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**Matching Markets with  
Signals**

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### Matching Markets with Signals

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#### Summary

A costless signaling mechanism has been proposed as a device to improve welfare in decentralized two-sided matching markets. An example of such an environment is a job market for new Ph.D. economists. We study a market game of incomplete information between firms and workers and show that costless signaling is actually harmful in some matching markets. Specifically, if agents have very similar preferences, signaling lessens the total number of matches and the welfare of firms, as well as it affects ambiguously the welfare of workers. These results run contrary to previous findings that costless signaling facilitates match formation.

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# Matching Markets with Signals

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## **Abstract**

A costless signaling mechanism has been proposed as a device to improve welfare in decentralized two-sided matching markets. An example of such an environment is a job market for new Ph.D. economists. We study a market game of incomplete information between firms and workers and show that costless signaling is actually harmful in some matching markets. Specifically, if agents have very similar preferences, signaling lessens the total number of matches and the welfare of firms, as well as it affects ambiguously the welfare of workers. These results run contrary to previous findings that costless signaling facilitates match formation.

# 1 Introduction.

Signaling as an actual mechanism design instrument was first implemented by the Ad Hoc Committee<sup>1</sup> of American Economic Association (AEA) in December 2006 to facilitate match formation in the job market of new Ph.D. economists. This market begins in the early Fall each year when economic departments advertise for open faculty positions and graduate students nearing completion of their dissertations apply to these positions according to their preferences. Additionally, each student has an opportunity to send two signals to two departments prior to the market<sup>2</sup>. Each signal states only that the student has indicated her interests to some department. Only the faculty of the chosen department knows that the student has sent her signal. The main part of the market happens afterwards when departments invite students for interviews and finally choose the best candidates, whom they make job offers. However, each department can interview only a small portion of students, which creates congestion. The decision which candidates to invite for interviews is a strategic one. An average department does not want to find itself interviewing the same candidates who are being interviewed by the elite departments.

The Ad Hoc Committee introduced the signals in order to alleviate the congestion on the interviewing stage. Signaling is essentially a costless communication, or cheap talk. There is no penalty attaches for lying, and claims do not directly affect payoffs. Therefore, the signals can only enlarge the set of equilibria. Crawford and Sobel (1982) show that cheap talk can be credible in an equilibrium if parties have common interests. Moreover, costless communication leads to new equilibria that are Pareto-superior to the one without communication. Therefore, one may conjecture that cheap talk should be also beneficial for decentralized matching markets. Roth (2008b) also suggests that the limited number of signals can credibly transmit information about students' preference, which could help to reduce the coordination failures faced by the market participants and facilitate better match formation (see also "Signaling for Interviews in the Economics Job Market"<sup>3</sup> for the

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<sup>1</sup>The Ad Hoc Committee was established in 2005 in order to develop ways to facilitate the market for new economists. Its members are Alvin E. Roth (chair), John Cawley, Philip Levine, Muriel Niederle, and John Siegfried.

<sup>2</sup>This mechanism is implemented via AEA website <http://www.aeaweb.org/joe/signal/>.

<sup>3</sup>The document has been created by the Ad Hoc Committee (AEA) to provide advice for participants of Job Market for new Economists, <http://www.aeaweb.org/joe/signal/signaling.pdf>.

discussion). Recently, Coles and Niederle (2007) obtain results that support this intuition<sup>4</sup>. They consider one-to-one matching market between firms and workers, whose preferences are ex-ante uniformly distributed over the range of all possible preference order lists (preferences are equally likely). Each worker can send only one signal to some firm. They show that the introduction of signals increases the expected number of matches and the welfare of workers in equilibria<sup>5</sup>.

We consider a more realistic assumption (than a uniform distribution) on agents' preferences and show that signals impede match formation in our environment. Though signals transmit information about agents' types truthfully, they also introduce information asymmetry. The information asymmetry facilitates coordination failures that decrease the expected number of matches and the welfare of agents. This decrease of agents' welfare in new cheap talk equilibria is in the line of Farrell and Gibbons (1989)'s results, though it differs in its intuition<sup>6</sup>. Costless communication in their two-agent bargaining model gives the buyer an opportunity to pretend to have a lower value and the seller an opportunity to pretend to have a higher value (compare to truthful information transmission in our model). This enhances their bargaining positions at the cost of the risk of no trade. New cheap talk equilibria are characterized by both less trade and a reduction in the expected gains from trade.

We analyze one-to-one matching market between workers and firms in this paper. We examine an environment when workers have almost aligned preferences. Each worker has either "typical" commonly known preferences with probability close to one or "atypical" preferences taken from some distribution with complementary probability close to zero. The preferences of workers are ex-ante independently distributed. Firms have identical and commonly known preferences over workers. We consider a decentralized matching game with three stages. First, each worker chooses some firm, which she sends her signal to. Each worker can send at most one signal<sup>7</sup>. Workers send signals simultaneously. Only firms that

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<sup>4</sup>See also Paredo (2007) for related work.

<sup>5</sup>Coles and Niederle (2007)'s analysis is limited by the assumption that each firm can see only the signal from its most preferred worker. Kushnir (2008) relaxes this assumption and extends their results to the case of many-to-one matching market with many signals. He also shows explicitly that the introduction of signals ambiguously affects the welfare of firms.

<sup>6</sup>We are thankful for Lones Smith who drew our attention to this comparison.

<sup>7</sup>A worker can abstain from sending any signal.

receive signals observe them. Second, firms make decisions about job offers taking into account signals received at the first stage. Each firm can make only one offer. Finally, each worker chooses some offer to accept among available offers. Each worker can accept at most one offer.

We show that if firms respond to signals in this environment, i.e. treat signals informatively, the introduction of signals decreases the expected number of matches and the welfare of firms. The effect of signals on the welfare of workers is ambiguous. Intuitively, they help workers with "atypical" preferences to obtain better matches, but at the same time, they deprive some workers with "typical" preferences of their matches. Overall, our analysis reveals two important roles of signals. On the one hand, signals reduce coordination failures because they transmit previously unavailable information about workers' preferences. On the other hand, signals introduce information asymmetry. They transmit information about the preferences of students to the limited number of firms, leaving the other firms uninformed. This information asymmetry facilitates coordination failures.

Let us illustrate why signals can facilitate coordination failures by a simple example with three firms and three workers. The firms rank the workers in the same order  $(w_1, w_2, w_3)$ , i.e. they strictly prefer worker  $w_1$  to worker  $w_2$  to worker  $w_3$ . Each worker's preference is either "typical"  $(f_1, f_2, f_3)$  with probability  $1 - \varepsilon$ ,  $\varepsilon \ll 1$  or "atypical" with complementary probability  $\varepsilon$ <sup>8</sup>. "Atypical" preferences are uniformly distributed among all possible preference order lists. All workers are acceptable to all firms and vice versa.

If signals are not allowed, the only possible match in an equilibrium is the assortative match, in which each firm is matched to the corresponding worker. If signals are allowed, we consider the following equilibrium strategies of agents. Each worker with "typical" preferences sends her signal to the corresponding firm (worker  $w_i$  sends her signal to firm  $f_i$ ). Each worker with "atypical" preferences send her signal to the best firm worse or equal to the corresponding one (according to "typical" preferences). Each firm makes her offer to a worker better or equal to the corresponding one only if it receives a signal from her. Each firm ignores all signals from workers worse than the corresponding one. If a firm receives no signals, it makes an offer to the best worker worse than the corresponding one.

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<sup>8</sup>See section 2 for the exact assumption on  $\varepsilon$ .

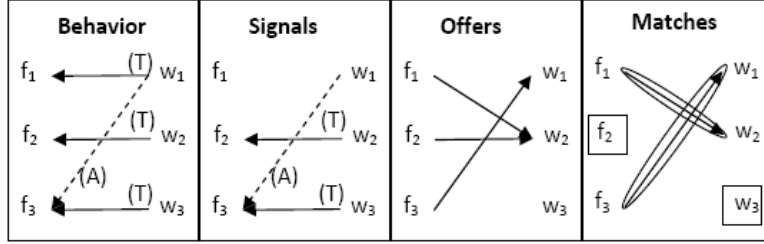


Figure I.

