 5 leaves a larger class of orderings than that identified by Theorems 3 or 4, because in the latter results two additional axioms are employed, but it is strikingly similar to the characterisation in the finite context.

Finally, it is worth noting that the Weak Harm Principle (**WHP***) can also be used to characterise the intergenerational version of the leximin SWO recently proposed by Sakai ([18]), which drops transitivity but retains completeness. In particular, if one replaces Hammond Equity with **WHP***, a modified version of his characterisation results ([18], Lemma 6, p.17; and Theorem 5, p.18) can easily be proved.

4.2 The Maximin SWR

In this subsection, Rawls's difference principle is analysed in the context of economies with an infinite number of agents. First of all, the analysis focuses on the subset of utility streams that reach a minimum in a finite period. Formally, define the following subset Y of X :

$$Y = \{ {}_1u \in X \mid \exists T' \geq 1: {}_1u_T = {}_1u_{T'} \forall T \geq T' \}.$$

The maximin SWR can be formally defined as follows.

Definition 3 For all ${}_1u, {}_1v \in Y$, ${}_1u \sim^{M*} {}_1v \Leftrightarrow \min({}_1u) = \min({}_1v)$, and ${}_1u \succ^{M*} {}_1v \Leftrightarrow \min({}_1u) > \min({}_1v)$.

Let $\succsim^{M*} = \succ^{M*} \cup \sim^{M*}$. It is easy to show that \succsim^{M*} is a quasi-ordering on X and that \succsim^{M*}

is complete for any ${}_1u, {}_1v \in Y$. In the framework proposed by Asheim and Tungodden ([2]), Definition 3 has equivalent reformulations.

Proposition 6 *For all ${}_1u, {}_1v \in Y$, the following statements are equivalent:*

- (a) *either $\min({}_1u) = \min({}_1v)$ or $\min({}_1u) > \min({}_1v)$.*
- (b) *$\exists \tilde{T} \in \mathbb{N}$: either ${}_1\underline{u}_T = {}_1\underline{v}_T \forall T \geq \tilde{T}$ or ${}_1\underline{u}_T > {}_1\underline{v}_T \forall T \geq \tilde{T}$.*
- (c) *$\exists \tilde{T} \in \mathbb{N}$: either ${}_1\underline{u}_T = {}_1\underline{v}_T$ and $[{}_1\underline{u}_T = {}_1\underline{v}_{\tilde{T}} = {}_1\underline{v}_T \forall T \geq \tilde{T}]$ or ${}_1\underline{u}_T > {}_1\underline{v}_T$ and $[{}_1\underline{u}_T = {}_1\underline{u}_{\tilde{T}}$ and ${}_1\underline{v}_T = {}_1\underline{v}_{\tilde{T}} \forall T \geq \tilde{T}]$.*

Proof. Obvious, so omitted. ■

It is worth noting that the relevant \tilde{T} in part (b) may be different from that in part (c).

In order to prove the main characterisation result, the following four standard axioms are imposed, which are similar to those used in the finite setting, and need no further comment, except noting that **WC** is a weakening of standard continuity axioms. Continuity requires that if ${}_1u$ is strictly better than ${}_1v$, then *any* vector sufficient close to ${}_1u$ should be strictly better than any vector sufficient close to ${}_1v$. **WC** only requires the existence of *some* vectors with the latter property.

FINITE ANONYMITY, **FA***: $\forall {}_1u \in Y$ and $\forall \pi \in \Pi \Rightarrow \pi({}_1u) \sim {}_1u$.

WEAK PARETO OPTIMALITY, **WPO***: $\forall {}_1u, {}_1v \in Y, {}_1u \gg {}_1v \Rightarrow {}_1u \succ {}_1v$.

WEAK HARM PRINCIPLE, **WHP****: $\forall {}_1u, {}_1v, {}_1u', {}_1v' \in Y, {}_1u \succ {}_1v$ and ${}_1u', {}_1v'$ are such that $\exists i \in \mathbb{N}$,

$$\begin{aligned} u'_i &< u_i, \\ v'_i &< v_i, \\ u'_j &= u_j \forall j \neq i \\ v'_j &= v_j \forall j \neq i \end{aligned}$$

implies ${}_1v' \not\succeq {}_1u'$ whenever $u'_i > v'_i$.

WEAK CONTINUITY, **WC**: $\forall {}_1u, {}_1v \in Y, {}_1u \succ {}_1v \Rightarrow \exists \epsilon > 0 : {}_1u \succ {}_1v + \text{con}\epsilon, \exists \epsilon' > 0 : {}_1u - \text{con}\epsilon' \succ {}_1v$.

In addition to the above requirements, following again Asheim and Tungodden ([2]), a weak consistency requirement on the symmetric part of \succ is imposed.

WEAK INDIFFERENCE CONTINUITY, **WIC**: $\forall {}_1u, {}_1v \in Y, \exists T' \geq 1$ such that ${}_1u \sim ({}_1v_{T, T+1} u) \forall T \geq T' \Rightarrow {}_1u \sim {}_1v$.

In analogy with **WPC**, the latter axiom represents a mainly technical requirement that provides a link to the standard finite setting of distributive justice. **WIC** imposes that “an infinite utility

path should be considered indifferent to another infinite utility path if the head of the former is considered indifferent to the latter at every point in time beyond a certain initial phase" ([2], p.229). Axioms similar to **WIC** are common in the literature (see, e.g., [5], [1]).

Finally, the next axiom requires that \succsim be complete at least when comparing elements of Y with the same tail: this is an even weaker completeness requirement than **MC**, and it seems uncontroversial, for it is obviously desirable to be able to rank as many vectors as possible.

