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Working Paper

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Manchester Business School Working Paper, No. 603

Provided in Cooperation with:
Manchester Business School, The University of Manchester

Suggested Citation: Chen, Yu-wang et al. (2010) : Multi-asset portfolio optimisation using a belief rule-based system, Manchester Business School Working Paper, No. 603, The University of Manchester, Manchester Business School, Manchester

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https://hdl.handle.net/10419/50667

Working Paper Series

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Manchester Business School Working Paper No. 603
July 2010

Manchester Business School

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+44(0)161 306 1320
http://www.mbs.ac.uk/research/working-papers/default.aspx
ISSN 0954-7401

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How to quote or cite this document

Chen, Yu-Wang, Yang, Jian-Bo, Xu, Dong-Ling, Zhang, Dongxu, Acomb, Simon and Poon, Ser-Huang. (2010). Multi-asset class portfolio optimisation using a belief rule-based system. *Manchester Business School Working Paper, Number 603,* available: http://www.mbs.ac.uk/research/working-papers.aspx.

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1. Introduction

Portfolio optimisation is concerned with implementing investment strategy and information technology to maximise the wealth of investors and manage the risk of an integrated portfolio over a fixed-term period. In the process of implementing an investment strategy, multi-asset class allocation is the most crucial decision required to achieve the investment goals of high returns and low risks (Kritzman, 1999; Gratcheva and Falk, 2003).

In the past decades, many studies have shown that strategic asset allocation has more influence on the aggregate portfolio return than other investment decisions and usually explains most of its return variations over a long-term period (Brinson et al., 1986; Hensel et al., 1991; Ibbotson and Kaplan, 2000). Asset allocation usually produces a set of portfolio weights for multiple asset classes such as stocks and bonds, and this diversification strategy is beneficial to both return enhancement and volatility reduction.

As the basis of modern portfolio theory, the seminal Markowitz’s mean-variance model provides a useful theoretical framework to solve strategic asset allocation problem (Markowitz, 1952). It suggested that an investor should make a trade-off between return and risk in determining the allocation of assets. Among an infinite number of investment portfolios with a specific return objective, the investor should choose the portfolio that has the smallest variance or risk, and all other portfolios are “inefficient” because they have higher variance at a given level of return.

Although Markowitz's optimisation model provides a parametric methodology for asset allocation and portfolio diversification, it has not been widely used in practice due to a number of practical and theoretical problems associated with it. In mean-variance analysis, the assumption of normal distribution of returns may not be accurate under some scenarios, and it is difficult to measure the necessary inputs, namely expected returns, variances and co-variances of risky assets (Bawa, 1975; Jorion, 1991; Kan and Zhou, 2007). In addition, the original Markowitz model ignores environmental factors (e.g., tax and transaction cost) and assumes a single-period investment horizon. However, in reality, investments are usually multi-period, and optimal portfolio decisions, especially for long-term investors, depend on the details of the changing investment environment, such as the available financial assets, the taxation, and the expected returns and preferences of investors (Campbell and Viceira, 2002). Furthermore, multi-period portfolio optimisation problem needs scenario simulation analysis to forecast expected returns (Markowitz and Perold, 1981; Koskosidis and Duarte, 1997). As such, a series of theoretical work has been conducted to improve portfolio optimisation models on the basis of mean-variance analysis (Black and Litterman, 1992; Tütüncü and Koenig, 2004; Ehrgott et al., 2004; Jacobs et al., 2005; Kim et al., 2005; Idzorek, 2006; Çelikyurta and Özekici, 2007). Simultaneously, various simulation techniques using historical data or Monte Carlo algorithm have been used to forecast the portfolios with nonlinear instruments such as options (RiskMetrics, 1999; 2004; Palmquist et al., 2001; Glasserman, 2004; Brandt et al., 2005).

In allocating investment to a number of asset classes, the optimisation objective depends on investors' financial needs and actual investment scales. The relationship between the proportion of each asset class in a portfolio and the distribution of its return is highly nonlinear (Campbell and Viceira, 2002; RiskMetrics, 2004). To identify the optimal combination of asset classes in order to maximise returns and minimise risks, there is a need to develop a method which can model such a highly nonlinear relationship.

This paper is dedicated to developing a novel portfolio modelling and optimisation method using a belief rule-based (BRB) system. The BRB system was developed on the basis of the concept of belief structures and the evidential reasoning (ER) approach (Yang and Singh, 1994; Yang et al., 2006; 2007). It has been widely applied in fields of nonlinear system identification and decision support systems (Xu et al, 2007; Zhou et al., 2009; Chen et al., 2010).

