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## Agame-theoretical approach to network capacity planning under competition

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**Title: A Game-Theoretical Approach to Network Capacity Planning  
under Competition**

**Abstract:** The paper discusses the dimensioning strategies of two network providers (operators) that supply channels to the same population of users in a competitive environment. Users are assumed to compete for best service (lowest blocking probability of new request), while operators wish to maximize their profits. This setting gives rise to two interconnected, non-cooperative games: a) a users game, in which the partition of primary traffic between operators is determined by the operators' channel capacities and by the users' blocking-avoidance strategy; and b) a network dimensioning game between operators in which the players alternate dimensioning decisions that maximize their profit rate under the current channel capacity of his/her opponent. At least for two plausible users' blocking avoidance strategies discussed in the paper, the users game will always reach some algorithmic equilibrium. In the operators' game, the player strategies are given by their numbers of deployed channels, limited by their available infrastructure resources. If the infrastructure is under-dimensioned with respect to the traffic rate, the operators game will reach a Nash equilibrium when both players reach full use of their available infrastructures. Otherwise, a Nash equilibrium may also arise if both operators incur the same deployment costs. If costs are asymmetric, though, the alternating game may enter a loop. If the asymmetry is modest, both players may then try to achieve a competitive monopoly in which the opponent is forced to leave the game or operate with a loss (negative profit). However, if the asymmetry is high enough, only the player with the lower costs can force his opponent to leave the game while still holding a profitable operation.

**JEL codes:** C72 – Noncooperative Games

**Keywords:** network dimensioning, game theory, duopoly, Nash equilibrium, circuit switching, blocking probability.

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# A Game-Theoretical Approach to Network Capacity Planning under Competition

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## I - Introduction

Competition in telecommunication networks, particularly in their core, raises questions regarding their dimensioning, costs and profitability in an uncertain environment [1]. Such questions were ignored in the classical approach to dimension a network, which consisted of deploying a barely sufficient number of channels to assure a small enough probability of a call being blocked [2].

Clearly, the classic approach assumed that a single channel provider (operator) was allowed to operate, thus precluding any competition. If more than one operator were allowed to operate, no single operator could be held responsible for the probability of a user being blocked by all operators, hereby called the hard blocking probability. If users are free to request a channel from any operator, then both the hard blocking probability and the blocking probabilities of each operator would depend only on the numbers of channels made available to the users by all operators.

Given the competitive business environment of current networks, a new approach is now needed to discuss the dimensioning of networks that share a physical infrastructure but compete for customers in a channel provisioning market. In this paper, we propose a game-theoretical approach to this problem. The underlying games are played by users that wish to minimize their blocking probabilities and operators that wish to maximize their profits.

In [3], we have shown that profit maximization may be compatible with acceptable blocking performance for high enough traffic intensities, even under a monopolistic setting. We now discuss a situation in which two operators provide channels on the same link to a large population of users. Any new request for a channel will always be submitted by the user to its primary operator. Whenever such request is blocked by the primary operator, it will then be submitted by the user to another (secondary) operator. For this reason, blocking of a request by a primary operator is hereby called soft blocking.

Given the total traffic generated by the user population, the hard blocking probability depends only on the sum of the channels provided by all operators. Therefore, all other service attributes being the same, users are likely to choose a primary operator with minimal soft blocking probability. We assume that users play a

game in which they occasionally switch their primary operator in search of lower blocking probabilities of their future requests.

In Section II, we introduce a Markovian model that yields the soft blocking probabilities of two operators of a duopoly, given their numbers of deployed channels and the intensities of the primary traffic bound to each one. In Section III, we let the primary traffic intensities float under a given users game until they reach equilibrium, thus obtaining the primary traffic partition between the operators under such game. Based on such equilibrium traffic partitions, Section IV analyzes a channel capacity dimensioning game between two profit-seeking operators and variations thereof that may arise from a motivation to stabilize the game. Finally, Section V ends the paper with concluding remarks.

## II – A Soft Blocking Model

In [5], we have proposed a Markovian model that captures the behaviors described above and yields the resulting soft blocking probabilities of each operator when the intensities of the primary traffic bound to each one are given. Let us consider the case of a duopoly run by operators 1 and 2. Let  $\nu_m$  be the traffic intensity of the primary requests addressed to operator  $m$  and let  $C_m$  be the number of channels deployed by operator  $m$ . In a duopoly,  $m \in \{1,2\}$ , so the total traffic intensity generated by the users is given by:

$$\nu = \nu_1 + \nu_2, \quad (1)$$

and the total number of channels made available by all operators is:

$$C = C_1 + C_2. \quad (2)$$

Fig. 1a illustrates the proposed Markov chain for a duopoly when  $(C_1, C_2) = (3,2)$ . The system is in state  $(i,j)$  when operator 1 has  $i \leq C_1$  active channels and operator 2 has  $j \leq C_2$  active channels. Notice that the transition rate from state  $(i,j)$  to state  $(i+1,j)$  is given by  $\nu_1$  only when operator 2 is not in a blocking state ( $j < C_2$ ), and by  $\nu$  when operator 2 is in a blocking state ( $j = C_2$ ), so that its primary traffic is forwarded to operator 1. Analogously, the transition rate from state  $(i,j)$  to state  $(i,j+1)$  is given by  $\nu_2$  when operator 1 is not in a blocking state ( $i < C_1$ ) and by  $\nu$  when operator 1 is in a blocking state ( $i = C_1$ ). The downward transition rate from  $(i,j)$  to  $(i-1,j)$  is given by  $i$  for any positive  $i$ , and from  $(i,j)$  to  $(i,j-1)$  by  $j$  for any positive  $j$ , reflecting the standard assumption that all services are independent, exponentially distributed processes with unit mean, so that traffic rates are expressed in Erlang. All remaining transition rates are zero because all requests are assumed to demand single channels.