MINIMAL COMPLETENESS, **MC***: $\forall_1 u, {}_1 v \in Y, {}_1 u \neq {}_1 v \Rightarrow ({}_1 u_{T, T+1} v) \succsim {}_1 v$ or ${}_1 v \succsim ({}_1 u_{T, T+1} v)$.

In order to derive the main characterisation result concerning the maximin SWR, first two useful Lemmas are proved. The next result states that any SWR satisfying **WC**, **MC***, and **WPO*** also satisfies monotonicity.

Lemma 7 *Let \succsim on X be a SWR satisfying **WC**, **MC***, and **WPO***. Then, $\forall_1 u, {}_1 v \in Y : \exists T \geq 1$ ${}_1 u > ({}_1 v_{T, T+1} u) \Rightarrow {}_1 u \succsim ({}_1 v_{T, T+1} u)$.*

Proof. See appendix. ■

Lemma 8 provides sufficient conditions to implement some type of Pareto-rankings. In particular, it proves that, for a subset of allocations in Y , there is a finite number T' such that, starting from a given allocation, if the welfare of at least the first T' generations increases uniformly, then the new allocation is strictly preferred.

Lemma 8 *Let \succsim on X be a SWR satisfying **FA***, **WPO***, **WC**, **WHP****, **WIC**, and **MC***. Let ${}_1 u, {}_1 v \in Y$ be such that ${}_1 u \succ^{M*} {}_1 v$, let $\tilde{T} \in \mathbb{N}$ be such that ${}_1 v_{\tilde{T}} = {}_1 v_{T'}$ and ${}_1 u_{\tilde{T}} = {}_1 u_{T'}$ $\forall T' \geq \tilde{T}$, and let $\epsilon > 0$ be such that ${}_1 v_{T'} + \epsilon < {}_1 u_{T'}$. Then, $({}_1 v_{T'} + {}_1 \epsilon_{T', T'+1} u) \succ ({}_1 v_{T', T'+1} u)$.*

Proof. See appendix. ■

Given Lemmas 7 and 8, the next Theorem proves that the combination of Finite Anonymity (**FA***), Weak Pareto Optimality (**WPO***), Weak Harm Principle (**WHP****), Weak Continuity (**WC**), Weak Indifference Continuity (**WIC**), and Minimal Completeness (**MC***), characterises the maximin SWR.

Theorem 9 *\succsim on X is an extension of \succsim^{M*} on Y if and only if \succsim satisfies Finite Anonymity (**FA***), Weak Pareto Optimality (**WPO***), Weak Harm Principle (**WHP****), Weak Continuity (**WC**), Weak Indifference Continuity (**WIC**), and Minimal Completeness (**MC***).*

Proof. (\Rightarrow) It is easy to see that \succsim meets **FA***, **WPO***, **WHP****, **WC**, **WIC**, and **MC*** whenever \succsim is an extension of \succsim^{M*} .

(\Leftarrow) Suppose that \succ meets **FA***, **WPO***, **WHP****, **WC**, **WIC**, and **MC***. We show that $\succ^{M^*} \subseteq \succ$, that is, $\forall_{1u, 1v} v \in Y$,

$$1u \succ^{M^*} 1v \Rightarrow 1u \succ 1v \quad (4.3)$$

and

$$1u \sim^{M^*} 1v \Rightarrow 1u \sim 1v. \quad (4.4)$$

We prove (4.3) and (4.4) by showing: 1) that (4.3) holds for vector having the same T -tail; 2) that the implication (4.4) holds $\forall_{1u, 1v} v \in Y$; and, finally, 3) that the implication (4.3) is met $\forall_{1u, 1v} v \in Y$ too.

Proof of (1)

Take any $1u, 1v \in Y$ such that $\min(1u) > \min(1v)$. By proposition 6 it follows that $\exists \tilde{T} \geq 1$ such that $1u_{\tilde{T}} > 1v_{\tilde{T}}$ and $1u_T = 1u_{\tilde{T}}$ and $1v_T = 1v_{\tilde{T}} \forall T \geq \tilde{T}$. Take any $T \geq \tilde{T}$ and let $1w = (1v_{T, T+1} u)$. Observe that $1w \in Y$. We show that $1u \succ 1w$ since $1u \succ^{M^*} 1w$, by construction. Assume, to the contrary, that $1u \not\succ 1w$, so that $1w \succ 1u$ as the premises of **MC*** are met. We distinguish two cases.

Case 1: $1w \succ 1u$

As **FA*** holds, let $1\bar{w}, 1\bar{u}$ be such that $_{T+1}\bar{w} = _{T+1}w = _{T+1}u = _{T+1}\bar{u}$, and $1\bar{w}_T, 1\bar{u}_T$ are such that $\bar{w}_1 \leq \dots \leq \bar{w}_T$ and $\bar{u}_1 \leq \dots \leq \bar{u}_T$. If $1\bar{w}_T \ll 1\bar{u}_T$, Lemma 7 implies $1\bar{u} \succ 1\bar{w}$, a contradiction. Otherwise, let $\bar{w}_t \geq \bar{u}_t \exists t \leq T$. We proceed in two steps.

Step 1.