The rest of the paper is organised as follows: in the following section, the portfolio optimisation problem is modelled using a BRB system, and the detailed procedures are discussed. In Section 3, on the basis of the BRB system, two portfolio optimisation methods are presented to locate the optimal portfolio under different constraints. In Section 4, Numerical studies are conducted to demonstrate the effectiveness and efficiency of the developed BRB system to portfolio optimisation. The paper is concluded in Section 5.

2. Modelling portfolio optimisation with a BRB system

Portfolio optimisation is to build the allocation of asset classes which maximizes the return for a given level of risk or minimizes the risk for a given level of return (Bekkers et al., 2009). Generally, an asset class is a set of assets with some fundamental economic similarities (Greer, 1997). Portfolios are usually represented in terms of asset classes rather than assets because it is widely believed that the asset class returns can be forecasted more accurately than asset returns and asset returns tend to revert to asset class returns in the long-term simulation (RiskMetrics, 2004).

Suppose there are M asset classes, we can model an asset class portfolio by a vector of non-negative weights $(w_1, w_2, ..., w_M)$, where w_i represents the proportion of the total wealth to be invested in the *i*th asset class. In other word, if v_i represents the current market value of the total investment in the *i*th asset class, then $w_i = v_i / (v_1 + ... + v_M)$. Thus, the portfolio optimisation problem is transformed to determining the optimal weights of asset classes.

To apply a BRB system to model and solve the portfolio optimisation problem, we need to explain some basic theoretical concepts at first. Belief rules are the central constituents of BRB systems, and they usually are designed with an extended IF-THEN scheme, in which each possible consequent is associated with a belief degree. In portfolio optimisation, formally, the *k*th belief rule $R_k (k = 1,..., K)$ can be defined as follows,

$$R_k: \begin{array}{l} \text{IF}\,(w_1 \text{ is } A_1^k) \wedge (w_2 \text{ is } A_2^k) \wedge \cdots \wedge (w_M \text{ is } A_M^k), \\ \text{THEN}\,\{(D_1, \beta_{k,1}), (D_2, \beta_{k,2}), ..., (D_N, \beta_{k,N})\}, \left(\sum_{n=1}^{N} \beta_{k,n} \leq 1 \right), \\ \text{with rule weight } \theta_k, \\ \text{and attribute weight } \delta_{1,k}, \delta_{2,k}, ..., \delta_{M,k}, k \in \{1,..., K\}. \end{array} \tag{1}$$

where $w_1, w_2, ..., w_M$ denote the *antecedent attributes* in the *k*th rule. Each attribute takes values from $A = \{A_1,..., A_M\}$, in which each of the element $A_i (i = 1,..., M)$ is defined by a set of *referential values* $A_i = \{A_{i,j}; j = 1,..., NA_i\}$. In the *k*th rule, A_i^k represents the referential value taken by the attribute w_i, and $A_i^k \in \{A_{i,j}; j = 1,..., NA_i\}$. $\beta_{k,n}$ is the *belief degree* associated with *consequent* D_n given the logical relationship of $(w_1 \text{ is } A_1^k) \wedge (w_2 \text{ is } A_2^k) \wedge \cdots \wedge (w_M \text{ is } A_M^k)$. θ_k is the relative weight of the *k*th rule, and $\delta_{1,k}, \delta_{2,k}, ..., \delta_{M,k}$ are the relative weights of the M antecedent attributes in the *k*th rule.

In portfolio optimisation problem, the antecedent attributes w_i are the weights of asset classes, and the consequents are portfolio returns grouped by buckets (i.e., ranges of values). As such, we can use a BRB system to model and solve the optimisation problem with the following main steps: (1) constructing belief rules; (2) calculating activation weights; (3) generating new portfolios through BRB interpolation. Here, it is worth noting the BRB interpolated outputs are characterised by the belief distribution $(\beta_1, \beta_2, ..., \beta_N)$.

2.1 Constructing belief rules with linear bounds and unity constraint

In the portfolio optimisation problem, the decision variables are the weights of asset classes, which are usually subject to two types of constraints. First of all, the sum of all asset weights should be equal to 1 or 100%. Secondly, individual or institutional investors may also specify upper or lower boundaries of weights to some major asset classes for business reasons. For example, an investor may recommend portfolios with at least 20% wealth in large capital stocks, or wish that all portfolios have between 30% and 50% wealth invested in equities. Thus, we need to consider these bounds and unity constraints when constructing the initial belief rule base to model the portfolio optimisation problem.

