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Article

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Economics: The Open-Access, Open-Assessment E-Journal

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Suggested Citation: Gracia, Eduard (2011) : Uncertainty and capacity constraints: Reconsidering the aggregate production function, Economics: The Open-Access, Open-Assessment E-Journal, ISSN 1864-6042, Kiel Institute for the World Economy (IfW), Kiel, Vol. 5, Iss. 2011-19, pp. 1-50, <https://doi.org/10.5018/economics-ejournal.ja.2011-19>

This Version is available at:

<https://hdl.handle.net/10419/52688>

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Uncertainty and Capacity Constraints: Reconsidering the Aggregate Production Function

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Abstract The Cobb–Douglas function is today one of the most widely adopted assumptions in economic modeling, yet both its theoretical and empirical bases have long been under question. This paper builds an alternative function on very different (albeit also neoclassical) microfoundations aimed at both addressing those theoretical drawbacks and providing a better empirical fit than the Cobb–Douglas formula. The new model, unlike the Cobb–Douglas function, does not portray installed capacity as aggregate capital but as a sunk cost generating economic rents. An analysis of 1949–2008 annual U.S. growth data suggest this alternative model explains nearly 85 percent of GDP fluctuations and is empirically more robust than the Cobb–Douglas, whilst both contemporary and lagged aggregate capital are statistically rejected as explanatory variables. This lends support to the old “Cambridge Critique”, according to which using value-weighted capital aggregates to explain production simply makes no sense. At face value, these results not only pose a question on any macroeconomic model assuming a Cobb–Douglas function but also point towards an alternative interpretation of phenomena such as the way monetary policy impacts productivity.

JEL E22, E23

Keywords Production function; Cobb–Douglas; capital controversy

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Citation Eduard Gracia (2011). Uncertainty and Capacity Constraints: Reconsidering the Aggregate Production Function. *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 5, 2011-19. <http://dx.doi.org/10.5018/economics-ejournal.ja.2011-19>

1 Introduction

Few assumptions in economics are as widely adopted as the Cobb-Douglas production function. Not only is it taught as one of the first lessons in basic macroeconomics textbooks, but it also stands, in more or less complex forms, at the core of most of today's mainstream Dynamic Stochastic General Equilibrium (DSGE) models. Yet it has also been, almost since it was first put forward in Cobb and Douglas (1928), the object of various well-grounded criticisms questioning both its theoretical rationale and its empirical fit. Hence it makes sense to explore whether an alternative function could be built on a neoclassical foundation such that it is not subject to the same theoretical objections, and then test whether it would provide better empirical results than the Cobb-Douglas. This is the purpose of this paper.

Perhaps the most serious theoretical challenge to the Cobb-Douglas function is the one at the center of the so-called 'Cambridge Capital Critique' that started with Robinson (1953–54). The objection was that, if we define capital as a price-weighted aggregate of heterogeneous goods, it makes no sense to model its return as a marginal product for, in production processes requiring time, capital value is determined by that rate of return acting as a discount rate on the future cash flows it generates – which makes the reasoning circular.¹ As Cohen and Harcourt (2003) note, although capital aggregates continued to be used in modeling practice, the theoretical challenge was never satisfactorily addressed, and hence remains valid today.

The Cobb-Douglas function is also empirically weak. True, estimating GDP growth as a function of capital and labor growth rates through an OLS regression without a time trend (as in Cobb and Douglas 1928) yields results supporting the hypothesis that the regression parameters of the two input variables approximate their shares of output. Yet, when one includes a time trend in the mix (as in Lucas 1970, Romer 1987, Klette and Griliches 1996, Griliches and Mairesse 1998 or Felipe and Adams 2005), the regression coefficient associated to aggregate capital usually turns out to be either *negative or statistically insignificant*, which squarely

¹ This objection, and the debate that ensued, is known as the "Cambridge Capital Controversy, as it pitted economists from Cambridge, England against others from the MIT in Cambridge, MA.

contradicts the model. DGSE models usually sidestep this issue by calibrating capital and labor coefficients *a priori*, instead of estimating them empirically; yet this of course does not validate the function, since it assumes away what it should actually test.

Furthermore, the fact that the regression without a time trend displays such good results does not necessarily constitute real evidence because (as first pointed out by Phelps Brown 1957 and then by Simon and Levy 1963, Fisher 1971, Shaikh 1974, 1980, Samuelson 1979, Felipe and Fisher 2003 or Felipe and McCombie 2005, 2009) such a specification is formally identical to the Fundamental Growth Accounting Identity. Thus, as long as the factor shares of output remain stable over time (as is indeed the case), the empirical fit is bound to be good simply because it constitutes a test of the accounting identity itself, not of the underlying production function.

This paper puts forward the hypothesis that observed GDP fluctuations are better modeled by regarding capital, both physical and human, as a “sunk cost” (whose returns are therefore economic rents) than as a variable input whose reward is its marginal product, as the basic Cobb-Douglas function assumes. As capital investment requires a lead time to become productive, in an uncertain world the initial response to random demand changes will be subject to existing capacity constraints.² Importantly, since only variable inputs need to be included in the production function, if the time horizon is short enough then only one input is truly variable: labor time. Under these conditions, value-weighted factor aggregates are no longer necessary, which leads to an aggregate production function that is exempt from the Cambridge Critique.

This does not negate the role of installed capital in the production process – only its measurement. At the time an investment is made, of course, the investor expects it to yield at least the market return; yet in an uncertain world these expectations may or may not be fulfilled, so going forward past costs become irrelevant. Hence, the return on capital should not be modeled as a marginal product on a historical investment, but as an economic rent resulting from

² Lead investment time and capacity constraints have been invoked in production function models before (e.g. Jorgenson and Griliches 1967, Kydland and Prescott 1982, Basu and Fernald 2000 or Hansen and Prescott 2005), but nearly always as a refinement of the Cobb-Douglas function, as opposed to being the basis to build a different production function altogether.

available production capacity vs. output demand: the higher the spare capacity, the lower the ability of asset owners to charge a rent.

The structure of this paper is as follows. Section 2 provides an intuitive rationale for the alternative model, which is then developed analytically in Section 3. Section 4 describes the empirical strategy and presents the statistical test results for both the proposed model itself and for the standard Cobb-Douglas function. Finally, Section 5 ends with a summary of findings and conclusions.

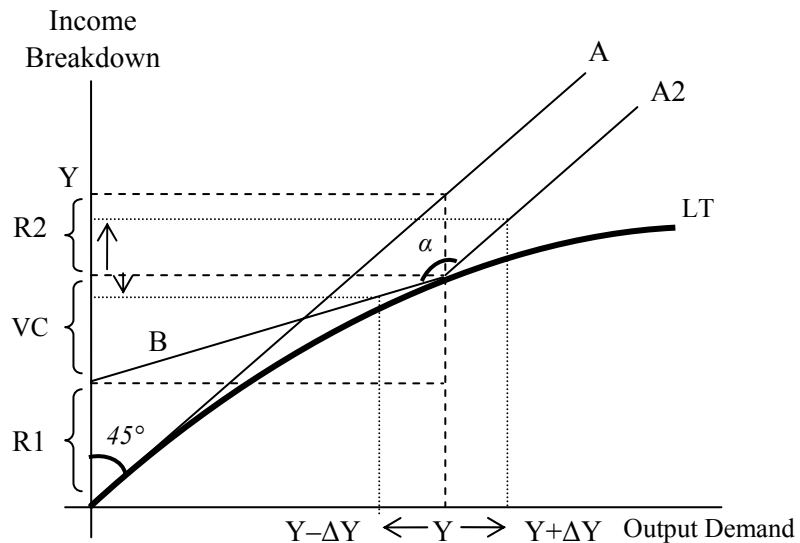
2 Model Rationale

Imagine an economy composed of many production units, each one devoted to transforming inputs into a given set of outputs. For every given output volume, there are multiple productive processes or “techniques” available, each one requiring a given fixed investment in plant capacity in addition to a variable cost per unit produced. We assume each one of these producers is rationally aiming to select the output volume and productive process that maximizes real profit (i.e. the difference between output value and input costs, measured in output units). We also assume that the optimal technology curve that results from their selecting the most profitable technique at each production level displays economies of scale i.e. that, given an increase in production volume, there is always a technique that would allow to reduce the overall cost per unit (always expressed in terms of output units) and therefore increase the real profit. This is represented graphically by curve ‘LT’ in Figure 1.

In this diagram, both the horizontal and vertical axes represent real output, whereas every line in the quadrant represents the cost structure of a given productive process or technique. Therefore, if we select an output demand level on the horizontal axis, its projection on the vertical axis according to the curve representing a given technique indicates how much of its output value would correspond to fixed cost (‘R1’) vs. variable cost (‘VC’) vs. “pure” profit (‘R2’). For example, curve ‘LT’ represents the optimal long-term production cost curve i.e. the lowest production cost possible for every given output level, regardless of how long it would take to deploy the associated production process. Every point along this curve (say, point ‘Y’) is associated to a given optimal production technique (for example ‘B’) requiring a fixed upfront cost (‘R1’) plus a certain

variable cost per unit (represented by the slope of 'B'), up to a plant capacity equal to 'Y'. Conversely, the straight line 'A' represents the cost profile of a production technique with constant variable costs per unit, no fixed costs and no barriers of

Figure 1: Long-term vs. Short-term Production Functions



entry, which would of course result in the unit price equating the marginal (i.e. variable) cost (hence it forms a 45° angle respective to the axes). Evidently, technique 'A' has a steeper slope than 'B' because its variable costs are higher, and is more inefficient for a given level of production 'Y' because the overall cost per unit for technique 'B', including both fixed and variable costs, is lower than that of technique 'A' – the difference being of course the "pure" profit R2.

In economics, prices are conventionally broken down into marginal costs (i.e. the incremental cost of producing the last output unit) and economic rents (that is, the difference between marginal cost and actual price). Evidently no rational, profit-maximizing producer would willingly sell for less than the marginal cost (i.e. in the case of a production level 'Y', for less than the length of segment 'VC'). Conversely, from a short-term perspective, fixed cost investments

constitute a “sunk cost” whose historical size is simply irrelevant for the maximization of future profit, and whose return therefore constitutes an economic rent (i.e. in this case the sum of segments ‘R1’ and ‘R2’) since the concept of “marginal cost” does not apply to it.

To be sure, if a production unit were considering whether to commit an upfront investment or not, it would require the present value of the expected return to equal or exceed the upfront cost. Hence, in a deterministic world it would make sense to model those future rents as equating marginal costs with marginal returns for that historical investment amount; yet, in an uncertain world, as soon as the conditions change so do those rents, after which no relationship may exist between them and the sunk costs.

The assumption behind the long-term curve ‘LT’ is that, when planning for the long run, producers can jump from one technique to the next as their output volumes change, choosing for every level of production the technique with the lowest cost. Conversely, when unexpected shocks hit demand, it is not possible to do this in the short run, for the upfront investments to expand capacity and deploy a more efficient production process cannot be deployed instantly, nor can installed capacity be easily divested, even if such a thing is possible at all. Hence, if demand, for example, drops unexpectedly by a magnitude ‘ ΔY ’ (i.e. down to ‘ $Y - \Delta Y$ ’), the producer will use the same technique to produce at less than full capacity, whereas, if demand increases instead by ‘ ΔY ’ (i.e. up to ‘ $Y + \Delta Y$ ’), the producer will have to resort to the less efficient technique A to produce the supplementary units required. This means that, although the long-term cost function is concave (i.e. has positive returns to scale), the short-term one is convex (that is, displays diseconomies of scale), for, given a planned output level ‘ Y ’, actual production follows segment B when it falls below ‘ Y ’ but segment ‘A2’ (parallel to ‘A’) when it raises above ‘ Y ’.