Let

$$k = \min \{l \leq T \mid \bar{w}_l \geq \bar{u}_l\}.$$

Let $1\hat{w}$ and $1\hat{u}$ be two finite permutations of \mathbb{N} such that $_{T+1}\bar{w} = _{T+1}\hat{w} = _{T+1}\hat{u} = _{T+1}\bar{u}$ and, for some $i \leq T$, $\hat{w}_i = \bar{w}_k$ and $\hat{u}_i = \bar{u}_1$. By **FA***, $1\hat{w} \sim 1\bar{w}$ and $1\hat{u} \sim 1\bar{u}$, so that $1\hat{w} \succ 1\hat{u}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors $1u', 1w'$ formed as follows: first, \hat{u}_i is lowered to $\hat{u}_i - d_1 > \bar{w}_1$; next, \hat{w}_i is lowered to $\hat{w}_i - d_2$ such that $\bar{u}_k > \hat{w}_i - d_2 > \hat{u}_i - d_1 > \bar{w}_1$; finally, all other entries of $1\hat{u}$ and $1\hat{w}$ are unchanged. It follows from **WHP**** and **MC***, that $1w' \succ 1u'$. Let $1\bar{w}'$ and $1\bar{u}'$ be two finite permutations of \mathbb{N} such that $_{T+1}\bar{w}' = _{T+1}w' = _{T+1}u' = _{T+1}\bar{u}'$ and $1\bar{w}'_T, 1\bar{u}'_T$ are such that $\bar{w}'_1 \leq \dots \leq \bar{w}'_T$ and $\bar{u}'_1 \leq \dots \leq \bar{u}'_T$. By construction, $\bar{u}'_j > \bar{w}'_j$ for all $j \leq k$. By **FA*** and transitivity, $1\bar{w}' \succ 1\bar{u}'$.

Step 2.

Let

$$0 < \epsilon < \min\{\bar{u}'_j - \bar{w}'_j \mid \forall j \leq k\}$$

and define ${}_1\bar{w}_T^\epsilon = {}_1\bar{w}'_T + {}_1\epsilon_T$, where ${}_1\epsilon_T$ is T -head of ${}_{con}\epsilon$. Let ${}_1\bar{w}^\epsilon \equiv ({}_1\bar{w}'_T, {}_{T+1}\bar{w}')$, and observe that ${}_1\bar{w}^\epsilon \in Y$. Lemma 8 implies ${}_1\bar{w}^\epsilon \succ {}_1\bar{w}'$. As ${}_1\bar{w}' \succcurlyeq {}_1\bar{u}'$, by *step 1*, transitivity of \succcurlyeq implies ${}_1\bar{w}^\epsilon \succcurlyeq {}_1\bar{u}'$.

If $\bar{u}'_j > \bar{w}'_j$ for all $j \leq T$, Lemma 7 implies ${}_1\bar{u}' \succcurlyeq {}_1\bar{w}^\epsilon$, a contradiction. Otherwise, let $\bar{w}'_l \geq \bar{u}'_l$ for some $T \geq l > k$. Then, let

$$k' = \min \{l \leq T \mid \bar{w}'_l \geq \bar{u}'_l\}$$

where $k' > k$.

The above steps 1-2 can be applied to ${}_1\bar{u}'$, ${}_1\bar{w}^\epsilon$ to derive vectors ${}_1\bar{u}''$, ${}_1\bar{w}''$ such that $\bar{u}''_j > \bar{w}''_j$ for all $j \leq k' \leq T$, whereas ${}_1\bar{w}'' \succcurlyeq {}_1\bar{u}''$. By Lemma 7, a contradiction is obtained whenever $\bar{u}''_j > \bar{w}''_j$ for all $j \leq T$. Otherwise, let $\bar{w}''_l \geq \bar{u}''_l$ for some $T \geq l > k'$. And so on. After a finite number s of iterations, two vectors ${}_1\bar{u}^s$, ${}_1\bar{w}^s$ can be derived such that ${}_1\bar{w}^s \succcurlyeq {}_1\bar{u}^s$, but ${}_1\bar{u}^s \succcurlyeq {}_1\bar{w}^s$, by Lemma 7, yielding a contradiction.

Case 2: ${}_1w \sim {}_1u$.

Since ${}_1w_T < {}_1u_T$, there exists $\epsilon > 0$ such that ${}_1w_T^\epsilon = {}_1w_T + \epsilon < {}_1u_T$. Then fix such $\epsilon > 0$ and let ${}_1w^\epsilon = ({}_1w_T + {}_1\epsilon_{T,T+1} w)$, where ${}_1\epsilon_T$ is the T -head of ${}_{con}\epsilon$. Thus, by construction, ${}_1u \succcurlyeq^{M^*} {}_1w^\epsilon$. By Lemma 8, ${}_1w^\epsilon \succcurlyeq {}_1w$, and thus by the transitivity of \succcurlyeq , ${}_1w^\epsilon \succcurlyeq {}_1u$ and so *Case 1* above obtains.

Proof of (2)