2.1.1 Generating referential rule points

Let lb_i and ub_i represent the lower and upper bound of the weight for the *i*th asset class respectively, and then the linear bounds and unity constraint can be represented as follows,

$$lb_i \le w_i \le ub_i \tag{2}$$

$$\sum_{i=1}^{M} w_i = 1 \tag{3}$$

Generally speaking, when constructing an initial belief rule base, it is necessary to sample some *referential rule points* in the feasible solution space. If the space is two or more dimensional, usually, we select a few referential values on each dimension, or equivalently, for each independent variable, and the referential rule points are generated by the combination of the referential values of all the independent variables. Those variables are input decision variables to the BRB system.

In portfolio optimisation, using the unity constraint of equation (3), we can reduce the number of dimensions of the solution space by one, and the weight w_M of the *M*th asset class can be represented as,

$$w_M = 1 - \sum_{i=1}^{M-1} w_i \tag{4}$$

That means the number of independent decision variables becomes $M-1$.

Furthermore, we check the conflicts among the bound constraints and adjust the initial lower and upper bounds for all asset classes. To illustrate the necessity of adjusting initial bounds, two examples are given below.

Example 1. Suppose there are 2 asset classes, and the bound constraints are given as follows.

$$0 \le w_1 \le 1$$

$$0.2 \le w_2 \le 1$$

We uniformly select three referential values from the feasible interval of w_1, and those are 0, 0.5 and 1. As shown in Fig.1(a), the referential value $w_1 = 1$ corresponds to the weight combination $(w_1, w_2) = (1,0)$, which is an infeasible referential point.

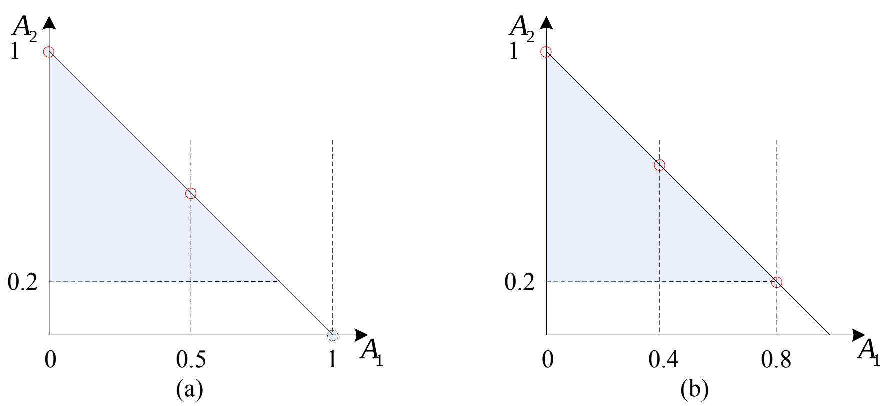

Fig. 1 Two asset class example with upper bound adjustment

To make sure that the referential points are in the feasible space, the upper bound of w_1 should be adjusted to 0.8 as shown in Fig. 1(b). Three feasible referential values 0, 0.4 and 0.8 can be selected for w_1 after the bound adjustment.

Example 2. Suppose there are 2 asset classes, and the bound constraints are given as follows.

$0 \le w_1 \le 1$

$0 \le w_2 \le 0.2$

If we also uniformly select three referential values from the feasible interval of w_1, two referential values 0 and 0.5 will be infeasible as shown in Fig. 2 (a).

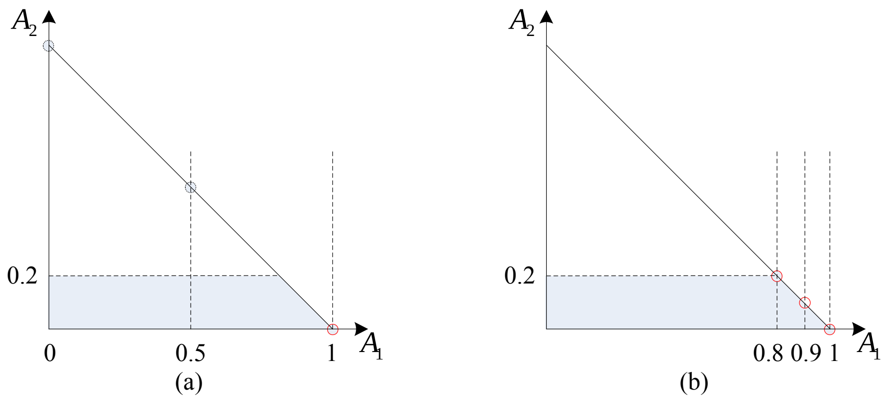

Fig. 2 Two asset class example with lower bound adjustment

Similarly, the lower bound of w_1 should be adjusted to 0.8, and three feasible referential values of 0.8, 0.9 and 1 can be sampled as Fig. 2 (b).