Let us consider the realization of preference profiles when only worker  $w_1$  is "atypical" and firm  $f_3$  is her favorite firm. Worker  $w_2$  and worker  $w_3$  are "typical". Then, worker  $w_1$  sends her signal to firm  $f_3$ . Worker  $w_2$  and worker  $w_3$  send their signals to firm  $f_2$  and firm  $f_3$  correspondingly. Firm  $f_3$  makes her offer to worker  $w_1$ , and firm  $f_1$  anticipates that worker  $w_1$  is "atypical" and makes her offer to worker  $w_2$ . However, the coordination failure arises because firm  $f_2$  has no information about worker  $w_1$ 's type and cannot anticipate firm  $f_1$ 's behavior. Firm  $f_2$  makes also her offer to worker  $w_2$ , and it eventually ends up unmatched because worker  $w_2$  accepts firm  $f_1$ 's offer. Thus, the number of matches for some realization of preferences is fewer than the number of matches when the signals are not allowed. Therefore, the expected number of matches is also fewer.

A substantial part of the literature on two-sided matching markets studies centralized markets that employ deferred acceptance algorithm proposed by Gale and Shapley (1962). The outcome of this algorithm is a "stable" matching, in which no agents is matched to an unacceptable agent of the other side of the market, and no pair of agents is unmatched if it prefers to be matched. Centralized clearinghouses organized around the deferred acceptance algorithm can deliver thickness to the market, help deal with the congestion, and make it safe to participate (Roth (2008b)). These desirable properties have allowed some failed markets to be successfully reorganized. Roth (2008a) and Roth and Sotomayor (1990) present an excellent overview of the main theoretical accomplishments in this area.

However, many labor markets are decentralized or at least preceded by decentralized opportunities for participants to match. Therefore, the analysis of decentralized matching markets outcomes and devices that facilitate match formation for these markets appear to be an important issue. Alcalde and Romero-Medina (2000) propose a simple mechanism in



form of a two-stage game that implements stable matches<sup>9</sup>. Pais (2006) models decentralized matching markets by means of a sequential game where firms are randomly given the opportunity to make job offers. She shows that every stable matching can be reached as the outcome of an equilibrium play of the game. Niederle and Yariv (2008) is one of the few papers that study decentralized matching markets under incomplete information. They show that strong assumptions are required for the existence of equilibrium strategies that yield a stable outcome in the presence of uncertainty and frictions.

Finally, we want to note that this paper does not analyze search for matches. Agents usually need to perform costly search to locate a better partner in decentralized matching markets. Contrary to search literature (see Chade and Smith (2006), Lee and Schwarz (2007), Kircher (2008)), we assume that agents perfectly know the payoffs from their matches.

The paper proceeds as follows. Section 2 outlines our general model. Our main result that costless signaling is actually harmful for some matching markets is presented in section 3. Section 4 derives two roles that signals play in matching markets: transmit information and facilitate asymmetry. Finally, section 5 concludes and outlines some directions for further research.

## 2 Model.

We consider a two-sided matching market with  $N$  workers and  $M$  firms,  $N \geq M$ . The set of workers and the set of firms are denoted as  $W$  and  $F$  correspondingly.  $W$  and  $F$  include the empty set  $\{\emptyset\}$ . Each worker  $w$  has a strict preference order  $\theta_w$  over firms. Similarly, each firm  $f$  has a strict preference order  $\theta_f$  over workers.  $\Theta_W$  and  $\Theta_F$  are the set of all possible workers' and firms' preference orders. Each agent knows only her/its preference order list and has some ex-ante common beliefs about the other agents' preferences. We assume that agents' preferences are independently distributed and denote the joint distribution of agents' types as  $g(\theta)$ , where  $\theta \in (\Theta_W)^N \times (\Theta_F)^M$ .

Though we mainly analyze ordinal utilities of agents, we employ the equilibrium concept

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<sup>9</sup>See also Alcalde (1996), Alcalde et al. (1998), and Haeringer and Wooders (2008).

which demands the specification of cardinal utilities. As well, we need cardinal utilities in order to analyze welfare properties of equilibria. Each agent  $a$  has cardinal utility compatible with her/its preference ranking  $\theta_a$ <sup>10</sup>. If worker  $w$  with ranking  $\theta_w$  is matched with firm  $f$ , she receives utility  $u_w(f, \theta_w)$ . Similarly, if firm  $f$  with ranking  $\theta_f$  is matched with worker  $w$ , it receives utility  $u_f(w, \theta_f)$ .

To make our exposition more comprehensible we assume that cardinal utility from being matched with an agent on the  $k$ th position in one's ranking is the same across agents. We denote this cardinal utility as  $\delta_k$ . Additionally, agent's cardinal utility from being unmatched is normalized to zero. We also assume that there is no worker which firms do not want to hire, and there is no worker who prefers being unemployed to being matched with some firm, i.e. for any  $k$ ,  $\delta_k > 0$ .

Our main goal is to model the influence of signals on the congested markets. In order to accomplish it, we consider one period model of workers and firms. Each worker can send at most one signal and accept at most one offer. Then, each firm has only one vacant position and can make at most one offer<sup>11</sup>. The timing of the game is the following.

1. *Agents' preferences are realized. Workers send signals to firms simultaneously. Only firms who receive signals observe them.*
2. *Firms make offers to workers simultaneously. Using knowledge of the received signals each firm makes its offer to some worker. A firm can make an offer to any worker independently on whether it receives a signal from this worker or not.*
3. *Workers choose offers to accept.*

We restrict our analysis to pure strategies<sup>12</sup>. A strategy of worker  $w$  is a duple  $v_w = (v_w^1, v_w^2)$  that represents her decisions on the first and third stages. A strategy of a worker on the first stage is to choose a firm she sends her signal to,  $v_w^1 : \Theta_W \rightarrow F$ . A strategy of

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<sup>10</sup>We employ cardinal utilities compatible with ordinal ranking similar to Bogomolnaia and Moulin (2001).

<sup>11</sup>In practice, some firms should rationally make several offers, anticipating that some workers probably reject their offers. We do not model this firms' decision.

<sup>12</sup>Mixed strategy does not add much to our analysis, though they make comprehension of our results more involved.

a worker on the last stage is to choose an offer among available to her,  $v_w^2 : \Theta_W \times 2^F \rightarrow F$ , where  $2^F = \{h : h \subset F\}$ .

A strategy of firm  $f$  is its decision on the second stage. Based on its preferences and a set of received signals firm  $f$  chooses which worker it makes an offer to,  $v_f : \Theta_F \times 2^W \rightarrow W$ , where  $2^W = \{h : h \subset W\}$ .

For a given strategy profile of agents  $v = (v_w, v_f)$  and realized agents' types  $\theta \in (\Theta_W)^N \times (\Theta_F)^M$  one can determine the final matching and agents' utilities. We denote the utility of agent  $a$  given a strategy profile  $v$  and a profile of types  $\theta$  as  $u_a(v, \theta)$ . In the same manner, the interim expected payoff of worker  $w$  with preferences  $\theta_w$  from strategy  $v_w$  when the other agents follow a strategy profile  $v_{-w}$  equals

$$u_w(v_w|v_{-w}, \theta_w) = \sum_{\theta_{-w}} g(\theta_{-w}) u_w((v_w, v_{-w}), (\theta_w, \theta_{-w}))$$

where  $g(\theta_{-w})$  is the joint distribution of all agents except worker  $w$ . A firm  $f$  interim expected payoff given its preferences  $\theta_f$ , a subset of received signals  $h \subset W$ , believes  $\mu_f(\cdot|h)$ , and others agents' strategy profile  $v_{-f}$  is

$$u_f(v_f|v_{-f}, h, \theta_f) = \sum_{\theta} \mu_f(\theta|h) u_f(v_f, v_{-f}, \theta)$$

We employ the concept of perfect Bayesian Equilibrium for multi-stage games with observed actions and incomplete information in order to solve the game(see Fudenberg and Tirole (1991)).

