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Abstract

Although the concept of monetary policy lag has historical roots deep in the monetary economics literature, relatively little attention has been paid to the idea. In this paper, we build on Svensson's (1997) inflation targeting framework by explicitly taking into account the lagged effect of monetary policy and characterize the optimal monetary policy reaction function both in the absence and in the presence of the zero lower bound on the nominal interest rate. We numerically show the function to be more aggressive and more pre-emptive with the lagged effect than without it. We also characterize the long-run stabilization cost to the central bank by explicitly taking into account the lagged effect of monetary policy. It turns out that, in the presence of the zero lower bound constraint, the long-run stabilization cost is higher with the lagged effect than the case without it. This result suggests that the central bank and/or the government should set a relatively high inflation target when confronted with a relatively long monetary policy lag. This can be interpreted as another justification for targeting a positive inflation rate in the long-run.

JEL classification: E52, E58, C63
Bank classification: Inflation targets; Monetary policy framework; Monetary policy implementation

Résumé

Le délai de transmission de la politique monétaire demeure une notion assez peu étudiée, même si elle plonge loin ses racines dans les travaux d'économie monétaire. Prenant pour point de départ le régime de cibles d'inflation modélisé par Svensson (1997), l'auteur de l'étude y intègre explicitement l'effet à retardement des mesures de politique monétaire. Il caractérise la fonction de réaction optimale de la politique monétaire tant avec que sans une borne inférieure limitant les taux d'intérêt nominaux à zéro. De manière numérique, il montre que la politique monétaire doit être mise en œuvre de façon plus énergique et plus préventive quand le délai de transmission de ses effets est pris en compte. L'intégration de ce délai modifie le coût de stabilisation à long terme supporté par la banque centrale : sous la contrainte de la borne du zéro, ce coût s'avère supérieur à ce qu'il serait si l'on faisait abstraction du délai de transmission de la politique monétaire. L'auteur en conclut qu'un délai de transmission relativement important devrait inciter les banquiers centraux ou les gouvernements à viser des taux d'inflation plutôt élevés. On peut y voir un argument supplémentaire en faveur de l'établissement d'une cible d'inflation supérieure à zéro dans le long terme.

Classification JEL : E52, E58, C63
Classification de la Banque : Cibles en matière d'inflation; Cadre de la politique monétaire; Mise en œuvre de la politique monétaire

"...monetary actions affect economic conditions only after a lag... ."
(Milton Friedman, 1961, p.447)

1 Introduction

Since Friedman's (1961) article titled "The Lag in Effect of Monetary Policy," most researchers and practitioners in the field of monetary policy have been aware that it takes a while for monetary policy action to have an effect on economic conditions such as output or inflation. For instance, the Monetary Policy Committee of the Bank of England (1999, p.3) states that "[a]s to timing, in the Bank's macroeconomic model ..., official interest rate decisions have their fullest effect on output with a lag of around one year, and their fullest effect on inflation with a lag of around two years." As another example, Bernanke (2004) states that "[b]ecause monetary policy works with a lag, the ability of policymakers to stabilize the economy depends critically on our ability to peer into our cloudy crystal balls and see something resembling the future." For nearly a half century since Friedman's (1961) work, the lagged effects of monetary policy action has remained a factor that central bankers take into account when conducting monetary policy. Though the lagged effect of monetary policy is an old issue, it is definitely not a moribund issue for central bankers.

Although the concept of monetary policy lag has historical roots deep in the monetary economic literature, there have been only a few studies that focus on this issue. In an earlier studies, Cagan and Gandolfi (1969) take the time pattern of monetary effects on interest rates as an indirect measure of the movement in expenditures and income. They find the lagged effect of monetary policy on income to be somewhere between six months to two years. Friedman (1972), following up on his earlier work, reported empirical evidence on the lag between monetary policy actions (i.e., change in the money stock) and the responses of inflation using US and UK data and confirmed his earlier hypothesis. Tanner (1979) tested the variability of the lag between monetary policy action and resulting changes in output. He found the length of the lag to be significantly and substantially variable. Duguay (1994) estimated the monetary policy lag for output to be 12 to 18 months and the lag for inflation to be 18 to 24 months using Canadian data. Batini and Nelson (2001) reaffirmed Friedman's (1972) empirical results using US and UK data from 1953 to 2001 and found the lag length to be 25 months for the US and 13 months for the UK. They also estimated the length of the lag using different sample periods and found little evidence of the lag becoming shorter in more recent years. Indeed, according to some estimates they reported, the length of the lag became *longer* in more recent years. Hafer et al. (2007) re-examined the role of money and reported a significant statistical relationship between the lagged change in the money stock and the changes in the output gap.

From this empirical evidence, the existence of a monetary policy lag in the economy is evident, both in the past and present, and remains a factor that influences the conduct of monetary policy. Yet, theoretical development in the optimal monetary policy literature in the line of Woodford (1999, 2003) or Svensson (1997, 1999) has put little or no emphasis[1] on the monetary policy lag, assuming that monetary policy action affects output or inflation instantly. Given the empirical evidence of a monetary policy lag and, further, being aware that central bankers are struggling with one-year to two-year lagged effects of monetary policy in reality, this apparent lack of interest in incorporating the lagged effect into the theory of optimal monetary policy seems odd.

In this paper, building on Svensson's (1997) inflation targeting framework, we explicitly take into account the lagged effect of monetary policy and characterize the optimal monetary policy reaction function. Further, for the sake of exemplifying the role of the monetary policy lag, we characterize the optimal monetary policy reaction functions both in the absence and in the presence of the zero lower bound constraint on the nominal interest rate.[2] Using a numerical method to characterize the optimal monetary policy reaction function following Orphanides and Wieland (2000) and Kato and Nishiyama (2005), the main findings are as follows. First, with a lagged effect of monetary policy, the optimal monetary policy reaction function will be more aggressive than in the case without a lag. Second, in the presence of the zero lower bound, the optimal monetary policy reaction function with a lagged effect will be more pre-emptive than in the case without a lag. Third, when lagged effect of monetary policy is present, the optimal monetary policy characterized in this paper reveals an offsetting behavior of current monetary policy vis-à-vis past monetary policy stance which is in apparent conflict with the stylized fact of interest rate smoothing behavior by the central banks reported by Sack and Wieland (2000) among others. Although this feature is a drawback, this result suggests that the interest rate smoothing motive of the central banks is not arising from the lagged effect of monetary policy.

Besides the monetary policy conduct in the short-run, another important issue for central banks is the level of the inflation target. In other words, where should the central bank (or the government, depending on who is responsible for setting the target) set its inflation target? Should it be positive, negative, or zero?[3] For any inflation targeting economy – most of which choose targets between 1% and 3% – this is a crucial question. One argument justifying a positive inflation target comes from Akerlof et al. (1996). They argue that because of downward nominal wage rigidity, the real wage cannot adjust properly in a deflationary environment. In order to avoid this inefficiency (or to keep the 'grease' in the labour market), they argue that it is optimal for the central bank to target a positive inflation rate in the long run.[4] Another case justifying a positive inflation target comes from Summers

[1] A notable exception is Amato and Laubach (2004) where habit formation is introduced in a general equilibrium model and the welfare-maximizing optimal monetary policy rule is developed.

[2] In this paper, we have confined our analysis to a closed-economy model where an exchange rate channel is absent. For an analysis of exchange rate policy in the presence of the zero lower bound, see, for instance, Coenen and Wieland (2003).

[3] For instance, Friedman's Rule, which is mainly stemming from the transaction motive for holding money, suggests the optimal rate of inflation to be negative. On the other hand, the arguments based on the unit-of-account function of money suggest the optimal rate of inflation to be zero.

[4] The hypothesis of downward nominal wage rigidity was faced by mixed empirical results. For instance, Crawford and Harrison (1998) found base wage rates in Canada to be quite flexible especially in the non-union

(1991). Summers (1991) was among the first to point out the risk stemming from the zero lower bound constraint on the nominal interest rate, followed by Fuhrer and Madigan (1997), Orphanides and Wieland (1998), Blinder (2000), Reifschneider and Williams (2000), Hunt and Laxton (2003), and Coenen et al. (2004) among others. According to Summers (1991), once the nominal interest rate (i.e., the policy instrument) binds at zero, the traditional monetary policy channel for stabilizing the economy becomes ineffective – i.e., the central bank will lose its main tool for controlling the economy. Consequently, the economy will have to bear the higher cost from volatile economic conditions – a scenario known as the 'liquidity trap' or 'deflationary trap.' In order to avoid the risk of being caught in the liquidity trap, Summers (1991) suggests setting a positive inflation target. Finally, there is a well-known issue of measurement bias in the inflation rate championed by Boskin et al. (1996). Clearly, measurement bias in the inflation rate is one of the primary reasons central banks set a positive inflation target in reality. The preceeding three arguments are all valid in justifying a positive inflation target. However, to the best of our knowledge, we are not aware of any study[5] that justifies a positive inflation target on account of a lag in the effect of monetary policy action.

In this paper, in addition to the analysis of the optimal monetary policy conduct in the short-run, we also investigate the implication of a monetary policy lag to the level of the inflation target in the long-run. Specifically, we numerically characterize the long-run stabilization cost for the central bank explicitly taking into account the lagged effect of monetary policy both in the absence and presence of the zero lower bound. Our findings can be summarized as follows. First, in the absence of the zero lower bound, the long-run stabilization cost for the central bank is invariant to both the level of the inflation target and the monetary policy lag. Second, in the presence of the zero lower bound, the long-run stabilization cost for the central bank is higher with monetary policy lag compared to the case without a lag. In reality, since there exists a zero lower bound constraint,[6] this result suggests that the central bank (or the government) should set a relatively high inflation target when they are confronted with a relatively long monetary policy lag. Combined with the

sector and at smaller firms. Smith (2000), using UK microdata, found only 1% of workers facing downwardly rigid nominal wage. Kimura and Ueda (2001), using aggregate time series data, found nominal wages in Japan to be downwardly rigid until 1998, but less so after then. Kuroda and Yamamoto (2003a, b), using Japanese microdata, found similar results. Lebow, Saks, and Wilson (2003), using the microdata underlying the Bureau of Labor Statistics' employment cost index, found stronger evidence of downward nominal wage rigidity compared to the studies using the Panel Study of Income Dynamics (PSID).

[5] For instance, Bernanke and Mishkin (1997) and Mishkin (2007, Ch.19) offer a selective review on the optimal level of inflation target, but there is no mentioning about the monetary policy lag. Bernanke (2002) only mentions the zero lower bound and inflation measurement bias in justifying a 'buffer zone' with regard to the inflation target, but does not mention the lagged effect of monetary policy. For a more thorough review on the optimal level of the inflation target, see Konieczny (1994), Shiratsuka (2001) and Fuchi, Oda, and Ugai (2007).

[6] This is evident from the Japanese experience in late 1990's and early 2000's. The overnight call rate, which is the policy instrument of the Bank of Japan (BOJ), became virtually zero on February 12th, 1999. Since then, with an exception of short-lived recovery period from 2000 to 2001, the BOJ commited to the zero interest rate policy. This zero interest rate policy was finally lifted on July 14th, 2006.

empirical evidence on monetary policy lag, which lasts one to two years, this result can be interpreted as yet another justification for targeting a positive inflation rate in the long-run.

The remainder of this paper is organized as follows. Section 2 describes the set up of the model with a monetary policy lag. Section 3 numerically characterizes the optimal monetary policy reaction function with a monetary policy lag both in the absence and presence of the zero lower bound constraint. Section 4 defines the concept of long-run stabilization cost for the central bank and numerically characterizes its relationship with regard to both the level of the inflation target and the monetary policy lag. Section 5 summarizes the main results of this paper and discusses possible extensions of the model.