The steady-state probability  $p_{i,j}$  of each state  $(i,j)$  of the system may then be obtained from standard Markovian analysis [4]. If  $P_{bi}$  is the soft blocking probability of operator  $i$ , the soft blocking probabilities of operators 1 and 2 may be expressed as:

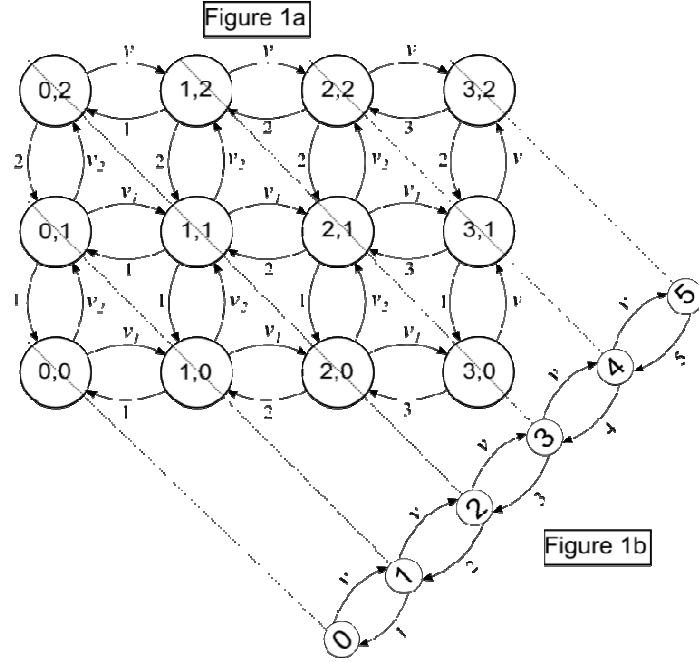
$$P_{b1} = \sum_{j=0}^{C_2} p_{C_1,j} \quad (3)$$

$$P_{b2} = \sum_{i=0}^{C_1} p_{i,C_2} \quad (4)$$

Hard blocking will occur whenever a request finds the system in state  $(C_1, C_2)$ . Therefore, the hard blocking probability is given by:

$$P_b = p_{C_1, C_2} \cdot \quad (5)$$

Let  $U_k$  be the set of all states where  $i + j = k$ . Then, it can be seen from Fig. 1a that the sum of all transition rates from any state in  $U_k$  to all states of  $U_{k+1}$  is  $v$ , and the sum of all rates from any state in  $U_k$  to all states of  $U_{k-1}$  is  $k$ . Therefore, a Markov process is obtained when all states in each set  $U_k$  are lumped into one state as shown in Fig. 1b.



**Figure 1** - (a) Markov blocking model for a duopoly; (b) classical blocking model.

This process yields the hard blocking probability in the form of the classical Erlang-B equation:

$$P_b = \text{prob}(k = C) = p_{C_1, C_2} = \frac{v^C / C!}{\sum_{k=0}^C v^k / k!} \quad (6)$$

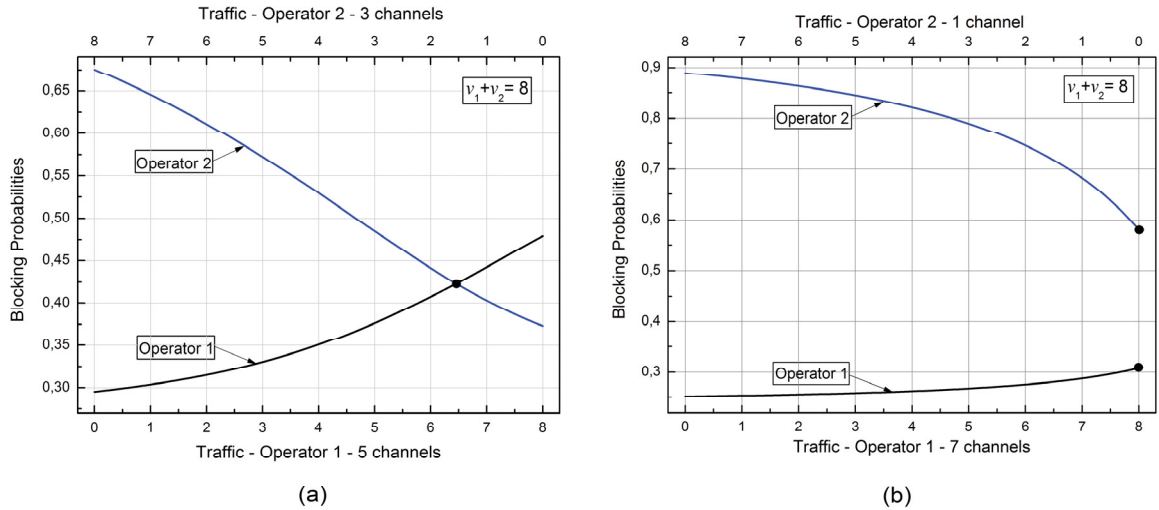
### III – Users games

Section II might be sufficient to describe a situation in which the operators were allocated to the users by a third party such as a broker. However, if users are free to choose their primary operators, the primary traffic partition  $(v_1, v_2)$  will result from the outcome of a game played by them. Let us start with a naïve approach, in which we assume that users have somehow access to information about the soft blocking probability of each operator or are willing to estimate its value by occasionally testing their secondary operator. Reliable estimates may require many requests, and thus a long time to be obtained, especially if the target blocking probability is low. After looking up this information or performing this test, the user will switch his primary operator if and only if he finds that the other (secondary) operator has a lower soft blocking probability.