**Definition 1.** A strategy profile  $(v_w^*, v_f^*)$  and posterior beliefs  $\mu_f(\cdot|h)$  for each firm  $f$  and each subset of workers  $h \subset W$  is a perfect Bayesian Equilibrium if:

$$\begin{aligned} \forall w \in W, \theta_w \in \Theta_W : v_w^1(\theta_w) &\in \arg \max_{\alpha} u_w(\alpha|v_{-w}^*, \theta_w) \\ \forall f \in F, h \subset W, \theta_f \in \Theta_F : v_f(h, \theta_f) &\in \arg \max_{\beta} u_f(\beta|v_{-f}^*, h, \theta_f) \\ \forall w \in W, \theta_w \in \Theta_W, h' \subset F : v_w^2(h', \theta_w) &\in \arg \max_{\gamma \in h'} u_w(\gamma, \theta_w) \end{aligned}$$

Also, for any  $f \in F$  and  $h \subset W$ ,  $\mu_f(\theta|h) = \prod_{w_i} \mu_f(\theta_{w_i}|h)$ , and beliefs are defined using

*Bayesian rule whenever possible.*

We present our main results in the next section. We consider a specification of the above model in which the preferences of agents are almost aligned and show that the signals have a negative influence on matching formation in this case.

### 3 Almost aligned preferences.

This section analyzes the case when agents' preferences are almost aligned. First, we describe the type of agents' preferences we analyze and introduce some definitions. Then, we analyze the benchmark model when signals are not allowed. Afterwards, we proceed with the analysis of the model with signals. In this case, there are two types of pure strategy perfect Bayesian equilibria: *babbling* and *informative*. All firms ignore signals in the former one, which leads to the same matching as in the case when signals are not allowed. However, if some firms respond to signals, which happens in an informative equilibrium, the introduction of signals changes the matching formation. In the end of the section, we discuss the roles of signals in match formation and their implication on the welfare of market participants.

We consider an environment when each worker can be of two types: "typical" or "atypical". A "typical" worker is denoted as  $w(T)$  and she has a fixed preference list  $\theta_0$ , the same for all typical workers. We enumerate firms according preference  $\theta_0$ , i.e.  $f_1$  is the best firm,  $f_2$  is the second best, etc. An "atypical" worker is denoted as  $w(A)$  and she has a preference list  $\theta_w^A$ , which is ex-ante distributed according to some distribution  $A(\Theta_W)$ . Each worker is ex-ante "typical" with probability  $1 - \varepsilon$  and "atypical" with probability  $\varepsilon$ .

$$\theta_w = \begin{cases} \theta_0 & \text{if } \gamma_w = 0 \\ \theta_w^A & \text{if } \gamma_w = 1 \end{cases}$$

where  $\gamma_w$ ,  $w \in W$ , are independent and identically distributed across workers. Also,  $\gamma_s \in \{0, 1\}$  and  $P(\gamma_s = 1) = \varepsilon$ . We assume that  $\varepsilon$  is small.

**Assumption E.**  $0 < \varepsilon < \min(\min_{j,l} \frac{|\delta_j - \delta_l|}{l\delta_l}, \min_l \frac{\delta_M}{\delta_1 + l\delta_l})$

We also assume that the distribution of "atypical" preferences,  $A(\Theta)$ , has a full support, i.e. each firm can be the top firm of an "atypical" worker with some positive probability.

**Assumption A.**  $A(\Theta)$  has a full support: for any  $f \in F$  and any  $w \in W$   $\Pr(f = \max_{\theta_w} \{f' : f' \in F\}) > 0$ .

We consider a model, where firms have the same fixed preferences list  $\theta_f$ <sup>13</sup>. Similar to firms, we also enumerate workers according to this preference list. To avoid unnecessary notation, we omit, henceforth, the dependence of firms' strategies on their preferences,  $v_f(h, \theta_f) \equiv v_f(h)$ .

Even the signals are voluntary in our model, they still can play negative role and draw away the offers of firms. In order to eliminate such equilibria, we assume that if firm  $f$  makes an offer to worker  $w$  when it does not receive her signal, firm  $f$  makes an offer to worker  $w$  when it receives her signal<sup>14</sup>.

**Assumption PRS (Positive Role of Signals).** For any firm  $f \in F$  and any worker  $w \in W$  and any  $h \subset W$ ,  $w \notin h$ , if  $v_f(h) = w$  then  $v_f(h \cup w) = w$ .

Now we introduce a couple of new notations that are helpful in our further discussion. We call a subset of workers  $h \subset W$  as *reached for firm  $f$  when agents follow strategy profile  $v$*  if ex-ante probability that only workers from set  $h$  send their signals to firm  $f$  strictly more than zero.

**Definition 2.** A subset of workers  $h \subset W$  is reached for firm  $f$  when agents follow strategy profile  $v$  if

$$\Pr_f(h) = \sum_{\theta} g(\theta) \prod_{w \in h} I_{v_w(\theta_w)=f} \prod_{w' \notin h} (1 - I_{v_{w'}(\theta_{w'})=f}) > 0$$

$$\text{where } I_{v_w(\theta_w)=f} = \begin{cases} 1 & \text{if } v_w(\theta_w) = f \\ 0 & \text{otherwise} \end{cases}.$$

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<sup>13</sup>We believe that our results are still valid for the model in which the preferences of firms are almost aligned.

<sup>14</sup>See example A1in Appendix for an example of an equilibrium when assumption PRS is violated in Appendix.

We call firm  $f$  responds to worker  $w$ 's signal if her signal changes the strategy of firm  $f$  with positive probability.

**Definition 3.** *Firm  $f$  responds to worker  $w$ 's signal if there exists a subset of workers  $h$ ,  $w \notin h$ , such that both  $h$  and  $h \cup w$  are reached for firm  $f$ , and  $v_f(h) \neq v_f(h \cup w)$ .*

Let us consider an environment when workers cannot send signals. Then, the outlined above model is a static game of incomplete information. Therefore, a perfect Bayesian equilibrium coincides with a Bayesian equilibrium and agents' beliefs are irrelevant.

If signals are not allowed and  $\varepsilon$  is small, the dominant strategy of firm  $f_1$  is to make an offer to worker  $w_1$ . The second top firm anticipates that worker  $w_1$  is likely to accept firm  $f_1$ 's offer. Hence, the dominant strategy of firm  $f_2$  is to make an offer to worker  $w_2$ . Sequentially, the dominant strategy of each firm is to make an offer to the corresponding worker.

**Proposition 1 (No signaling equilibrium)** *If assumption E holds and signals are not allowed then the only equilibrium match is the assortative match.*

Now, we analyze the set of equilibria in the matching market with signals. We distinguish two kinds of equilibria.

**Definition 4.**

- *An equilibrium is babbling if no firm responds to any signal.*
- *An equilibrium is informative if there is at least one firm that responds to some worker's signal.*

We proceed with the characterization of equilibria in the following two subsections.

### 3.1 Babbling equilibria

This section analyzes equilibria when firms do not respond to signals. We first show a result, which we will further frequently refer to. It states that each firm believes that each worker has "typical" preferences with probability more than  $1 - \varepsilon$  either when it receives her signal or it does not receive her signal.