Let’s imagine now that we have many industries in an economy, each producing a different set of goods and services but all subject to a cost function with the same characteristics. Then, if the structure of aggregate demand changes unexpectedly, so that demand for one product increases at the expense of another while the total consumer budget stays the same, the costs of those industries whose demand dropped will go down comparatively less than the costs of the industries with higher demand will go up. This will therefore result in an aggregate loss of productive efficiency:

1. The larger the variability of demand (i.e. ‘ ΔY ’ in the diagram) and/or
2. The smaller the angle ‘ α ’ between segments ‘B’ and ‘A2’

In turn, since any increment in the share of economic rents (‘ $R1+R2$ ’) over the total revenue ‘ Y ’ results in squeezing segment ‘VC’ and hence flattening segment ‘B’ and closing angle ‘ α ’, we may say that, under demand uncertainty, the higher the ratio $\frac{R1+R2}{Y}$, the lower the overall productive efficiency in the short run.

As a result, under uncertainty the link between economic rents and productivity in the short run is exactly opposite to the long run. In the long run what matters is curve ‘LT’ and, since segment ‘B’ represents the tangent to this curve, the larger the ratio $\frac{R1+R2}{Y}$, the flatter (i.e. the more concave) the curve will be at that point – i.e. the higher will be its economies of scale. Conversely, in the short run, the larger this ratio (i.e. the more concave the ‘LT’ curve), the more convex the angle ‘ α ’ will be. If the world were deterministic, only the long-term function would matter, since rational agents would plan only at one point in the beginning of time with a view in the long run, and never have to revise their expectations again. Yet, in a stochastic world, circumstances change continuously and, as the agents adapt every time to the new conditions, it is the short-term function that determines the initial response.

In sum, the model predicts that the short-term production is a function of *variable* inputs only (as opposed to both fixed and variable), combined with the percentage of economic rents over total output (which determines the angle ‘ α ’ and therefore the degree of convexity) as well as the variability of demand composition (which in essence represents the average shock ‘ ΔY ’). Furthermore, since all variable inputs translate at an aggregate level into capital plus labor, it can be argued that, if the time horizon is sufficiently short, the only “variable” input at a macro level is labor time.

Importantly, since the shape of the curve is directly dependent on the ratio $\frac{R1+R2}{Y}$, any policy aimed at changing the weight of economic rents over production would actually have the power to change the shape of the production function, and thus, by implication, to manipulate the rate of productivity growth. The implications are substantial, because the weight of economic rents can actually be modified through public intervention, be it by the central bank (through basic interest rates, which transfer income from borrowers to lenders) or by the government (e.g. through taxes and subsidies, or through legal monopolies and

other constraints on competition). This means that, even if the market were perfectly rational and efficient, so that money supply were absolutely neutral from a demand perspective, the central bank's discretionary control on monetary supply would have the power to modify the shape of the production function (and hence the rate of productivity growth), accelerating it with low real interest rates and slowing it down with higher ones. An important prediction of this model is therefore that there is a direct, positive correlation between cheap, abundant credit and observed productivity growth.

3 Analytical Framework

3.1 The Individual Production Function

Consider an economic system with $i = 1 \dots n$ types of goods and services and $j = 1 \dots m$ production units, each one of which transforms, at any given point in time t , a certain set of input quantities of goods and services $\mathbf{X} \equiv \{x_{1,j,t} \dots x_{n,j,t}\}$ into another set of output quantities $\mathbf{Y} \equiv \{y_{1,j,t} \dots y_{n,j,t}\}$. We refer to this transformation as a *technical* production function, and we represent it as a mapping $f_{j,t} : \mathbf{X} \rightarrow \mathbf{Y}$ or, what is the same, as a function $f_{j,t} \{ \dots \}$ such that:

$$f_{j,t} \{x_{1,j,t} \dots x_{n,j,t}\} \equiv \{y_{1,j,t} \dots y_{n,j,t}\} \quad (1)$$

We now define the real output flow (‘ $Y_{j,t}$ ’) of a production unit j at a given point in time t as the sum of all the component output quantities $\mathbf{Y} \equiv \{y_{1,j,t} \dots y_{n,j,t}\}$ multiplied by their market prices $\mathbf{P} \equiv \{p_{1,t} \dots p_{n,t}\}$ and expressed in terms of a given basket of consumable goods and services selected as *numéraire* (and whose unit price we conventionally represent as ‘ P_t ’) i.e.:

$$Y_{j,t} \equiv \frac{1}{P_t} \mathbf{P} \mathbf{Y} \equiv \frac{1}{P_t} \sum_{i=1}^n p_{i,t} y_{i,j,t} \quad (2)$$

On this basis, we define the *economic* production function $F_{j,t} \{x_{1,j,t} \dots x_{n,j,t}\}$ as the transformation $\mathfrak{R}^n \rightarrow \mathfrak{R}$ leading from the input quantities to the real output flow i.e.:

$$F_{j,t} \{x_{1,j,t} \dots x_{n,j,t}\} \equiv Y_{j,t} \quad (3)$$

Note that this *economic* production function differs from the *technical* one in its using output prices to weight the various (and otherwise heterogeneous) output quantities into a scalar metric, the real output flow. As this paper aims to develop an expression linking a set of inputs to a measure of GDP (i.e. ‘ $Y_{j,t}$ ’), it is the *economic* production function we will focus on going forward.

We now define the profit (‘ $\Pi_{j,t}$ ’) of the production unit j as the difference between its output flow at current prices and its total production cost (‘ $C_{j,t}$ ’) i.e. $\Pi_{j,t} \equiv P_t Y_{j,t} - C_{j,t}$, where $C_{j,t}$ is a function of the input quantities $\mathbf{X} \equiv \{x_{1,j,t} \dots x_{n,j,t}\}$. When this profit is expressed in terms of the basket of goods and services we have selected as real output *numéraire*, we refer to it as “real” profit (‘ $\Pi_{j,t}^*$ ’) i.e.:

$$\Pi_{j,t}^* \equiv Y_{j,t} - C_{j,t}^* \quad (4)$$

Where obviously $C_{j,t}^* \equiv \frac{C_{j,t}}{P_{j,t}}$.

At this point we introduce four assumptions that will be central to this section:

1. *Profit Maximization:*

Producers select their output quantities $Y_{j,t}$ as well as their corresponding demand for inputs to maximize their real profit flow $\Pi_{j,t}^* \equiv Y_{j,t} - C_{j,t}^*$.

2. *Existence of an Optimum:*

There is at least one finite maximum point for every real profit function $\Pi_{j,t}^*$.

3. *Continuity and Differentiability:*

The real profit function is continuous and partially-differentiable respective to all input and output quantities.

4. Homogeneous Production Function:

The economic production function $Y_{j,t} \equiv F_{j,t} \{x_{1,j,t} \dots x_{n,j,t}\}$ is such that, given a scalar $a \in \mathfrak{R}$, the expression $F_{j,t} \{ax_{1,j,t} \dots ax_{n,t}\} = a^{h_{j,t}} F_{j,t} \{x_{1,j,t} \dots x_{n,j,t}\}$ holds (where the degree of homogeneity is independent of $\{x_{1,j,t} \dots x_{n,j,t}\}$)

For convenience, we also introduce the following two definitions:

- We define the *combined input* (' $X_{j,t}$ ') as:

$$X_{j,t} \equiv \frac{1}{c_{1,j,t}^*} \mathbf{C} \mathbf{X} \equiv \frac{1}{c_{1,j,t}^*} \sum_{i=1}^n c_{i,j,t}^* x_{i,j,t} \quad (5)$$

Where $\mathbf{C} \equiv \{c_{1,j,t}^* \dots c_{n,j,t}^*\}$ represent the real marginal costs i.e. $c_{i,t}^* \equiv \frac{\partial C_t^*}{\partial x_{i,t}}$,

and $c_{1,j,t}^*$ represents the marginal cost of an input '1' selected as aggregation unit (and which, incidentally, does not need to coincide necessarily with the basket of goods we selected as output *numéraire*).

- We also define the *rent ratio* or *rent coefficient* (' $\rho_{j,t}$ ') as the output value premium over real marginal costs per unit of input i.e.:

$$\rho_{j,t} \equiv \frac{Y_{j,t}}{c_{1,j,t}^* X_{j,t}} - 1 \quad (6)$$

The four assumptions above suffice to build the following general expression:

PROPOSITION 1: *Under Assumptions 1 to 4, the economic production function of a production unit may be expressed as:*

$$Y_{j,t} = A_{j,t} X_{j,t}^{\frac{1}{1+\rho_{j,t}}} \quad (7)$$

Where both the integration coefficient $A_{j,t}$ (which we will refer to as “productivity coefficient”) and the rent ratio $\rho_{j,t}$, are functions independent of $X_{j,t}$ and $Y_{j,t}$.

PROOF: *See Appendix 1*

This is an important expression, for it represents the analytical equivalent of the broken line B-A2 in Figure 1 (only, now under the homogeneity condition imposed in Assumption 4, which the broken line B-A2 would obviously not fulfill due to its non-differentiability at the inflexion point). As one would expect on the basis of the reasoning in Section 2, the individual production function, represented here by the analytical expression (11), will display diseconomies of scale as long as the rent coefficient $\rho_{j,t}$ is positive.³

3.2 The Path of Output Growth

The purpose of this subsection is to determine the path of output growth over time under the conditions established in Subsection 3.1. We now introduce the following three additional assumptions:

5. *Neutral Price Breakdown:*

The way prices are broken down between marginal costs and economic rents (as represented by the rent ratio $\rho_{j,t}$) has no impact, other things being equal, on

output demand i.e. $\forall \rho_{j,t} \in \mathfrak{R} \ ; \ \frac{\partial Y_{j,t}}{\partial \rho_{j,t}} = 0 \quad (8)$

³ In some of the initial working paper discussions about this model, this conclusion caused some discomfort, as it was perceived as contradictory with the standard development as it appears in a number of papers (e.g. Basu and Fernald 2000). As shown in Appendix 2, however, there is no contradiction and, given the same assumptions, one can easily be derived from the other.

6. *Random Walk Perturbation on Technology Shocks:*

Technology shocks follow an Itô stochastic diffusion process subject to a Wiener perturbation (that is, a linear, continuous, normally-distributed random-walk process otherwise known as “Brownian motion”) i.e.:

$$\frac{dA_{j,t}}{A_{j,t}} = \gamma_{j,t} dt + s_{j,t} dW_{j,t} \quad (9)$$

Where $A_{j,t}$ represents the productivity coefficient in $Y_{j,t} = A_{j,t} X_{j,t}^{\frac{1}{1+\rho_{j,t}}}$, the symbols $\gamma_{j,t}$ and $s_{j,t}$ represent functions whose value is known at time t , and W_t is a Wiener process i.e. a stochastic process such that $W_0 \equiv 0$ and $dW_{j,t} \equiv \omega_{j,t} \sqrt{dt}$, where $\omega_{j,t}$ is a serially-uncorrelated, normally-distributed standardized white noise (i.e. analytically $\omega_{j,t} \sim N[0,1]$).

7. *Random Walk Perturbation on Output Demand:*

The growth rate of output demand also follows an Itô stochastic process subject to a Wiener perturbation, i.e.:

$$\frac{dY_{j,t}}{Y_{j,t}} = E_t \left[\frac{dY_{j,t}}{Y_{j,t}} \right] + s_{j,t} dW_{j,t} + \sigma_{j,t} dV_{j,t} \quad (10)$$

Where $\sigma_{j,t}$ is a function whose value is known at time t , the operator $E_t[\circ]$ indicates the expected value according to the information available at instant t , and $V_{j,t}$ is a Wiener process i.e. a serially-uncorrelated, normally-distributed standardized white noise just as $W_{j,t}$. For simplicity, we will also assume that $V_{j,t}$ and $W_{j,t}$ are independent random processes so that $dV_{j,t} dW_{j,t} = 0$.