In subsections III.1 and III.2, we discuss equilibrium conditions generated by two non-cooperative games, in which each user decides independently to remain with his current primary operator or switch to another one. In each case, a different user strategy is assumed for all users. In subsection III.3, we discuss a hypothetical situation in which all users cooperate to minimize the global rate of soft blockings.

**III.1 – A Nash equilibrium.** Using the model introduced in Section II, the soft blocking probabilities of the two operators were calculated as a function of  $v_1$  when  $(C_1, C_2) = (5, 3)$  and  $v = 8$ , and are shown on Fig. 2a. Notice that the two curves cross at a point where  $P_{b1} = P_{b2}$ . At the left of this point,  $P_{b1} < P_{b2}$ , so a primary customer of operator 2, when testing operator 1, would find a lower soft blocking probability and would switch his primary operator. This move would increase the primary traffic  $v_1$  bound to operator 1, so the system operating point would move slightly to the right, thus approaching the crossing point. Analogously, customers of operator 1 would switch to operator 2 whenever the system operates at the right of the crossing point, thus moving the operating point leftwards and closer to the crossing point. We conclude that the crossing point yields an algorithmic equilibrium where  $P_{b1} = P_{b2}$ , so that no user would be motivated to switch his primary operator for the purpose of minimizing his soft blocking probability. In game-theory parlance, this situation defines Nash equilibrium.

Fig. 2b shows the soft blocking probabilities of the two operators as a function of  $v_1$  when  $(C_1, C_2) = (7, 1)$  and  $v = 8$ . In this case, the two curves do not cross within the feasible range of partitions. For any  $v_1 \in [0, 8)$ , primary users of operator 2 would switch to operator 1 whenever testing their soft blocking probabilities. Therefore, Nash equilibrium occurs for  $(v_1, v_2) = (8, 0)$ , with  $P_{b1} < P_{b2}$ .



**Figure 2 - Soft blocking probabilities when: (a)  $(C_1, C_2) = (5, 3)$ ; (b)  $(C_1, C_2) = (7, 1)$ .**

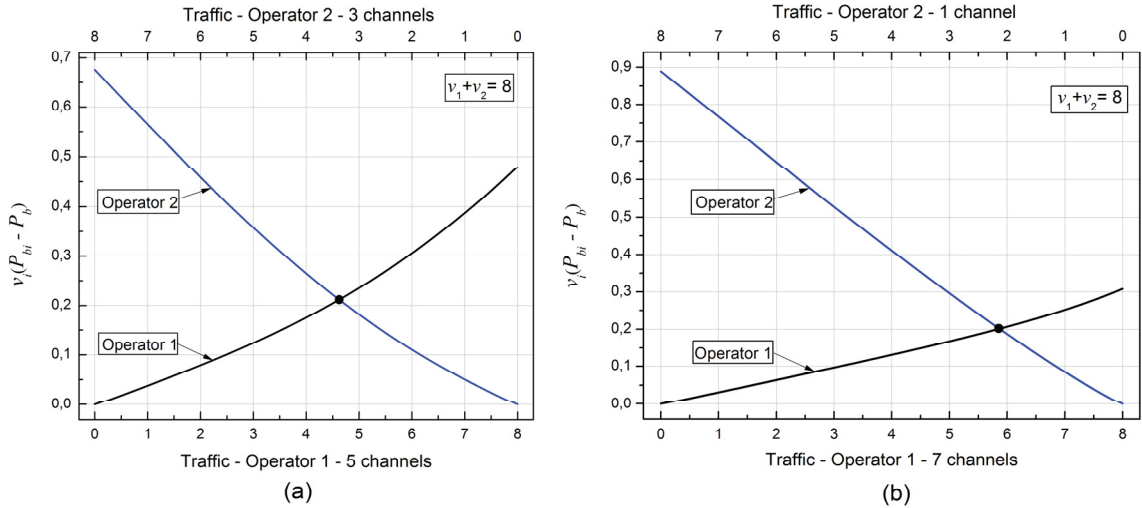
**III.2 – An equilibrium generated by impatient user behavior.** Information about the soft blocking probabilities of the operators is not likely to be available for consultation. Moreover, reliable estimation of these parameters may take a long time, prompting the users to switch operators on the basis of partial information. Taking an extreme behavior for comparison, we may consider the case of the impatient user, who



switches operators whenever having a request blocked by his current primary operator and accommodated by the other operator, which then becomes the new primary operator. If all users are impatient, then the rate of user switching events from operator 1 to 2 is given by  $\sigma_1 = v_1(P_{b1} - P_b)$ , while the rate of user switching events from operator 2 to 1 is given by  $\sigma_2 = v_2(P_{b2} - P_b)$ . Equilibrium will emerge when  $\sigma_1 = \sigma_2$ , or::

$$v_1(P_{b1} - P_b) = v_2(P_{b2} - P_b) \quad (7)$$

Figs. 3a and 3b show the variations of  $\sigma_1$  and  $\sigma_2$  with  $v_1$  for the same two cases discussed for Nash equilibrium in subsection III.1. Notice that the two curves will always cross now, and the crossing point defines the equilibrium generated by impatient users. This means that the operator with the smaller number of channels will always capture some primary traffic, but will always offer a higher soft blocking probability. The new equilibrium is not fair in the short term because some users will feel a higher soft blocking probability than others even in cases where Nash equilibrium would make all users have the same soft blocking probability. However, user impatience would still be fair in the long term, since all users would stay with each operator for some time, with the same mean sojourn time at each one for all users.