**Proposition 2** *For any worker  $w \in W$ , any firm  $f \in F$ , and any set of workers  $h \subset W$  either  $\mu_f(\theta_w = \theta_0|h \cup w) \geq 1 - \varepsilon$  or  $\mu_f(\theta_w = \theta_0|h \setminus w) \geq 1 - \varepsilon$ . Similarly, either  $\mu_f(\theta_w \neq \theta_0|h \cup w) \leq \varepsilon$  or  $\mu_f(\theta_w \neq \theta_0|h \setminus w) \leq \varepsilon$ .*

*Proof.*

Let us denote the probability that worker  $w(A)$  and worker  $w(T)$  send their signals to firm  $f$  as  $\alpha_A$  and  $\alpha_T$  correspondingly. Then, if worker  $w$  sends her signal to firm  $f$ ,  $(1 - \varepsilon)\alpha_T + \varepsilon\alpha_A > 0$ , we derive its beliefs using Bayesian rule

$$\begin{cases} \mu_f(\theta_w = \theta_0|h \cup w) = \frac{(1-\varepsilon)\alpha_T}{(1-\varepsilon)\alpha_T + \varepsilon\alpha_A} \\ \mu_f(\theta_w = \theta_0|h \setminus w) = \frac{(1-\varepsilon)(1-\alpha_T)}{(1-\varepsilon)(1-\alpha_T) + \varepsilon(1-\alpha_A)} \end{cases}$$

One can verify that

$$\begin{cases} \mu_f(\theta_w = \theta_0|h \cup w) \geq 1 - \varepsilon \Leftrightarrow \alpha_T \geq \alpha_A \\ \mu_f(\theta_w = \theta_0|h \setminus w) \geq 1 - \varepsilon \Leftrightarrow \alpha_T \leq \alpha_A \end{cases}$$

Hence, either  $\mu_f(\theta_w = \theta_0|h \cup w) \geq 1 - \varepsilon$  or  $\mu_f(\theta_w = \theta_0|h \setminus w) \geq 1 - \varepsilon$ .

If worker  $w$  never sends her signal to firm  $f$ ,  $(1 - \varepsilon)\alpha_T + \varepsilon\alpha_A = 0$ , firm  $f$ 's beliefs are  $\mu_f(\theta_w = \theta_0|h \setminus w) = 1 - \varepsilon$  and  $\mu_f(\theta_w = \theta_0|h \cup w)$  is arbitrary. The second statement follows from the first one.  $\square$

Now we are ready to characterize the set of possible matches in a babbling equilibrium. If  $\varepsilon$  is small (assumption E) and firms do not response to signals, signals play no role in equilibria. Hence, the only possible match in a babbling equilibrium is the assortative match as in our benchmark case.

**Theorem 1** *The only possible match in a babbling equilibrium is the assortative match.*

*Proof.*

Assume that no firm responds to any signal. Let us assume that there exists a realization of agents' preferences such that firm  $f_1$  is matched to some worker  $w_i$ ,  $i > 1$ , in an equilibrium. Hence, there exists  $h \subset W$ , reached in the equilibrium, such that  $v_{f_1}(h) = w_i$ . If  $w_1 \in h$  firm  $f_1$ 's belief equals  $\mu_{f_1}(\theta_{w_1} = \theta_0|h) < 1 - \varepsilon$ , otherwise firm  $f_1$  should make her offer to worker  $w_1$ . Hence,  $\mu_{f_1}(\theta_{w_1} = \theta_0|h \setminus w_1) > 1 - \varepsilon$  (see proposition 2) and worker  $w_1$  secures firm  $f_1$ 's offer in the equilibrium if she does not send her signal to it and firm  $f_1$  receives signals only from set  $h \setminus w_1$  of workers,  $v_{f_1}(h \setminus w_1) = w_1$ . Firm  $f_1$  never responds to signals. Therefore, this is possible only if  $h \setminus w_1$  is not reached for firm  $f_1$  in the equilibrium. However, the strategies of workers are not correlated. Hence, if set  $h$  is reached and set  $h \setminus w_1$  is not reached for firm  $f_1$ , worker  $w_1$  always sends her signal to firm  $f_1$ . However, this means that  $\mu_{f_1}(\theta_{w_1} = \theta_0|h) = 1 - \varepsilon$ . Hence, firm  $f_1$ 's strategy is suboptimal. Firm  $f_1$ 's payoff is maximized if it makes its offer to worker  $w_1$ . One may similarly show a contradiction if  $w_1 \notin h$ .

The other possibility that firm  $f_1$  is not matched to worker  $w_1$  in an equilibrium is that firm  $f_1$  is unmatched. Let us assume that this is the case for some profile of preferences in an equilibrium. Hence, worker  $w_1$  prefers some firm  $f_k$ ,  $k > 1$ , to firm  $f_1$  for this profile of preferences, i.e. she is "atypical". Therefore, there exists  $h' \subset W$ , reached for firm  $f_k$  in the equilibrium, such that  $v_{f_k}(h') = w_1$ . If set  $h' \setminus w_1$  is not reached for firm  $f_k$  in the equilibrium, its belief about worker  $w_1$  is  $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h') = \varepsilon$ , which contradicts the rationality of firm  $f_k$ 's behavior<sup>15</sup>. Therefore, set  $h' \setminus w_1$  is reached for firm  $f_k$  in the equilibrium. Firm  $f_k$  should make an offer to worker  $w_1$ , i.e.  $v_{f_k}(h) = v_{f_k}(h \setminus w_1) = w_1$ , because firm  $f_k$  does not respond to signals. Hence, its beliefs according to proposition 2 are either  $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h) \leq \varepsilon$  or  $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h \setminus w_1) \leq \varepsilon$ . Assume the former one,  $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h) \leq \varepsilon$ <sup>16</sup>. If firm  $f_k$  makes her offer to worker  $w_1$ ,  $v_{f_k}(h \setminus w_1) = w_1$ , it should believe that  $\mu_{f_k}(\theta_{w_k} = \theta_0|h) < 1 - \varepsilon$ , otherwise firm  $f_k$  plays suboptimal strategy. Hence, firm  $f_k$ 's belief is  $\mu_{f_k}(\theta_{w_k} = \theta_0|h \cup w_k) \geq 1 - \varepsilon$

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<sup>15</sup>The argument about unreachable sets is omitted further. If either  $h' \cup w_1$  or  $h' \setminus w_1$  is not reached for firm  $f_k$ , either worker  $w$  always sends her signal to firm  $f_k$  or worker  $w_1$  never sends her signal to firm  $f_k$  correspondingly. Hence, the below arguments follow trivially.

<sup>16</sup>If we assume the latter one, the argument is similar.



(if  $h \cup w_k$  is not reached for firm  $f_k$  in the equilibrium the argument is trivial) and firm  $f_k$ 's strategies are  $v_{f_k}(h \cup w_k) = v_{f_k}(h) = w_1$ . However, this strategy cannot be optimal, because firm  $f_k$ 's payoff from making an offer to worker  $w_k$ , when worker  $w_k$  sends her signal to it, equals at least  $(1 - \varepsilon)\delta_k$ . Taking into account assumption E, we obtain that  $(1 - \varepsilon)\delta_k > \varepsilon\delta_1$ . Contradiction.

We argued above that the best feasible worker for firm  $f_2$  is worker  $w_2$ . Hence, using similar speculations one may show that firm  $f_2$  can be matched only with worker  $w_2$  in a babbling equilibrium. Therefore, the only feasible match in a babbling equilibrium is the assortative match. It is a routine to show that the assortative match can be supported by some system of beliefs in a perfect Bayesian equilibrium.  $\square$

### 3.2 Informative equilibria

This subsection considers equilibria in which at least one firm responds to some worker's signal. There is a great multiplicity of such equilibria. For example, each firm can believe that it is the worst program in worker's preferences if it receives her signal. Taking into account this belief this worker rationally never sends her signal to this firm in equilibrium. Therefore, these beliefs are off-equilibrium path, and they are consistent with the notion of perfect Bayesian equilibrium.

In order to make our analysis more focused we consider in this subsection the equilibria when each firm responds to signals from any worker better or equal to the corresponding one.<sup>17</sup> According to assumption A, each firm is the best firm for each worker with positive probability. Hence, each worker prefers this firm to all other firms with positive probability. Hence, if she could use her signal to receive its offer, she would use her signal. In the same time, each firm would prefer the match with each worker better than the corresponding one. The assumption that each firm responds to signals from any worker better or equal to the corresponding one guarantees that each firm should make its offer to such workers if it does not receive a better signal<sup>18</sup>.