Take any ${}_1u, {}_1v \in Y$ such that $\min({}_1u) = \min({}_1v)$. By proposition 6 it follows that $\exists \tilde{T} \geq 1$ such that ${}_1u_{\tilde{T}} = {}_1v_{\tilde{T}}$ and ${}_1u_T = {}_1u_{\tilde{T}} = {}_1v_T \forall T \geq \tilde{T}$. If ${}_1v = \pi({}_1u)$ for some $\pi \in \Pi$, **FA*** implies ${}_1u \sim {}_1v$. Otherwise, let ${}_1v \neq \pi({}_1u)$ for all $\pi \in \Pi$. Take any $T \geq \tilde{T}$ and let ${}_1w = ({}_1v_{T,T+1} u)$. We show that ${}_1u \sim {}_1w$. Observe that ${}_1u \sim^{M^*} {}_1w$, by construction, so that $\exists \tilde{T} \geq 1$ such that ${}_1u_{\tilde{T}} = {}_1w_{\tilde{T}}$ and ${}_1u_T = {}_1u_{\tilde{T}} = {}_1w_T \forall T \geq \tilde{T}$. Assume, to the contrary, that ${}_1u \not\sim {}_1w$, so that either ${}_1w \succcurlyeq {}_1u$ or ${}_1u \succcurlyeq {}_1w$ holds by **MC***. Without loss of generality, suppose ${}_1u \succcurlyeq {}_1w$. As \succcurlyeq meets **WC** it follows that $\exists \epsilon > 0$ such that ${}_1u \succcurlyeq {}_1w + {}_{con}\epsilon \equiv {}_1w^\epsilon$. **WC** implies, $\exists \alpha > 0$, ${}_1u - {}_{con}\alpha \succcurlyeq {}_1w^\epsilon$. By another application of **WC**, it follows that, $\exists \beta > 0$, ${}_1u - {}_{con}\alpha - {}_{con}\beta \succcurlyeq {}_1w^\epsilon$. Let us consider the feasible utility stream $({}_1u_T - {}_1\alpha_{T,T+1} u + {}_{T+1}\epsilon) \in Y$. As ${}_1w^\epsilon \succcurlyeq^{M^*} ({}_1u_T - {}_1\alpha_{T,T+1} u + {}_{T+1}\epsilon)$, we have that ${}_1w^\epsilon \succcurlyeq ({}_1u_T - {}_1\alpha_{T,T+1} u + {}_{T+1}\epsilon)$, by (1) proved above. Furthermore, $({}_1u_T - {}_1\alpha_{T,T+1} u + {}_{T+1}\epsilon) \gg {}_1u - {}_{con}\alpha - {}_{con}\beta$, and therefore $({}_1u_T - {}_1\alpha_{T,T+1} u + {}_{T+1}\epsilon) \succcurlyeq {}_1u - {}_{con}\alpha - {}_{con}\beta$ by **WPO***. It then follows from the transitivity of \succcurlyeq that ${}_1w^\epsilon \succcurlyeq {}_1u - {}_{con}\alpha - {}_{con}\beta$, a contradiction. Therefore, ${}_1u \sim {}_1w \equiv ({}_1v_{T,T+1} u)$. Since ${}_1u \sim {}_1w \equiv ({}_1v_{T,T+1} u)$ holds for any $T \geq \tilde{T}$, **WIC** implies ${}_1u \sim {}_1v$.

Proof of (3)

Take any ${}_1u, {}_1v \in Y$ such that $\min({}_1u) > \min({}_1v)$. As ${}_1u, {}_1v \in Y$, $\exists \tilde{T} \geq 1$ such that ${}_1u_{\tilde{T}} > {}_1v_{\tilde{T}}$ and ${}_1u_T = {}_1u_{\tilde{T}}$ and ${}_1v_{\tilde{T}} = {}_1v_T \forall T \geq \tilde{T}$. Take any $T \geq \tilde{T}$ and let ${}_1w = ({}_1v_{T, T+1} u)$. Observe that ${}_1w \in Y$. We show that ${}_1u \succ {}_1v$. Observe that ${}_1u \succ^{M^*} {}_1w$, by construction. By (1) proved above, it follows that ${}_1u \succ {}_1w$. As ${}_1w \sim^{M^*} {}_1v$, it follows from (2) proved above that ${}_1w \sim {}_1v$. Transitivity of \succ implies ${}_1u \succ {}_1v$, as desired. ■

The properties in Theorem 9 are easily shown to be tight (see Appendix).

Theorem 9 provides an original characterisation of the maximin SWR in the context of societies with an infinite number of agents. This result is interesting per se, as compared to alternative characterisations of the maximin. For example, Lauwers ([11]) characterises the maximin SWO by an anonymous social welfare function (SWF) defined over the set of bounded infinite utility streams, by imposing a strong version of **HE** according to which for any two bounded infinite vectors ${}_1u, {}_1v$ such that $u_i \geq v_i \geq v_j \geq u_j$ for some $i, j \in \mathbb{N}$ and $u_k = v_k \forall k \in \mathbb{N} \setminus \{i, j\}$, then ${}_1v \succcurlyeq {}_1u$. The main focus of this paper is different and so the question of the characterisation of the maximin SWO by an anonymous and liberal SWF remains open. It is worth noting, however, that Theorem 9 does characterise the maximin SWR on a different set of infinite utility streams, which can be unbounded above, and to this aim neither the continuity condition, nor to the so-called “repetition approximation principle” imposed by Lauwers ([11], p.146) are necessary. Indeed, subject to the domain restriction, and except for the rather mild condition **WIC**, the axioms are strikingly similar to those used to characterise the maximin SWO in finite economies.³

Perhaps more importantly, Theorem 9 provides further support to the main theoretical arguments of this paper. For it confirms that the main intuitions concerning the role of the liberal notion of noninterference embodied in the Harm Principle are robust and they do not depend on the specific definition of the maximin and leximin SWR adopted to rank infinite utility streams (pioneered by Swensson, [19]).

5 Conclusions

This paper analyses Rawls’s celebrated difference principle, and its lexicographic refinement, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise a family of egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed in [12]. This is quite surprising, because the Harm Principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic

³This is even more evident in the light of the discussion in footnote 2 above.

refinement are derived, including in the intergenerational context with an infinite number of agents and using different definitions of the relevant social welfare relations proposed in the literature. The results presented in this paper have two main sets of implications from a theoretical viewpoint. First, they shed new light on the ethical foundations of the egalitarian approaches stemming from Rawls's difference principle. In fact, both the leximin and the maximin are characterised by some standard axioms (such as Anonymity and the Pareto Principle) together with a liberal principle incorporating only a noninterfering view of society. No axiom with an explicitly egalitarian content is necessary in order to derive the main liberal egalitarian principles. Second, from the viewpoint of liberal approaches emphasising a notion of individual autonomy, or freedom, they have a rather counterintuitive implication. For they prove that, in a number of different contexts, liberal noninterfering principles lead straight to welfare egalitarianism.