**Figure 3** – Rates of impatient user switching events when: (a)  $(C_1, C_2) = (5, 3)$ ; (b)  $(C_1, C_2) = (7, 1)$ .

III.3 – *Cooperative welfare maximization.* The users might also be collectively interested in cooperating to produce an “efficient” traffic partition with minimal aggregate rate of soft blockings. The total rate of soft blockings is:

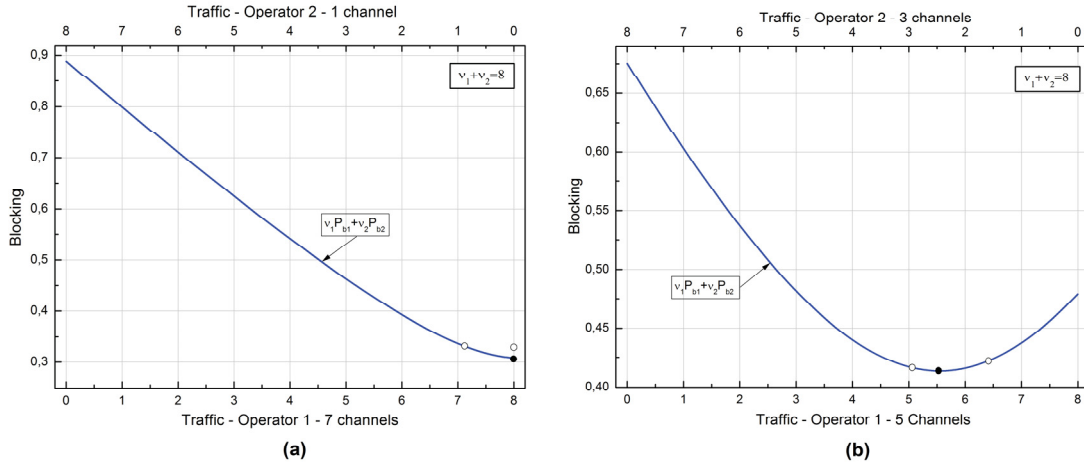
$$\sigma = \sigma_1 + \sigma_2 = v_1 P_{b1} + v_2 P_{b2} - v P_b \quad (8)$$

Given the total traffic  $v$  generated by all users, the total rate of hard blockings  $v P_b$  is invariant with respect to  $v_1$ . Therefore, efficiency is produced by the minimization of the total rate of primary blockings:

$$\pi = v_1 P_{b1} + v_2 P_{b2} \quad (9)$$

Fig. 4 plots the variations of  $\pi$  with  $v_1$  for the same two cases discussed in subsections III.1 and III.2. The minimal value of  $\pi$  is marked with a full dot. Empty

dots mark the operation points of the equilibria generated by the two non-cooperative games discussed in previous subsections. When  $(C_1, C_2) = (7, 1)$ , Nash equilibrium is efficient and impatient user behavior is not. If  $(C_1, C_2) = (5, 3)$ , however, the most efficient partition is somewhere in between the partitions generated by the two user behaviors discussed above. This suggests that there might be some moderately impatient user behavior resulting in the efficient partition in this case. Alternatively, efficiency may be obtained by enforcing the efficient partition. This could be done by submitting all requests to a broker, who would assign each request independently to operator 1 with probability  $(\frac{v_{1eff}}{v})$  and to operator 2 with probability  $(1 - \frac{v_{1eff}}{v})$ , where  $v_{1eff}$  is the value of  $v_1$  that minimizes  $\pi$ .



**Figure 4** – Total rate of primary blockings when: (a)  $(C_1, C_2) = (7, 1)$ ; (b)  $(C_1, C_2) = (5, 3)$ .

#### IV – Operators games

We normalize all revenues and costs with respect to the revenue generated per unit time by any active channel, which is then taken to be 1. The total cost incurred by operator  $m$  to deploy a channel per unit time is given by  $s_m < 1$ . All channels deployed by the same operator are assumed to generate the same cost, but only active channels generate revenue at any given time. When the system is in state  $(i, j)$ , the instantaneous profit rates of operators 1 and 2 are then given respectively by:

$$T_1 = i - s_1 \cdot C_1 \quad (10)$$

$$T_2 = j - s_2 \cdot C_2 \quad (11)$$

The mean profit rates, or profitabilities, of operators 1 and 2 are then given respectively by:

$$R_1 = E(i) - s_1 C_1 \quad (12)$$

$$R_2 = E(j) - s_2 C_2 \quad (13)$$

From Little's Law [7], we know that the mean number of ongoing services in operator 1 is its total rate (in Erlang) of incoming traffic. From an inspection of Fig. 1a, we then get:

$$E(i) = v_1(1 - P_{b1}) + v_2(P_{b2} - P_b). \quad (14)$$



Likewise for operator 2:

$$E(j) = v_2(1 - P_{b2}) + v_1(P_{b1} - P_b) \quad (15)$$

The profitabilities of operators 1 and 2 may then be expressed as:

$$R_1 = v_1(1 - P_{b1}) + v_2(P_{b2} - P_b) - s_1 C_1 \quad (16)$$

$$R_2 = v_2(1 - P_{b2}) + v_1(P_{b1} - P_b) - s_2 C_2 \quad (17)$$

We consider a game in which each operator  $m$  chooses a number  $C_m$  of channels to be deployed with the purpose of maximizing its profitability, or payoff,  $R_m$ . For each strategy profile  $(C_1, C_2)$ , the aggregate traffic  $v$  generated by the user population will face a hard blocking probability  $P_b$  determined by  $C = C_1 + C_2$  as in Eq. 6. A users game will determine the primary traffic partition  $(v_1, v_2)$  and the soft blocking probabilities  $P_{b1}$  and  $P_{b2}$ , so the profitabilities may be calculated from Eqs. 16 and 17.