Assumption 5 is fairly intuitive: demand may be impacted by changes in output prices, but what portion of that price is devoted to paying for marginal costs as opposed to economic rents has no relevance for consumers, and should therefore have no impact on their demand – which we are implicitly assuming to always equal supply. Note that the condition $\forall \rho_{j,t} \in \Re$ means that price

composition is neutral throughout the whole real domain i.e. that $\frac{\partial Y_{j,t}}{\partial \rho_{j,t}} = 0$ holds

for every possible value of $\rho_{j,t}$, and thus means that $\frac{\partial^2 Y_{j,t}}{\partial \rho_{j,t}^2} = 0$ and, in general,

$$\frac{\partial^2 Y_{j,t}}{\partial \rho_{j,t} \partial V_{j,t}} = 0 \text{ (where } V_{j,t} \text{ represents any other variable in the model).}$$

The other two assumptions introduce a stochastic element into the model. The assumption that technology progress follows a random walk (Assumption 6) is consistent with the standard literature (e.g. in DSGE models), except perhaps for its being expressed in continuous instead of discrete time. Similarly, Assumption 7, which postulates that output growth perturbations are the sum of technology and non-technology shocks, should also be fairly intuitive (except perhaps for the fact that, for the sake of analytical simplicity, we assume these two sources of uncertainty to be independent from each other), although the second source of uncertainty is sometimes assumed to be nil in standard DGSE models.

Under these conditions, if we differentiate expression (7) respective to time (see the detailed derivation in Appendix 3) we obtain the following output growth function:

PROPOSITION 2: *Under Assumptions 1 to 7, the output growth rate of a production unit may be expressed as:*

$$\frac{dY_{j,t}}{Y_{j,t}} = \gamma_{j,t} dt + s_{j,t} dW_{j,t} - \frac{\sigma_{j,t}^2}{2} \rho_{j,t} dt + \frac{1}{1 + \rho_{j,t}} \frac{dX_{j,t}}{X_{j,t}} \quad (11)$$

PROOF: *See Appendix 3*

There are two key takeaways from this expression (11). One is obvious: the output fluctuations that are not matched by technology change (as represented by $dW_{j,t}$) must be reflected on input change (represented by $dX_{j,t}$). The other, however, is more important: the higher the non-technology variance ($\sigma_{j,t}^2$) and/or the rent ratio ($\rho_{j,t}$), the slower will output growth be, even under the same levels of input

growth and technology progress. This is exactly the conclusion anticipated in Section 2.

3.3 Aggregation

To extend expression (11) to a macroeconomic scale, we still need to aggregate output across all the production units $j = 1 \dots m$. In this context, we introduce only one additional assumption:

8. *Single Marginal Cost for the Reference Input:*

The marginal cost $c_{1,\varepsilon,t}^*$ of the input ‘1’ selected as aggregation unit is the same for all producers i.e. $c_{1,1,t}^* = \dots = c_{1,\mu,t}^*$ (12)

This is actually quite a weak assumption, as it only requires to find *one* commodity that displays the same marginal cost for all producers (for example a currency trading in a highly liquid, competitive market), and then take it as the accounting unit of $X_{i,t}$.

For convenience we need of course to introduce a naming convention for the aggregate variables. A detailed definition of each one of the aggregate variables is provided in Appendix 4. As a general rule, the aggregate versions of the variable introduced in Subsections 3.1 and 3.2 are always indicated with a superscript bar

(e.g. \bar{Y}_t represents the aggregate output i.e. $\bar{Y}_t \equiv \sum_{j=1}^m Y_{j,t}$).

In this context there is however an important distinction to be made between the overall non-technology variance $\bar{\sigma}_t^2 dt \equiv \left(\frac{d\bar{X}_t}{\bar{Y}_t} \right)^2$ (expressed as a percentage of

output) and the weighted sum of individual non-technology variances $\bar{\delta}_t \equiv \sum_{j=1}^m \alpha_{j,t} \delta_{j,t}$ (where $\delta_{j,t} \equiv \sigma_{j,t}^2 \frac{\alpha_{j,t}}{\beta_{j,t}}$, and where $\alpha_{j,t}$ and $\beta_{j,t}$ represent the

relative shares of, respectively, output and input of the various productive units i.e.

$\alpha_{j,t} \equiv \frac{Y_{j,t}}{\bar{Y}_t}$ and $\beta_{j,t} \equiv \frac{X_{j,t}}{\bar{X}_t}$). Needless to say, these two magnitudes would

necessarily be equivalent for an individual production unit, but not for an aggregate.

Under these conditions the following proposition and its corollary hold:

PROPOSITION 3: *Under Assumptions 1 to 8, the aggregate output growth rate may be expressed as:*

$$\frac{d\bar{Y}_t}{\bar{Y}_t} = \bar{\Gamma}_t dt + \bar{s}_t d\bar{W} - \frac{\bar{\delta}_t}{2} \bar{\rho}_t dt + \frac{1}{1 + \bar{\rho}_t} \frac{d\bar{X}_t}{\bar{X}_t} \quad (13)$$

PROOF: *See Appendix 4*

COROLLARY 1: *Under Assumptions 1 to 8, aggregate output may be expressed as:*

$$\bar{Y}_t = \bar{A}_t \bar{X}_t^{\frac{1}{1+\bar{\rho}_t}} \quad (14)$$

Where the aggregate productivity coefficient \bar{A}_t is a martingale such that:

$$\frac{d\bar{A}_t}{\bar{A}_t} = \left(\bar{\Gamma}_t + \frac{\bar{\sigma}_t^2 - \bar{\delta}_t}{2} \bar{\rho}_t \right) dt + \bar{s}_t d\bar{W}_t \quad (15)$$

PROOF: *This simply results from integrating expression (13) per Itô's lemma⁴*

Importantly, notice that expression (15) is *NOT* just a weighted aggregate of the individual productivity growth expression (9), as one might expect. This is because the aggregate productivity coefficient \bar{A}_t reflects not just the impact of technology changes but also that of changes in the share of output of different

⁴ To prove that expression (14) is the integral of (13), simply differentiate (13) according to Itô's lemma (i.e. the stochastic version of the chain rule), which will of course yield expression (14).

units whose individual production functions, as depicted graphically in Section 2 and analytically in Subsection 3.1, are not necessarily linear.

The question remaining now is how to measure \bar{X}_t : after all, this metric is defined as the sum of the individual producers' input quantities weighted by their own marginal costs, which may be different from one produce to the next. The definition of "variable input", however, depends on the time horizon of analysis: the shorter it is, the fewer inputs can be regarded as variable. Hence, there must be a period short enough for only one input to be genuinely regarded as variable.

Which input would this be? At an aggregate level all inputs are a combination of labor time and a stock of capital resulting from investments made (and sunk) at some earlier point in time (be it under the form of physical fixed assets, human capital or any other). Hence, if we define the time horizon as short enough, we may assume that:

9. Labor Hours are the only Variable Input within the Target Time Horizon:

According to this assumption, if we represent the total sum of labor hours as \bar{H}_t , then obviously expression (14) may be rewritten as:

$$\bar{Y}_t = \bar{A}_t \bar{H}_t^{\frac{1}{1+\beta}} \quad (16)$$

Where \bar{H}_t represents a non-weighted sum of work hours of all persons (as opposed to the usual Tornquist labor input aggregate, where input time is weighted by the share of output of each type of worker).

In short: this model's main prediction is that GDP growth is better explained by taking non-weighted labor time and an estimate of the rent ratio as explanatory variables than by a Cobb-Douglas function depending on value-weighted capital and labor aggregates. This is the hypothesis that will be tested in Section 4.

3.4 Comparison with the Cobb-Douglas Function

For reference, it may now make sense to establish what sort of assumptions would derive a Cobb-Douglas production function within the same analytical framework. This requires introducing two additional assumptions. The first is the following:

10. Market Prices Equal Marginal Costs:

The marginal costs of all inputs equal their market prices i.e. $c_{i,j,t} = p_{i,t}$.

This assumption implies that:

- The rent ratios are always zero (i.e. $\rho_{i,t} = 0$), as there are no economic rents
- Assumption 9 becomes indefensible, as all inputs are now instantly variable

Under this assumption, the aggregate production function obviously takes the form:

$$\bar{Y}_t = \bar{A}_t \bar{X}_t \quad (17)$$

Where the aggregate productivity coefficient \bar{A}_t is a martingale such that:

$$\frac{d\bar{A}_t}{\bar{A}_t} = \bar{\gamma}_t dt + \bar{s}_t d\bar{W}_t \quad (18)$$

And where \bar{X}_t can now be calculated as a sum of inputs multiplied by their market

prices i.e. $\bar{X}_t \equiv \frac{1}{p_{1,t}} \sum_{i=1}^n \sum_{j=1}^m p_{i,t} x_{i,j,t}$.

The impact of Assumption 10 is therefore substantial: not only does the function become linear (i.e. exhibits constant returns of scale), but also, as a result, the growth rate of the productivity coefficient \bar{A}_t becomes a direct aggregation of the individual expression (9), which, per Assumption 6, is only determined by technology factors.

The second key assumption is related to aggregate output elasticity. Specifically, given the usual Tornquist aggregates for labor (\bar{L}_t) and capital (\bar{K}_t), it assumes:

11. Constant Elasticity of Aggregate Capital vs. Output:

The elasticity $\frac{\partial \bar{Y}_t}{\partial \bar{K}_t} \frac{\bar{K}_t}{\bar{Y}_t} = \alpha_K$ is a constant (19)

Since, according to expression (17), this is a homogeneous function of degree one

(because $\bar{\rho}_t = 0$), we know that $\bar{Y}_t = \alpha_K \frac{\partial \bar{Y}_t}{\partial \bar{K}_t} \bar{K}_t + (1 - \alpha_K) \frac{\partial \bar{Y}_t}{\partial \bar{L}_t} \bar{L}_t$, which means

that the production function may be rewritten as:

$$\bar{Y}_t = \bar{A}_t \bar{K}_t^{\alpha_K} \bar{L}_t^{\alpha_L} \quad (\text{where } \alpha_L = 1 - \alpha_K) \quad (20)$$

Which is of course the constant-returns Cobb-Douglas function.

Note that, as shown in Appendix 2, by dropping Assumption 1 (i.e. profit maximization) this result can be extended to non-constant returns i.e. $\alpha_K + \alpha_L \neq 1$.