Appendix

6 Proofs of Lemmas

Lemma 7. Let \succsim on X be a SWR satisfying **WPO***, **WC**, and **MC***. Then, $\forall_{1u, 1v \in Y} : \exists T \geq 1$ $1u > (1v_{T, T+1} u) \Rightarrow 1u \succsim (1v_{T, T+1} u)$.

Proof. Let \succsim on X be a SWR satisfying **WPO***, **WC**, and **MC***. Let $1u, 1v \in Y$ be such that $\exists T \geq 1$ $1u > (1v_{T, T+1} u)$. We show that $1u \succsim (1v_{T, T+1} u)$. Assume, to the contrary, that $1u \not\succeq (1v_{T, T+1} u)$. **MC*** implies $(1v_{T, T+1} u) \succ 1u$. It follows from **WC** that $\exists \epsilon > 0$, $(1v_{T, T+1} u) - \text{con}\epsilon \succ 1u$. However, since $1u > (1v_{T, T+1} u)$ we have that $1u \gg (1v_{T, T+1} u) - \text{con}\epsilon$, so that **WPO*** implies $1u \succ (1v_{T, T+1} u) - \text{con}\epsilon$, a contradiction. ■

In order to prove Lemma 8, some preliminary results are derived, which are interesting per se. They are Lemma A.1 and Lemma A.2. The former Lemma proves a property of the SWR \succsim in ranking a subset of vectors with the same tail. The latter Lemma provides sufficient conditions to implement some type of Pareto-rankings. In particular, it proves that, for a subset of allocations in Y , there is a finite number T such that, starting from a given allocation, if the welfare of at least the first T generations increases uniformly, then the new allocation is strictly preferred.

Lemma A.1. Let \succsim on X be a SWR satisfying **FA***, **WPO***, **WHP****, **WC**, and **MC***. Let $1u, 1v \in Y$ be such that $1u \succ^{M*} 1v$, and let $\tilde{T} \in \mathbb{N}$ be such that $1v_{\tilde{T}} = 1v_T$ and $1u_{\tilde{T}} = 1u_T \forall T \geq \tilde{T}$. Then, $(1v_{T, T+1} u) \sim (1v_{T+1, T+2} u) \forall T \geq \tilde{T}$.

Proof. Take any $T \geq \tilde{T}$. If $u_{T+1} = v_{T+1}$, then the result follows by **FA***. Otherwise, let $u_{T+1} \neq v_{T+1}$. Lemma 7 implies

$$(1v_{T+1, T+2} u) \succsim (1v_{T, T+1} u) \text{ whenever } v_{T+1} > u_{T+1}, \quad (6.1)$$

or

$$(1v_{T, T+1} u) \succsim (1v_{T+1, T+2} u) \text{ whenever } u_{T+1} > v_{T+1}. \quad (6.2)$$

Let $v_{T+1} > u_{T+1}$, and suppose that $(1v_{T+1, T+2} u) \succ (1v_{T, T+1} u)$. It follows from **WC** that $(1v_{T+1, T+2} u) - \text{con}\alpha \succ (1v_{T, T+1} u) \exists \alpha > 0$. If $v_{T+1} - \alpha < u_{T+1}$, **WPO*** implies $(1v_{T, T+1} u) \succ (1v_{T+1, T+2} u) - \text{con}\alpha$, a contradiction. Otherwise, let $v_{T+1} - \alpha \geq u_{T+1}$. As $1u, 1v \in Y$, are such that $1u \succ^{M*} 1v$ and $T \geq \tilde{T}$, then $\exists t \leq T$ such that $v_t < u_{T+1}$ and, obviously, $v_t - \alpha < v_t$. For the sake of notational simplicity, let $(1v_{T, T+1} u) \equiv 1x$ and $(1v_{T+1, T+2} u) - \text{con}\alpha \equiv 1y$, so that $1y \succ 1x$. Let $1\hat{x}$ be a finite permutation of $1x$ such that $\hat{x}_{T+1} = v_t$ and $\hat{x}_t = u_{T+1}$, while

all other entries are unchanged. By **FA***, ${}_1\hat{x} \sim {}_1x$, so that ${}_1y \succ {}_1\hat{x}$, by the transitivity of \succ . Then, let two real numbers $d_1, d_2 > 0$, and consider vectors ${}_1y', {}_1\hat{x}'$ formed from ${}_1\hat{x}$ and ${}_1y$ as follows: first, \hat{x}_{T+1} is lowered to $\hat{x}_{T+1} - d_1 > v_t - \alpha$; next, y_{T+1} is lowered to $y_{T+1} - d_2$ such that $u_{T+1} > y_{T+1} - d_2 > \hat{x}_{T+1} - d_1$; finally, all other entries of ${}_1\hat{x}$ and ${}_1y$ are unchanged. It follows from **WHP**** that ${}_1\hat{x}' \not\sim {}_1y'$. Let ${}_1x'$ be a finite permutation of ${}_1\hat{x}'$ such that $x'_{T+1} = u_{T+1}$ and $x'_t = \hat{x}_{T+1} - d_1$. **FA*** and the transitivity of \succ imply ${}_1x' \not\sim {}_1y'$, but **WPO*** implies ${}_1x' \succ {}_1y'$, a contradiction. Therefore, $({}_1v_{T+1, T+2} u) \not\sim ({}_1v_{T, T+1} u)$ whenever $v_{T+1} > u_{T+1}$. **MC*** implies that $({}_1v_{T, T+1} u) \succ ({}_1v_{T+1, T+2} u)$, and it follows from (6.1) that $({}_1v_{T, T+1} u) \sim ({}_1v_{T+1, T+2} u)$.