Table 1 shows the normal form of the operators game when  $v = 8$ ,  $s_1 = s_2 = .2$  (symmetric costs), and users are free to minimize their individual soft blocking probabilities in a non-cooperative game. Rows are indexed by  $C_1$  and columns are indexed by  $C_2$ . Maximal values for  $C_1$  and  $C_2$  are set by the buried infrastructures of each operator. Each cell shows the corresponding value of  $(R_1, R_2)$ . For the sake of readability, the full accuracy of the numbers is not shown. A Nash equilibrium is said to be generated by a pure strategy profile  $(C_1, C_2)$  if the corresponding cell in the normal form shows a value of  $R_1$  that is maximal in its column and a value of  $R_2$  that is maximal in its row. Under this condition, no operator will be motivated to change his strategy if it believes his opponent will not change his either. In Table 1, a Nash equilibrium emerges in cell (11, 11) with profit profile (0.3, 0.3).

Table 2 shows the same normal form when  $s_2 = 0.9s_1 = .18$ , but with a larger infrastructure. For this reason, odd rows and columns are omitted in order to keep a manageable table size. No cell satisfies the condition for Nash equilibrium between pure strategies in this form. What will happen in real life will depend on the players' ability, but a hint may be obtained by first assuming a hypothetical game in which players alternate moves in which each one maximizes his/her profit after looking at Table 2. Let the game start at cell (0, 0), and let player 1 have the first move. He will deploy 8 channels, moving the system to cell (8,0), where he enjoys a maximal-profit monopoly. Operator 2 then enters the game with a slightly smaller (10%) channel deployment cost and deploys 16 channels, taking the network to cell (8, 16) where her profit is maximized under current conditions. Operator 1 will then look at Table 2 and see that he is incurring a loss, and the only way for him to minimize his loss (i.e. maximize his "profit") is to leave the game, leading the system to cell (0, 16), where operator 2 enjoys a monopoly.

If operator 2 keeps looking for maximal profit, she would then reduce the dimension of her network to 9 channels, thus raising her profit to 3.5. This would prompt operator 1 to re-enter the game with 12 channels, and operator 2 to raise her number of channels to 18, thus forcing operator 1 to leave the game again. So, if each operator makes his/her next move in search of maximal profit, the game will enter the

loop given by  $(0,18) \rightarrow (0,9) \rightarrow (12,9) \rightarrow (12,18) \rightarrow (0,18) \rightarrow \dots$ . However, the players' behavior is short-sighted, since it only aims at maximal profit immediately after the next move. Eventually, next time the system goes through cell  $(0, 18)$ , operator 2 will observe that if she reduces her number of channels to 14 instead of 9, operator 1 will not have any incentive to re-enter the game, and she can enjoy a stable, competitive monopoly, although with profit 2.9 instead of 3.5: a small price (17%) to pay for stability. On the other hand, this is still a weak stability, since it is based on the short-sightedness of operator 1. A close examination of row 18 of Table 2 will tell operator 1 that he may also force operator 2 to leave the game by deploying 18 channels, if she cannot stand a lossy operation. However, this would let him with only 1.4 of profit, while the same strategy would yield profit 2.9 to operator 2. Both players are able to hurt each other. Each can achieve a competitive monopoly only if the other one is not willing to withstand losses.

Table 3 is a description of the same normal form when  $s_2 = 0.75s_1 = .15$ . The same analysis applies, leading initially to the loop  $(0,10) \rightarrow (6,10) \rightarrow (6,24) \rightarrow (0,24) \rightarrow (0,10) \rightarrow \dots$ . Operator 2 would then choose between an unstable, maximal-profit monopoly with 10 channels and profit 3.9; or a stable, competitive monopoly with 14 channels and profit 3.6. An important change will now arise in the situation of operator 1, though. Now, in order to force operator 2 to leave the game or incur a loss, operator 1 must deploy 26 channels and have a negative profit (-.2) even if he succeeds. Since this can hardly be considered a competitive behavior, we may say that there is no real (or legal) possibility of operator 1 achieving a competitive monopoly. Operator 1 can hurt operator 2 only by hurting himself, and this is not to be allowed. Therefore, a cost advantage of 25% would seem to give operator 2 a much more decisive chance of challenging the initial move of operator 1 than just 10%.

#### IV – Concluding Remarks

Two interconnected games were discussed: the users game and the operators game. In the users game, a uniform switching behavior is assumed for all users. For example, users may switch their primary operator whenever they estimate the blocking probability of their secondary operator to be smaller than their current primary operator. Or, if they are too impatient to calculate good estimates of the blocking probabilities of each operator, they may switch whenever blocked by their primary operator and accommodated by the secondary one. The users switching strategy defines the users game, which will lead to an equilibrium that determines the primary traffic captured by each operator. The equilibria resulting from the two user strategic behaviors described above are compared with respect to their efficiency and fairness.

In [6], the operators' game was discussed for the situation when their channel deployment costs are symmetric. In this paper, we have extended these results to the case where deployment costs are asymmetric. The game-theoretical model is tested to determine the degree of cost asymmetry needed to produce plausible equilibria between two operators, thus challenging the notion of a natural monopoly. The results provide insight on the ability of innovative entrants to challenge the market dominance of an incumbent. Moreover, the proposed game-theoretical model provides a framework for future studies of interest to both regulators and companies.