2 Model Description

2.1 Definition of Monetary Policy Lag

Before we describe the model in this paper, we start this section with a definition of monetary policy lag. In this paper, we adopt the following definition for the monetary policy lag.

Definition 1 *(Monetary Policy Lag)*
A monetary policy lag is the length of the period that is required for a monetary policy action (i.e., change in nominal interest rate) to take maximum effect on output gap or inflation rate.

The above definition is more or less akin to Friedman's (1961) original notion[7] of 'monetary policy lag', except that the monetary policy action is presumed to be a change in nominal interest rate rather than a change in money stock. Since most of today's central banks use a short-term nominal interest rate as the policy instrument, in this paper, we regard changes in the nominal interest rate as a monetary policy action.[8]

2.2 Setting up the Model with Monetary Policy Lag

In this paper, we adopt the Svensson (1997, 1999) and Ball (1999) type model as a base model for our analysis and extend it by incorporating the monetary policy lag and the zero lower bound on the nominal interest rate. The Svensson-Ball type model assumes a standard quadratic loss function for the central bank and linear state transition equations for the inflation rate and output gap. With these features, the dynamic optimization problem for the central bank becomes a standard stochastic linear-quadratic regulator problem. As was shown by Svensson (1997), this problem leads to a quadratic value function and a linear monetary policy reaction function in the absence of a monetary policy lag and the zero

[7] For instance, Friedman (1961) notes "monetary policy actions that produce a peak in the rate of change of the stock of money can be expected on the average to be followed by a peak in general business some sixteen months later" (p.457) in the subsection titled "The Meaning of 'the' Lag."

[8] Indeed, Batini and Nelson (2001) which attempts to replicate and update Friedman 's (1972) results includes change in short term nominal interest as monetary policy measure.

lower bound for the nominal interest rate. We modify the Svensson-Ball type model by incorporating a monetary policy lag and the zero lower bound on the nominal interest rate, and then characterize optimal monetary policy.

First, following Svensson (1997), we assume that a central bank's period-by-period loss function to be

$$L_t = \frac{1}{2}\left\{(\pi_t - \pi^*)^2 + \lambda y_t^2\right\}, \tag{1}$$

where π and y denote the inflation rate and the output gap, respectively, and π^* is the target inflation rate of the central bank. λ is a positive weight which represents the preference of the central bank.

The specification of the AS equation (or Phillips curve) also follows Svensson (1997),

$$\pi_{t+1} = \pi_t + \alpha y_t + \varepsilon_{t+1}. \tag{2}$$

As can be seen from eq. (2), next period's inflation rate depends on the current inflation rate, the output gap, and the AS shock, ε, where ε is assumed to be a normally distributed *i.i.d.* error with mean zero and standard deviation σ_ε. The parameter, α, pins down the short-run sacrifice ratio between the output gap and the inflation rate. Given this formulation, expected inflation can be fully characterized by current variables, i.e.,

$$E_t\pi_{t+1} = \pi_t + \alpha y_t. \tag{3}$$

Next, we specify the IS equation. Unlike the Svensson-Ball type model where monetary policy action can affect output gap without policy lag, we introduce policy lag. The IS equation and the monetary policy lag equation are as follows:

$$y_{t+1} = \rho y_t - (1-\omega)\delta_1(i_t - E_t\pi_{t+1} - rr^*) - \omega\delta_2 rrlag_t + \nu_{t+1} \quad \text{and} \tag{4}$$

$$rrlag_{t+1} = i_t - E_t\pi_{t+1} - rr^*, \tag{5}$$

where ρ is an autoregressive coefficient for the output gap, δ_1 and δ_2 are the parameters governing the elasticity of the output gap vis-à-vis the nominal interest rate, ω stands for the lag weight, and the AD shock ν is assumed to be *i.i.d.* normally distributed with mean zero and standard deviation σ_ν and uncorrelated with the AS shock ε. Next, rr^* stands for the neutral the real interest rate and $rrlag_t$ stands for the deviation of real interest rate from rr^* in the previous period. Thus, $rrlag_t$ can be considered as the previous period's monetary policy stance. When $rrlag_t$ is positive (negative), it implies that last period's monetary policy stance was contractionary (expansionary) and when zero, it implies that last period's policy stance was neutral.

Now, as can be seen from eq. (4), next period's output gap (i.e., y_{t+1}) is affected by the current nominal interest rate (i.e., i_t), but also by past the monetary policy stance (i.e., $rrlag_t$). As such, under this setup, the current nominal interest rate is not only a control variable affecting the output gap this period, but also becomes a state variable via $rrlag_{t+1}$ (see eq. (5)) which affects the output gap next period. When considering the optimal monetary policy, the central bank needs to take into account the current and lagged effects arising from a monetary policy action. Finally, it should be noted that when the lag weight ω is set equal to zero, eq. (4) reduces back to Svensson's (1997) IS equation. The lagged monetary policy effect via the state variable $rrlag$ will vanish.

The central bank's problem is formulated as a dynamic optimization problem where it minimizes the following objective function,

$$E_t \sum_{j=0}^{\infty} \beta^j L_{t+j}, \tag{6}$$

by controlling the nominal interest rate subject to eq. (2), eq. (4), and eq. (5) with the zero lower bound constraint on the nominal interest rate,

$$i_t \geq 0. \tag{7}$$

We follow the treatment of the non-negativity constraint as in Kato and Nishiyama (2005) and apply Kuhn-Tucker conditions in this dynamic optimization problem. Since this problem can be interpreted as a conventional optimal bounded control problem with a linear-quadratic system, we can set up a Bellman equation[9] as follows,

$$\begin{aligned}
V\left(y_t, \pi_t, rrlag_t\right) &= \min_{i_t \geq 0} \left\{ \frac{1}{2}\left(\left(\pi_t - \pi^*\right)^2 + \lambda y_t^2 \right) + \psi_t i_t + \beta E_t V\left(y_{t+1}, \pi_{t+1}, rrlag_{t+1}\right) \right\} \\
s.t. \quad y_{t+1} &= \rho y_t - (1-\omega)\delta_1 (i_t - \pi_t - \alpha y_t - rr^*) - \omega \delta_2 rrlag_t + \nu_{t+1} \\
\pi_{t+1} &= \pi_t + \alpha y_t + \varepsilon_{t+1} \\
rrlag_{t+1} &= i_t - \pi_t - \alpha y_t - rr^*
\end{aligned} \tag{8}$$

where ψ_t is a Lagrangian multiplier for the non-negativity constraint on i_t. From the Kuhn-Tucker conditions, ψ_t takes positive value when the non-negativity constraint is binding and zero otherwise. Also, note that we have substituted expected inflation rate for current state variables using eq. (3). With this substitution, the RHS of the state transition equations are now expressed in terms of current state variables – i.e., y_t, π_t, and $rrlag_t$ – and stochastic shocks, whereas the LHS of the state transition equations are all in next period's state variables. As such, Bellman equation (8) is recursive and well-defined. Finally, note that the value function $V(\cdot)$ in eq. (8) is a real-valued function, $f : \mathbb{R}^3 \to \mathbb{R}$, whose arguments consist of the output gap, the inflation rate, and the past monetary policy stance. In Svensson (1997), the value function was a real-valued function, $f : \mathbb{R}^2 \to \mathbb{R}$, whose arguments were the output gap and the inflation rate. Thus, the introduction of the monetary policy lag adds a dimension to the state space.[10] In other words, in the presence of a monetary policy

[9] Here we assume a central bank's discount factor β to be sufficiently small for the Contraction Mapping Theorem to hold. For the list of regularity conditions regarding the Contraction Mapping Theorem, see Stokey and Lucas (1989). Throughout this paper, we simply assume that regularity conditions are satisfied.

[10] In this paper, we have introduced a one-period monetary policy lag in the IS equation (4). In principle, it is possible to introduce higher-order monetary policy lags, such as two-period or three-period lags. However, introduction of higher-order monetary policy lags will cause the dimensions of the state space to increase proportionately (e.g. a two-period lag will add two dimensions) and one will inevitably face the 'curse of dimensionality' (see for instance, Miranda and Fackler (2002, p.166)). In this paper, in order to avoid the 'curse of dimensionality', we have confined our analysis to a one-period monetary policy lag.

lag, the central bank's present-valued total loss will depend not only on the output gap and inflation rate, but also on the past monetary policy stance. This implication is true even if the period-by-period loss function (1) depends only on the output gap and inflation rate.

2.3 Parameter Settings

Table 1 describes the benchmark settings of the key parameters in the model. The meaning of each parameters are provided in Table 1. It should be noted that parameters λ, ρ, δ_1, δ_2, α, σ_ν, σ_ε, and rr^* will be kept fixed throughout the paper and parameters ω and π^* will become variable during the sensitivity analysis exercise later in this paper.

Table 1: Benchmark Parameter Settings

Parameter	Benchmark Value	Meaning
λ	0.5	Central bank's preference for output volatility
ρ	0.6	AR(1) parameter for output gap
δ_1 and δ_2	0.8	Elasticity of output gap w.r.t. nominal interest rate
α	0.4	Parameter controlling sacrifice-ratio
ω	0.7	Lag weight on past monetary policy stance
σ_ν	0.5	IS shock volatility
σ_ε	0.25	AS shock volatility
π^*	2	Inflation target
rr^*	2	Neutral real interest rate

3 Optimal Monetary Policy with a Monetary Policy Lag

In this section, we characterize the optimal monetary policy reaction function in the presence of a monetary policy lag and the zero lower bound on nominal interest rates. For the sake of comparison, we start this section by characterizing the optimal monetary policy reaction function *without* the zero bound constraint.

3.1 Case 1: Without the Zero Lower Bound Constraint

Consider the dynamic optimization problem (8) without the zero bound constraint on the nominal interest rate. Without a constraint on the control variable, the problem will reduce to a standard linear-quadratic control problem. Using vector and matrix notation, the problem can be restated as follows;

$$\begin{aligned} V^{LQ}(s_t) &= \min_{i_t} \left\{ (s_t - s^*)'\mathbf{R}(s_t - s^*) + \beta E_t V^{LQ}(s_{t+1}) \right\} \\ s.t.\ s_{t+1} &= \mathbf{\Gamma} + \mathbf{A}s_t + \mathbf{B}i_t + \mathbf{\Sigma}e_{t+1} \end{aligned} \tag{9}$$

where s_t is a 3-by-1 vector of state variables $(y_t, \pi_t, rrlag_t)'$, s^* is a 3-by-1 vector of constants $(0, \pi^*, 0)'$ which represents the target state for the central bank, and $\mathbf{R}$ is a 3-by-3 symmetric matrix where the elements are given as

$$\mathbf{R} = \begin{bmatrix} \lambda/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that the third row of matrix $\mathbf{R}$ is given as zeros. This is because the volatility of the nominal interest rate – and consequently the past monetary policy stance $rrlag_t$ – does not enter into the loss function of a central bank. Also, notice that the first element of vector s^* is zero. This is due to an implicit assumption in the loss function (1) that a central bank targets output gap to be zero in addition to targeting inflation. The relative importance of inflation targeting (toward π^*) and output gap targeting (toward zero) is captured by the parameter λ.