4 Empirical Strategy and Testing

4.1 Empirical Model

Following the traditional approach to testing macroeconomic production functions empirically, this paper resorts primarily to an OLS regression against a GDP growth series. This requires rearranging expression (14) in logarithmic form:

$$\ln \bar{Y}_t = \ln \bar{A}_t + \frac{1}{1 + \bar{\rho}_t} \ln \bar{H}_t \quad (21)$$

Where, since we know from expression (15) that \bar{A}_t follows an Itô diffusion process:

$$d \ln \bar{A}_t = \frac{d\bar{A}_t}{\bar{A}_t} - \frac{1}{2} \left(\frac{d\bar{A}_t}{\bar{A}_t} \right)^2 = \frac{d\bar{A}_t}{\bar{A}_t} - \frac{\bar{s}_t^2}{2} dt \quad (22)$$

And therefore we can rewrite expression (21) as:

$$\ln \bar{Y}_t = \int \left(\bar{\Gamma}_t + \frac{\bar{\sigma}_t^2 - \bar{\delta}_t}{2} \bar{\rho}_t - \frac{\bar{s}_t^2}{2} \right) dt + \int \bar{s}_t d\bar{W}_t + \frac{\ln \bar{H}_t}{1 + \bar{\rho}_t} \quad (23)$$

Which, taking discrete time increments, becomes:

$$\Delta \ln \bar{Y}_t = \Delta \int \left(\bar{\Gamma}_t + \frac{\bar{\sigma}_t^2 - \bar{\delta}_t}{2} \bar{\rho}_t - \frac{\bar{s}_t^2}{2} \right) dt + \Delta \int \bar{s}_t d\bar{W}_t + \Delta \frac{\ln \bar{H}_t}{1 + \bar{\rho}_t} \quad (24)$$

To formulate the model in linear form we introduce two ancillary assumptions:

- a) The coefficients $\bar{\Gamma}$, \bar{s} , $\bar{\sigma}$ and $\bar{\delta}$ are constant
- b) Time increments are short enough to allow the approximation

$$\Delta \int \bar{\rho}_t dt \approx \bar{\rho}_t \Delta t \quad \text{and} \quad \Delta \frac{\ln \bar{H}_t}{1 + \bar{\rho}_t} \approx \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$$

Under these assumptions we can rewrite expression (24) as:

$$\Delta \ln \bar{Y}_t = \left(\bar{\Gamma} - \frac{\bar{s}^2}{2} \right) \Delta t + \frac{\bar{\sigma}^2 - \bar{\delta}}{2} \bar{\rho}_t \Delta t + \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t} + \bar{s} \Delta \bar{W}_t \quad (25)$$

Which, conventionally taking the discrete time increments in the series as unity (i.e. $\Delta t \equiv 1$), can be estimated as the regression:

$$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \bar{\rho}_t + \beta_2 \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t} + u_t \quad (26)$$

Where α , β_1 and β_2 are regression parameters whose correspondences with the model are the following:

$$\alpha \equiv \bar{\Gamma} - \frac{\bar{s}^2}{2}$$

$$\beta_1 \equiv \frac{\bar{\sigma}^2 - \bar{\delta}}{2}$$

$$\beta_2 \equiv 1$$

$$u_t \equiv \bar{s} \Delta \bar{W}_t$$

Note that the definition of β_2 imposes an additional test of consistency, because only the value $\beta_2 = 1$ is compatible with the theoretical model, and therefore the regression results must be such that this hypothesis cannot be rejected.

In addition, if the regression is performed against the data of an economy large enough for the foreign sector to be proportionally small (e.g. the USA), one should also expect the value of β_1 to be lower than zero. The reason is that, although in

principle the definition of $\beta_1 \equiv \frac{\bar{\sigma}^2 - \bar{\delta}}{2}$ is compatible with any sign, in a closed or

nearly-closed economic system, where the total number of labor hours available can be taken as a given, increases in the demand of hours by some producers can only be fulfilled through reallocation from other producers, so the variance

$\bar{\sigma}^2 dt = \left(\frac{d\bar{H}_t}{(1 + \bar{\rho}_t)\bar{H}_t} \right)^2$ will logically be zero, whereas this does not apply to the

aggregation of individual variances $\bar{\delta}$ (for, unlike $\bar{\sigma}^2$, this metric excludes the covariances across units, which in this case would be negative). Hence, although a positive value for β_1 is in principle possible, given that imports represent a relatively small portion of the U.S. economy the result should be similar to that of a closed system, and thus a result where $\beta_1 \geq 0$ ought to pose a question mark on the model.

4.2 Data and Approach

The variables of the regression are obtained on the basis of the following dataset:

- As estimators of **aggregate output** (\bar{Y}_t), **total labor time** (\bar{H}_t), **aggregate labor input** (\bar{L}_t), and **capital input** (\bar{K}_t) we use *aggregate U.S. private nonfarm growth data for, respectively, real GDP, hours of all persons, labor input and capital services from 1949 to 2008* according to the data provided by the *U.S. Bureau of Labor Statistics*.⁵

The difference between ' \bar{H}_t ' and ' \bar{L}_t ' is that the first is a non-weighted sum of all workers' time, whereas the second is the standard Tornquist aggregate

⁵ The source data are in <http://stats.bls.gov/mfp/> and have been reproduced in Appendix 5.

that is used for macroeconomic Cobb-Douglas estimations. In truth, both variables are so highly correlated ($R^2=97.77\%$ between the two in this particular sample) that it would matter little which one we used in a model – but the distinction is important from a conceptual viewpoint.

- As for the **rent ratio** $\bar{\rho}_t$, there is unfortunately no single, all-encompassing indicator of the portion of economic rents embedded in the overall price of productive factors. Nevertheless, as in this model all capital investment is regarded as “sunk” costs, any return it generates must be a rent flow. Hence, a way to measure at least part of this flow would be to look into the process of purchasing power transfer between lenders and borrowers in the money market. In this context, the risk-free interest rate represents a flow of purchasing power from debtors to creditors, whilst any additional credit funds represent a flow of purchasing power in the opposite direction.⁶ The balance of interests generated less additional credit provided is thus equivalent to a rent paid by producers, under the form of a transfer of purchasing power, for their use of capital invested.

In addition, since expression (25) takes as an input its value at instant t instead of its increment between instants t and $t + \Delta t$, we need to take the value of this indicator at the start of the period – which, as we deal with annual data, means at the start of the year. Of course this is a fairly crude measure, as there is nothing inherently “special” in the rent ratio in the month of January, or in the calendar year as a time horizon, but this should allow us to test if the rent ratio today does indeed help to predict the rate of output growth tomorrow.

As a result of all this, we take as an estimator of the rent ratio $\bar{\rho}_t$, the *U.S. nominal risk-free interest rate (specifically, the Federal Reserve Prime Rate) at the start of every year from 1949 to 2008, less the annualized rate of growth of USD money supply (measured as M2) during the month of January of that year*, always according to the *Federal Reserve Bank of St. Louis*.⁷

⁶ Credit flows would logically be associated to the expectation by creditors of repayment at a later date, but in a non-deterministic world this expectation may or may not be fulfilled, so all we know for certain is that credit increments represent purchasing power transfers from lenders to borrowers.

⁷ The source data are <http://research.stlouisfed.org/fred2/> and <http://research.stlouisfed.org/aggreg/> and have also been reproduced in Appendix 5.

Under these conditions, the model is subjected to four empirical tests:

- **TEST 1:** First, the proposed model is estimated through an OLS regression in order to analyze its empirical fit and its consistency with the theory.
- **TEST 2:** Next, a Cobb-Douglas production function is estimated under the same conditions, and its empirical fit compared with the previous one.⁸
- **TEST 3:** The two models are then compared through a Davidson-MacKinnon test (following Davidson and MacKinnon 1981) to assess whether one should be preferred to the other, or if they should be regarded as complementary.
- **TEST 4:** Finally, the regression in test 1 is repeated, only now adding capital input changes (both contemporary and lagged) as an explanatory variable to test whether aggregate capital is better or worse an indicator of installed capacity than financial rents, or if the two are complementary.

4.3 Empirical Results

4.3.1 Test 1

Here we estimate the parameters of the model as formulated in expression (26) by means of an OLS regression.⁹ The results appear in Table A.

The R^2 value suggests that this expression can explain nearly 85 percent of the total variability of the series. The results are also consistent with the model's predictions regarding parameters β_1 and β_2 . The t -test suggests both parameters are different to zero at over 99% confidence, the estimated value of β_1 is negative

⁸ The source data come from <http://stats.bls.gov/mfp/> and have been reproduced in Appendix 5.

⁹ In this and the following tables, the asterisks beside the estimated values indicate their significance according to Student's t -test: one asterisk indicates significance at 90% confidence or more; two asterisks, significance at 95% confidence or more; three asterisks, significance at 99% confidence or more. For further clarity, standard deviations are always represented between brackets.

(as expected) and the t -ratio for the null hypothesis $\beta_2 = 1$ is equal to $\frac{0.993380 - 1}{0.0648514} = -0.10208$, which does not allow to reject the hypothesis.

Moreover, the robustness of this result is confirmed by both the Durbin-Watson and the Breusch-Godfrey tests: in both cases, the result indicates that there is no first-order autocorrelation in the residuals.

Table A: Results of an OLS Regression on the Model Proposed in this Paper

<i>Analytical Expression:</i>		$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \bar{\rho}_t + \beta_2 \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t} + u_t$
<i>Independent Variables:</i>		
<i>Constant</i>	<i>Coefficient ('$\hat{\alpha}$')</i>	2.41653 ***
	<i>Standard Deviation</i>	(0.184043)
$\bar{\rho}_t$	<i>Coefficient ('$\hat{\beta}_1$')</i>	-0.160799 ***
	<i>Standard Deviation</i>	(0.0288181)
$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$	<i>Coefficient ('$\hat{\beta}_2$')</i>	0.993380 ***
	<i>Standard Deviation</i>	(0.0648514)
<i>Statistical Fit Metrics:</i>		
	<i>Degrees of Freedom</i>	57
	R^2	82.9808 %
	<i>F-statistic</i>	138.9580
	<i>Akaike Information Criterion</i>	201.8077
	<i>Bayesian (a.k.a. Schwarz) Information Criterion</i>	208.0907
	<i>Hannan-Quinn Information Criterion</i>	204.2653
	<i>Durbin-Watson Test</i> ¹⁰ (<i>critical values 1.51442 and 1.65184</i>)	2.045814
	<i>Breusch-Godfrey Test (with a lag of order one)</i>	0.0886961
	<i>Breusch-Godfrey P-value for $P(F(1,56) > 0.0886961)$</i>	76.6943 %

4.3.2 Test 2

Next we follow the same procedure to test a Cobb-Douglas function as expressed in (20) i.e. $\bar{Y}_t = \bar{A}_t \bar{K}_t^{\beta_1} \bar{L}_t^{\beta_2}$ (only, for the sake of generality, allowing $\beta_1 + \beta_2 \neq 1$) and where we assume the magnitudes $\bar{\gamma}_t$ and \bar{s}_t in (18) are constant. The results are in Table B.

Table B: Results of an OLS Regression on a Cobb-Douglas Function

<i>Analytical Expression:</i>		
	$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \Delta \ln \bar{K}_t + \beta_2 \Delta \ln \bar{L}_t + u_t$	
<i>Independent Variables:</i>		
<i>Constant</i>	<i>Coefficient ('$\hat{\alpha}$')</i>	2.57462 ***
	<i>Standard Deviation</i>	(0.814243)
K_t	<i>Coefficient ('$\hat{\beta}_1$')</i>	-0.150151
	<i>Standard Deviation</i>	(0.195456)
L_t	<i>Coefficient ('$\hat{\beta}_2$')</i>	1.06501 ***
	<i>Standard Deviation</i>	(0.0952961)
<i>Statistical Fit Metrics:</i>		
	<i>Degrees of Freedom</i>	57
	R^2	69.9311 %
	<i>F-statistic</i>	66.28242
	<i>Akaike Information Criterion</i>	235.9566
	<i>Bayesian (a.k.a. Schwarz) Information Criterion</i>	242.2396
	<i>Hannan-Quinn Information Criterion</i>	238.4142
	<i>Durbin-Watson Test (critical values 1.51442 and 1.65184)</i>	1.566992
	<i>Breusch-Godfrey Test (with a lag of order one)</i>	2.831592
	<i>Breusch-Godfrey P-value for $P(F(1,56) > 2.831592)$</i>	9.8 %

The statistical fit is clearly much worse than in Table A. Not only is the R^2 lower in Table B, but the F -statistic is also smaller, and all the information criteria (Akaike, Schwarz and Hannan-Quinn), which signal higher likelihood the smaller they are, take much larger values. Furthermore, unlike in Table A, the Durbin-Watson test takes a value which suggests that there might be an autocorrelation issue: indeed, the value of this test falls right between the Durbin-Watson lower and upper critical values for 5% significance (1.51442 and 1.65184, respectively), and therefore its result is inconclusive. A more powerful tool, the Breusch-Godfrey test, allows to reject the autocorrelation hypothesis at 5% significance – but only just, because at 10% it would not. These inconclusive results suggest a possibility that t -ratio values might be somewhat overestimated, thus reducing their explanatory value even further.