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**Table 1:** The normal form of the operators game when  $v = 8$ ,  $s_1 = s_2 = .2$

| Column<br>Line | 0    |      | 1    |      | 2    |      | 3    |      | 4    |      | 5    |      | 6    |      | 7    |      | 8    |      | 9    |      | 10   |      | 11   |      | 12   |      | 13   |      | 14   |      |
|----------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|                | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 |
| 0              | 0.0  | 0.0  | 0.0  | 0.6  | 0.0  | 1.2  | 0.0  | 1.8  | 0.0  | 2.2  | 0.0  | 2.6  | 0.0  | 2.8  | 0.0  | 3.0  | 0.0  | 3.0  | 0.0  | 3.0  | 0.0  | 2.9  | 0.0  | 2.8  | 0.0  | 2.6  | 0.0  | 2.4  | 0.0  | 2.2  |
| 1              | 0.6  | 0.0  | 0.6  | 0.6  | 0.5  | 1.2  | 0.5  | 1.8  | 0.4  | 2.2  | 0.3  | 2.6  | 0.2  | 2.8  | 0.1  | 3.0  | 0.0  | 3.0  | -0.1 | 3.0  | -0.1 | 2.9  | -0.2 | 2.8  | -0.2 | 2.6  | -0.2 | 2.4  | -0.2 | 2.2  |
| 2              | 1.2  | 0.0  | 1.2  | 0.5  | 1.1  | 1.1  | 0.9  | 1.7  | 0.7  | 2.1  | 0.5  | 2.5  | 0.3  | 2.8  | 0.1  | 2.9  | -0.1 | 3.0  | -0.2 | 2.9  | -0.2 | 2.8  | -0.3 | 2.7  | -0.3 | 2.5  | -0.4 | 2.4  | -0.4 | 2.2  |
| 3              | 1.8  | 0.0  | 1.8  | 0.5  | 1.7  | 0.9  | 1.4  | 1.4  | 1.2  | 1.8  | 0.9  | 2.2  | 0.6  | 2.4  | 0.4  | 2.5  | 0.2  | 2.6  | 0.0  | 2.6  | 0.0  | 2.6  | -0.3 | 2.5  | -0.4 | 2.4  | -0.4 | 2.2  | -0.5 | 2.1  |
| 4              | 2.2  | 0.0  | 2.2  | 0.4  | 2.1  | 0.7  | 1.8  | 1.2  | 1.5  | 1.5  | 1.2  | 1.8  | 0.9  | 2.0  | 0.6  | 2.1  | 0.4  | 2.2  | 0.2  | 2.2  | 0.0  | 2.2  | -0.2 | 2.2  | -0.3 | 2.1  | -0.4 | 2.0  | -0.5 | 1.9  |
| 5              | 2.6  | 0.0  | 2.6  | 0.3  | 2.5  | 0.5  | 2.2  | 0.9  | 1.8  | 1.2  | 1.5  | 1.5  | 1.1  | 1.6  | 0.8  | 1.8  | 0.5  | 1.9  | 0.3  | 1.9  | 0.1  | 1.9  | -0.1 | 1.9  | -0.2 | 1.8  | -0.4 | 1.8  | -0.5 | 1.7  |
| 6              | 2.8  | 0.0  | 2.8  | 0.2  | 2.8  | 0.3  | 2.4  | 0.6  | 2.0  | 0.9  | 1.6  | 1.1  | 1.3  | 1.3  | 1.0  | 1.4  | 0.7  | 1.5  | 0.4  | 1.6  | 0.2  | 1.6  | 0.0  | 1.6  | -0.2 | 1.6  | -0.3 | 1.5  | -0.5 | 1.5  |
| 7              | 3.0  | 0.0  | 3.0  | 0.1  | 2.9  | 0.1  | 2.5  | 0.4  | 2.1  | 0.6  | 1.8  | 0.8  | 1.4  | 1.0  | 1.1  | 1.1  | 0.8  | 1.2  | 0.5  | 1.3  | 0.3  | 1.3  | 0.1  | 1.3  | -0.1 | 1.3  | -0.3 | 1.3  | -0.4 | 1.2  |
| 8              | 3.0  | 0.0  | 3.0  | 0.0  | 3.0  | -0.1 | 2.6  | 0.2  | 2.2  | 0.4  | 1.9  | 0.5  | 1.5  | 0.7  | 1.2  | 0.8  | 0.9  | 0.9  | 0.6  | 1.0  | 0.4  | 1.0  | 0.2  | 1.0  | 0.0  | 1.0  | -0.2 | 1.0  | -0.4 | 1.0  |
| 9              | 3.0  | 0.0  | 3.0  | -0.1 | 2.9  | -0.2 | 2.6  | 0.0  | 2.2  | 0.1  | 1.9  | 0.3  | 1.6  | 0.4  | 1.3  | 0.5  | 1.0  | 0.6  | 0.7  | 0.7  | 0.5  | 0.7  | 0.2  | 0.8  | 0.0  | 0.8  | -0.2 | 0.8  | -0.4 | 0.8  |
| 10             | 2.9  | 0.0  | 2.9  | -0.1 | 2.8  | -0.3 | 2.6  | -0.2 | 2.2  | 0.0  | 1.9  | 0.1  | 1.6  | 0.2  | 1.3  | 0.3  | 1.0  | 0.4  | 0.7  | 0.5  | 0.5  | 0.5  | 0.3  | 0.5  | 0.1  | 0.5  | -0.1 | 0.5  | -0.3 | 0.5  |
| 11             | 2.8  | 0.0  | 2.8  | -0.2 | 2.7  | -0.3 | 2.5  | -0.3 | 2.2  | -0.2 | 1.9  | -0.1 | 1.6  | 0.0  | 1.3  | 0.1  | 1.0  | 0.2  | 0.8  | 0.2  | 0.5  | 0.3  | 0.3  | 0.3  | 0.1  | 0.3  | -0.1 | 0.3  | -0.3 | 0.3  |
| 12             | 2.6  | 0.0  | 2.6  | -0.2 | 2.5  | -0.3 | 2.4  | -0.4 | 2.1  | -0.3 | 1.8  | -0.2 | 1.6  | -0.2 | 1.3  | -0.1 | 1.0  | 0.0  | 0.8  | 0.0  | 0.5  | 0.1  | 0.3  | 0.1  | 0.1  | 0.1  | -0.1 | 0.1  | -0.3 | 0.1  |
| 13             | 2.4  | 0.0  | 2.4  | -0.2 | 2.4  | -0.4 | 2.2  | -0.4 | 2.0  | -0.4 | 1.8  | -0.4 | 1.5  | -0.3 | 1.3  | -0.3 | 1.0  | -0.2 | 0.8  | -0.2 | 0.5  | -0.1 | 0.3  | -0.1 | 0.1  | -0.1 | -0.1 | -0.1 | -0.3 | -0.1 |
| 14             | 2.2  | 0.0  | 2.2  | -0.2 | 2.2  | -0.4 | 2.1  | -0.5 | 1.9  | -0.5 | 1.7  | -0.5 | 1.5  | -0.5 | 1.2  | -0.4 | 1.0  | -0.4 | 0.8  | -0.4 | 0.5  | -0.3 | 0.3  | -0.3 | 0.1  | -0.3 | -0.1 | -0.3 | -0.3 | -0.3 |