For a multiple input system, or a portfolio with more than two asset classes, it is difficult to adjust the lower and upper bounds manually. The following equations, however, can be used systematically to produce the bounds $\overline{lb}_i$ and $\overline{ub}_i$ for each asset class so that the conflicts among bound constraints can be eliminated.

$$\overline{lb}_i = \max(lb_i, 1 - \sum_{j=1, j\neq i}^{M} ub_j) \quad (5)$$

$$\overline{ub}_i = \min(ub_i, 1 - \sum_{j=1, j\neq i}^{M} lb_j) \quad (6)$$

Furthermore, if there are more than 2 asset classes, we need to consider the unity constraint, i.e., $\sum_{i=1}^{M} w_i = 1$, when selecting referential values for the weight of each asset class in order to generate feasible referential points. An illustrative example is given as follows.

Example 3. Suppose there are 3 asset classes, the bound constraints are without conflicts and given as follows.

$0 \le w_1 \le 0.6$

$0.2 \le w_2 \le 0.8$

$0.1 \le w_3 \le 0.8$

As discussed above, the weight w_3 can be decided by the sum of w_1 and w_2 using equation (4). Therefore, we can use the combination of the referential values of w_1 and w_2 to construct the initial belief rule base.

Assume the referential values of w_1 are {0, 0.3, 0.6} and that of w_2 are {0.2, 0.5, 0.8}. Without the unity constraint, 3×3 referential points will be generated as shown in Fig. 3(a).

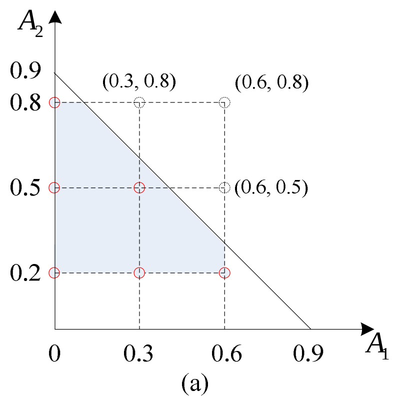

Fig. 3 Three asset class example with projecting new rule points

However, some of the points, (0.3, 0.8), (0.6, 0.5) and (0.6, 0.8) in Fig. 3(a), are not within the feasible solution space which is constrained by the following inequality constraints,

$$\sum_{i=1}^{M-1} w_i \le 1 - \overline{lb}_M \tag{7}$$

$$\sum_{i=1}^{M-1} w_i \ge 1 - \overline{ub}_M \tag{8}$$

If we delete the infeasible points directly, the BRB inference accuracy near those deleted referential points will decrease since fewer referential belief rules are available. As such, some new referential points should be generated to replace them as shown in Fig. 3(b).

To replace the infeasible referential point (0.3, 0.8), two new referential points (0.1, 0.8) and (0.3, 0.6) are generated by projecting it to the boundary of the feasible space. Similarly, two new referential points (0.4, 0.5) and (0.6, 0.3) are generated to replace the infeasible referential point (0.6, 0.5). No new referential points need to be generated for the infeasible referential point (0.6, 0.8), since it projects overlapped referential points (0.1, 0.8) and (0.6, 0.3).

To a multi-asset class portfolio optimisation problem, a generic method needs to be proposed to generate new projected referential points for replacing infeasible points effectively. Here, it is worth noting that we need to consider the downward projection onto the feasible boundary $\sum_{i=1}^{M-1} w_i = 1 - \overline{lb}_M$ and the upward projection onto $\sum_{i=1}^{M-1} w_i = 1 - \overline{ub}_M$ simultaneously in generating projected referential points.

We first consider the downward projection onto $\sum_{i=1}^{M-1} w_i = 1 - \overline{lb}_M$. Suppose A_{i,m_i} is the referential value taken by w_i in an infeasible referential point with

$\sum_{i=1}^{M-1} A_{i,m_i} > 1-\overline{lb}_M$, where $A_{i,m_i} \in \{A_{i,j}; j=1,...,NA_i\}$, and $m_i (m_i \geq 1)$ is an integer and denotes the order number in the vector of referential values. In Example 3, the vector of referential values for w_1 is (0, 0.3, 0.6). For the referential point (0.3, 0.8), m_1 is equal to 2 since the second referential value 0.3 is taken by w_1.

With this definition of m_i, we can generate new projected referential points to replace any infeasible referential point with $\sum_{i=1}^{M-1} A_{i,m_i} > 1-\overline{lb}_M$ by reducing the weights of the $M-1$ asset classes one by one. For w_i , if $m_i > 1$ and $A_{i,m_i-1} + \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} < 1-\overline{lb}_M$, a new referential point will be generated in which the referential value of w_i is projected to be $1-\overline{lb}_M - \sum_{j=1, j\neq i}^{M-1} A_{j,m_j}$; otherwise, no new referential point will be projected.