Much more importantly, the regression results simply do not support the Cobb-Douglas specification. In particular, the estimated regression coefficient associated to capital input (‘ β_1 ’) is *negative* (i.e. it portrays capital is a factor of value *destruction*) which is contradictory to the model, although with such a large standard deviation it is not even possible to reject the hypothesis that the actual coefficient be $\beta_1 = 0$ (which would still contradict the Cobb-Douglas framework). Nevertheless, the estimated value is so far away from the theoretical one (i.e. from capital’s share of output, which is approximately $\beta_1 = 0.32$) that we can reject it as a null hypothesis, for its t -ratio would be $\frac{-0.150151 - 0.32}{0.195456} = -2.40541$, which allows to reject it with a 95% confidence margin. The conclusion for the parameter (‘ β_2 ’) associated to labor is similar to that of capital (although, in this case, its regression coefficient is statistically different from zero): its t -ratio respective to the null hypothesis that it be equal to labor’s share of output (i.e. approximately $\beta_2 = 0.68$) is $\frac{1.06501 - 0.68}{0.0952961} = 4.04014$, which allows to reject the hypothesis with over 99% confidence.

As discussed in the introduction, this is by no means a novel finding. If it is not highlighted more often is probably because most mainstream models are designed to work around it, be it by performing the regression without a constant time trend factor (i.e., in terms of Table B, by imposing *a priori* that $\alpha = 0$) and then

estimating parameters ‘ $\hat{\beta}_1$ ’ and ‘ $\hat{\beta}_2$ ’, be it by calibrating the theoretical weights of capital and labor (i.e. by defining *a priori* the values of ‘ β_1 ’ and ‘ β_2 ’) and then estimating only ‘ α ’ as a residual growth rate. These restrictions may well be useful for other purposes, but do not help to estimate the production function, for they assume away what should actually be proven. The results in Table B are clear-cut: neither can parameter ‘ α ’ be equated to zero (because the *t*-statistic rejects such hypothesis with a confidence margin of over 99%) nor, once this time trend element is introduced, can the hypothesis of parameters ‘ β_1 ’ and ‘ β_2 ’ representing factor shares be sustained.

4.3.3 Test 3

Although all the criteria we have reviewed so far support the regression in Table A (i.e. the model postulated in this paper) against the one in Table B (the standard Cobb-Douglas function), this comparison alone does not suffice to reject the latter, for there is still a possibility that both models be complementary and a “hybrid” one combining the explanatory variables of the two offer an even better result.

To distinguish between these two possibilities we resort to a Davidson/MacKinnon test (following Davidson and MacKinnon 1981). This test requires to re-run each one of the two original regression models, only now including as explanatory variables, in addition to the original ones, the fitted values of the alternative model. A rejection of one of the two models will take place when its fitted values come out as statistically insignificant (on the basis of a *t*-test) when included in the alternative model, and at the same time the fitted values of the alternative model come out as significant when included as an independent variable in the first one. As a result, this test does not “force” the identification of a single winner: it may happen that none of the models is rejected, in which case we would probably need to consider a “hybrid” model, or that both are, which might point towards a more fundamental specification issue. Hence, if we *do* end up finding a clear winner, this will arguably provide a very strong piece of evidence in favor of adopting it at the expense of the other model.

As shown in Tables C and D, the outcome of this test is clear. Table C shows the result of performing the same regression as in Table A but including now the fitted values from the one in Table B as an explanatory variable (represented by ‘ \hat{B}_t ’):

Table C: Davidson/Mac Kinnon Test 1

Analytical Expression:
$$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \bar{\rho}_t + \beta_2 \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t} + \beta_3 \hat{B}_t + u_t$$

Independent Variables:

<i>Constant</i>	<i>Coefficient (' $\hat{\alpha}$ ')</i>	2.05381 **
	<i>Standard Deviation</i>	(0.896827)
$\bar{\rho}_t$	<i>Coefficient (' $\hat{\beta}_1$ ')</i>	-0.163373 ***
	<i>Standard Deviation</i>	(0.0296904)
$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$	<i>Coefficient (' $\hat{\beta}_2$ ')</i>	0.845029 **
	<i>Standard Deviation</i>	(0.364774)
\hat{B}_t	<i>Coefficient (' $\hat{\beta}_3$ ')</i>	0.151979
	<i>Standard Deviation</i>	(0.367651)

Statistical Fit Metrics:

<i>Degrees of Freedom</i>	56
R^2	83.0326 %
<i>F-statistic</i>	91.34813

This result proves that, under the conditions of this test, the fitted values from the regression in Table B (i.e. from the Cobb-Douglas model) *lose* their statistical significance, whereas all the explanatory variables of the model postulated in this paper remain significant.

Table D: Davidson/MacKinnon Test 2

<i>Analytical Expression:</i>		$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \Delta \ln \bar{K}_t + \beta_2 \Delta \ln \bar{L}_t + \beta_3 \hat{A}_t + u_t$
<i>Independent Variables:</i>		
<i>Constant</i>	<i>Coefficient (‘$\hat{\alpha}$’)</i>	0.0780094
	<i>Standard Deviation</i>	(0.725731)
K_t	<i>Coefficient (‘$\hat{\beta}_1$’)</i>	-0.00931168
	<i>Standard Deviation</i>	(0.149858)
L_t	<i>Coefficient (‘$\hat{\beta}_2$’)</i>	0.0319070
	<i>Standard Deviation</i>	(0.173350)
\hat{A}_t	<i>Coefficient (‘$\hat{\beta}_3$’)</i>	0.975332 ***
	<i>Standard Deviation</i>	(0.148739)
<i>Statistical Fit Metrics:</i>		
	<i>Degrees of Freedom</i>	56
	R^2	82.9912 %
	<i>F-statistic</i>	91.08028

Conversely, Table D displays the results of the opposite exercise, namely, performing the same regression as in Table B, but including the fitted values from the one in Table A (represented by variable ‘ \hat{A}_t ’) as an input variable, and proves that, under these conditions, *only* the fitted values from that regression Table A retain their statistical significance as an explanatory variable. In short, these results reinforce the conclusions from Tests 1 and 2, i.e. *the model in this paper constitutes a better-fitting specification than the traditional Cobb-Douglas production function* and, furthermore, that *a hybrid combining the explanatory variables of both models would not provide any additional explanatory power.*

4.3.4 Test 4

Although the results of Tests 1, 2 and 3 are statistically robust and clear-cut, one could still wonder whether they should simply be interpreted as rejection of the Cobb-Douglas specification or, more fundamentally, of the significance of the variables it is built upon. The question is almost a moot point in the case of Tornquist aggregate labor (\bar{L}_t) because it is so highly correlated to the non-weighted sum of work hours (\bar{H}_t) – although the latter is statistically more closely correlated to GDP,¹¹ just as the model in this paper would predict. In the case of aggregate capital, conversely, the question of whether it is a better or worse measure of installed capacity than economic rents is relevant for two reasons:

- The production function proposed in this paper takes production capacity as a constraint, but lacks a mechanism to explain how this capacity evolves in the long run. To the extent capacity is built up through investment, one would expect today's net investments (i.e. changes in aggregate capital) to determine future capacity expansions.
- On the flip side, if (as the Cambridge Critique would predict) it makes no sense to measure capital input as a value-weighted capital aggregate, then one would expect to find that net investment plays no role in capacity expansion.

To test these alternative hypotheses, we perform a regression on an encompassing model (following the method proposed by Mizon and Richard 1986) including both the production function in (26) and the change in aggregate capital, lagged n periods. To cover a wide range of possibilities, the regression is repeated for aggregate capital lags ranging from zero (i.e. contemporary) to nine years, as shown in Table E.

The outcome is unambiguous. Both the rent ratio and the labor input variables retain their significance in every single case, while none of the regressions with lagged net investment provides any clear improvement in explanatory power (as measured by the R^2 and F statistics). Conversely, for net investments with up to seven years' lag not only is the coefficient negative in most of the cases, but in none of them is it possible to reject the null hypothesis of it being zero... and for

Table E: Results of OLS Regressions on a Comprehensive Model Including Both Rent Ratio and Lagged Net Investment

Analytical Expression:
$$\Delta \ln \bar{Y}_t = \alpha + \beta_1 \bar{\rho}_t + \beta_2 \frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t} + \beta_3 \Delta \ln \bar{K}_{t-n} + u_t$$

Lag (n)		$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	Degrees of Freedom	R^2	F
0	Coefficient	2.42695 ***	-0.160649 ***	0.993808 ***	-0.00263317	56	82.9809%	91.01400
	St. Dev.	(0.647701)	(0.0304137)	(0.0702052)	(0.156788)			
1	Coefficient	2.50365 ***	-0.160649 ***	0.993808 ***	-0.00263317	55	82.6823%	87.53122
	St. Dev.	(0.666359)	(0.0304137)	(0.0702052)	(0.156788)			
2	Coefficient	1.76982 ***	-0.149895 ***	1.00602 ***	0.127561	54	84.5758%	98.69962
	St. Dev.	(0.611189)	(0.0267011)	(0.0640098)	(0.134288)			
3	Coefficient	2.18125 ***	-0.150875 ***	0.988664 ***	0.0330864	53	83.8601%	91.79266
	St. Dev.	(0.606440)	(0.0271605)	(0.0649257)	(0.134746)			
4	Coefficient	2.14379 ***	-0.153336 ***	0.985094 ***	0.0471138	52	84.0698%	91.47464
	St. Dev.	(0.599357)	(0.0272911)	(0.0648228)	(0.134932)			
5	Coefficient	3.04774 ***	-0.154033 ***	0.985779 ***	-0.163779	51	84.4047%	92.00753
	St. Dev.	(0.616606)	(0.0272029)	(0.0647449)	(0.137464)			
6	Coefficient	2.99013 ***	-0.154498 ***	0.970040 ***	-0.141506	50	83.4339%	83.94044
	St. Dev.	(0.642998)	(0.0275939)	(0.0682558)	(0.141171)			
7	Coefficient	3.26846 ***	-0.146056 ***	0.951912 ***	-0.206147	49	83.1418%	80.55323
	St. Dev.	(0.653130)	(0.0278211)	(0.0687830)	(0.142637)			
8	Coefficient	3.62372 ***	-0.137721 ***	0.968504 ***	-0.282509 **	48	85.7281%	96.10870
	St. Dev.	(0.605256)	(0.0262653)	(0.0634209)	(0.134347)			
9	Coefficient	3.88621 ***	-0.138796 ***	0.984766 ***	-0.352919 **	47	86.2609%	98.36325
	St. Dev.	(0.603281)	(0.0256572)	(0.0630930)	(0.137118)			

lags of eight or nine years, bizarrely, the test indicates statistical significance for a *negative* coefficient: in other words, it suggests that investments made eight or nine years ago have over 95% probability of resulting in a *reduced* productive capacity today (!).