**Table 2:** The normal form of the operators game when  $v = 8$ ,  $s_1 = .2$ ,  $s_2 = .18$

| Column | 0    |      | 2    |      | 4    |      | 6    |      | 8    |      | 9    |      | 10   |      | 12   |      | 14   |      | 16   |      | 18   |      |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| Line   | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 |
| 0      | 0    | 0    | 0    | 1.3  | 0    | 2.4  | 0    | 3.1  | 0    | 3.4  | 0    | 3.5  | 0    | 3.4  | 0    | 3.2  | 0    | 2.9  | 0    | 2.6  | 0    | 2.3  |
| 2      | 1.2  | 0    | 1.1  | 1.2  | 0.7  | 2.3  | 0.3  | 3.1  | -0.1 | 3.4  | -0.2 | 3.4  | -0.3 | 3.3  | -0.3 | 3.1  | -0.4 | 2.9  | -0.4 | 2.6  | -0.4 | 2.3  |
| 4      | 2.2  | 0    | 2.1  | 0.8  | 1.5  | 1.7  | 0.9  | 2.3  | 0.4  | 2.6  | 0.2  | 2.7  | 0    | 2.7  | -0.3 | 2.7  | -0.5 | 2.6  | -0.6 | 2.4  | -0.7 | 2.2  |
| 6      | 2.8  | 0    | 2.8  | 0.4  | 2    | 1.1  | 1.3  | 1.6  | 0.7  | 1.9  | 0.4  | 2    | 0.2  | 2.1  | -0.2 | 2.2  | -0.5 | 2.2  | -0.7 | 2.1  | -0.9 | 2    |
| 8      | 3    | 0    | 3    | 0    | 2.2  | 0.6  | 1.5  | 1    | 0.9  | 1.3  | 0.6  | 1.4  | 0.4  | 1.5  | 0    | 1.6  | -0.4 | 1.7  | -0.7 | 1.7  | -0.9 | 1.6  |
| 10     | 2.9  | 0    | 2.8  | -0.2 | 2.2  | 0.2  | 1.6  | 0.5  | 1    | 0.8  | 0.7  | 0.9  | 0.5  | 1    | 0.1  | 1.1  | -0.3 | 1.2  | -0.7 | 1.3  | -0.9 | 1.2  |
| 12     | 2.6  | 0    | 2.5  | -0.2 | 2.1  | -0.1 | 1.6  | 0.1  | 1    | 0.4  | 0.8  | 0.5  | 0.5  | 0.6  | 0.1  | 0.7  | -0.3 | 0.8  | -0.7 | 0.9  | -1   | 0.9  |
| 14     | 2.2  | 0    | 2.2  | -0.3 | 1.9  | -0.3 | 1.5  | -0.2 | 1    | 0    | 0.8  | 0.1  | 0.5  | 0.2  | 0.1  | 0.3  | -0.3 | 0.4  | -0.7 | 0.5  | -1   | 0.5  |
| 16     | 1.8  | 0    | 1.8  | -0.3 | 1.6  | -0.4 | 1.3  | -0.4 | 0.9  | -0.3 | 0.7  | -0.2 | 0.5  | -0.2 | 0.1  | -0.1 | -0.3 | 0    | -0.7 | 0.1  | -1   | 0.1  |
| 18     | 1.4  | 0    | 1.4  | -0.3 | 1.3  | -0.5 | 1.1  | -0.6 | 0.7  | -0.5 | 0.5  | -0.5 | 0.3  | -0.4 | 0    | -0.4 | -0.4 | -0.3 | -0.8 | -0.2 | -1.1 | -0.2 |