Let $u_{T+1} > v_{T+1}$ and suppose that $({}_1v_{T, T+1} u) \succ ({}_1v_{T+1, T+2} u)$. It follows from **WC** that $({}_1v_{T, T+1} u) \succ ({}_1v_{T+1, T+2} u) + {}_{con}\alpha \exists \alpha > 0$. If $v_{T+1} + \alpha > u_{T+1}$, **WPO*** implies $({}_1v_{T+1, T+2} u) + {}_{con}\alpha \succ ({}_1v_{T, T+1} u)$, a contradiction. Otherwise, let $v_{T+1} + \alpha \leq u_{T+1}$. For the sake of notational simplicity, let $({}_1v_{T, T+1} u) \equiv {}_1x$ and $({}_1v_{T+1, T+2} u) + {}_{con}\alpha \equiv {}_1y$, so that ${}_1x \succ {}_1y$. Again, note that $\exists t \leq T$ such that $v_t < u_{T+1}$. Let ${}_1\hat{y}$ be a finite permutation of ${}_1y$ such that $\hat{y}_{T+1} = y_t$ and $\hat{y}_t = y_{T+1}$, while all other entries are unchanged. By **FA***, ${}_1\hat{y} \sim {}_1y$, so that ${}_1x \succ {}_1\hat{y}$, by the transitivity of \succ . Then, let two real numbers $d_1, d_2 > 0$, and consider vectors ${}_1\hat{y}', {}_1x'$ formed from ${}_1\hat{y}$ and ${}_1x$ as follows: first, $\hat{y}_{T+1} = y_t$ is lowered to $\hat{y}_{T+1} - d_1 > x_t = v_t$; next, x_{T+1} is lowered to $x_{T+1} - d_2$ such that $y_{T+1} > x_{T+1} - d_2 > \hat{y}_{T+1} - d_1$; finally, all other entries of ${}_1\hat{y}$ and ${}_1x$ are unchanged. It follows from **WHP**** that ${}_1\hat{y}' \not\sim {}_1x'$. Let ${}_1y'$ be a finite permutation of ${}_1\hat{y}'$ such that $y'_{T+1} = y_{T+1}$ and $y'_t = \hat{y}_{T+1} - d_1$. **FA*** and the transitivity of \succ imply ${}_1y' \not\sim {}_1x'$, but **WPO*** implies ${}_1y' \succ {}_1x'$, a contradiction. Therefore, $({}_1v_{T, T+1} u) \not\sim ({}_1v_{T+1, T+2} u)$ whenever $u_{T+1} > v_{T+1}$. **MC*** implies that $({}_1v_{T+1, T+2} u) \succ ({}_1v_{T, T+1} u)$, and it follows from (6.2) that $({}_1v_{T, T+1} u) \sim ({}_1v_{T+1, T+2} u)$. ■

Lemma A.2. Let \succ on X be a SWR satisfying **FA***, **WPO***, **WHP****, **WC**, **WIC**, and **MC***. Let ${}_1u, {}_1v \in Y$ be such that ${}_1u \succ^{M^*} {}_1v$, let $\tilde{T} \in \mathbb{N}$ be such that ${}_1v_{\tilde{T}} = {}_1v_{T'}$ and ${}_1u_{\tilde{T}} = {}_1u_{T'} \forall T' \geq \tilde{T}$, and let $\epsilon > 0$ be such that ${}_1v_{T'} + \epsilon < {}_1u_{T'}$. Then, $({}_1v_T + {}_1\epsilon_{T, T+1} u) \succ ({}_1v_{T, T+1} u) \exists T \geq T'$.

Proof. Let the premises hold and assume, to the contrary, that $\exists \epsilon \in \mathbb{R}_{++}$ such that ${}_1v_{T'} + \epsilon < {}_1u_{T'}$, $\nexists T \geq T'$ such that $({}_1v_T + {}_1\epsilon_{T, T+1} u) \succ ({}_1v_{T, T+1} u)$. **MC*** implies that, $\forall T \geq T'$, $({}_1v_{T, T+1} u) \succ ({}_1v_T + {}_1\epsilon_{T, T+1} u)$. By Lemma 7, we can rule out $({}_1v_{T, T+1} u) \succ ({}_1v_T + {}_1\epsilon_{T, T+1} u) \exists T \geq T'$. Therefore let $({}_1v_{T, T+1} u) \sim ({}_1v_T + {}_1\epsilon_{T, T+1} u) \forall T \geq T'$. By Lemma A.1, $({}_1v_{T, T+1} u) \sim ({}_1v_{T+1, T+2} u)$, $\forall T \geq \tilde{T}$, and so, $\forall T \geq T'$, we have that

$$({}_1v_T + {}_1\epsilon_{T, T+1} u) \sim ({}_1v_{T, T+1} u) \sim ({}_1v_{T+1, T+2} u) \sim ({}_1v_{T+1} + {}_1\epsilon_{T+1, T+2} u)$$