In the above example, the referential point (0.3, 0.8) is infeasible, since the sum of referential values is $\sum_{i=1}^{M-1} A_{i,m_i} = 0.3+0.8 = 1.1$, which is larger than $1-\overline{lb}_M = 1-0.1 = 0.9$. Therefore, we use the above-mentioned method to generate new projected referential points to replace it. For w_1 , we have $m_1 = 2$ and $A_{i,m_i-1} + \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} = 0+0.8 = 0.8$, which is less than 0.9. Its projected weight for w_1 can be calculated by $1-\overline{lb}_M - \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} = 1-0.1-0.8 = 0.1$, and a new referential point (0.1, 0.8) can be generated from it. For w_2 , we have $m_2 = 3$ and $A_{i,m_i-1} + \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} = 0.5+0.3 = 0.8$, which is less than 0.9. Its projected weight can be calculated by $1-\overline{lb}_M - \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} = 1-0.1-0.3 = 0.5$, and another new referential point (0.3, 0.6) can be obtained. As a result, two new referential points (0.1, 0.8) and (0.3, 0.6) are used to replace the infeasible referential point (0.3, 0.8).

Similarly, we can generate new projected referential points to replace any infeasible referential point with $\sum_{i=1}^{M-1} A_{i,m_i} < 1-\overline{ub}_M$. For w_i , if $m_i \leq NA_i$ and $A_{i,m_i+1} + \sum_{j=1, j\neq i}^{M-1} A_{j,m_j} > 1-\overline{ub}_M$, a new referential point will be generated in which the referential value of w_i is projected to be $1-\overline{ub}_M - \sum_{j=1, j\neq i}^{M-1} A_{j,m_j}$; otherwise, no new referential point will be projected.

With this method, it is obvious that the maximum number of new projected referential points for replacing each infeasible rule point is no more than $M-1$ (i.e., the number of asset classes minus 1), and the minimum number is 0. For example, in Fig. 3(b), the number of projected referential points for the infeasible referential points (0.3, 0.8) and (0.6, 0.5) is $3-1=2$, and that for the infeasible referential point (0.6, 0.8) is 0.

2.1.2 Constructing belief degrees for referential portfolios

In applying BRB systems to portfolio optimisation, we use belief distribution to approximate the distribution of the consequential portfolio returns. Belief degrees on consequents can be obtained with historical and simulated data. In this paper, we use RiskMetrics WealthBench (RM-WB) platform (RiskMetrics, 2004) to simulate a set of $L=500$ observed returns $R_{P,l}^{k}, l=1,\ldots,L$ for the kth set of portfolio weights that has the following mean and variance:

$$E(R_P^k)=\frac{1}{L}\sum_{l=1}^{L}R_{P,l}^{k} \tag{9}$$

$$\operatorname{var}(R_P^k)=\frac{1}{L}\sum_{l=1}^{L}(R_{P,l}^{k})^2-(\frac{1}{L}\sum_{l=1}^{L}R_{P,l}^{k})^2 \tag{10}$$

It is worth noting that this pair of mean and variance is not used in portfolio optimisation apart from serving as a check for the bucket distribution below. As shown in Fig. 4, we group the simulated portfolio returns into $N-1$ buckets of consequents using N referential values $D_1, D_2,\ldots,D_N$ within the range of portfolio returns. Note that the buckets could be non-uniformly distributed.

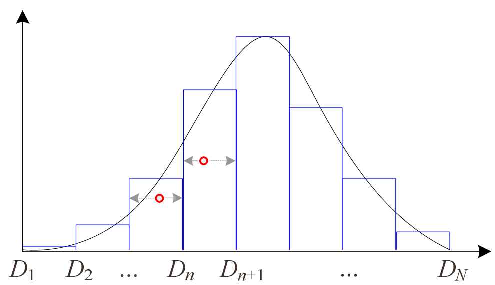

Fig. 4 Belief distribution on buckets of consequents

For the kth belief rule, we can simply put $R_{P,l}^{k}$ into the buckets and obtain the probability of observed returns falling into each bucket, which is further used to be the belief degree on the corresponding buckets of consequents. However, since the actual value of observed returns in each bucket has not been considered in this way, the expected return can not be calculated accurately with the belief distribution. As a consequence, in this study we use the following equation to calculate the belief degree,