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<sup>17</sup>This refinement makes the results more comprehensible. All the results are still true if there is enough programs that responds to students' signals.

<sup>18</sup>See proposition A2 in Appendix for more detailed explanation.

We postpone the explanation of patterns of agents' behavior in informative equilibria to Appendix. It appears that a firm never makes its offer to a worker better than the corresponding one if it does not receive a signal from her in an equilibrium. In addition, if a firm responds to some worker's signal and her signal is the best signal it receives, the firm makes its offer to this worker. We concentrate here only on the discussion of our main result about existence and uniqueness of equilibria.

Let us denote the set of workers worse or equal to worker  $w_j$  as  $\Delta_j = \{w_1, \dots, w_j\} \subset W$ . Then, there is a unique equilibrium such that each firm responds to all signals from workers better or equal to the corresponding one.

**Theorem 2** *If each firm responds to all signals from workers better or equal to the corresponding one, the set of strategies*

$$\begin{aligned}
 & - v_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}} (f' \in F : f' \preceq_{\theta_0} f_i) \\
 & - v_{f_j}(h) = \begin{cases} \max_{\theta_{f_j}} (w : w \in h) & \text{if } h \cap \Delta_j \neq 0 \\ w_{j+1} & \text{if } h \cap \Delta_j = 0 \end{cases}
 \end{aligned}$$

*and set of firms' beliefs consistent with agents' strategies constitute a unique equilibrium.*

Basically, each worker sends her signal to the best firm (according to her preferences) among the firms worse or equal to the corresponding one (according to "typical" preference). If firm  $f_j$  receives at least one signal from corresponding worker  $w_j$  or better one, it makes an offer to the best such a worker, otherwise it makes an offer to worker  $w_{j+1}$ .

The above result completes our description of agents' behavior in equilibria. If we do not assume that each firm responds to signals from any worker better or equal to the corresponding one, there is a great multiplicity of equilibria in our model. However, it is still possible to derive the welfare comparison and evaluate the role of signals for all equilibria. We proceed with this analysis in the next subsection.

### 3.3 Welfare properties of equilibria

We evaluate the effect of signals from ex-ante perspective. We mainly use the following quantitative characteristics: the expected number of matches, the expected total welfare of firms and the expected total welfare of workers. However, theorem 4 provides also some results for the effect of signals on the individual firms.

Let us denote the ex-post number of matches when agents follow profile of strategies  $v$ , and the realized profile of preferences is  $\theta \in \Theta_W \times \Theta_F$  as  $m(v, \theta)$ . The expected number of matches when agents follow profile of strategies  $v$  can be expressed as

$$EM(v) = \sum_{\theta} g(\theta) m(v(\theta), \theta)$$

Similarly, the expected total welfare of workers and firms equal

$$\begin{aligned} EW_{\text{firm}}(v) &= \sum_f \sum_{\theta} g(\theta) Pr_f(h_f(\theta)) u_f(v_f(\theta) | v_{-f}(\theta), h_f(\theta), \theta_f) \\ EW_{\text{worker}}(v) &= \sum_w \sum_{\theta} g(\theta) u_w(v_w(\theta_w) | v_{-w}(\theta_{-w}), \theta_w) \end{aligned}$$

correspondingly.

Proposition 1 shows that the expected number of matches in any "no signaling" equilibrium equals  $M$  (under assumption that  $N \geq M$ ), which is the maximum one. Hence, it is impossible that the expected number of matches in any informative equilibrium exceeds the expected number of matches in any "no signaling" equilibrium. Example 1 and example 2 demonstrate the case of equality and strict inequality for this welfare criterion.

The preferences of firms are the same. Firms are also matched to the best workers in any "no signaling" equilibrium. Hence, the decrease in the expected number of matches leads to the decrease of the expected total welfare of firms.

The result about total workers' welfare is not such straightforward and depends on their cardinal utility. The intuition is that signals help "atypical" workers. In the same time, they decrease the expected number of matches, which makes worse some workers. Example 3 illustrates this point.

**Example 1 (Equal expected number of matches)**

Let us consider the following equilibrium strategies.

- for any  $l < M$ ,  $v_{w_l}(\theta_{w_l}) = \max_{\theta_{w_l}}(f : f \in \{f_l, f_{l+1}\})$
- for any  $l \geq M$   $v_{w_M}(\theta_0) = f_M$
- if  $v_{f_l}(h) = \begin{cases} \max_{\theta_f}(w : w \in h) & \text{if } h \cap \Delta_l \neq \emptyset \\ w_{l+1} & \text{if } h \cap \Delta_l = \emptyset \end{cases}$

If for any  $h \subset W$  and for any  $1 \leq l \leq M$ ,  $1 \leq k \leq N$ ,  $k \neq l, l - 1$ , we denote  $\Theta_{f_l} = \{\theta \in \Theta : f_l = \min_{\theta}(f : f \in F)\}$ . Then, firm  $f_l$ 's off-equilibrium path beliefs are such that for any  $h \subset W$   $\mu_{f_l}(\theta_{w_k} \in \Theta_{f_l} | h \cup w_k) = 1$ . As one may check, the described strategies constitute an equilibrium. Moreover, the introduction of signals does not lead to the loss of the number of matches. If worker  $w_l$  is "atypical" and prefers firm  $f_{l+1}$  to firm  $f_l$ , she sends her signal to firm  $f_{l+1}$ . Firm  $f_{l+1}$  makes her offer to worker  $w_l$ . In the same time, firm  $f_l$  makes her offer to worker  $w_{l+1}$ . There is no loss in the number of matches, because firm  $f_l$  and firm  $f_{l+1}$  exchange their matches. The total welfare of firms neither changes. The expected welfare of workers increases, because they end up with better matches than in any "no signaling" equilibrium.

**Example 2 (Fewer expected number of matches)**

The fact that the expected number of matches decreases is illustrated perfectly by the example in the introduction. One may extend this example to the case with more workers and firms. We present only figure II to illustrate the idea. The explanation is similar to the one in the introduction.

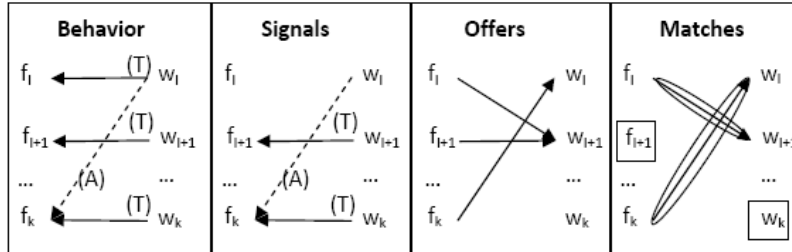


Figure II.

### Example 3 (Welfare of workers)

Let us consider three firms and three workers and the uniform distribution of "atypical" preferences  $A(\Theta) = U(\Theta)$ . Workers' cardinal utilities:  $(\delta_1, \delta_2, \delta_3) = (\delta + \lambda, \delta, \delta - \lambda)$ . The expected total welfare of workers when signals are not allowed equals

$$EW_{\text{worker}}^{\text{nosignals}} = \sum_{i=1}^3 [(1 - \varepsilon) \delta_i + \varepsilon \frac{1}{3} \sum_{l=1}^3 \delta_l] = 3\delta$$

Let us consider a perfect Bayesian equilibrium, described in theorem 2. One may check that the expected total welfare of workers equals<sup>19</sup>

$$EW_{\text{worker}}^{\text{signals}} = 3\delta + \left(-\frac{1}{3}\delta + \frac{19}{6}\lambda\right) \varepsilon$$

Hence, the expected total welfare of workers increases if and only if the difference in utilities between adjacent firms is large enough,  $\lambda > \frac{2}{19}\delta$ . This illustrates that the introduction of signals is beneficial for a matching market according to egalitarian welfare criterion if and only if the decrease in the total number of matches is offset by better matches of "atypical" workers.

The below theorem summarize the results derived above.