**Table3:** The normal form of the operators game when  $v = 8$ ,  $s_1 = .2$ ,  $s_2 = .15$

| Column | 0    |      | 2    |      | 4    |      | 6    |      | 8    |      | 10   |      | 12   |      | 14   |      | 16   |      | 18   |      | 20   |      | 22   |      | 24   |      | 26   |      |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| Line   | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 | Op.1 | Op.2 |
| 0      | 0    | 0    | 0    | 1.4  | 0    | 2.6  | 0    | 3.4  | 0    | 3.8  | 0    | 3.9  | 0    | 3.8  | 0    | 3.6  | 0    | 3.4  | 0    | 3.2  | 0    | 3    | 0    | 2.8  | 0    | 2.6  | 0    | 2.4  |
| 2      | 1.2  | 0    | 1.1  | 1.3  | 0.7  | 2.5  | 0.3  | 3.4  | -0.1 | 3.8  | -0.3 | 3.8  | -0.3 | 3.7  | -0.4 | 3.6  | -0.4 | 3.4  | -0.4 | 3.2  | -0.4 | 3    | -0.4 | 2.8  | -0.4 | 2.6  | -0.4 | 2.4  |
| 4      | 2.2  | 0    | 2.1  | 0.9  | 1.5  | 1.9  | 0.9  | 2.6  | 0.4  | 3    | 0    | 3.2  | -0.3 | 3.3  | -0.5 | 3.3  | -0.6 | 3.2  | -0.7 | 3.1  | -0.8 | 3    | -0.8 | 2.8  | -0.8 | 2.6  | -0.8 | 2.4  |
| 6      | 2.8  | 0    | 2.8  | 0.5  | 2    | 1.3  | 1.3  | 1.9  | 0.7  | 2.3  | 0.2  | 2.6  | -0.2 | 2.8  | -0.5 | 2.9  | -0.7 | 2.9  | -0.9 | 2.9  | -1   | 2.8  | -1.1 | 2.7  | -1.1 | 2.5  | -1.1 | 2.3  |
| 8      | 3    | 0    | 3    | 0.1  | 2.2  | 0.8  | 1.5  | 1.3  | 0.9  | 1.7  | 0.4  | 2    | 0    | 2.2  | -0.4 | 2.4  | -0.7 | 2.5  | -0.9 | 2.5  | -0.9 | -1.1 | 2.5  | 2.4  | -1.4 | 2.4  | -1.4 | 2.2  |
| 10     | 2.9  | 0    | 2.8  | -0.1 | 2.2  | 0.4  | 1.6  | 0.8  | 1    | 1.2  | 0.5  | 1.5  | 0.1  | 1.7  | -0.3 | 1.9  | -0.7 | 2.1  | -0.9 | 2.1  | -1.2 | 2.2  | -1.4 | 2.2  | -1.5 | 2.1  | -1.6 | 2    |
| 12     | 2.6  | 0    | 2.5  | -0.1 | 2.1  | 0.1  | 1.6  | 0.4  | 1    | 0.8  | 0.5  | 1.1  | 0.1  | 1.3  | -0.3 | 1.5  | -0.7 | 1.7  | -1   | 1.8  | -1.2 | 1.8  | -1.5 | 1.9  | -1.7 | 1.9  | -1.8 | 1.8  |
| 14     | 2.2  | 0    | 2.2  | -0.2 | 1.9  | -0.1 | 1.5  | 0.1  | 1    | 0.4  | 0.5  | 0.7  | 0.1  | 0.9  | -0.3 | 1.1  | -0.7 | 1.3  | -1   | 1.4  | -1.3 | 1.5  | -1.5 | 1.5  | -1.8 | 1.6  | -2   | 1.6  |
| 16     | 1.8  | 0    | 1.8  | -0.2 | 1.6  | -0.2 | 1.3  | -0.1 | 0.9  | 0.1  | 0.5  | 0.3  | 0.1  | 0.5  | -0.3 | 0.7  | -0.7 | 0.9  | -1   | 1    | -1.3 | 1.1  | -1.6 | 1.2  | -1.9 | 1.3  | -2.1 | 1.3  |
| 18     | 1.4  | 0    | 1.4  | -0.2 | 1.3  | -0.3 | 1.1  | -0.3 | 0.7  | -0.1 | 0.3  | 0.1  | 0    | 0.2  | -0.4 | 0.4  | -0.8 | 0.6  | -1.1 | 0.7  | -1.4 | 0.8  | -1.7 | 0.9  | -2   | 1    | -2.2 | 1    |
| 20     | 1    | 0    | 1    | -0.2 | 1    | -0.4 | 0.8  | -0.4 | 0.5  | -0.3 | 0.2  | -0.2 | -0.2 | 0    | -0.5 | 0.1  | -0.9 | 0.3  | -1.2 | 0.4  | -1.5 | 0.5  | -1.8 | 0.6  | -2.1 | 0.7  | -2.3 | 0.7  |
| 22     | 0.6  | 0    | 0.6  | -0.2 | 0.6  | -0.4 | 0.5  | -0.5 | 0.2  | -0.4 | 0    | -0.4 | -0.3 | -0.3 | -0.7 | -0.1 | -1   | 0    | -1.3 | 0.1  | -1.6 | 0.2  | -1.9 | 0.3  | -2.2 | 0.4  | -2.4 | 0.4  |
| 24     | 0.2  | 0    | 0.2  | -0.2 | 0.2  | -0.4 | 0.1  | -0.5 | 0    | -0.6 | -0.3 | -0.5 | -0.5 | -0.5 | -0.8 | -0.4 | -1.1 | -0.3 | -1.4 | -0.2 | -1.7 | -0.1 | -2   | 0    | -2.3 | 0.1  | -2.6 | 0.2  |
| 26     | -0.2 | 0    | -0.2 | -0.2 | -0.2 | -0.4 | -0.2 | -0.6 | -0.4 | -0.6 | -0.5 | -0.7 | -0.8 | -0.6 | -1   | -0.6 | -1.3 | -0.5 | -1.6 | -0.4 | -1.9 | -0.3 | -2.2 | -0.2 | -2.5 | -0.1 | -2.7 | -0.1 |