Turning to the state transition equation, $\mathbf{\Gamma}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{\Sigma}$ are a 3-by-1 vector, 3-by-3 matrix, 3-by-1 vector, 3-by-2 matrix, respectively, and given as

$$\begin{aligned} \mathbf{\Gamma} &= \begin{bmatrix} (1-\omega)\delta_1 rr^* \\ 0 \\ -rr^* \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \rho + (1-\omega)\delta_1\alpha & (1-\omega)\delta_1 & -\omega\delta_2 \\ \alpha & 1 & 0 \\ -\alpha & -1 & 0 \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} -(1-\omega)\delta_1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Finally, e_{t+1} is a 2-by-1 vector of stochastic shocks $(\nu_{t+1}, \varepsilon_{t+1})'$ where the mean is zero and the variance-covariance matrix,[11] $\mathbf{\Omega}$, is given by

$$\mathbf{\Omega} = E_t(e_{t+1}e'_{t+1}) = \begin{bmatrix} \sigma_\nu^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix}.$$

Since the dynamic optimization problem (9) is in the class of standard linear-quadratic control problem, the optimal monetary policy reaction function for this problem can be characterized analytically to some extent.[12] It can be shown that the value function in (9) to be linear-quadratic function as,

$$V^{LQ}(s_t) = P_0 + \mathbf{P}_1 s_t + s'_t \mathbf{P}_2 s_t, \tag{10}$$

[11] By the certainty equivalence property of the linear-quadratic control problem, the existence of stochastic shocks will not affect the characterization of the optimal reaction function. However, it does affect the characterization of the value function and will be used in Section 4.

[12] When the number of state variables is less than or equal to two, the optimal reaction function can be fully characterized by the closed-form expression as in Svensson (1997). Unfortunately, when the number of state variables is more than two (in our case three), as we will see later in this section, the optimal reaction function cannot be characterized by the closed-form expression (for instance, see Ljungqvist and Sargent (2004)). However, by exploiting the nature of the linear-quadratic problem, it is still possible to *indirectly* characterize the optimal reaction function and this is what we do in this subsection.

where P_0 is a scalar, $\mathbf{P}_1$ is a 3-by-1 vector, and $\mathbf{P}_2$ is a 3-by-3 symmetric matrix. (See Appendix for derivation of P_0, $\mathbf{P}_1$, and $\mathbf{P}_2$.) Substituting eq. (10) into Bellman eq. (9) yields the following relationship,

$$\begin{aligned} & P_0 + \mathbf{P}_1 s_t + s_t' \mathbf{P}_2 s_t \\ = \; & \min_{i_t} \left\{ (s_t - s^*)' \mathbf{R} (s_t - s^*) + \beta E_t \left[\begin{array}{c} P_0 + \mathbf{P}_1 (\mathbf{\Gamma} + \mathbf{A} s_t + \mathbf{B} i_t + \mathbf{\Sigma} e_{t+1}) \\ +(\mathbf{\Gamma} + \mathbf{A} s_t + \mathbf{B} i_t + \mathbf{\Sigma} e_{t+1})' \mathbf{P}_2 (\mathbf{\Gamma} + \mathbf{A} s_t + \mathbf{B} i_t + \mathbf{\Sigma} e_{t+1})' \end{array} \right] \right\}. \end{aligned}$$

Taking the FOC with respect to the nominal interest rate, i_t, the optimal monetary policy reaction function can be characterized as

$$i_t^{LQ} = -(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}(\frac{1}{2}\mathbf{B}'\mathbf{P}_1' - \mathbf{B}'\mathbf{P}_2\mathbf{\Gamma}) - (\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}s_t \tag{11}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are given as

$$\mathbf{P}_2 = \mathbf{R} + \beta\mathbf{A}'\mathbf{P}_2\mathbf{A} - \beta\mathbf{A}'\mathbf{P}_2\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A} \tag{12}$$

$$\mathbf{P}_1 = 2\left[\beta\mathbf{\Gamma}'(\mathbf{P}_2 - \mathbf{P}_2\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2)\mathbf{A} - s^{*\prime}\mathbf{R}\right] \cdot \left[\mathbf{I} - \beta\mathbf{A} + \beta\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}\right]^{-1} \tag{13}$$

As can be seen from eq. (12), the parameter matrix $\mathbf{P}_2$ is defined recursively by the non-linear Riccati equation and does not have a closed-form solution whenever the matrix is larger than 2-by-2. In Svensson (1997), it was possible to derive the closed-form expression for the optimal monetary reaction function because there were only two state variables – i.e., the output gap and the inflation rate. Unfortunately, the introduction of a monetary policy lag increases the number of state variables beyond two so that it is no longer possible to derive the exact analytical expression for the optimal monetary policy reaction function even in the linear-quadratic environment.

In practice, given the parameter values, matrix $\mathbf{P}_2$ is numerically evaluated by function iteration. Once $\mathbf{P}_2$ is numerically evaluated, vector $\mathbf{P}_1$ is evaluated sequentially and then the coefficients of the optimal reaction function (11) are evaluated. In what follows, we characterize the optimal monetary policy reaction function with two numerical examples; a benchmark example where there is no monetary policy lag ($\omega = 0$) and another example with a monetary policy lag ($\omega > 0$). By comparing the two cases, we aim to illustrate the nature of the optimal monetary policy reaction function with and without monetary policy lag when there is no zero lower bound constraint.

3.1.1 Example A: No Monetary Policy Lag ($\omega = 0$)

In this example, we shut off the monetary policy lag effect in the model by setting $\omega = 0$ and then we numerically characterize the optimal monetary policy reaction. All other parameter settings are from Table 1. With no monetary policy lag effect (i.e., irrelevance of past monetary policy stance, $rrlag_t$, within the system), the model setup (9) reduces to the original Svensson (1997) setup and, thus, the optimal monetary policy reaction function is identical to that in Svensson (1997).[13] A numerical illustration of this case is shown in Figure 1.

Figure 1 Here

As can be seen in Figure 1, in the absence of the zero lower bound constraint, the optimal monetary policy reaction function is linear in its state variables. Further, without a monetary policy lag, the optimal reaction function is $f : \mathbb{R}^2 \to \mathbb{R}$ and the optimal nominal interest rate depends only on the inflation rate and the output gap. In this sense, in an environment where there is no zero lower bound constraint and no monetary policy lag, the optimal monetary policy reaction function will be similar to the specification of Taylor's (1993) rule.

3.1.2 Example B: With a Monetary Policy Lag ($\omega > 0$)

In this example, we numerically characterize the optimal monetary policy reaction function with a monetary policy lag. Specifically, we set the monetary policy lag parameter $\omega = 0.7$. Again, all other parameter settings are from Table 1. With a monetary policy lag, the optimal monetary policy reaction function now departs from Svensson's (1997) canonical policy function. A numerical illustration of this case is shown in Figure 2.

Figure 2 Here

In the presence of a monetary policy lag, the optimal monetary policy reaction function will be $f : \mathbb{R}^3 \to \mathbb{R}$ rather than $f : \mathbb{R}^2 \to \mathbb{R}$. Thus, it is not possible to capture the entire functional space in three-dimensional figure. As such, we present multiple figures to characterize the policy function. On the left-most panel of Figure 2, holding past monetary policy stance "loose" at -3% (i.e., $rrlag_t = -3$) the relationship between the optimal nominal interest rate and the inflation rate and output gap is depicted. In the middle panel, the policy function is depicted holding past monetary policy stance "neutral" at 0% (i.e., $rrlag_t = 0$). Finally, at the right-most panel of Figure 2, the policy function is depicted holding past monetary policy stance "tight" at 3% (i.e., $rrlag_t = 3$). Also, it should be noted from eq. (11) that, in the absence of the zero lower bound constraint, the optimal monetary policy reaction function is linear in its state variables. This property is visually confirmed in Figure 2.

3.1.3 Observations

From the numerical characterizations of Example A (no monetary policy lag) and Example B (with monetary policy lag), we can make the following observations. By comparing Figure 1 and the middle panel of Figure 2, we observe that the optimal monetary policy reaction

[13] However, one qualification applies with regard to the constant term, $\mathbf{\Gamma}$, in the state transition equation. The existence of a constant term in state transition equation in this paper is due to consideration of the neutral real interest rate, rr^*. In Svensson's (1997) canonical model, the presence of the neutral real interest rate is implicitly assumed away to keep the analysis simple.

function is 'steeper' in the middle panel of Figure 2.[14] From this observation, we infer that the existence of monetary policy lag requires monetary policy to be more reactive (or aggressive) compared to the case without lag.

It is worthwhile to understand why the existence of a monetary policy lag entails a more aggressive conduct of monetary policy. The mechanism of aggressiveness comes two-fold. First, with a monetary policy lag in IS equation (4), it will take at most two periods for a central bank to exert control over the output gap. Consequently, the uncontrolled portion of the output gap will affect the inflation rate through the AS equation (eq. (2)) twice – i.e., through y_t and y_{t+1} – and contribute to a larger movement in the inflation rate, ceteris paribus. In order to counter this larger movement in inflation rate, naturally, the central bank is forced to react to the shock more aggressively. Second, recall from the AS equation in eq. (2) and eq. (3) that expected inflation depends on the current inflation rate and output gap. In other words, in this economy, the expectation of future inflation is formed in a backward-looking manner. With a larger movement in the current inflation rate, expected inflation is also in a larger movement compared to the case without a monetary policy lag. Since Taylor's principle (see Woodford (2003, Ch.2)) dictates that the central bank should adjust the nominal interest rate more than one-for-one for changes in expected inflation to stabilize the economy, this larger movement in expected inflation will require the central bank to adjust the nominal interest rate more aggressively than would be the case without the monetary policy lag.

Next, by comparing the left, middle, and right panels of Figure 2, we observe that the level of the optimal nominal interest rate declines as the past monetary policy stance moves from loose (i.e., $rrlag_t = -3\%$) to tight (i.e., $rrlag_t = 3\%$). This implies that, as for the characterization of the optimal policy function (11), the coefficient on $rrlag_t$ has a negative value. From this observation, we infer that the presence of a monetary policy lag requires monetary policy to offset its lagged effect by taking the opposite stance this period. In other words, when past monetary policy stance was expansionary, the central bank offset its effect by taking a contractionary stance this period, and vice versa. Another inference we make is the absence of interest smoothing. As was reported by Sack and Wieland (2000), there is empirical evidence that the central bank (FRB in Sack and Wieland's (2000) case) tends to smooth changes in policy instruments over time and, whenever a change is made, it is likely for a central bank to keep in the same direction as previous changes. An abrupt reversal of the direction of monetary policy is unlikely in reality. Thus, the tendency toward offsetting behavior in settling the current stance vis-à-vis the past stance as observed in Figure 2 is in apparent conflict with the stylized fact of interest rate smoothing. We concede that this is one of the major drawbacks of the model considered in this paper. However, what we can infer from this result is that the interest smoothing behaviors by central banks – the

[14] Indeed, from eq. (11), we know that the coefficients for the output gap and the inflation rate are independent of past monetary policy stance, $rrlag_t$. Thus, the 'steepness' observed in the left, middle, and right panels of Figure 2 are all equal regardless of the state $rrlag_t$. In this sense, we can safely rephrase the statement here by saying that each panel of Figure 2 depicts a 'steeper' optimal monetary policy reaction function compared to Figure 1.

motivation for which is still being debated[15] – are *not* arising from the monetary policy lag factor. Although this result itself is quite interesting and potentially can cast a new light on the debate regarding the interest rate smoothing behavior, we will not pursue this topic further in the rest of this paper.

In sum, from the numerical characterizations of the optimal monetary policy reaction functions with and without monetary policy lag, we observe the following points.

Observation 1: In the model considered in this paper, the presence of a monetary policy lag requires monetary policy to be aggressive.