**Theorem 3** - *The expected number of matches in any informative equilibrium is weakly fewer than in any "no signaling" equilibrium.*

- *The expected total welfare of firms in any informative equilibrium is weakly less than in "no signaling" equilibrium.*
- *The effect of signals on the expected total welfare of workers is ambiguous.*

We have compared an informative and "no signaling" equilibrium using aggregate welfare criteria. The following theorem provides more strict results for individual firms.

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<sup>19</sup>The terms  $\varepsilon^2$  and  $\varepsilon^3$  are ignored.

**Theorem 4** *If firm  $f_{l-1}$  and at least one firm among  $\{f_{l+1}, \dots, f_M\}$  responds to worker  $w_{l-1}$ 's signal in an informative equilibrium, then*

- *firm  $f_l$ 's expected number of matches is strictly fewer in any "informative" equilibrium than in any "no signaling" equilibrium.*

*Moreover, if no worker among  $\{s_1, \dots, s_{l-1}\}$  sends her signal to firm  $f_l$  in any "informative" equilibrium, then*

- *firm  $f_l$ 's expected welfare is strictly less in an "informative" equilibrium than in any "no signaling" equilibrium.*

## 4 Role of signals in matching markets

The previous study by Coles and Niederle (2007) shows that the introduction of signals facilitates match formation. They consider a model similar to the model of this paper except that agents preferences are uniformly distributed,  $\theta_s \sim U(\Theta_s)$  and  $\theta_p \sim U(\Theta_p)$ . They show that the introduction of signals increase the expected number of matches and the welfare of workers<sup>20</sup>. As shown in the previous section, these results, however, highly rely on the assumption that the preferences of agents are uniformly distributed.

If the preferences of workers are almost aligned and the preferences of firms are the same, the introduction of signals decreases the expected number of matches and the expected total welfare of firms. The influence of signals on the welfare of workers is ambiguous. Overall, table I presents the effects from the introduction of the signals for the two different environments: almost aligned and uniform distribution of preferences.

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<sup>20</sup>Kushnir (2008) relaxes one of their assumptions and extends their results to the case of many-to-one matching market with many signals. He also shows explicitly that the introduction of signals ambiguously affects the welfare of firms.

Preferences	No signals	EMatches	EW <sub>worker</sub>	EW <sub>firm</sub>
Almost aligned	0	–	±	–
Uniform distribution	0	+	+	±

Table I. Almost aligned VS uniform distribution preferences

A natural question is why signals influence matching markets in different ways. We argue that the signals play two different roles: transmit information and facilitate information asymmetry. On the one hand, the introduction of signals helps "atypical" workers to transmit the information about their preferences and locate a better match. On the other hand, signals transmit information to small number of firms, which facilitate information asymmetry. This information asymmetry leads to coordination failures, which decrease the number of matches in the market.

Preferences	Transmit information	Facilitate information asymmetry
Almost aligned	Small	<b>Large</b>
Uniform distribution	<b>Large</b>	Small

Table II. The roles of signals

When there is ex-ante small amount of information about agents' preferences, the information transmission plays more important role in match formation. This happens when agents' preferences are ex-ante uniformly distributed as in Coles and Niederle (2007). However, when there is almost complete information about agents' preferences (as in this paper), the introduction of signals leads to coordination failures. Overall, the signals play the controversial roles in match formation process. This makes them a less powerful tool than it was previously anticipated.

## 5 Conclusion

We exemplify an environment when the introduction of signals harms matching markets: lessens the expected number of matches, decreases the welfare of firms, and affects am-

biguously the welfare of workers. Based on this example, we argue that the signals play controversial roles in match formation. Though they help to transmit information about participants' preferences, they also facilitate the information asymmetry among them. While the former one reduces coordination failures and facilitates better match formation, the latter one acts in the opposite direction. We leave here, as an open empirical question, what effect dominates in real matching markets.

We consider a one-period model where each worker sends at most one signal and each firm has only one vacant position. However, these assumptions are not crucial for our results. Several (finite) periods would allow firms to secure a better match, but signals would still introduce information asymmetry. If each worker sent several signals, it would transmit information to greater number of firms, but each signal would be less informative. Several vacant positions would make only the preferences of firms more complicated and would not influence the results. Overall, the roles of signals in match formation robust to these modifications.

One interesting question for further theoretical research is to analyze the effect of signals for a larger range of preference distributions. Though an analytical analysis can be a cumbersome for other cases, a computer simulation seems a perfectly suitable tool to accomplish this goal.

Another interesting question is to explore the effect of public signals. Do the signals benefit matching markets if all firms can observe them? One potential problem with public signals is that they may induce some complicated strategic behavior by agents, which may render them useless.



## A Appendix

The first two propositions are devoted to the description of agents' strategies in informative equilibria. The first proposition shows that it is too risky for a firm to make an offer to a worker better than the corresponding one if it does not receive a signal from her.

**Proposition A1 (Offers to better workers)** *If firm  $f_j$  does not receive a signal from worker  $w_i$ ,  $i < j$ , it does not make an offer to her in an equilibrium.*

*Proof.*

We proof this proposition by induction. Let us show the validity of the claim for  $j = 2$ . We consider two possibilities: either worker  $w_1(T)$  does not send a signal to firm  $f_1$ ,  $v_{w_1}(\theta_0) \neq f_1$ , or she sends a signal to firm  $f_1$ ,  $v_{w_1}(\theta_0) = f_1$ .

Worker  $w_1$  employs strategy  $v_{w_1}(\theta_0) \neq f_1$  in an equilibrium only if firm  $f_1$  makes its offer to worker  $w_1$  with probability equals to one. Firm  $f_2$  has a chance to be matched with worker  $w_1$  only if she is "atypical". Similar to proposition 2, we denote the probability that worker  $w_1(A)$  and worker  $w_1(T)$  send a signal to firm  $f_2$  as  $\alpha_A$  and  $\alpha_T$  correspondingly.

Let us assume that worker  $w_1(T)$  sends her signal to firm  $f_2$  in an equilibrium, i.e.  $\alpha_T = 1$ . If firm  $f_2$  does not receive a signal from worker  $w_1$ , it believes that worker  $w_1$  is "atypical". According to assumption PRS, if firm  $f_2$  makes an offer to worker  $w_1$  when it does not receive her signal, it should make an offer to her when it receives her signal. However, firm  $f_2$ 's beliefs are such in this case that for any  $h \subset W$

$$\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \cup w_1) = \frac{\alpha_A \varepsilon}{\alpha_A \varepsilon + \alpha_T (1 - \varepsilon)} \leq \varepsilon$$

Now, if worker  $w_1(T)$  does not send her signal to firm  $f_2$ , ( $\alpha_T = 0$ ), firm  $f_2$ 's belief is such that for any  $h \subset W$

$$\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \setminus w_1) = \frac{(1 - \alpha_A) \varepsilon}{(1 - \alpha_A) \varepsilon + (1 - \alpha_T)(1 - \varepsilon)} \leq \varepsilon$$

For both cases,  $\alpha_T = 1$  and  $\alpha_T = 0$ , firm  $f_2$ 's payoff from making her offer to worker  $w_1$  is less than  $\varepsilon \delta_1$ . According to assumption  $M \leq N$ , firm  $f_2$  can secure a match with some

worker  $w_t$ ,  $t > 2$ , with probability at least  $1 - \varepsilon$ . Hence, firm  $f_2$  does not make an offer to worker  $w_1$  in an equilibrium.

If worker  $w_1$  employs strategy  $v_{w_1}(\theta_0) = f_1$ , similar to the above discussion firm  $f_2$ 's belief is  $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \setminus w_1) \leq \varepsilon$ . Hence, it is again suboptimal for firm  $f_2$  to make an offer to worker  $w_1$ .