Two straightforward interpretations spring to mind: either there is really a mechanism by which net investment is not only irrelevant but even detrimental to production (which seems contrary to common sense) or using aggregate capital as a measure of non-labor input simply makes no sense, as the Cambridge Critique authors defended so long ago. Faced with this choice, one must let Occam's razor rule and conclude with the Cambridge (England) school that *value-weighted aggregate capital metrics play no measurable role in the macroeconomic production function*.

5 Summary and Conclusions

Already more than four decades ago, Fisher (1965, 1968, 1969) and Gorman (1968) proved that a macroeconomic production function cannot be built on the basis of just any set of individual production functions, even if behind them are rational, profit-maximizing agents. One can always add up individual production functions into an aggregation, to be sure, but then its aggregate input results from adding up individual input quantities weighted by their marginal costs, and these may have different productivities depending on which individual producer uses them. Additional assumptions are therefore required to develop a usable function.

One such assumption is perfect competition, i.e. the postulate that all marginal costs equal market prices. This postulate is central to the Cobb-Douglas function, and is often justified by supposing that, in the long run, most inputs are variable and so economic rents may be discounted as a mere form of short-term friction. The Cobb-Douglas function, however, poses a number of theoretical and empirical issues which suggest there is room for exploring other specifications. This paper has put forward an alternative based on the exact opposite assumption: that it is the "short term" that matters most to understand output fluctuations, and that, within this time horizon, only one major aggregate input (labor time) may be regarded as variable. From a theoretical perspective, this approach presents a number of advantages, among them not being subject to the Cambridge Critique.

Nevertheless, one could equally think of some strong theoretical points to support the Cobb-Douglas function, so this alone cannot drive the choice: to select a model instead of the other we need to resort to empirical data.

We perform this empirical analysis in Section 4, with the following results:

- The model proposed in this paper is empirically robust, and can explain nearly 85 percent of the U.S. GDP fluctuation from 1949 to 2008.
- Against the same data set, the Cobb-Douglas function proves to be a lot less robust, with substantially lower fit and likelihood metrics, and rejecting the hypothesis that the capital and labor elasticity parameters are equal to their shares of GDP with a confidence margin of over 95%.
- A non-nested model comparison (specifically, a Davidson/MacKinnon test) concludes that the model put forward in this paper represents a better specification than the Cobb-Douglas function and, furthermore, that a hybrid of the two would not add any meaningful explanatory power either.
- Last but not least, the estimated impact of capital, both contemporary and lagged, on production is nearly always negative (although the variance around it does not allow to exclude its simply being nil), which suggests that, as the Cambridge Critique postulated long ago, aggregate capital makes no sense as a measure of non-labor productive inputs.

Why does it matter which model we select? It matters because their implications are very different. A Cobb-Douglas world where output depends on the quantities of capital and labor and their marginal productivities, which also happen to be their prices, is very different from one where every income above the marginal product of basic unqualified labor constitutes a rent, and where these rents in turn determine how overall productivity behaves. In this latter world, for example, central bank policy would not require any sort of irrationality or monetary illusion to be effective as long as the balance of interest rates and credit flows acts as an economic rent and therefore drives the behavior of the aggregate production function. Conversely, in such a world it might not be licit to model the choice between investment and consumption as most DSGE papers do, for aggregate capital (i.e. accumulated net investment) could not be represented as a productive input... It would go well beyond the scope of this paper to analyze each one of these implications; but it is probably fair to say here that the predictions

from models based on the production function put forward in this paper would probably diverge substantially from the standard ones.

Acknowledgement

A very special thank you to Bernarda Zamora as well as two anonymous referees, for providing numerous and extremely helpful comments to this paper.

Appendices

Appendix 1

The purpose of this appendix is to prove Proposition 1 in Subsection 3.1.

Following Euler's homogeneous functions theorem, Assumption 4 implies that:

$$h_{j,t} = \sum_{i=1}^n \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} \frac{x_{i,j,t}}{Y_{j,t}} \quad (1.1)$$

Since producers fine-tune their demand for inputs in order to maximize their real profit (Assumption 1), if a finite maximum profit point exists (Assumption 2), and if the real output and cost functions are continuous and partially-differentiable (Assumption 3), then the real profit maximum must be such that, for every input $x_{i,j,t} \in \{x_{1,j,t} \dots x_{n,j,t}\}$, the following expression holds:

$$\begin{aligned} \frac{\partial \Pi_{j,t}^*}{\partial x_{i,j,t}} &= \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} - \frac{\partial C_{j,t}^*}{\partial x_{i,j,t}} = 0 \\ \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} &= \frac{\partial C_{j,t}^*}{\partial x_{i,j,t}} = c_{i,j,t}^* \end{aligned} \quad (1.2)$$

Which, replacing into the definition of rent ratio, yields:

$$\begin{aligned} \rho_{j,t} &\equiv \frac{Y_{j,t}}{c_{1,j,t}^* X_{j,t}} - 1 \\ X_{j,t} &\equiv \frac{1}{c_{1,j,t}^*} \sum_{i=1}^n c_{i,j,t}^* x_{i,j,t} \\ \frac{1}{1 + \rho_{j,t}} &\equiv c_{1,j,t}^* \frac{X_{j,t}}{Y_{j,t}} \equiv \sum_{i=1}^n c_{i,j,t}^* \frac{x_{i,j,t}}{Y_{j,t}} = \sum_{i=1}^n \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} \frac{x_{i,j,t}}{Y_{j,t}} \end{aligned} \quad (1.3)$$

Thus, by comparing expressions (1.1) and (1.3), we can conclude that:

$$h_{j,t} = \frac{1}{1 + \rho_{j,t}} \quad (1.4)$$

This implies that, since $h_{j,t}$ is independent of the input, so is $\rho_{j,t}$. Hence:

$$\begin{aligned} \frac{1}{1 + \rho_{j,t}} &\equiv c_{1,j,t}^* \frac{X_{j,t}}{Y_{j,t}} = \frac{\partial Y_{j,t}}{\partial x_{1,j,t}} \frac{X_{j,t}}{Y_{j,t}} = \frac{\partial Y_{j,t}}{\partial X_{j,t}} \frac{X_{j,t}}{Y_{j,t}} \\ \frac{\partial Y_{j,t}}{Y_{j,t}} &= \frac{1}{1 + \rho_{j,t}} \frac{\partial X_{j,t}}{X_{j,t}} \end{aligned} \quad (1.5)$$

And, integrating both sides, we obtain:

$$Y_{j,t} = A_{j,t} X_{j,t}^{\frac{1}{1+\rho_{j,t}}} \quad (1.6)$$

Where both the integration coefficient $A_{j,t}$ (which we will refer to as “productivity coefficient”) and the rent ratio $\rho_{j,t}$, are functions independent of $X_{j,t}$ and $Y_{j,t}$.

This is the exactly expression we introduced as Proposition 1 in Subsection 3.1.

Q.E.D.

Appendix 2

The purpose of this appendix is to show the development of a widely-used derivation of a production function’s degree of homogeneity (see for example Varian 1986 for a textbook example, or Basu and Fernald 2000 and 2002 for empirical applications), and so to highlight where its assumptions differ from those in Subsection 3.1.

We define the marginal cost respective to the output quantity $C'_{j,t} \equiv \frac{\partial C_{j,t}}{\partial Y_{j,t}}$.

Combining this with the definition of marginal cost respect to input

$c_{i,j,t} \equiv \frac{\partial C_{j,t}}{\partial x_{i,j,t}}$, if we adopt Assumption 3 (i.e. continuity and differentiability),

then the relationship between these two variables may be expressed as

$$c_{i,j,t} = C'_{j,t} \frac{\partial Y_{j,t}}{\partial x_{i,j,t}}.$$

Thus, if we also adopt Assumption 4 (i.e. homogeneity) in terms of expression (5) and combine it with the above we obtain:

$$h_{j,t} = \sum_{i=1}^n \frac{c_{i,j,t}}{C'_{j,t}} \frac{x_{i,j,t}}{Y_{j,t}} = \frac{1}{C'_{j,t} Y_{j,t}} \sum_{i=1}^n c_{i,j,t} x_{i,j,t} \quad (2.1)$$

It is common at this point to introduce an assumption to the effect that the cost function $C_{j,t}$ is homogeneous of degree one respective to the inputs $\{x_{1,j,t} \dots x_{n,j,t}\}$

so that $C_{j,t} = \sum_{i=1}^n c_{i,j,t} x_{i,j,t}$, and then to establish the following definitions:

- Mark-up ratio $m_{j,t} \equiv \frac{P_t}{C'_{j,t}}$
- Profit ratio as $\pi_{j,t} \equiv \frac{P_t Y_{j,t} - C_{j,t}}{P_t Y_{j,t}}$

Which allow to rewrite expression (2.1) in the more familiar form:

$$h_{j,t} = \frac{1}{C'_{j,t} Y_{j,t}} \underbrace{\sum_{i=1}^n c_{i,j,t} x_{i,j,t}}_{=C_{j,t}} = \frac{C_{j,t}}{C'_{j,t} Y_{j,t}} = \frac{C_{j,t}}{C'_{j,t} Y_{j,t}} \frac{P_t}{P_t} = \underbrace{\frac{P_t}{C'_{j,t}}}_{m_{j,t}} \underbrace{\frac{C_{j,t}}{P_t Y_{j,t}}}_{1-\pi_{j,t}}$$

$$h_{j,t} = m_{j,t} (1 - \pi_{j,t}) \quad (2.2)$$

Note, however, that, up to this point, we have resorted to Assumptions 3 and 4 (and also implicitly to Assumption 2, at least to the extent that, if no optimal point

existed, the whole exercise would be pointless) – yet Assumption 1 (i.e. maximization of the real profit function $\Pi_{j,t}^*$) has played no role so far. If we now formally introduce this assumption, then evidently:

$$\frac{\partial \Pi_{j,t}^*}{\partial Y_{j,t}} \equiv \frac{\partial}{\partial Y_{j,t}} (Y_{j,t} - C_{j,t}^*) \equiv 1 - C'_{j,t}^* = 0 \quad (2.3)$$

Which means that $m_{j,t} \equiv \frac{P_t}{C'_{j,t}} \equiv \frac{1}{C'_{j,t}^*} = 1$. On the other hand, since applying

definition (6) to the profit ratio leads to $\pi_{j,t} = \frac{\rho_{j,t}}{1 + \rho_{j,t}}$, the degree of

homogeneity must be $h_{j,t} = 1 - \pi_{j,t} = \frac{1}{1 + \rho_{j,t}}$. This is exactly the conclusion we reached in Proposition 1.

Q.E.D.

Appendix 3

The purpose of this appendix is to show the analytical derivation of Proposition 2 in Subsection 3.2.