By the transitivity of \succ , it follows that, $\forall T \in \mathbb{N} \setminus \{1, \dots, T' - 1\}$,

$$({}_1v_{T', T'+1} u) \sim ({}_1v_{T'} + {}_1\epsilon_{T', T'+1} u) \sim ({}_1v_{T, T+1} u) \sim ({}_1v_T + {}_1\epsilon_{T, T+1} u),$$

so that, by the transitivity of \succ , it follows that, $\forall T \in \mathbb{N} \setminus \{1, \dots, T' - 1\}$,

$$({}_1v_{T',T'+1} u) \sim ({}_1v_T + {}_1\epsilon_{T,T+1} u) \quad (6.3)$$

and

$$({}_1v_{T'} + {}_1\epsilon_{T',T'+1} u) \sim ({}_1v_{T,T+1} u). \quad (6.4)$$

WIC and (6.3) imply $({}_1v_{T',T'+1} u) \sim {}_1v + {}_1\epsilon$, whereas **WIC** and (6.4) imply that $({}_1v_{T'} + {}_1\epsilon_{T',T'+1} u) \sim {}_1v$. By the transitivity of \succ , it follows that ${}_1v + {}_1\epsilon \sim {}_1v$, but **WPO*** implies ${}_1v + {}_1\epsilon \succ {}_1v$, a contradiction. ■

Lemma 8. Let \succ on X be a SWR satisfying **FA***, **WPO***, **WHP****, **WC**, **WIC**, and **MC***. Let ${}_1u, {}_1v \in Y$ be such that ${}_1u \succ^{M^*} {}_1v$, let $\tilde{T} \in \mathbb{N}$ be such that ${}_1v_{\tilde{T}} = {}_1v_{T'}$ and ${}_1u_{\tilde{T}} = {}_1u_{T'} \forall T' \geq \tilde{T}$, and let $\epsilon > 0$ be such that ${}_1v_{T'} + \epsilon < {}_1u_{T'}$. Then, $({}_1v_{T'} + {}_1\epsilon_{T',T'+1} u) \succ ({}_1v_{T',T'+1} u)$.

Proof. By Lemma A.2, $\exists \bar{T} \geq T'$ such that $({}_1v_{\bar{T}} + {}_1\epsilon_{\bar{T},\bar{T}+1} u) \succ ({}_1v_{\bar{T},\bar{T}+1} u)$. By Lemma A.1

$$({}_1v_{T,T+1} u) \sim ({}_1v_{T+1,T+2} u) \forall T \geq \tilde{T} \quad (6.5)$$

and

$$({}_1v_T + {}_1\epsilon_{T,T+1} u) \sim ({}_1v_{T+1} + {}_1\epsilon_{T+1,T+2} u) \forall T \geq \tilde{T}. \quad (6.6)$$

The transitivity of \succ and (6.5) imply $({}_1v_{T',T'+1} u) \sim ({}_1v_{\bar{T},\bar{T}+1} u)$. And again, the transitivity of \succ and (6.6) imply $({}_1v_{T'} + {}_1\epsilon_{T',T'+1} u) \sim ({}_1v_{\bar{T}} + {}_1\epsilon_{\bar{T},\bar{T}+1} u)$. As $({}_1v_{\bar{T}} + {}_1\epsilon_{\bar{T},\bar{T}+1} u) \succ ({}_1v_{\bar{T},\bar{T}+1} u)$ it follows from the transitivity of \succ that $({}_1v_{T'} + {}_1\epsilon_{T',T'+1} u) \succ ({}_1v_{T',T'+1} u)$. ■

7 Independence of Axioms

The proofs of the independence of the axioms used to characterise the maximin and leximin SWO are obvious and therefore they are omitted. It is worth noting, however, that some of the examples below can be easily adapted to apply to the finite context.

7.1 Independence of axioms used in Theorem 3

In order to complete the proof of Theorem 3, we show that the axioms are tight.

For an example violating *only* **FA**, define \succ on X in the following way: $\forall {}_1u, {}_1v \in X$,

$$\begin{aligned} {}_1u = {}_1v &\Rightarrow {}_1u \sim {}_1v \\ \exists T \in \mathbb{N} : u_t = v_t \forall t < T \text{ and } u_T > v_T &\Rightarrow {}_1u \succ {}_1v \end{aligned}$$

The SWR \succsim on X is not an extension of the leximin SWR \succsim^{LM^*} . The SWR \succsim on X satisfies all properties except **FA**.

For an example violating *only* **SPO**, define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X, {}_1 u \sim {}_1 v$. The SWR \succsim on X is not an extension of the leximin SWR \succsim^{LM^*} . Clearly, the SWR \succsim on X satisfies all properties except **SPO**.

For an example violating *only* **WHP***, define \succsim on X be the leximax SWR, i.e., $\forall_1 u, {}_1 v \in X$,

$${}_1 u \sim^{LX} {}_1 v \Leftrightarrow \exists \tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T} : {}_1 \bar{u}_T = {}_1 \bar{v}_T,$$

and

$${}_1 u \succ_1^{LX} {}_1 v \Leftrightarrow \exists \tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T}, \exists t \in \{1, \dots, T\} \text{ with } \bar{u}_s = \bar{v}_s \text{ (} \forall t < s \leq T \text{) and } \bar{u}_t > \bar{v}_t.$$

It follows that the SWR \succsim on X is not an extension of the leximin SWR. The SWR \succsim on X satisfies all properties except **WHP***.

For an example violating *only* **WC**, define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$,

$${}_1 u \geq \pi({}_1 v) \exists \pi \in \Pi \Rightarrow {}_1 u \succsim {}_1 v.$$

The relation \succsim on X is the *Suppes-Sen grading principle*. The SWR \succsim on X satisfies all properties except **WC**.