Observation 2a: In the model considered in this paper, the presence of a monetary policy lag requires monetary policy to offset its lagged effect by taking an opposite monetary policy stance in the current period.

Observation 2b: The interest rate smoothing motive is not observed in the model considered in this paper.

3.2 Case 2: With the Zero Lower Bound Constraint

So far, we have characterized the optimal monetary policy reaction function without the zero lower bound constraint on the nominal interest rate. In this subsection, following Orphanides and Wieland (2000) and Kato and Nishiyama (2005), we numerically characterize the optimal monetary policy reaction function taking into account the presence of the zero lower bound constraint. Consider the dynamic optimization problem (8). Using vector and matrix notation, the problem can be restated as follows,

$$\begin{aligned} V^{ZLB}(s_t) &= \min_{i_t \geq 0} \left\{ (s_t - s^*)' \mathbf{R} (s_t - s^*) + \psi_t i_t + \beta E_t V^{ZLB}(s_{t+1}) \right\} \\ s.t.\ s_{t+1} &= \mathbf{\Gamma} + \mathbf{A} s_t + \mathbf{B} i_t + \mathbf{\Sigma} e_{t+1} \end{aligned} \tag{14}$$

where ψ_t is a Lagrangian multiplier for the non-negativity constraint on i_t. Due to this Lagrangian multiplier, as shown by Chmielewski and Manousiouthakis (1996), the value function in (14) will no longer be a linear-quadratic function and, further, it will no longer have an analytical form. Consequently, the optimal monetary policy reaction function in the presence of zero lower bound,

$$i_t^{ZLB} = g(y_t, \pi_t, rrlag_t), \tag{15}$$

will be highly non-linear and no longer has an analytical expression. As such, rather than pursuing an analytical approach, we adopt a numerical method known as the Collocation method[16] to characterize the optimal monetary policy reaction function. A description of

[15] Sack and Wieland (2000) point out three possible motives for the central bank to smooth interest rate: forward-looking behavior by market participants, measurement errors associated with macroeconomic variables, and uncertainty regarding structural parameters. Besides these motives, several other explanations have been proposed in the literature such as the maintenance of central bank reputation, motivation for not disrupting the financial markets, among others. For more details, see Sack and Wieland (2000).

[16] For detailed explanation on the Collocation method, see, for instance, Judd (1998) and Miranda and Fackler (2002).

how we apply the Collocation method to the dynamic optimization problem (14) is provided in the Appendix.

3.2.1 Numerical Characterization with the Zero Lower Bound Constraint

Based on the benchmark parameter settings in Table 1, the optimal monetary policy reaction function in the presence of the zero lower bound is numerically characterized and is shown in Figure 3.

Figure 3 Here

Again, in the presence of a monetary policy lag, the optimal monetary policy reaction function will be $f : \mathbb{R}^3 \to \mathbb{R}$ and requires multiple figures in capturing its nature. Following the format as in Figure 2, the left panel of Figure 3 depicts the relationship of the optimal nominal interest rate vis-à-vis the inflation rate and the output gap holding past monetary policy stance expansionary at -3%; the middle panel depicts the policy function holding past monetary policy stance neutral at 0%; and the right panel depicts the function holding past monetary policy stance contractionary at 3%. In contrast to the case in Figure 2, the nominal interest rate will no longer take a negative value in Figure 3. This is especially evident in the right panel of Figure 3 when the past monetary policy stance was excessively tight.

Now, to illustrate the differences between the optimal monetary policy reaction function with and without the zero lower bound and, further, to vividly capture the non-linearity of the policy function, we define the concept of the 'central bank's pre-emption motive' following Kato and Nishiyama (2005).

Definition 2 *(Central Bank's Pre-emption Motive)*
Let a monetary policy reaction function, $i^{naïve}$, be defined as $i^{naïve} \equiv \max\left\{0, i^{LQ}\right\}$ where the function i^{LQ} is the optimal monetary policy reaction function in the absence of the zero lower bound constraint as in eq. (11). Let i^{ZLB} be the optimal monetary policy reaction function in the presence of the zero lower bound constraint as in eq. (15). Then, in the presence of the zero lower bound constraint, the central bank's pre-emption motive is defined as the difference between $i^{naïve}$ and i^{ZLB} – i.e.,

$$\textit{Pre-emption Motive} \equiv i^{naïve}(y_t, \pi_t, rrlag_t) - i^{ZLB}(y_t, \pi_t, rrlag_t).$$

As shown in Kato and Nishiyama (2005), in the presence of the zero lower bound, it is in the interest of a central bank to set a lower nominal interest rate compared to the case where there is no zero lower bound to pre-empt the risk of being caught in the liquidity trap in the future. Note that the auxiliary definition of a monetary policy reaction function, $i^{naïve}$, is made in order to eliminate negative values implied by the policy function, i^{LQ}. It is worthwhile to mention that since the parameter settings and the model environment are exactly the same, except for the presence of the zero lower bound constraint, the difference in $i^{naïve}$ and i^{ZLB} is solely attributable to the central bank's pre-emption motive in containing the risk stemming from the zero lower bound (or liquidity trap). Nothing else, other than this pre-emption motive against the liquidity trap, is causing the difference between $i^{naïve}$ and i^{ZLB}.

Figure 4 depicts the central bank's pre-emption motive for various states of the economy.

Figure 4 Here

Again, following the format of Figure 2 and Figure 3, the left panel of Figure 4 depicts the central bank's pre-emption motive vis-à-vis the inflation rate and the output gap, holding the past monetary policy stance expansionary at -3%; the middle panel depicts the pre-emption motive holding past monetary policy stance neutral at 0%; and the right panel depicts the pre-emption motive holding past monetary policy stance contractionary at 3%. The first aspect to note in Figure 4 is the non-linearity of the pre-emption motive which, in turn, implies that the optimal monetary policy reaction function is non-linear in the presence of the zero lower bound. Second, as can be seen from the left panel, when past monetary policy stance is loose, the pre-emption motive can be as high as 2%. In contrast, as can be seen from the right panel, when past monetary policy stance is tight, the level of the optimal nominal interest rate is already very close to zero (see the right panel of Figure 3) and there is virtually no latitude for a central bank in exerting the pre-emption motive. Overall, we do observe a positive pre-emption motive regardless of past monetary policy stances, although the magnitude of the pre-emption motive varies quite a bit depending on the past monetary policy stance.

In sum, from the numerical characterization of the optimal monetary policy reaction function in the presence of the zero lower bound, we observe the following points.

Observation 3a: In the model considered in this paper, the optimal monetary policy reaction function will be highly non-linear in the presence of the zero lower bound constraint on the nominal interest rate.

Observation 3b: In the model considered in this paper, the pre-emption motive is positive regardless of the past monetary policy stance, although the magnitude varies substantially.

3.2.2 Sensitivity Analysis: Pre-emption Motive and Monetary Policy Lag

Based on the numerical characterization above, we next conduct sensitivity analysis with respect to the monetary policy lag. In this exercise, we are interested to see how the pre-emption motive is affected by the magnitude of monetary policy lag. In the previous numerical characterizations (i.e., Figure 3 and Figure 4), the monetary policy lag weight parameter, ω, was fixed at 0.7. In this sensitivity analysis, we vary this lag weight parameter, ω, to see how the pre-emption motive is affected. Figure 5 shows the results from the sensitivity analysis exercise.

Figure 5 Here

The left panel of Figure 5 depicts the pre-emption motive when the lag weight, ω, is set equal to 0 – i.e., no monetary policy lag. The middle panel depicts the pre-emption motive when the lag weight is set equal to 0.25. Finally, the right panel of Figure 5 depicts the pre-emption motive when the lag weight is set equal to 0.5. Here, it should be noted that all results in Figure 5 are reported holding the past monetary policy stance neutral at 0% (note the difference in reporting format from Figures 2, 3, and 4). As can be seen from Figure 5, the pre-emption motive is getting larger as the lag weight gets larger. This implies that when there is a monetary policy lag, a larger pre-emptive action is required for a central bank to contain the future risk of being caught in the liquidity trap, compared to the case when there is no monetary policy lag. These observations from the sensitivity analysis exercise can be summarized as follow.

Observation 4 In the model considered in this paper, the pre-emption motive becomes larger as the weight on monetary policy lag gets larger.

4 Inflation Target and Monetary Policy Lag

So far we have discussed the design of monetary policy rule in the presence of a monetary policy lag and the zero lower bound constraint, taking the level of the inflation target as given. A natural next question is then, "What if the inflation target is variable? Can a central bank further reduce stabilization cost by raising the inflation target?" As has been pointed out by several researchers,[17] there is potential for a central bank to reduce the risk of falling into the liquidity trap by targeting some positive inflation rate, which Bernanke (2002) coined as the 'buffer'. This role of a buffer can be considered a social benefit, rather than a social cost of inflation, in the sense that a central bank can reduce the stabilization cost by targeting a positive inflation rate in the long run. The purpose of this section is to analyze the buffer role of the inflation target in an environment where there is a monetary policy lag.

4.1 Methodology to Assess Stabilization Cost

To demonstrate the relationship between long-run stabilization cost and the level of the inflation target in the presence of the zero lower bound, several simulation studies have been conducted. For instance, Coenen et al. (2004), employing a stochastic simulation approach based on an estimated US model, showed that the frequency of the nominal interest rate binding at zero is high when the inflation target is set low and vice versa. Hunt and Laxton (2003), using the IMF's MULTIMOD simulation model, showed that targeting too low an inflation rate will induce a central bank to be susceptible to a deflationary spiral and suggested to target the inflation rate higher than 2% in the long run. Lavoie and Pioro

[17] For instance, see Summers (1991), Orphanides and Wieland (1998), Blinder (2000), Reifschneider and Williams (2000), Hunt and Laxton (2003), Nishiyama (2003), Teranishi (2003), Coenen et al. (2004) and Lavoie and Pioro (2007) among others.

(2007) conducted a simulation study based on the Bank of Canada's ToTEM[18] model and showed that a 2% inflation target regime has lower stabilization costs (i.e., lower variance in output gap and inflation rate) compared to a 0% inflation target regime. Although these studies have demonstrated the predicament stemming from the zero lower bound, a drawback of the simulation approach is its reliance to an ad-hoc monetary policy rule (most of them being linear or piece-wise linear functions) in assessing stabilization costs. Since an ad-hoc monetary policy rule is not the same as the optimal monetary policy reaction function, the assessment of the stabilization cost for the central bank (as evaluated using the central bank's value function) may be misguided. Further, when the monetary policy lag is present as in this paper, it is not clear what kind of functional form to adopt for the monetary policy rule when conducting simulations.

In this section, rather than relying on a simulation method, we utilize a numerically interpolated value function, V^{ZLB}, defined in (14) to assess the stabilization cost for the central bank. The strength of this approach is that since the central bank's value function defined in (14) is consistent with the optimal monetary policy reaction function (15), the assessment of the central bank's stabilization cost will be quite accurate.