We have shown that it is not optimal for firm  $f_2$  to make an offer to a better student if it does not send her signal to it. Let us assume that the claim is valid for any  $f_j$ ,  $j < k$  and show that the claim is valid for firm  $f_k$ . Let us consider some worker  $w_i$ ,  $i < k$ . Firm  $f_i$  makes her offer to workers better than worker  $w_i$  with probability less than  $\varepsilon(i - 1)$ . In addition, worker  $w_i$  is "atypical" with probability  $\varepsilon$ . Hence, firm  $f_k$  can secure a match with worker  $w_i$  with probability less than  $i\varepsilon$ . Assumption E guarantees that  $\varepsilon$  is small enough that firm  $f_k$ 's offer to worker  $w_i$  is suboptimal.  $\square$

Now we assume that each firm responds to a signal from any worker better or equal to the corresponding one. The following proposition shows that a firm makes its offer to some worker better or equal to the corresponding one if the worker's signal is the best signal the firm receives.

**Proposition A2 (Response to signals)** *Assume that  $M > N$ , and each firm responds to a signal from any worker better or equal to the corresponding one. Then, for any firm  $f_j$ , worker  $w_i$ ,  $i \leq j$ , and any  $h \subset W$   $v_{f_j}(h \cup w_i \setminus \Delta_{i-1}) = w_i$  in an equilibrium<sup>21</sup>.*

*Proof.*

We proof this statement by induction. Let us consider first firm  $f_1$  and worker  $w_1$ . If worker  $w_1$  employs strategy  $v_{w_1}(\theta_0) \neq f_1$ , for any  $h \subset W$  firm  $f_1$ 's belief is  $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \setminus w_1) \geq 1 - \varepsilon$ . Hence, firm  $f_1$  always makes an offer to worker  $w_1$ , which contradicts to our assumption that it responds to worker  $w_1$ 's signal. Therefore, the only possible optimal strategy of worker  $w_1$  is  $v_{w_1}(\theta_0) = f_1$ . In this case, for any  $h \subset W$  firm  $f_1$ 's belief is  $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \cup w_1) \geq 1 - \varepsilon$ . Hence, firm  $f_1$ 's the highest expected payoff when it receives

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<sup>21</sup>If  $M = N$  the claim is still valid with the same assumption for all firms except firm  $f_M$ . Firm  $f_M$  should respond to a signal from any worker strictly better than the corresponding one.

worker  $w_1$ 's signal is from making an offer to worker  $w_1$ . Hence, for any  $h \subset W$ , firm  $f_1$ 's strategy  $v_{f_1}(h \cup w_i \setminus \Delta_{i-1}) = w_1$  is optimal.

Assume now that for any  $t \leq j < k$ , and for any  $h \subset W$ , firm  $f_j$  employs strategy  $v_{f_j}(h \cup w_t \setminus \Delta_{t-1}) = w_t$ . We prove below that firm  $f_k$ 's optimal strategy is for any  $h \subset W$  and any  $i \leq k$ ,  $v_{f_k}(h \cup w_i \setminus \Delta_{i-1}) = w_i$ .

Let us consider some  $i \leq k$ . There are two possibilities: either  $v_{w_k}(\theta_0) \neq f_k$  or  $v_{w_k}(\theta_0) = f_k$ . For the former case, for any  $h \subset W$ ,  $\mu_{f_k}(\theta_{w_k} = \theta_0 | h \setminus w_k) \geq 1 - \varepsilon$ . Hence, it is optimal for firm  $f_k$  to make an offer to worker  $w_k$  when it receives no signals from workers from  $\Delta_k$ , i.e. for any  $h' \subset W$   $v_{f_k}(h' \setminus w_k \setminus \Delta_{k-1}) = w_k$ . Hence, it is also optimal for firm  $f_k$  to make an offer to when worker  $w_k$ 's signal is the best signal it receives, i.e.  $v_{f_k}(h \cup w_k \setminus \Delta_{k-1}) = w_k$ . In addition, if firm  $f_k$  receives a signal from some worker, which belongs to set  $\Delta_k$ , it makes an offer to it independently on worker  $w_k$ 's signal according to the induction assumption. Therefore, firm  $f_k$  does not respond to signal from  $w_k$ . Contradiction.

For the latter case,  $v_{w_k}(\theta_0) = f_k$ , if firm  $f_k$  does not receive a signal from worker  $w_k$ , it anticipates that she is "atypical", and it does not make an offer to her in an equilibrium. If firm  $f_j$  receives signals from the set  $h \cup w_i \setminus \Delta_{i-1}$  (for some  $h \subset W$ ), no other firm  $f_p$ ,  $p \neq j$  and  $p > i$ , makes its offer to worker  $w_i$  according to proposition. The only competing with firm  $f_j$ 's offer could be the offers from firms in set  $\{f_p, p < i\}$ . However, any firm  $f_p$ ,  $p < i$ , could make an offer to worker  $w_i$  only if worker  $w_p$  is "atypical", which happens with probability  $\varepsilon$ . Hence, the interim expected payoff for firm  $f_j$  from making its offer to worker  $w_i$  equals at least  $(1 - (i-1)\varepsilon)\delta_k$  which the highest interim payoff it could receive. Therefore, firm  $f_j$  makes its offer to worker  $w_i$ ,  $v_{f_j}(h \cup w_i \setminus \Delta_{i-1}) = w_i$ .  $\square$

### **Proof of theorem 2.**

We first prove that the set of strategies, stated in the theorem, constitute an equilibrium. Then, we show that given assumption E, assumption A, and the assumption that all firms respond to a signal from any worker better or equal to the corresponding one, the equilibrium is unique.

Let us show that, if all agents, except firm  $f_l$ , follow the strategies, stated in the theorem, firm  $f_l$ 's strategy is optimal given its believes consistent with the other agents' strategies.

If firm  $f_l$  receives a signal from worker  $w_t$ ,  $t < l$ , firm  $f_l$  believes that itself is the best firm among set  $\{f' \in F : f' \preceq_{\theta_0} f_t\}$  for worker  $w_t$ . Let us assume that worker  $w_t$  is the best worker that sends a signal to firm  $f_l$ . Worker  $w_t$  does not accept firm  $f_l$ 's offer only if she receives an offer from firm  $f_{t-1}$ . However, it happens only if worker  $w_{t-1}$  is atypical, i.e. with probability less than  $\varepsilon$ . Hence, firm  $f_l$  interim expected payoff from making an offer to worker  $w_t$  equals at least  $(1 - \varepsilon) \delta_t$ .

Firm  $f_l$ 's expected payoff from making an offer to a worker better than worker  $w_t$  is less than  $(1 - \varepsilon) \delta_t$  according to proposition A1 and assumption E. Firm  $f_l$ 's expected payoff from making an offer to worker  $w_k$ ,  $k > t$ , is less than making an offer to worker  $w_t$ , because  $(1 - \varepsilon) \delta_t > \delta_k$ . Overall, firm  $f_l$ 's strategy is optimal.

Let us show that, if all agents, except worker  $w_t$ , follow the strategies, stated in the theorem, worker  $w_t$ 's strategy is optimal. Firm  $f_t$  does not make an offer to worker  $w_t$  when it receives a signal from a better worker. Therefore, if worker  $w_t$  is "typical", her payoff from sending a signal to firm  $f_t$  equals at least  $[1 - (l - 1)\varepsilon] \delta_t$ . If worker  $w_t$  does not send her signal to firm  $f_t$  it loses her offer. Her payoff from sending a signal to a firm worse than firm  $f_t$  is less or equal to  $\delta_{t-1}$ , which is less than  $[1 - (l - 1)\varepsilon] \delta_t$  according to assumption E. There is also no reason for worker  $w_t$  to send a signal to a firm better than firm  $f_t$ , because this firm does not respond to it. Hence, worker  $w_t(T)$ 's strategy is optimal. Using similar logic one may show that worker  $w_t(A)$ 's strategy is also optimal.