$$\beta_{k,n}=\begin{cases}\frac{1}{L}\sum_{l_n\in L_n}(D_{n+1}-R_{P,l_n}^{k})/(D_{n+1}-D_n), & n=1\\ \frac{1}{L}\left(\sum_{l_{n-1}\in L_{n-1}}(R_{P,l_{n-1}}^{k}-D_{n-1})/(D_n-D_{n-1})+\sum_{l_n\in L_n}(D_{n+1}-R_{P,l_n}^{k})/(D_{n+1}-D_n)\right), & 1<n<N\\ \frac{1}{L}\sum_{l_{n-1}\in L_{n-1}}(R_{P,l_{n-1}}^{k}-D_{n-1})/(D_n-D_{n-1}), & n=N\end{cases} \tag{11}$$

where $D_n\le R_{P,l_n}^{k}<D_{n+1}, \forall l_n\in L_n$. Obviously, we have $\sum_{n=1}^{N}\beta_{k,n}=1$, and it is possible that some $\beta_{k,j}=0$. Using the belief degrees, the mean and variance of expected returns represented by the belief distribution in the kth belief rule can be calculated as follows,

$$E(P_k)=\sum_{n=1}^{N}D_n\beta_{k,n} \tag{12}$$

$$\operatorname{var}(P_k)=\sum_{n=1}^{N}(D_n)^2\beta_{k,n}-\left(\sum_{n=1}^{N}D_n\beta_{k,n}\right)^2 \tag{13}$$

Combined with the equation (11), we can conclude that the returns calculated in equations (9) and (12) will be equal, but the variances in equations (10) and (13) will not necessarily be equal. The deviation between the statistical variance and approximated variance depends heavily on the number of the buckets of consequents. As such, we need make a trade-off between computational complexity and inference accuracy.

2.2 Calculating activation weights

On the basis of the constructed belief rule base, the belief degrees embedded in belief rules can be used to perform inference for new inputs (portfolios with new and different set of weights in our case). In belief rules, the "$\wedge$" connective is used to represent the "and" logical relationship of antecedent attributes. This means the consequents of a belief rule is not believed to be true unless all the antecedents of the rule are matched with the input to some extent. To represent the matching degree between a set of values of an input and the referential values of antecedent attributes in the *k*th rule, the activation weight λ_k need to be calculated (Yang et al., 2006).

In portfolio optimisation problem, since some initial referential points violate the constraints (7) and (8) as discussed in Section 2.1.1, new projected referential points are added to the rule base, which results in some new referential values for each asset class. As shown in Fig. 3(b), the infeasible rule points (0.3, 0.8), (0.6, 0.5) and (0.6, 0.8) are removed, and new referential rule points (0.1, 0.8), (0.3, 0.6), (0.4, 0.5) and (0.6, 0.3) are added to the rule base. The referential values for the two asset classes have also been updated from (0, 0.3, 0.6) and (0.2, 0.5, 0.8) to (0, 0.1, 0.3, 0.4, 0.6) and (0.2, 0.3, 0.5, 0.6, 0.8) respectively. As such, the method proposed by Yang et al. (2006) is not suitable for the calculation of activation weighs in portfolio optimisation.

To a portfolio *P* with a set of weights $(w_1, w_2, ..., w_{M-1}, 1-\sum_{i=1}^{M-1} w_i)$, in the study, we use the following normalized Euclidian distance to represent the activation weight of the *k*th belief rule.

$$\lambda_k = \frac{\sqrt{\sum_{i=1}^{M-1}\left(w_i - A_i^k\right)^2}}{\sum_{l\in K_P}\left[\sqrt{\sum_{i=1}^{M-1}\left(w_i - A_i^l\right)^2}\right]} \tag{14}$$

where K_p is the set of activated belief rules. After defined a set of referential rule points in Section 2.1, the feasible solution space of asset weight combination is separated into granular polyhedrons. Any new feasible weight combination will fall into a specific polyhedron and the referential points on the polyhedral vertex will be activated. For example, in Example 3, if a new portfolio has the asset weights (0.4, 0.3, 1-0.4-0.3), the five referential points (0.3, 0.2), (0.3, 0.5), (0.4, 0.5), (0.6, 0.2) and (0.6, 0.3) will be activated.

2.3 Generating new portfolios through BRB interpolation

Once the activations weights are obtained, the belief rule base established using belief rules can then be summarized using a belief rule expression matrix (Yang et al., 2007), as show in Table 1.