4.2 Definition of Long-run Stabilization Cost

Here, we define the concept of long-run stabilization cost to the central bank. The value of $V^{ZLB}(y_t, \pi_t, rrlag_t)$ defined in (14) can be interpreted as the central bank's 'cost-to-go'[19] provided that the current state is $(y_t, \pi_t, rrlag_t)'$. In other words, the value of V^{ZLB} evaluated at an arbitrary state $(y_t, \pi_t, rrlag_t)'$ represents the expected discounted value of the future stream of losses, provided that the initial state is $(y_t, \pi_t, rrlag_t)'$. Now, it should be noted that when V^{ZLB} is evaluated at any state other than the steady state, then the central bank's cost-to-go includes the expected transitional losses that will be incurred while converging from $(y_t, \pi_t, rrlag_t)'$ to the steady state $(y_{ss}, \pi_{ss}, rrlag_{ss})'$. In order to separate out these transitional losses from the pure stabilization cost, we need to evaluate V^{ZLB} very close[20] to the steady state. As such, we define the long-run stabilization cost to the central

[18] ToTEM is a large-scale DSGE model developed at the Bank of Canada. It has the features such as monopolistic competition, multi-sector production, open-economy characteristics, various types of nominal rigidities including sticky price and wage among others. See Murchison and Rennison (2006) for a more detailed description regarding ToTEM.

[19] The term 'cost-to-go' has a specific meaning in the optimal control literature. In order to have an understanding of this concept, let us consider the deterministic case. Let the sequence $\{s_t^*\}_{t=0}^{\infty}$ and $\{x_t^*\}_{t=0}^{\infty}$ be the optimal path for the state variable and control variable, respectively, given the initial state s_0. Let $f(s_t, x_t)$ be the period-by-period loss function. Then the cost-to-go at state s_0 is given as

$$V(s_0) = \sum_{t=0}^{\infty} \beta^t f(s_t^*, x_t^*),$$

where β is a discount factor. For more precise definition, see, for instance, Bertsekas (2005).

[20] Ideally speaking, in order to completely seperate out the transitional losses from the pure stabilization cost, we would like to evaluate V^{ZLB} at the (stochastic) steady state – i.e., $V^{ZLB}(y_{ss}, \pi_{ss}, rrlag_{ss})$. However, as was shown in Nishiyama (2003), the (stochastic) steady state of inflation rate will be slightly above

bank as follows. In a similar fashion, we also define the long-run stabilization cost without the zero lower bound.

Definition 3 *(Long-run Stabilization Cost)*
Let V^{ZLB} be the value function in the presence of the zero lower bound constraint as given in (14). Let s^* be the target state and π^* be the level of the inflation target. Then the long-run stabilization cost in the presence of the zero lower bound constraint is defined as

$$V^{ZLB}(s^*) = V^{ZLB}(0, \pi^*, 0).$$

Let V^{LQ} be the value function in the absence of the zero lower bound constraint as given in (9). Then the long-run stabilization cost in the absence of the zero lower bound constraint is defined as

$$V^{LQ}(s^*) = V^{LQ}(0, \pi^*, 0).$$

Based on the above definition, we next investigate the relationship between long-run stabilization cost and the monetary policy lag.

4.3 Sensitivity Analysis: Long-run Stabilization Cost and Monetary Policy Lag

In the previous section, we have fixed the inflation target, π^*, at 2%. In this section, we now characterize the relationship bewteen the long-run stabilization cost and the inflation targets. Further, we conduct a sensitivity analysis with respect to the monetary policy lag (i.e., lag weight parameter, ω) and illustrate how it will affect the long-run stabilization cost for the central bank. Figure 6 shows the results from the sensitivity analysis exercise.

Figure 6 Here

Figure 6 depicts the relationship between the long-run stabilization cost and the inflation target. The range of inflation targets varies from 0% to 4%. Solid lines depict the relationship between the long-run stabilization cost and inflation target with various values of lag weights (ω being set equal to 0, 0.25, 0.5, and 0.7) in the presence of the zero lower bound constraint. A dashed line represents the relationship between the long-run stabilization cost and the level of the inflation target in the absence of the zero lower bound and is depicted in the same figure. To facilitate comparison, the long-run stabilization cost in the absence of the zero lower bound is normalized to one.

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the inflation target that they will not take a same value in the presence of the zero lower bound constraint. To avoid a techinical difficulty in finding the exact values for the (stochastic) steady state, as a compromise, we evaluate V^{ZLB} at the target state s^* (i.e., $y_t = 0$, $\pi_t = \pi^*$, and $rrlag_t = 0$), which is supposed to be very close to the steady state, and call it the 'long-run' stabilization cost, rather than the 'steady state' stabilization cost. The long-run stabilization cost will include some portion of the transitional losses, but the magnitude is deemed to be small because of its proximity to the steady state.

As can be seen from Figure 6, solid lines reveal a downward-sloping feature which implies that, in the long-run, the stabilization cost for the central bank diminishes as the level of inflation target is set higher in the presence of the zero lower bound. As the level of the inflation target gets large enough, the long-run stabilization cost in the presence of the zero lower bound asymptotes to that without the zero lower bound. The intuition is as follows. Suppose the inflation target is set at an extremely low value, say, at 0%. Since the risk of being caught in the liquidity trap will be high in this case, the long-run stabilization cost is also high. On the other hand, setting a high inflation target will mitigate the risk of being caught in the liquidity trap that the long-run stabilization cost becomes lower. Further, as the target gets high enough, the risk becomes virtually zero that the long-run stabilization cost with and without the zero lower bound virtually become the same.

Next, let us shift our attention to the lag weights. As can be seen from Figure 6, the long-run stabilization cost becomes higher as the lag weight is set higher. This is the main feature of this section and deserves some close attention. Consider the case where there is no monetary policy lag (i.e., lag weight, ω, is set equal to zero). Then it is relatively easy for the central bank to stabilize the movement of the state variables, in the sense that they can immediately affect the output gap and inflation rate through a change in nominal interest rate. Consequently, the central bank can contain the risk, if not completely, of being caught by the liquidity trap with a relative ease, resulting to a relatively low long-run stabilization cost. In contrast, consider the case where there exists a monetary policy lag – say the value of lag weight, ω, is 0.7. In this case, it will take some time for the central bank to affect the output gap and inflation rate, making it more difficult to exert control over the movements of state variables. As such, it will be more difficult for the central bank to contain the risk of being caught in the liquidity trap, resulting in a higher long-run stabilization cost compared to the case when there is no monetary policy lag. This is essentially the reason why the long-run stabilization cost for the central bank increases as the monetary policy lag gets longer as we see in the figure.

Finally, let us turn to the case where there is no zero lower bound, which is shown by a dashed line in Figure 6. As can be seen, the line is essentially horizontal implying that the long-run stabilization cost is constant regardless of the level of inflation target. The reason is quite simple. In the absence of the zero bound constraint, there will be no risk of being caught in the liquidity trap. Therefore the long-run stabilization cost is now composed of the standard stabilization cost arising from IS-AS shocks only. Since the variance of IS-AS shocks is invariant with respect to the choice of inflation target, the stabilization cost is constant regardless of the level of the inflation target. Now, as it turns out, in the absence of zero lower bound, the long-run stabilization cost is also constant regardless of the length of the monetary policy lag. In other words, the long-run stabilization cost is unaffected by the length of the monetary policy lag and will be exactly the same as in the case where there is no lag (i.e., $\omega = 0$). In comparison to the case with the zero lower bound, this is a strikingly different outcome. This result tells us that, in the absence of the zero lower bound, the central bank can perfectly offset the cost arising from the lag by conducting aggressive monetary policy (as in Observation 1) in the short-run. By conducting such monetary policy, the central bank will be able keep the stabilization cost as low as in the case without a monetary policy lag in the long-run. In this sense, the existence of the monetary policy lag itself does not pose any threat to the central bank. However, the existence of the monetary policy lag becomes threatening to the central bank *when combined with the zero lower bound* and aggravates the long-run stabilization cost as the lag gets longer.

In sum, we can make the following observations from the sensitivity analysis exercise conducted in this section.

Observation 5a In the model considered in this paper, in the presence of the zero lower bound constraint, the long-run stabilization cost to the central bank diminishes as the level of inflation target is set higher.

Observation 5b In the model considered in this paper, in the presence of the zero lower bound constraint, the long-run stabilization cost to the central bank increases as the lag weight of the monetary policy lag gets larger.

Observation 5c In the model considered in this paper, in the absence of the zero lower bound constraint, the long-run stabilization cost is invariant regardless of the level of the inflation target or the lag weight of the monetary policy lag.

5 Discussion: Policy Implications and Caveats

5.1 Policy Implications

So what are the policy implications from this analysis? In the short run, as we analyzed in Section 3, the monetary policy conduct should be more reactive (or aggressive) and pre-emptive for the central bank confronted with a monetary policy lag. By conducting monetary policy aggressively and pre-emptively, the central bank can curtail the risk of falling into the liquidity trap and this is even more so when the monetary policy lag is longer. In the long run, as we analyzed in Section 4, an obvious policy implication is that the central bank (or government) should set a higher inflation target in the presence of a monetary policy lag. Or putting it differently, for an economy confronted with a longer monetary policy lag, a central bank, other things being equal, should set a higher inflation target so that they have enough 'buffer' to protect themselves from the liquidity trap.

The literature on the optimal level of the inflation target has emphasized the factors such as downward nominal wage rigidity, the measurement bias of inflation, and the risk of the liquidity trap in justifying a positive inflation target. However, as far as the author of this paper is concerned, there has been no study that linked the length of monetary policy lag to the level of the inflation target. As we have seen in this paper, when considering the level of the inflation target, one should take into account the factor of monetary policy lag quite seriously – i.e., the longer the lag, the higher should be the target. Setting the level of inflation target without taking heed of the monetary policy lag may lead to the setting of a suboptimal target with strong negative consequences for the economy. Based on the results in this paper, we suggest that literature and policy discussion surrounding the optimal level of the inflation targeting take into account the monetary policy lag.

5.2 Caveats

Having stated the policy implications, some caveats are in order. Firstly, the above policy implications are conditional upon the existence of the zero lower bound on nominal interest rates. If, in reality, the zero lower bound does not exist, then there is no need to conduct pre-emptive[21] monetary policy and there is no need to set a high inflation target – i.e., no need for 'buffer'. However, since the zero lower bound on nominal interest rates seems realistic,[22] our premise on the existence of the zero lower bound and our policy implications appear to be cogent in this regard.

Secondly, the analysis in this paper has been based on the assumption that the economy is backward-looking.[23] Contrary to the backward-looking assumption, if the actual economy is indeed forward-looking as in Woodford (2003), perhaps the policy implications discussed above should be somewhat weakened. In other words, in the forward-looking economy, optimal monetary policy need not be as reactive or pre-emptive as in the backward-looking case. Further, the level of the inflation target need not be as high as in the backward-looking case. The reason is as follow. As shown by Eggertsson and Woodford (2003), Jung et al. (2005), and Adam and Billi (2006), under forward-looking economy, the central bank possesses the additional monetary policy channel – i.e., the expectation channel – on top of the usual nominal interest rate channel (which is the only monetary policy channel in backward-looking economy). Thus, even if the central bank is trapped by the zero lower bound, they can still exploit the expectation channel to lower the real interest rate[24] and stimulate the economy. As a result, since the zero lower bound is no longer a serious threat to the central bank under forward-looking economy, there is no need for excessively reactive or pre-emptive monetary policy as in the backward-looking case. Consequently, the level of inflation target need not be high because there is no need for an excessive 'buffer' as in the backward-looking case. Thus, when considering the level of the inflation target in reality, it is important to first examine whether the actual economy is well captured by the forward-looking model or the backward-looking model (or hybrid model). The policy

[21] However, the monetary policy conduct should be more aggressive as the monetary policy lag gets longer. This policy implication remains intact even if there is no zero lower bound as we have seen in Section 3.1.

[22] Again, reacall the Japanese experience in late 1990's and early 2000's.