Now we show that the above strategies constitute the unique equilibrium. Under assumption that all firms respond to signals from workers better or equal to the corresponding one, proposition A1 and proposition A2 imply that each firm  $f_l$ ,  $l = 1, \dots, M$  has to follow the following strategies in equilibrium: for any  $h \subset W$ ,  $v_{f_l}(h \setminus \Delta_l) \neq w_l$  and  $v_{f_l}(h \cup w_l \setminus \Delta_{l-1}) = w_l$ . Straightforwardly, the only worker  $w_l(T)$ 's optimal strategy is to send her signals to firm  $f_l$ ,  $v_{w_l}(\theta_0) = s_l$ , otherwise, firm  $f_l$ 's does not make an offer to student  $w_l$ .

Let us consider firm  $f^* = \max_{\theta_{w_l}} (f' \in F : f' \preceq_{\theta_0} f_l)$ . It responds to signals from workers better or equal than the corresponding one and firm  $f^*$ 's belief about worker  $w_l$ 's type is

$$\mu_{f^*}(\theta_{w_l} = \theta_0 | h \setminus w_l) \geq 1 - \varepsilon \text{ and } \mu_{f^*}(\theta_{w_l} \neq \theta_0 | h \cup w_l) = 1$$

Therefore, if firm  $f^*$  does not receive a signal better than worker  $w_l$ 's one, it's optimal strategy is to make an offer to worker  $w_l$ . Taking into account that firm  $f^*$  can receive a signal from a better worker with probability less than  $(l - 1)\varepsilon$ , worker  $w_l(A)$ 's optimal strategy is to send her signals to firm  $f^*$  (assumption E). Hence, the strategies, stated in the theorem, constitute a unique equilibrium.  $\square$

**Proof of theorem 4.**

Assume that some firm  $f_j$ ,  $j \geq l + 1$ , responds to worker  $w_{l-1}$ 's signal. We prove the first statement of the theorem in two steps. First, we show that firm  $f_l$  makes an offer to worker  $w_l$  with positive probability in any equilibrium. Second, we show that firm  $f_{l-1}$  also makes an offer to worker  $w_l$  with positive probability in any equilibrium. The offer decisions of firm  $f_l$  and firm  $f_{l-1}$  appear to be uncorrelated and independent on worker  $w_l$  type. Therefore, firm  $f_l$  is unmatched with positive probability.

We show that firm  $f_l$  makes an offer to worker  $w_l$  with positive probability in any equilibrium by induction. Let us consider  $l = 1$ . There are two possibilities:  $v_{w_1}(\theta_0) = f_1$  or  $v_{w_1}(\theta_0) \neq f_1$ . For the former case, for any  $h \subset W$   $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \cup w_1) \geq 1 - \varepsilon$ . Hence, it is optimal for firm to make an offer to worker  $w_1$  when it receives a signal from her. Worker  $w_1$  is "typical" with positive probability, which means that firm  $f_1$  makes an offer to worker  $w_1$  with positive probability. The latter case can happen in an equilibrium only when firm  $f_1$  always makes an offer to worker  $w_1$  independently on whether it receives a signal from her (assumption PRS).

Let us assume that for any  $i < l$ , firm  $f_i$  makes an offer to worker  $w_i$  with positive probability in an equilibrium. Let us show that firm  $f_l$  also makes an offer to worker  $w_l$  with positive probability. Firm  $f_l$  responds to worker  $w_i$ 's signal,  $i < l$ , only if  $v_{w_i}(\theta_0) \neq f_l$ . Hence, firm  $f_l$  makes an offer to some worker in set  $\Delta_{l-1}$  with probability less than one (see also proposition A1).

Assume worker  $w_l$  employs strategy  $v_{w_l}(\theta_0) = f_l$ . Hence, if firm  $f_l$  does not receive a signal from a worker better than the corresponding one (or it does not responds to signals it receives) and firm  $f_l$  receives a signal from worker  $w_l$ , then firm  $f_l$  should rationally make an offer to worker  $w_l$ .

Assume worker  $w_l$  employs strategy  $v_{w_l}(\theta_0) \neq f_l$ . Firm  $f_l$  makes an offer to worker  $w_i$ ,  $i < l$ , in an equilibrium with probability equal at most  $\varepsilon$ . Therefore, firm  $f_l$  is not available to worker  $w_l$  with probability at most  $(l-1)\varepsilon$ . Assumption  $E$  guarantees that worker  $w_l$  employs strategy  $v_{w_l}(\theta_0) \neq f_l$  in an equilibrium only if firm  $f_l$  follows the following strategy: for any  $h \subset W$   $v_{f_l}(h \cup w_l \setminus \Delta_{l-1}) = w_l$  (or it employs this strategy if it receives some signals from workers better than the corresponding one, which it does not respond). Hence, firm  $f_l$  makes also an offer to worker  $w_l$  with positive probability in this case.

Similar to the above speculations firm  $f_{l-1}$  does not receive a signal from a worker better than worker  $w_{l-1}$  (or it does not responds to better signals it receives) with positive probability. The fact that firm  $f_{l-1}$  responds to worker  $w_{l-1}$ 's signal guarantees that  $v_{f_{l-1}}(h \setminus \Delta_{l-1}) \neq w_{l-1}$ . Therefore, if worker  $w_{l-1}$  sends to firm  $f_{l-1}$  a signal with positive probability, then for any  $h \subset W$  firm  $f_{l-1}$  employs strategy  $v_{f_{l-1}}(h \cup w_l \setminus \Delta_l) = w_l$  in an equilibrium. If worker  $w_l$  never sends a signal to firm  $f_{l-1}$ , for any  $h \subset W$  firm  $f_{l-1}$  employs strategy  $v_{f_{l-1}}(h \cup w_l \setminus \Delta_l) = w_l$  in an equilibrium. For both cases, firm  $f_{l-1}$  makes an offer to worker  $w_l$  with positive probability in an equilibrium.

Both firm  $f_{l-1}$  and firm  $f_l$  make their offers to worker  $w_l$  with positive probability. Moreover, firm  $f_{l-1}$  makes offer only if it does not receive a signal from worker  $w_{l-1}$ . However, if worker  $w_{l-1}$  sends her signal to firm  $f_j$ , firm  $f_l$  has no information about worker  $w_{l-1}$  type and cannot predict behavior of firm  $f_{l-1}$ . Therefore, both firms make their offers to worker  $w_l$  simultaneously with positive probability in an equilibrium, which makes firm  $f_l$  sometimes unmatched.

The second statement follows directly from the first one if one considers how firm  $f_l$  can benefit from the introduction of signals. It may benefit if it ends up matched with some worker better than its corresponding one. However, this happens according to proposition A1 only if such a worker sends her a signal. Otherwise, it has only disadvantage from the introduction of signals (being unmatched or being match to a worse worker) as was shown above.  $\square$

**Example A1 (An equilibrium when assumption PRS does not hold.)**

Let us consider two firms and two workers. Assume that each "typical" worker has preferences  $\theta_0 = (f_1, f_2)$  and each "atypical" worker has preferences  $\theta_A = (f_2, f_1)$  with probability equal to one. Firms prefer worker  $w_1$  to worker  $w_2$ . Agents employ the following strategies:

- $v_{w_1}(\theta_0) = f_2, v_{w_1}(\theta_A) = f_1$
- $v_{w_2}(\theta_0) = f_1, v_{w_2}(\theta_A) = f_2$
- for any  $h \subset W$   $v_{f_1}(h) = \begin{cases} w_1 & \text{if } w_1 \notin h \\ w_2 & \text{if } w_1 \in h \end{cases}$ ,  $v_{f_2}(h) = \begin{cases} w_1 & \text{if } w_1 \notin h \\ w_2 & \text{if } w_1 \in h \end{cases}$

Agents' beliefs are:

- for any  $h \subset W$   $\mu_{f_j}(\theta_{w_i} : f_j = \max_{\theta_{w_i}}(f \in F)|h \setminus w_i) = 1$  and  $\mu_{f_j}(\theta_{w_i} : f_j = \min_{\theta_{w_i}}(f \in F)|h \cup w_i) = 1$

It is easy to show that the above strategies and the set of beliefs constitute a perfect Bayesian equilibrium. One may extend this example for the environment with more firms and workers.

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