Table 1. Belief rule expression matrix for the BRB system

Input	Belief output					
	D_1	D_2	$\cdots$	D_n	$\cdots$	D_N
$A^1(\lambda_1)$	$\beta_{1,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{1,n}$		$\beta_{1,N}$
$A^2(\lambda_2)$	$\beta_{2,1}$	$\beta_{2,2}$	$\cdots$	$\beta_{2,n}$		$\beta_{2,N}$
$\vdots$						
$A^k(\lambda_k)$	$\beta_{k,1}$	$\beta_{k,2}$	$\cdots$	$\beta_{k,n}$		$\beta_{k,N}$
$\vdots$						
$A^K(\lambda_K)$	$\beta_{K,1}$	$\beta_{K,2}$	$\cdots$	$\beta_{K,n}$		$\beta_{K,N}$

In the matrix, A^k represent the packet antecedents $(A_1^k, A_2^k, ..., A_M^k)$. $\beta_{k,n}$ is the belief degree on the consequents, and λ_k is the activation weight of the kth rule as discussed above. Based on the constructed belief rule expression matrix, the ER approach can then be used to combine activated rules and infer the belief distribution of portfolio returns. The ER approach is a generic evidence-based multi-criteria decision analysis (MCDA) approach for dealing with problems having both quantitative and qualitative criteria (Yang and Singh, 1994). The kernel of the ER approach is a recursive reasoning algorithm which is developed on the basis of Dempster-Shafer (D-S) theory (Dempster, 1968; Shafer, 1976), fuzzy set theory (Zadeh, 1965), and decision theory (Yoon and Hwang, 1995; Giovanni and Lurdes, 2009).

The implementing procedure of the ER approach is summarized as follows. Firstly, transform the basic belief degree $\beta_{k,n}$ in the belief rule expression matrix into basic probability mass $m_{k,n}$, which represents the degree to which the kth activated rule supports the hypothesis that D_n is the consequent. Let $m_{k,D}$ be the remaining probability mass unassigned to any known consequents. $m_{k,n}$ and $m_{k,D}$ can be calculated from the basic belief degree $\beta_{k,n}$ as follows (Yang and Xu, 2002),

$$m_{k,n} = \lambda_k \beta_{k,n}, \quad n = 1,..., N; k = 1,..., K$$

$$m_{k,D} = 1 - \sum_{n=1}^{N} m_{k,n} = 1 - \lambda_k \sum_{n=1}^{N} \beta_{k,n}, \quad k = 1,..., K$$

Decompose $m_{k,D}$ into $\overline{m}_{k,D}$ and $\tilde{m}_{k,D}$ as follows,

$$\overline{m}_{k,D} = 1 - \lambda_k, \quad \tilde{m}_{k,D} = \lambda_k \left(1 - \sum_{n=1}^{N} \beta_{k,n}\right), \quad k = 1,..., K$$

with $m_{k,D} = \overline{m}_{k,D} + \tilde{m}_{k,D}$.

Then, the final output distribution can be inferred using the analytical ER algorithm (Wang et al., 2006),

$$\{D_n\}: \quad m_n = \mu \left[\prod_{k=1}^{K} (m_{k,n} + \overline{m}_{k,D} + \tilde{m}_{k,D}) - \prod_{k=1}^{K} (\overline{m}_{k,D} + \tilde{m}_{k,D}) \right], \quad n = 1,..., N$$

$$\{D\}:\quad \tilde{m}_D = \mu\left[\prod_{k=1}^{K}(\bar{m}_{k,D}+\tilde{m}_{k,D}) - \prod_{k=1}^{K}(\bar{m}_{k,D})\right]$$

$$\{D\}:\quad \bar{m}_D = \mu\left[\prod_{k=1}^{K}(\bar{m}_{k,D})\right]$$

$$\mu = \left[\sum_{i=1}^{N}\prod_{k=1}^{K}(m_{k,i}+\bar{m}_{k,D}+\tilde{m}_{k,D}) - (N-1)\prod_{k=1}^{K}(\bar{m}_{k,D}+\tilde{m}_{k,D})\right]^{-1}$$

$$\{D_n\}:\quad \beta_n = \frac{m_n}{1-\bar{m}_D},\quad n=1,\dots,N$$

$$\{D\}:\quad \beta_D = \frac{\tilde{m}_D}{1-\bar{m}_D},\quad n=1,\dots,N$$

β_n represents the combined belief degree to which the output is assessed to D_n, and β_D represents the remaining belief degree unassigned to any known consequent. In portfolio optimisation problem, we have $\beta_D = 0$ due to $\sum_{n=1}^{N}\beta_{k,n} = 1$.

After the substitute of intermediate variable, the combined belief degree $\beta_n (n=1,\dots,N)$ can be analytically represented as follows,

$$\beta_n = \frac{\prod_{k=1}^{K}\left(\lambda_k\beta_{k,n}+1-\lambda_k\sum_{i=1}^{N}\beta_{k,i}\right) - \prod_{k=1}^{K}\left(1-\lambda_k\sum_{i=1}^{N}\beta_{k,i}\right)}{\sum_{j=1}^{N}\prod_{k=1}^{K}\left(\lambda_k\beta_{k,j}+1-\lambda_k\sum_{i=1}^{N}\beta_{k,i}\right) - (N-1)\prod_{k=1}^{K}\left(1-\lambda_k\sum_{i=1}^{N}\beta_{k,i}\right) - \prod_{k=1}^{K}(1-\lambda_k)} \tag{15}$$

The logic behind the approach is that, if the kth rule is activated by inputs and its consequents include D_n with $\beta_{k,n} > 0$, then combined belief degree β_n must be larger than 0, and it value mainly depends on $\beta_{k,n}$ and the activation weight λ_k.