[23] This can be seen from backward-looking Phillips curve (2), where one period ahead inflation depends upon current inflation and output gap, and backward-looking IS equation (4), where one period ahead output gap depends upon current output gap. In contrast, under the New Keynesian framework as in Woodford (2003), current inflation and output gap depend upon expected future inflation and output gap. Thus, the New Keynesian framework is based on the assumption of forward-looking economy.

[24] For instance, the central bank can commit to deliver higher inflation rate in the future as in Eggerstson and Woodford (2003). Assuming that this commitment is perfectly credible, the expected inflation will be higher and, thus, the real interest rate will be lower.

implications derived in this paper are conditional upon the assumption that the economy is backward-looking and should be perceived with this caveat.

6 Conclusion

Since Friedman's (1961) seminal work, researchers as well as practitioners in the monetary policy field have been aware of the monetary policy lag. Indeed, almost all researchers and practitioners seem to agree with Friedman's (1961, p.447) phrase, "monetary actions affects economic conditions only after a lag." Although there have been several empirical attempts to estimate the length of the monetary policy lag (Friedman (1972) and Batini and Nelson (2001) among others), there have been few attempts to characterize the optimal monetary policy rule or the optimal level of the inflation target taking into account the monetary policy lag.

One of the complications pertaining to the monetary policy lag is that the existence of the lag in a model will increase the number of the state variables proportionately to the order of the lag. Indeed, when the number of the state variables for the central bank exceeds a certain level, the optimal monetary policy reaction function does not have a closed-form expression even under the linear-quadratic model environment. This complication entails a numerical method in characterizing the optimal monetary policy when there is a lagged effect of monetary policy.

In this paper, adopting Svensson's (1997) inflation targeting framework, we numerically characterized the optimal monetary policy reaction function in the presence of a monetary policy lag. Further, we characterized the buffer role of an inflation target in the presence of a monetary policy lag as well. The key findings in this paper are as follows. First, in the presence of a monetary policy lag, it is optimal for the central bank to conduct more aggressive monetary policy compared to the case without the monetary policy lag. Second, in the presence of a monetary policy lag and the zero lower bound, it is optimal for the central bank to conduct monetary policy more pre-emptively compared to the case without the monetary policy lag. Third, in the presence of a monetary policy lag and the zero lower bound, the long-run stabilization cost to the central bank is higher compared to the case without the monetary policy lag. The third finding suggests that the central bank should set a higher (lower) inflation target when faced with longer (shorter) monetary policy lags. This finding casts new light on the discussion of the optimal level of inflation target, which, so far, has overlooked the effects stemming from the monetary policy lag. Based on the results in this paper, we suggest that future discussions take into account the monetary policy lag.

There are two remarks to make before we conclude this paper. First, as pointed out by Sack and Wieland (2000), actual conduct of monetary policy reveals a tendency towards interest rate smoothing over time. In this paper, the monetary policy lag was not compatible with the interest rate smoothing feature. On the contrary, the existence of monetary policy lag rendered the central bank to take an opposite monetary policy stance vis-à-vis the past policy stance. This result is in conflict with the empirical findings of interest rate smoothing and can be considered as a drawback of the model in this paper. However, what we can draw from this result is that the interest smoothing behavior of central banks is *not* a result of the monetary policy lag. Second, the model adopted in this paper is a 'backward-looking' model. Although a backward-looking model is useful, especially in capturing the inflation persistence property of an economy, it is vulnerable to the Lucas (1976) critique. To characterize the monetary policy rule in a fashion immune to the Lucas critique, it is important to model the monetary policy lag in the context of forward-looking economy à la Woodford (2003). Perhaps the main results of this paper will be weakened quantitatively in the context of forward-looking economy, but, at the same time, the author of this paper conjectures that the qualitative implications will remain intact. Nevertheless, this is a mere conjecture at this point and calls for a thorough investigation which will be left for a future research.

A Appendix: Derivation of the Optimal Monetary Policy Function under LQ Stochastic Control Problem

In this appendix, we derive analytical expressions for the value function and the optimal monetary policy reaction function in the absence of the zero lower bound constraint. For convenience, let us restate the Bellman equation (9) here.

$$\begin{aligned} V^{LQ}(s_t) &= \min_{i_t} \left\{ (s_t - s^*)'\mathbf{R}(s_t - s^*) + \beta E_t V^{LQ}(s_{t+1}) \right\} \\ s.t.\ s_{t+1} &= \mathbf{\Gamma} + \mathbf{A}s_t + \mathbf{B}i_t + \mathbf{\Sigma}e_{t+1} \end{aligned} \tag{A.1}$$

Now, since this problem is a standard linear-quadratic control problem, we conjecture that the value function has quadratic form as follows,

$$V^{LQ}(s_t) = P_0 + \mathbf{P}_1 s_t + s_t'\mathbf{P}_2 s_t. \tag{A.2}$$

Substituting eq. (A.2) to eq. (A.1), the Bellman equation becomes,

$$\begin{aligned} & P_0 + \mathbf{P}_1 s_t + s_t'\mathbf{P}_2 s_t \\ = & \min_{i_t} \left\{ \begin{array}{c} (s_t - s^*)'\mathbf{R}(s_t - s^*) \\ +\beta E_t \left[\begin{array}{c} P_0 + \mathbf{P}_1(\mathbf{\Gamma} + \mathbf{A}s_t + \mathbf{B}i_t + \mathbf{\Sigma}e_{t+1}) \\ +(\mathbf{\Gamma} + \mathbf{A}s_t + \mathbf{B}i_t + \mathbf{\Sigma}e_{t+1})'\mathbf{P}_2(\mathbf{\Gamma} + \mathbf{A}s_t + \mathbf{B}i_t + \mathbf{\Sigma}e_{t+1})' \end{array} \right] \end{array} \right\} \end{aligned} \tag{A.3}$$

Taking FOC of the RHS of eq. (A.3) with respect to i_t, the optimal monetary policy reaction function can be expressed as,

$$i_t^{LQ} = -\underbrace{(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}(\frac{1}{2}\mathbf{B}'\mathbf{P}_1' - \mathbf{B}'\mathbf{P}_2\mathbf{\Gamma})}_{\equiv \mathbf{E}} - \underbrace{(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}}_{\equiv \mathbf{F}} s_t. \tag{A.4}$$

Substituting eq. (A.4) back to eq. (A.3), it becomes

$$\begin{aligned} P_0 + \mathbf{P}_1 s_t + s_t'\mathbf{P}_2 s_t &= s^{*\prime}\mathbf{R}s^* + \beta P_0 + \beta\mathbf{P}_1(\mathbf{\Gamma} - \mathbf{BE})'\mathbf{P}_2(\mathbf{\Gamma} - \mathbf{BE}) + \beta tr(\mathbf{P}_2\mathbf{\Sigma\Omega\Sigma}') \\ &\quad -2s^{*\prime}\mathbf{R}s_t + \beta\mathbf{P}_1(\mathbf{A} - \mathbf{BF})s_t + 2\beta(\mathbf{\Gamma} - \mathbf{BE})'\mathbf{P}_2(\mathbf{A} - \mathbf{BF})s_t \\ &\quad + s_t'\mathbf{R}s_t + \beta s_t'(\mathbf{A} - \mathbf{BF})'\mathbf{P}_2(\mathbf{A} - \mathbf{BF})s_t. \end{aligned} \tag{A.5}$$

Matching the coefficients on both sides of eq. (A.5), P_0, $\mathbf{P}_1$, and $\mathbf{P}_2$ can be solved, respectively, as follows:

$$\begin{aligned} P_0 &= \frac{1}{1-\beta}s^{*\prime}\mathbf{R}s^* - \frac{\beta}{4(1-\beta)}\mathbf{P}_1\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}B'P_1' + \frac{\beta}{1-\beta}\mathbf{P}_1\left[\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\right]\mathbf{\Gamma} \\ &\quad + \frac{\beta}{1-\beta}\mathbf{\Gamma}'\left[\mathbf{P}_2 + 3\mathbf{P}_2\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\right]\mathbf{\Gamma} + \frac{\beta}{1-\beta}tr(\mathbf{P}_2\mathbf{\Sigma\Omega\Sigma}'), && \text{(A.6)} \\ \mathbf{P}_1 &= 2\left[\beta\mathbf{\Gamma}'(\mathbf{P}_2 - \mathbf{P}_2\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2)\mathbf{A} - s^{*\prime}\mathbf{R}\right] \\ &\quad \times\left[\mathbf{I} - \beta\mathbf{A} + \beta\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}\right]^{-1},\ \text{and} && \text{(A.7)} \\ \mathbf{P}_2 &= \mathbf{R} + \beta\mathbf{A}'\mathbf{P}_2\mathbf{A} - \beta\mathbf{A}'\mathbf{P}_2\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}. && \text{(A.8)} \end{aligned}$$

Notice that when $\mathbf{\Gamma}$ is equal to zero, in other words when the neutral real interest rate, rr^*, is equal to zero (which is the case in Svensson (1997)), the above expression for P_0 and $\mathbf{P}_1$ will simplify to

$$\begin{aligned}
P_0 &= \frac{1}{1-\beta}s^{*\prime}\mathbf{R}s^* - \frac{\beta}{4(1-\beta)}\mathbf{P}_1\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}B'P_1' + \frac{\beta}{1-\beta}tr(\mathbf{P}_2\mathbf{\Sigma}\mathbf{\Omega}\mathbf{\Sigma}') \text{ and} \\
\mathbf{P}_1 &= -2s^{*\prime}\mathbf{R}\left[\mathbf{I} - \beta\mathbf{A} + \beta\mathbf{B}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}\right]^{-1}.
\end{aligned}$$

Consequently, the optimal monetary policy reaction function will simplify to

$$i_t^{LQ} = -\frac{1}{2}(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_1' - (\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}s_t.$$

Further, notice that when s^* is equal to zero in addition to $\mathbf{\Gamma} = 0$, in other words when the inflation target, π^*, is equal to zero, the expression for P_0 and $\mathbf{P}_1$ will further simplify to

$$\begin{aligned}
P_0 &= \frac{\beta}{1-\beta}tr(\mathbf{P}_2\mathbf{\Sigma}\mathbf{\Omega}\mathbf{\Sigma}') \text{ and} \\
\mathbf{P}_1 &= 0.
\end{aligned}$$

The constant term of the value function will be trivially equal to the discounted present value of the variance of IS and AS shocks and the coefficient for the linear term will vanish. Consequently, the optimal monetary policy reaction function will be simplified as

$$i_t^{LQ} = -(\mathbf{B}'\mathbf{P}_2\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_2\mathbf{A}s_t.$$

It should be noted that the expression for $\mathbf{P}_2$ is independent from $\mathbf{\Gamma}$ or s^*.

B Appendix: Description of Collocation Method

In this appendix, we explain the numerical algorithm in approximating the value function and optimal policy reaction function in described in problem (14). Specifically, we employ the numerical method known as the collocation method in solving the functional fixed-point problem posed by the Bellman equation. For complete and detailed explanation regarding the collocation method, see Judd (1998, Ch.11 and 12) and Miranda and Fackler (2002, Ch.8 and 9). The treatment in this appendix follows that of Miranda and Fackler (2002).