As a first step, we need to differentiate the production function under the form introduced in expression (7), that is, $Y_{j,t} = A_{j,t} X_{j,t}^{1+\rho_{j,t}}$. According to Itô's lemma (i.e. the stochastic version of the chain rule) the differential of this expression is as follows (note that, due to its length, this expression spans over two lines of text):

$$\begin{aligned} dY_{j,t} = & \frac{\partial Y_{j,t}}{\partial t} dt + \frac{\partial Y_{j,t}}{\partial W_{j,t}} dW_{j,t} + \frac{\partial Y_{j,t}}{\partial X_{j,t}} dX_{j,t} + \frac{\partial Y_{j,t}}{\partial \rho_{j,t}} d\rho_{j,t} + \frac{1}{2} \frac{\partial^2 Y_{j,t}}{\partial W_{j,t}^2} (dW_{j,t})^2 + \frac{1}{2} \frac{\partial^2 Y_{j,t}}{\partial X_{j,t}^2} (dX_{j,t})^2 + \dots \\ & \dots + \frac{1}{2} \frac{\partial^2 Y_{j,t}}{\partial \rho_{j,t}^2} (d\rho_{j,t})^2 + \frac{\partial^2 Y_{j,t}}{\partial W_{j,t} \partial X_{j,t}} dW_{j,t} dX_{j,t} + \frac{\partial^2 Y_{j,t}}{\partial W_{j,t} \partial \rho_{j,t}} dW_{j,t} d\rho_{j,t} + \frac{\partial^2 Y_{j,t}}{\partial X_{j,t} \partial \rho_{j,t}} dX_{j,t} d\rho_{j,t} \end{aligned}$$

Which, as we know from Assumption 5 that all the differentials respective to ρ_t are zero, can be simplified into the following expression (still expanding over two lines):

$$\begin{aligned}
 dY_{j,t} &= \frac{\partial Y_{j,t}}{\partial t} dt + \frac{\partial Y_{j,t}}{\partial W_{j,t}} dW_{j,t} + \frac{\partial Y_{j,t}}{\partial X_{j,t}} dX_{j,t} + \frac{1}{2} \frac{\partial^2 Y_{j,t}}{\partial W_{j,t}^2} (dW_{j,t})^2 + \frac{1}{2} \frac{\partial^2 Y_{j,t}}{\partial X_{j,t}^2} (dX_{j,t})^2 + \dots \\
 &\dots + \frac{\partial^2 Y_{j,t}}{\partial W_{j,t} \partial X_{j,t}} dW_{j,t} dX_{j,t} \quad (3.1)
 \end{aligned}$$

If we now divide this formula by $Y_{j,t}$ and replace everything with its equivalent according to expression $Y_{j,t} = A_{j,t} X_{j,t}^{1+\rho_{j,t}}$, then (3.1) turns into the following (still requiring two lines of text):

$$\begin{aligned}
 \frac{dY_{j,t}}{Y_{j,t}} &= \frac{1}{A_{j,t}} \left(\frac{\partial A_{j,t}}{\partial t} dt + \frac{\partial A_{j,t}}{\partial W_{j,t}} dW_{j,t} + \frac{1}{2} \frac{\partial^2 A_{j,t}}{\partial W_{j,t}^2} (dW_{j,t})^2 \right) + \frac{1}{1+\rho_{j,t}} \frac{dX_{j,t}}{X_{j,t}} + \dots \\
 &\dots + \frac{1}{1+\rho_{j,t}} \frac{\partial A_{j,t}}{\partial W_{j,t}} \frac{dW_{j,t} dX_{j,t}}{A_{j,t} X_{j,t}} - \frac{\rho_{j,t}}{2(1+\rho_{j,t})^2} \left(\frac{dX_{j,t}}{X_{j,t}} \right)^2 \quad (3.2)
 \end{aligned}$$

Now, since we know from Assumption 5 that the perturbations on the increments of A_t are distributed according to a Wiener process, then we can also use Itô's lemma to decompose it as follows:

$$\frac{dA_{j,t}}{A_{j,t}} = \frac{1}{A_{j,t}} \left(\frac{\partial A_{j,t}}{\partial t} dt + \frac{\partial A_{j,t}}{\partial W_{j,t}} dW_{j,t} + \frac{1}{2} \frac{\partial^2 A_{j,t}}{\partial W_{j,t}^2} (dW_{j,t})^2 \right) = \gamma_{j,t} dt + s_{j,t} dW_{j,t} \quad (3.3)$$

Which, applied to expression (3.2), leads to:

$$\frac{dY_{j,t}}{Y_{j,t}} = \gamma_{j,t} dt + s_{j,t} dW_{j,t} + \frac{1}{1+\rho_{j,t}} \left(\frac{dX_{j,t}}{X_{j,t}} - \frac{\rho_{j,t}}{2(1+\rho_{j,t})} \left(\frac{dX_{j,t}}{X_{j,t}} \right)^2 \right) + s_{j,t} \frac{dW_{j,t} dX_{j,t}}{X_{j,t}} \quad (3.4)$$

Or, rearranging terms:

$$\frac{dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} = \frac{dY_{j,t}}{Y_{j,t}} - \gamma_{j,t}dt - s_{j,t}dW_{j,t} - s_{j,t} \frac{dW_{j,t}dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} + \frac{\rho_{j,t}}{2(1+\rho_{j,t})^2} \left(\frac{dX_{j,t}}{X_{j,t}} \right)^2 \quad (3.5)$$

If we replace $\frac{dY_t}{Y_t}$ per Assumption 7, this expression becomes:

$$\frac{dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} = E_t \left[\frac{dY_{j,t}}{Y_{j,t}} \right] + \sigma_{j,t}dV_{j,t} - \gamma_{j,t}dt - s_{j,t} \frac{dW_{j,t}dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} + \frac{\rho_{j,t}}{2(1+\rho_{j,t})^2} \left(\frac{dX_{j,t}}{X_{j,t}} \right)^2 \quad (3.6)$$

If we now square both sides of this expression and, as usual in Wiener processes, eliminate all terms $dt dW_{j,t}$, $dt dV_{j,t}$ or $dt dX_{j,t}$ as well as those under the form $(dt)^n$ or $(E_t[dY_{j,t}])^n$ or $(dX_{j,t})^{n+1}$ where $n > 1$, and then replace $(dV_{j,t})^2 = dt$, we obtain:

$$\left(\frac{dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} \right)^2 = \sigma_{j,t}^2 dt \quad (3.7)$$

Thus, replacing this back into expression (3.6) we obtain:

$$\frac{1}{1+\rho_{j,t}} \frac{dX_{j,t}}{X_{j,t}} = E_t \left[\frac{dY_{j,t}}{Y_{j,t}} \right] + \sigma_{j,t}dV_{j,t} - \gamma_{j,t}dt - s_{j,t} \frac{dW_{j,t}dX_{j,t}}{(1+\rho_{j,t})X_{j,t}} + \frac{\sigma_{j,t}^2}{2} \rho_{j,t}dt \quad (3.8)$$

Given that the stochastic component of $\frac{dX_{j,t}}{(1+\rho_{j,t})X_{j,t}}$ is $\sigma_{j,t}dV_{j,t}$, since Assumption 7 postulated that this variable would be independent from technology progress i.e. $dV_{j,t}dW_{j,t} = 0$, we can further simplify expression (3.8) into:

$$\frac{1}{1+\rho_{j,t}} \frac{dX_{j,t}}{X_{j,t}} = E_t \left[\frac{dY_{j,t}}{Y_{j,t}} \right] + \sigma_{j,t} dV_{j,t} - \gamma_{j,t} dt + \frac{\sigma_{j,t}^2}{2} \rho_{j,t} dt$$

$$\frac{dY_{j,t}}{Y_{j,t}} = \gamma_{j,t} dt + s_{j,t} dW_{j,t} - \frac{\sigma_{j,t}^2}{2} \rho_{j,t} dt + \frac{1}{1+\rho_{j,t}} \frac{dX_{j,t}}{X_{j,t}} \quad (3.9)$$

Which is equal to Proposition 2 in Subsection 3.2.

Q.E.D.

Appendix 4

The purpose of this appendix is to show the analytical derivation of Proposition 3 in Subsection 3.3.

For convenience we introduce the following definitions.

- The aggregate output (‘ \bar{Y}_t ’) as the sum of individual outputs (i.e. $\bar{Y}_t \equiv \sum_{j=1}^m Y_{j,t}$)
- The aggregate variable input (‘ \bar{X}_t ’) as the sum of inputs for the individual production units weighted by their marginal costs (i.e. $\bar{X}_t \equiv \sum_{j=1}^m X_{j,t}$)
- The average rent coefficient (‘ $\bar{\rho}_t$ ’) as the ratio $\bar{\rho}_t \equiv \frac{\bar{Y}_t}{c_{1,t}^* \bar{X}_t} - 1$
- The relative output share (‘ $\alpha_{j,t}$ ’) of a production unit j as the percentage of its production over the total (i.e. $\alpha_{j,t} \equiv \frac{Y_{j,t}}{\bar{Y}_t}$)
- The combined input share (‘ $\beta_{j,t}$ ’) of a production unit j as the percentage of its combined input consumption over the total (i.e. $\beta_{j,t} \equiv \frac{X_{j,t}}{\bar{X}_t}$)

Combining Assumption 8 with the identity $Y_{j,t} \equiv (1 + \rho_{j,t})c_{1,j,t}^* X_{j,t}$ we obtain that:

$$\frac{Y_{j,t}}{(1 + \rho_{j,t})X_{j,t}} = \bar{c}_{1,t}^* = \frac{\bar{Y}_t}{(1 + \bar{\rho}_t)\bar{X}_t} \quad (4.1)$$

Where $\bar{c}_{1,t}^*$ represents the marginal cost for any of the production units (since, according to Assumption 8, it is the same for all of them).

Rearranging terms we can also rewrite this expression as the following:

$$1 + \rho_{j,t} = (1 + \bar{\rho}_t) \frac{Y_{j,t}}{\bar{Y}_t} \frac{\bar{X}_t}{X_{j,t}} = (1 + \bar{\rho}_t) \frac{\alpha_{j,t}}{\beta_{j,t}} \quad (4.2)$$

Hence, if we replace $\rho_{j,t}$ into expression (11) (i.e. Proposition 2) we obtain:

$$\frac{dY_{j,t}}{Y_{j,t}} = \left(\gamma_{j,t} + s_{j,t} \frac{dW_{j,t}}{dt} - \frac{\sigma_{j,t}^2}{2} \left((1 + \bar{\rho}_t) \frac{\alpha_{j,t}}{\beta_{j,t}} - 1 \right) \right) dt + \frac{\bar{Y}_t}{1 + \bar{\rho}_t} \frac{dX_{j,t}}{\bar{X}_t Y_{j,t}} \quad (4.3)$$

If we now multiply both sides of the equation by $\alpha_{j,t}$ and aggregate for all $j = 1 \dots m$:

$$\begin{aligned} \sum_{j=1}^m \frac{dY_{j,t}}{\bar{Y}_t} &= \sum_{j=1}^m \alpha_{j,t} \left(\gamma_{j,t} + s_{j,t} \frac{dW_{j,t}}{dt} - \frac{\sigma_{j,t}^2}{2} \left((1 + \bar{\rho}_t) \frac{\alpha_{j,t}}{\beta_{j,t}} - 1 \right) \right) dt + \sum_{j=1}^m \frac{1}{1 + \bar{\rho}_t} \frac{dX_{j,t}}{\bar{X}_t} \\ \frac{d\bar{Y}_t}{\bar{Y}_t} &= \sum_{j=1}^m \alpha_{j,t} \left[\gamma_{j,t} + s_{j,t} \frac{dW_{j,t}}{dt} - \frac{\sigma_{j,t}^2}{2} \left((1 + \bar{\rho}_t) \frac{\alpha_{j,t}}{\beta_{j,t}} - 1 \right) \right] dt + \frac{1}{1 + \bar{\rho}_t} \frac{d\bar{X}_t}{\bar{X}_t} \end{aligned} \quad (4.4)$$

To further simplify the notation, we introduce now the following definitions:

- The aggregate technology perturbation $\bar{s}_t d\bar{W}_t \equiv \sum_{j=1}^m \alpha_{j,t} s_{j,t} dW_{j,t}$, which obviously follows a Gaussian distribution i.e. $\bar{s}_t d\bar{W}_t \sim N(0, \bar{s}_t^2)$ and therefore $d\bar{W}_t \sim N(0, 1)$

- The overall non-technology variance $\bar{\sigma}_t^2 dt \equiv \left(\frac{1}{1+\bar{\rho}_t} \frac{d\bar{X}_t}{\bar{X}_t} \right)^2$ i.e. the equivalent of expression (3.7) in Appendix 3, only calculated here at a macroeconomic level
- The aggregate expected technology growth rate $\bar{\gamma}_t \equiv \sum_{j=1}^m \alpha_{j,t} \gamma_{j,t}$
- The composite rate $\Gamma_{j,t} \equiv \gamma_{j,t} + \frac{\sigma_{j,t}^2 - \delta_{j,t}}{2}$ and its aggregate $\bar{\Gamma}_t \equiv \sum_{j=1}^m \alpha_{j,t} \Gamma_{j,t}$
- The composite coefficient $\delta_{j,t} \equiv \sigma_{j,t}^2 \frac{\alpha_{j,t}}{\beta_{j,t}}$ and its aggregate $\bar{\delta}_t \equiv \sum_{j=1}^m \alpha_{j,t} \delta_{j,t}$

Applying these definitions, expression (4.4) can be rewritten as follows:

$$\frac{d\bar{Y}_t}{\bar{Y}_t} = \bar{\Gamma}_t dt + \bar{s}_t d\bar{W} - \frac{\bar{\delta}_t}{2} \bar{\rho}_t dt + \frac{1}{1+\bar{\rho}_t} \frac{d\bar{X}_t}{\bar{X}_t} \quad (4.5)$$

Which is equal to Proposition 3 in Subsection 3.3.