For an example violating *only* **WPC**, $\forall T \in \mathbb{N}$, let the leximin ordering on X_T be denoted as \succsim_T^{LM} . Define \succsim_T^L on X as in (4.1). Then, let $\succsim = \bigcup_{T \in \mathbb{N}} \succsim_T^L$. By definition, this relation is reflexive and transitive. The SWR \succsim on X satisfies all properties but **WPC**. [To see that **WPC** is violated consider the following vectors, ${}_1 v = (3, \text{con}0)$ and ${}_1 u = (2, \text{con}1)$. Then, $({}_1 u, {}_1 v) \notin \succsim$ and $(({}_1 u_{T, T+1} v), {}_1 v) \in \succsim_T^L \forall T \geq 2$].

Independence of axioms used in Theorem 4

In order to complete the proof of Theorem 4, we show that the axioms are tight.

As Strong Preference Continuity (**SPC**) implies Weak Preference Continuity (**WPC**), the above examples show that the axioms used in Theorem 4 are tight as well.

Independence of axioms used in Theorem 9

In order to complete the proof of Theorem 9, we show that the axioms are tight.

For an example violating *only* **FA***, define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$, $i) u_1 = v_1 \Leftrightarrow {}_1 u \sim {}_1 v$; $ii) u_1 > v_1 \Leftrightarrow {}_1 u \succ {}_1 v$. The SWR \succsim on X is not an extension of the maximin SWR \succsim^{M*} . The SWR \succsim on X satisfies all properties except **FA***.

For an example violating *only* **WPO***, define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$, ${}_1 u \sim {}_1 v$. The SWR \succsim on X is not an extension of the maximin SWR \succsim^{M*} . Clearly, the SWR \succsim on X satisfies all properties except **WPO***.

For an example violating *only* **WC**, fix $\mathcal{T} \in \mathbb{N}$ such that $1 < \mathcal{T} < \infty$. For ${}_1 u \in X$, let ${}_1 \tilde{u}$ denote a permutation of ${}_1 u$ such that $\tilde{u}_1 \leq \tilde{u}_2 \leq \dots \leq \tilde{u}_{\mathcal{T}}$ and $\tilde{u}_t \geq \tilde{u}_{\mathcal{T}} \forall t > \mathcal{T}$. Let us define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$

$$\begin{aligned} {}_1 u \succ {}_1 v & \text{ if } \exists t \leq \mathcal{T} : \tilde{u}_s = \tilde{v}_s \ (\forall s < t) \ \& \ \tilde{u}_t > \tilde{v}_t \\ {}_1 u \sim {}_1 v & \text{ if } {}_1 \tilde{u}_{\mathcal{T}} = {}_1 \tilde{v}_{\mathcal{T}}. \end{aligned}$$

The SWR \succsim on X is not an extension of the maximin SWR \succsim^{M*} . Clearly, the SWR \succsim on X satisfies all properties except **WC**.

For an example violating *only* **HP***, fix $\mathcal{T} \in \mathbb{N}$ such that $1 < \mathcal{T} < \infty$. For ${}_1 u \in X$, let ${}_1 \tilde{u}$ denote a permutation of ${}_1 u$ such that $\tilde{u}_1 \leq \tilde{u}_2 \leq \dots \leq \tilde{u}_{\mathcal{T}}$ and $\tilde{u}_t \geq \tilde{u}_{\mathcal{T}} \forall t > \mathcal{T}$. Let us define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$

$${}_1 u \succ {}_1 v \text{ if } \sum_{i=1}^{\mathcal{T}} \tilde{u}_i > \sum_{i=1}^{\mathcal{T}} \tilde{v}_i$$

and

$${}_1 u \sim {}_1 v \text{ if } \sum_{i=1}^{\mathcal{T}} \tilde{u}_i = \sum_{i=1}^{\mathcal{T}} \tilde{v}_i$$

The SWR \succsim on X is not an extension of the maximin SWR \succsim^{M*} . Clearly, the SWR \succsim on X satisfies all properties except **HP***.

For an example violating *only* **MC***, let σ be a permutation of \mathbb{N} , Let Σ be the set of all permutations of \mathbb{N} . Define \succsim on X in the following way: $\forall_1 u, {}_1 v \in X$,

$$\begin{aligned} {}_1 u & = \pi({}_1 v) \exists \pi \in \Pi \Rightarrow {}_1 u \sim {}_1 v \\ {}_1 u & \gg \sigma({}_1 v) \exists \sigma \in \Sigma \Rightarrow {}_1 u \succ {}_1 v \end{aligned}$$

The SWR \succsim on X is not an extension of the maximin SWR \succsim^{M^*} . Clearly, the SWR \succsim on X satisfies all properties except **MC***.

For an example violating *only* **WIC**, define \succsim on Y in the following way: $\forall_1 u, {}_1 v \in Y$,

$$\begin{aligned} {}_1 u &\sim^{M^*} {}_1 v, \exists T \geq 1 : Tu = Tv \Rightarrow {}_1 u \sim {}_1 v \\ {}_1 u &\succ^{M^*} {}_1 v \Rightarrow {}_1 u \succ {}_1 v \\ {}_1 u &\sim^{M^*} {}_1 v, \nexists T \geq 1 : Tu = Tv \Rightarrow {}_1 u \not\sim {}_1 v \text{ and } {}_1 v \not\sim {}_1 u. \end{aligned}$$

The SWR \succsim on Y is not an extension of the maximin SWR \succsim^{M^*} . Clearly, the SWR \succsim on Y satisfies all properties except **WIC**.

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