For convenience, let us restate the Bellman equation (14) suppressing the time subscripts as follows,

$$V(\pi, y, rrlag) = \min_{x\geq 0}\{f(y,\pi) + \beta EV(g(y,\pi,rrlag,x,\nu,\varepsilon)\}, \tag{A.9}$$

where $f(y,\pi)$ stands for the period-by-period loss function and $g(y,\pi,rrlag,x,\nu,\varepsilon)$ stands for the state transition function. Note that the nominal interest rate, denoted by x in this appendix, is constrained by the zero lower bound. The state transition function is linear in the state variables and the coefficient matrix is time-invariant, i.e.,

$$g(y,\pi,rrlag,x,\upsilon,\varepsilon) = \mathbf{\Gamma} + \mathbf{A}\begin{bmatrix} y \\ \pi \\ rrlag \end{bmatrix} + \mathbf{B}x + \mathbf{\Sigma}e_{t+1}.$$

Given the above specification of the Bellman equation and the state transition function, our goal is to interpolate the value function $V(y, \pi, rrlag)$ in the interval of $-3 \leq y \leq 3$, $-1 \leq \pi \leq 5$, and $-3 \leq rrlag \leq 3$.

The collocation method proceeds in the following steps. First, we discretize the state space by the set of interpolation nodes such that $Node = \{(y_{n_y}, \pi_{n_\pi}, rrlag_{n_{lag}})|\ n_y = 1, 2, \cdots, N_y,\ n_\pi = 1, 2, \cdots, N_\pi, \text{ and } n_{lag} = 1, 2, \cdots, N_{lag}\}$. Thus, we discretize the state space into the total of $N_y \times N_\pi \times N_{lag}$ interpolation nodes. Then we interpolate the value function $V(\cdot)$ using a cubic spline function over these interpolation nodes as follows

$$V(y_{n_y}, \pi_{n_\pi}, rrlag_{n_{lag}}) = \sum_{i=1}^{N_y} \sum_{j=1}^{N_\pi} \sum_{k=1}^{N_{lag}} c_{ijk} \gamma_i^y(y_{n_y}) \gamma_j^\pi(\pi_{n_\pi}) \gamma_k^{lag}(rrlag_{n_{lag}}) \tag{A.10}$$

where the basis functions $\gamma_i^\pi(\pi_{n_\pi})$, $\gamma_j^y(y_{n_y})$, and $\gamma_k^{lag}(rrlag_{n_{lag}})$ take the form of cubic spline functions.

Interpolation equations (A.10) could be expressed compactly using the tensor product notation as follows,

$$\mathbf{v} = [\mathbf{\Xi}_y \otimes \mathbf{\Xi}_\pi \otimes \mathbf{\Xi}_{lag}] \cdot \mathbf{c}, \tag{A.11}$$

where $\mathbf{v}$ stands for $N_y N_\pi N_{lag} \times 1$ vector of the values of $V(y_{n_y}, \pi_{n_\pi}, rrlag_{n_{lag}})$ for each interpolation node, Ξ_y stands for $N_y \times N_y$ matrix of the basis functions $\gamma_i^y(y_{n_y})$, $\mathbf{\Xi}_\pi$ stands for $N_\pi \times N_\pi$ matrix of the basis functions $\gamma_j^\pi(\pi_{n_\pi})$, $\mathbf{\Xi}_{lag}$ stands for $N_{lag} \times N_{lag}$ matrix of the basis functions $\gamma_k^{lag}(rrlag_{n_{lag}})$, and $\mathbf{c}$ stands for $N_y N_\pi N_{lag} \times 1$ vector of the basis coefficients c_{ijk}.

Next, we turn to the right-hand side of the Bellman equation (A.9). In approximating the expected value function, i.e., $E[V(g(y, \pi, rrlag, x, \nu, \varepsilon)]$, we assume the distribution of the error terms (ν, ε) to be *i.i.d.* multivariate normal. We adopt Gaussian-Hermite quadrature method in discretizing the random space with the set of quadrature nodes such that $QNode = \{(\nu_{h_\nu}, \varepsilon_{h_\varepsilon})|h_\nu = 1, 2, \cdots, M_\nu \text{ and } h_\varepsilon = 1, 2, \cdots, M_\varepsilon\}$ with corresponding quadrature weights $\omega_{h_\nu h_\varepsilon}$. Thus, we discretize the random space into a total of $M_\nu \times M_\varepsilon$ quadrature nodes. Then by substituting the interpolation equation (A.10) for the value function $V(g(y, \pi, rrlag, x, \nu, \varepsilon)$, the right-hand side of the Bellman equation can be approximated as

$$\begin{aligned} & RHS_{n_y n_\pi n_{lag}}(\mathbf{c}) \\ = \; & \min_{x \geq 0} \left\{ f(y_{n_y}, \pi_{n_\pi}) + \beta \sum_{h_\nu=1}^{M_\nu} \sum_{h_\varepsilon=1}^{M_\varepsilon} \sum_{i=1}^{N_y} \sum_{j=1}^{N_\pi} \sum_{k=1}^{N_{lag}} \omega_{h_\nu h_\varepsilon} c_{ijk} \gamma_{ijk} \right\} \end{aligned} \tag{A.12}$$

for each $(y_{n_y}, \pi_{n_\pi}, rrlag_{n_{lag}}) \in Node$ where γ_{ijk} stands for the cross products of the basis function. The minimization of the above problem with respect to x can be attained using a standard Quasi-Newton optimization method. It should be noted that when implementing this minimization problem, one should pay attention to the corner solution of the minimization problem due to the zero lower bound constraint on the control variable x.

Finally, by equating eq. (A.10) and eq. (A.12) for each interpolation node, we obtain the following approximation of the Bellman equation (A.9);

$$\sum_{i=1}^{N_y}\sum_{j=1}^{N_\pi}\sum_{k=1}^{N_{lag}} c_{ijk}\gamma_i^y(y_{n_y})\gamma_j^\pi(\pi_{n_\pi})\gamma_k^{lag}(rrlag_{n_{lag}}) \tag{A.13}$$

$$= \quad RHS_{n_y n_\pi n_{lag}}(\mathbf{c}) \quad \text{for each } (y_{n_y}, \pi_{n_\pi}, rrlag_{n_{lag}}) \in Node. \tag{16}$$

Using the tensor product notation, the above equation can be compactly expressed as

$$[\boldsymbol{\Xi}_y \otimes \boldsymbol{\Xi}_\pi \otimes \boldsymbol{\Xi}_{lag}]\mathbf{c} = \mathbf{RHS}(\mathbf{c}), \tag{A.14}$$

where $\mathbf{RHS}(\mathbf{c})$ stands for $N_y N_\pi N_{lag} \times 1$ vector of the values of $RHS_{n_y n_\pi n_{lag}}(\mathbf{c})$. The nonlinear equation system can be solved by an appropriate iterative nonlinear root-finding algorithm.

Once we solve for the nonlinear equation system for vector $\mathbf{c}$, the interpolation of the value function $V(y, \pi, rrlag)$ is now attained. As a by-product of interpolating the value function, the approximation of the optimal policy function $x^*(y, \pi, rrlag)$ will also be attained at the same time.

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Figure 1: Optimal Monetary Policy Reaction Function
Case: No Bound, No Lag

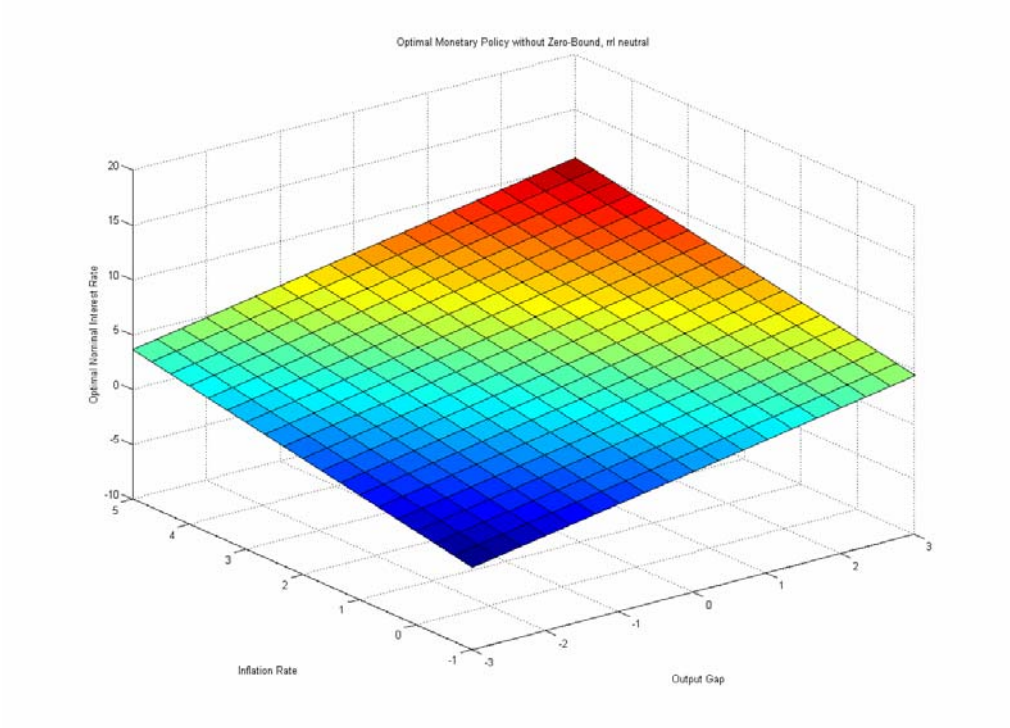
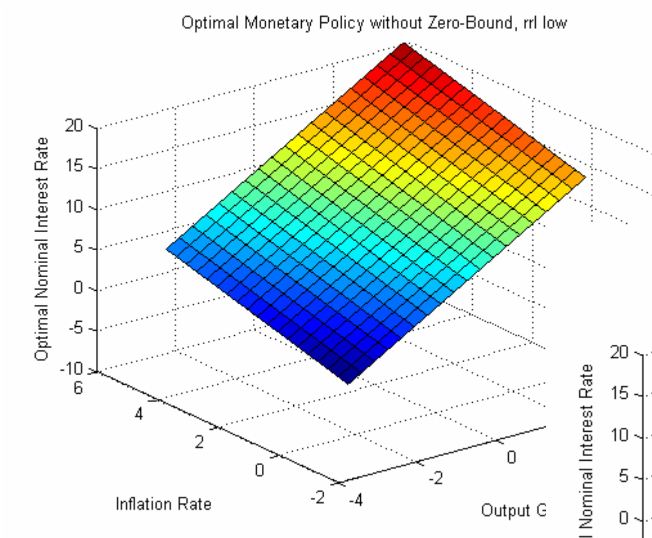
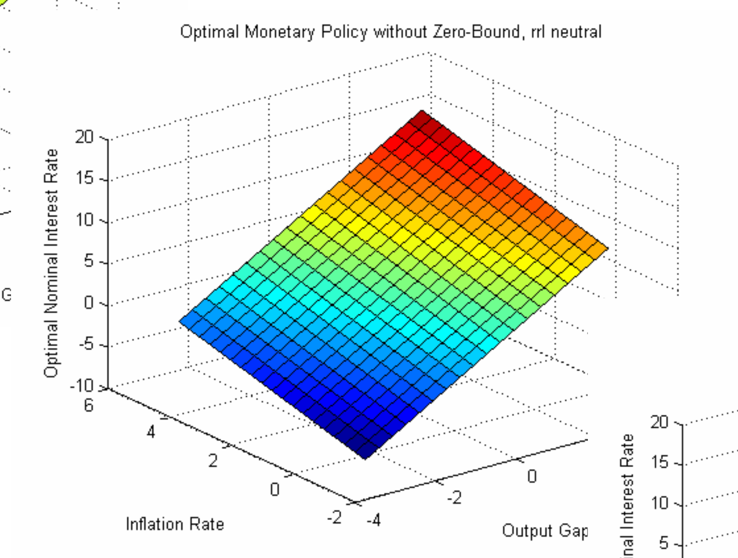