Q.E.D.

Appendix 5

The purpose of this appendix is to display the dataset supporting the empirical tests in Section 4.

Year	<i>Source: U.S. Bureau of Labor Statistics</i>				<i>Source: Federal Reserve Bank of St. Louis</i>			<i>Calculated Variables</i>	
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
	Real Value-Added Output Growth Rate	Sum of Work Hours of All Persons, Growth Rate	Aggregate Labor Growth (Tornquist Aggregate)	Aggregate Capital Growth Rate	Federal Reserve Prime Rate on January 1	Money Supply (M2) on January 1	Money Supply (M2) on February 1	Rent Ratio ¹²	Work Hours Growth Weighted by Rent Ratio ¹³
t	$\Delta \ln \bar{Y}_t$	$\Delta \ln \bar{H}_t$	$\Delta \ln \bar{L}_t$	$\Delta \ln \bar{K}_t$	-	-	-	$\bar{\rho}_t$	$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$
1949	-0.7	-4.3	-4.2	2.9	2.00	171.5	171.7	0.5	-4.3
1950	10.2	3.1	3.6	4.1	2.00	174.6	175.5	-4.4	3.2
1951	7.6	4.8	4.8	5.4	2.44	182.3	182.6	0.4	4.8
1952	2.9	0.9	1.7	3.4	3.00	193.7	194.8	-4.0	0.9
1953	4.9	2.5	3.3	3.3	3.00	205.9	206.6	-1.2	2.5
1954	-1.6	-3.6	-3.0	2.4	3.25	216.6	217.6	-2.4	-3.7
1955	8.5	4.0	4.2	3.7	3.00	230.8	232.4	-5.6	4.2
1956	1.7	2.5	2.6	4.1	3.50	241.7	242.3	1.0	2.5
1957	2.0	-0.7	-0.1	3.3	4.00	252.7	253.7	-0.8	-0.7
1958	-2.2	-4.4	-4.3	2.3	4.34	263.2	265.7	-7.5	-4.8
1959	8.7	4.5	4.6	2.6	4.00	286.6	287.7	-0.7	4.5
1960	1.7	0.4	0.4	3.1	5.00	298.2	298.5	3.8	0.4
1961	2.0	-1.4	-0.4	2.6	4.50	314.1	316.5	-5.0	-1.5

Source: U.S. Bureau of Labor Statistics

Source: Federal Reserve Bank of St. Louis

Calculated Variables

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
Year	Real Value-Added Output Growth Rate	Sum of Work Hours of All Persons, Growth Rate	Aggregate Labor Growth (Tornquist Aggregate)	Aggregate Capital Growth Rate	Federal Reserve Prime Rate on January 1	Money Supply (M2) on January 1	Money Supply (M2) on February 1	Rent Ratio ¹⁴	Work Hours Growth Weighted by Rent Ratio ¹⁵
t	$\Delta \ln \bar{Y}_t$	$\Delta \ln \bar{H}_t$	$\Delta \ln \bar{L}_t$	$\Delta \ln \bar{K}_t$	-	-	-	$\bar{\rho}_t$	$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$
1962	6.9	2.2	3.3	3.3	4.50	337.5	340.1	-5.1	2.3
1963	4.7	1.0	1.2	3.7	4.50	365.2	367.9	-4.7	1.0
1964	6.8	2.4	2.4	4.0	4.50	395.2	397.6	-3.0	2.5
1965	7.1	3.8	3.7	5.0	4.50	427.5	430.4	-3.9	4.0
1966	7.2	3.5	3.3	6.1	5.00	462.0	464.6	-1.9	3.6
1967	1.7	-0.1	-0.2	5.9	5.96	481.6	485.1	-3.1	-0.1
1968	5.3	2.1	1.8	4.6	6.00	527.4	530.4	-1.0	2.1
1969	3.0	2.9	3.3	5.2	6.95	569.3	571.9	1.3	2.9
1970	-0.1	-1.7	-1.3	4.9	8.50	589.6	586.3	15.0	-1.5
1971	3.8	-0.4	-0.7	4.2	6.29	633.0	641.0	-9.9	-0.4
1972	6.8	3.5	3.4	4.5	5.18	717.7	725.7	-9.0	3.8
1973	7.4	4.0	3.9	5.7	6.00	810.3	814.1	0.2	4.0
1974	-1.4	0.0	0.6	5.5	9.73	859.7	864.2	3.3	0.0
1975	-1.7	-4.4	-4.4	3.9	10.05	906.3	914.1	-0.7	-4.4
1976	7.1	3.4	3.1	3.2	7.00	1026.6	1040.3	-10.1	3.8

Source: U.S. Bureau of Labor Statistics

Source: Federal Reserve Bank of St. Louis

Calculated Variables

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
Year	Real Value-Added Output Growth Rate	Sum of Work Hours of All Persons, Growth Rate	Aggregate Labor Growth (Tornquist Aggregate)	Aggregate Capital Growth Rate	Federal Reserve Prime Rate on January 1	Money Supply (M2) on January 1	Money Supply (M2) on February 1	Rent Ratio ¹⁶	Work Hours Growth Weighted by Rent Ratio ¹⁷
t	$\Delta \ln \bar{Y}_t$	$\Delta \ln \bar{H}_t$	$\Delta \ln \bar{L}_t$	$\Delta \ln \bar{K}_t$	-	-	-	$\bar{\rho}_t$	$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$
1977	5.7	4.1	4.2	3.9	6.25	1165.2	1177.6	-7.2	4.4
1978	6.6	5.2	5.2	4.6	7.93	1279.7	1285.5	2.4	5.1
1979	3.2	3.5	3.2	5.5	11.75	1371.6	1377.8	6.2	3.3
1980	-1.1	-0.9	-0.5	5.5	15.25	1482.7	1494.5	5.3	-0.9
1981	2.2	0.7	1.5	5.4	20.16	1606.9	1618.7	11.0	0.6
1982	-3.1	-2.3	-1.3	4.6	15.75	1770.4	1774.5	12.9	-2.0
1983	6.6	1.9	2.3	3.8	11.16	1962.4	1999.6	-13.9	2.2
1984	8.3	6.1	6.2	5.0	11.00	2140.4	2160.5	-0.8	6.1
1985	4.3	2.6	2.9	5.4	10.61	2334.9	2356.4	-1.0	2.6
1986	3.9	0.8	1.4	4.8	9.50	2504.8	2515.3	4.4	0.8
1987	3.6	3.1	3.2	4.1	7.50	2747.1	2750.8	5.9	2.9
1988	4.6	2.9	3.5	3.7	8.75	2851.8	2874.8	-1.3	2.9
1989	3.5	2.8	3.3	3.9	10.50	2997.2	2997.6	10.3	2.5
1990	1.4	-0.4	0.1	3.4	10.11	3172.4	3184.8	5.3	-0.4
1991	-0.9	-2.5	-1.1	2.9	9.52	3292.7	3309.5	3.2	-2.4

Source: U.S. Bureau of Labor Statistics

Source: Federal Reserve Bank of St. Louis

Calculated Variables

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
Year	Real Value-Added Output Growth Rate	Sum of Work Hours of All Persons, Growth Rate	Aggregate Labor Growth (Tornquist Aggregate)	Aggregate Capital Growth Rate	Federal Reserve Prime Rate on January 1	Money Supply (M2) on January 1	Money Supply (M2) on February 1	Rent Ratio ¹⁸	Work Hours Growth Weighted by Rent Ratio ¹⁹
t	$\Delta \ln \bar{Y}_t$	$\Delta \ln \bar{H}_t$	$\Delta \ln \bar{L}_t$	$\Delta \ln \bar{K}_t$	-	-	-	$\bar{\rho}_t$	$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$
1992	3.8	-0.2	1.0	2.5	6.50	3385.7	3404.7	-0.4	-0.2
1993	3.6	2.9	3.3	3.2	6.00	3424.7	3420.4	7.5	2.7
1994	4.7	3.6	4.1	3.7	6.00	3482.0	3482.7	5.8	3.4
1995	3.3	2.8	2.7	4.5	8.50	3502.2	3499.5	9.4	2.6
1996	4.4	1.8	2.2	4.9	8.50	3659.0	3673.0	3.8	1.7
1997	5.1	3.6	4.0	5.5	8.25	3833.3	3844.6	4.7	3.4
1998	5.2	2.2	2.4	6.6	8.50	4055.9	4088.0	-1.4	2.2
1999	5.6	2.3	2.6	7.1	7.75	4398.1	4423.4	0.6	2.3
2000	4.5	1.0	1.2	6.8	8.50	4656.4	4673.6	4.0	1.0
2001	1.0	-2.1	-1.6	5.1	9.05	4964.3	4999.9	0.1	-2.1
2002	1.9	-2.6	-2.0	3.5	4.75	5446.7	5473.1	-1.2	-2.6
2003	3.1	-0.6	-0.4	2.9	4.25	5789.6	5826.2	-3.6	-0.6
2004	4.2	1.3	1.2	2.4	4.00	6058.9	6096.9	-3.8	1.4
2005	3.4	1.7	1.9	3.3	5.25	6402.7	6415.7	2.8	1.7
2006	3.2	2.2	2.5	3.4	7.26	6709.8	6729.3	3.7	2.1

	<i>Source: U.S. Bureau of Labor Statistics</i>				<i>Source: Federal Reserve Bank of St. Louis</i>			<i>Calculated Variables</i>	
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
Year	Real Value-Added Output Growth Rate	Sum of Work Hours of All Persons, Growth Rate	Aggregate Labor Growth (Tornquist Aggregate)	Aggregate Capital Growth Rate	Federal Reserve Prime Rate on January 1	Money Supply (M2) on January 1	Money Supply (M2) on February 1	Rent Ratio ²⁰	Work Hours Growth Weighted by Rent Ratio ²¹
t	$\Delta \ln \bar{Y}_t$	$\Delta \ln \bar{H}_t$	$\Delta \ln \bar{L}_t$	$\Delta \ln \bar{K}_t$	-	-	-	$\bar{\rho}_t$	$\frac{\Delta \ln \bar{H}_t}{1 + \bar{\rho}_t}$
2007	2.4	0.5	1.2	3.3	8.25	7067.1	7073.8	7.1	0.5
2008	-0.2	-2.2	-1.7	2.8	6.98	7461.7	7536.9	-5.7	-2.3

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