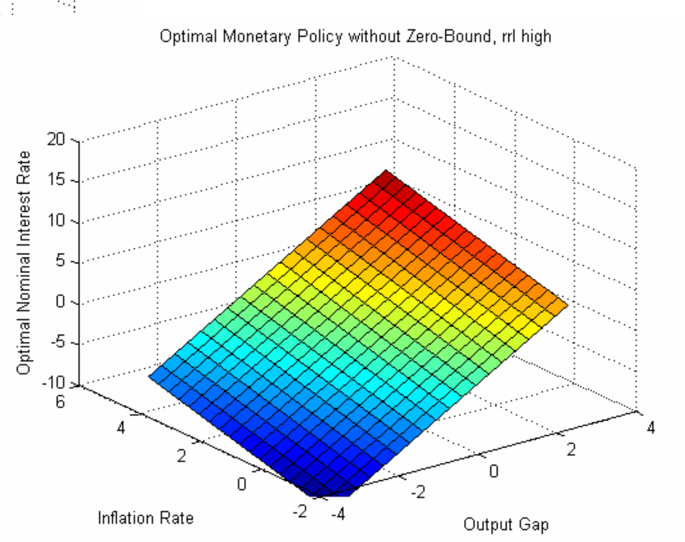
Figure 2: Optimal Monetary Policy Reaction Function
Case: No Bound, Lag weight = 0.7, various rrlag's



rrlag = -3% (loose)

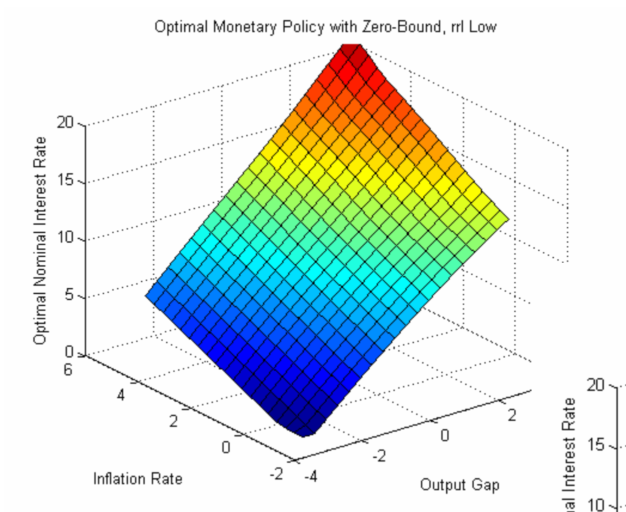


rrlag = 0% (neutral)

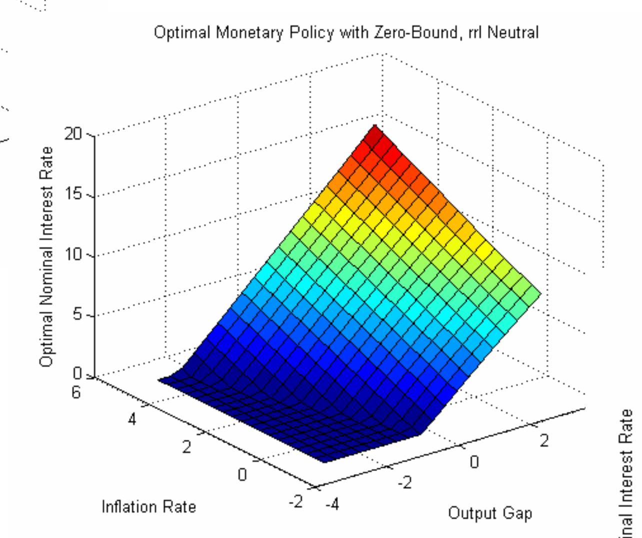


rrlag = 3% (tight)

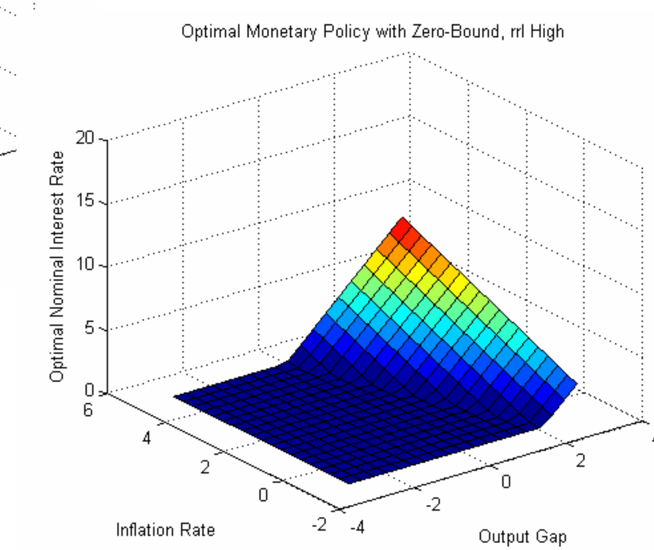
Figure 3: Optimal Monetary Policy Reaction Function
Case: Zero-bound, Lag weight = 0.7, various rrlag's



rrlag = -3% (loose)

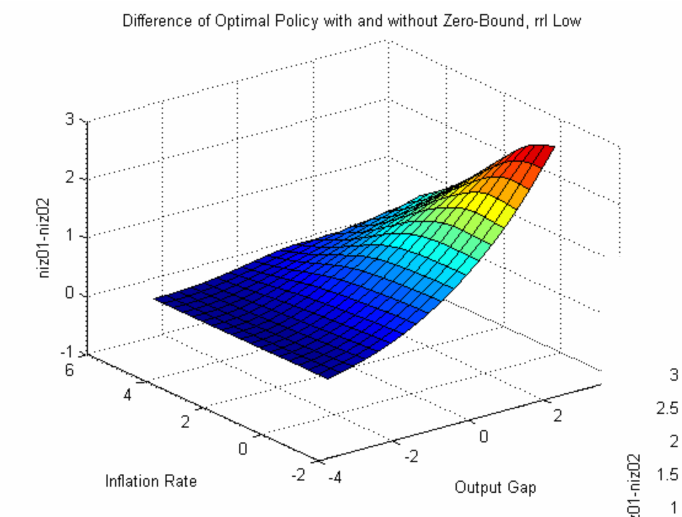


rrlag = 0% (neutral)

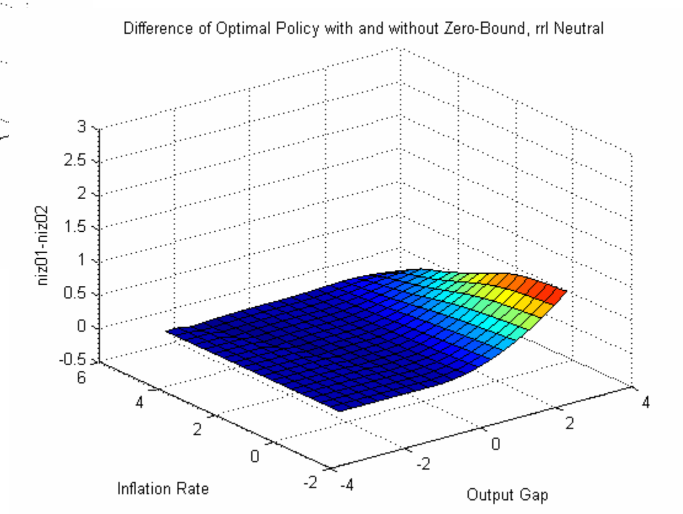


rrlag = 3% (tight)

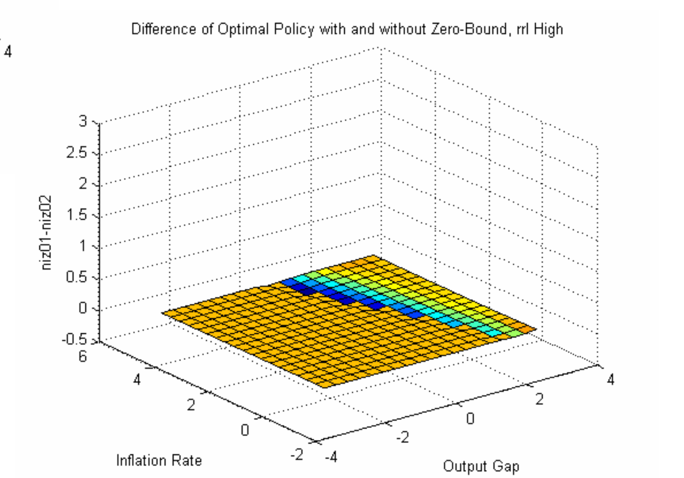
Figure 4: Pre-emption Motive in Monetary Policy
Case: Lag weight = 0.7, various rrlag's



rrlag = -3% (loose)

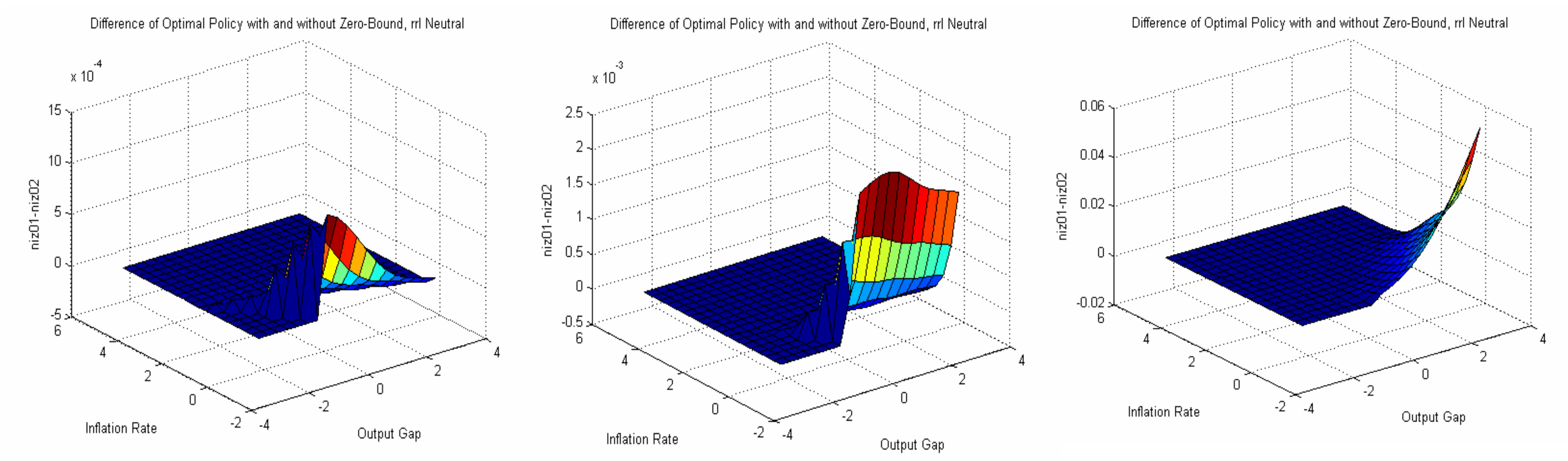


rrlag = 0% (neutral)



rrlag = 3% (tight)

Figure 5: Sensitivity Analysis of Pre-emption Motive and Lag Weights
Case: rrlag = 0% (neutral), various lag weights



Lag weight = 0 | Lag weight = 0.25 | Lag weight = 0.5

Figure 6: Sensitivity Analysis of Long-run Stabilization Cost with respect to Lag Weights