As a consequence, the distribution of the returns for the new portfolio can be represented by the following belief distribution,

$$S(P) = \{(D_n, \beta_n), n=1,\dots,N\} \tag{16}$$

with below mean and variance,

$$E(P) = \sum_{n=1}^{N} D_n\beta_n(P) \tag{17}$$

$$\mathrm{var}(P) = \sum_{n=1}^{N}(D_n)^2\beta_n(P) - \left(\sum_{n=1}^{N} D_n\beta_n(P)\right)^2 \tag{18}$$

where D_n is defined in Section 2.1.2, and $\beta_n(P)$ can be calculated from the equation (15) above.

3. Portfolio optimisation methods

The above belief rule-based system can be used to solve the portfolio optimisation problem in two different ways. Section 3.1 locates the optimal portfolio by first constructing the efficient frontier using the portfolios generated by BRB interpolation in Section 2.3 and then searching along the efficient frontier using the objective function supplied by the investors. Section 3.2 finds the optimal portfolio directly using the BRB models in conjunction with a non-linear optimiser.

3.1 Constructing efficient frontier from BRB interpolated portfolios

In Markowitz's mean-variance analysis, generally, the set of optimal or efficient portfolios is plotted on two-dimensional graph called efficient frontier or Markowitz frontier in which the expected returns are plotted against their standard deviations as a measure of their risks (Tütüncü and Koenig, 2004). Portfolios on the efficient frontier have maximum return for a given level of risk or, alternatively, minimum risk for a given level of return.

Undoubtedly, a rational investor will select a portfolio on the efficient frontier. To determine the entire efficient frontier ranging from the portfolio with the smallest variance to that with the highest expected return, the parametric Markowitz's mean-variance model is used to the portfolio optimisation. However, in practice, it is difficult to obtain such an efficient frontier directly for the following two reasons (RiskMetrics, 2004):

- The optimisation involves a high-dimensional spaces of which dimensionality is determined by the number of asset classes.
- The distribution of portfolio returns is obtained by simulation with a given set of portfolio weights under various environmental factors (e.g., taxes and cash flow requirements) and constraints.

Thus, in this study we generate the efficient frontier using the referential portfolios from Section 2.1 and the portfolios generated through BRB interpolation in Section 2.3. A portfolio P is called long-term efficient if there is no other portfolio P' with both higher mean and lower variance of returns than those of portfolio P. Under the investor's specified portfolio weight constraints, we eliminate all inefficient portfolios, and then we can obtain the set of efficient portfolios for constructing the efficient frontier. Along the efficient frontier, the optimal frontier can be located once the investor's objective function is specified. Two common objective functions are listed below:

- Maximizing expected return – Since all the investment weights and case flow constraints are already incorporated into portfolio simulation, the optimal portfolio is the portfolio on the efficient frontier that has the highest return and satisfies the risk tolerance constraint which is usually stated in terms of the variance of expected return.
- Minimizing risk – Suppose there are two return distributions with the same mean, but different variances. The distribution with less variance has higher probability of being close to or above the mean.

It is worth noting that the optimality of the solution found along an efficient frontier depends on the granularity of interpolation points in the solution space. We can also use large granularity to roughly locate the optimal portfolio and then apply small granularity to further interpolate the target zone in order to produce more accurate solution.

3.2 Optimisation based on belief rules

Using the BRB system in equations (14) to (18), we can develop the following portfolio optimisation procedures for different objective functions.

(1) Mean-variance efficient set of portfolios

For an investor who wishes to find the portfolio P with the highest expected return under given objective function as,

$$\text{Max } E(P) \quad \text{subject to } \operatorname{var}(P) \le \sigma \tag{19}$$

From equations (17)-(18), the equation (19) can be translated into

$$\max_{w_i} \sum_{n=1}^{N} D_n \beta_n(P)$$

s.t.

$$\overline{lb}_i \le w_i \le \overline{ub}_i$$

$$\sum_{i=1}^{M-1} w_i \le 1 - \overline{lb}_M$$

$$\sum_{i=1}^{M-1} w_i \ge 1 - \overline{ub}_M$$

$$\sum_{n=1}^{N} (D_n)^2 \beta_n(P) - (\sum_{n=1}^{N} D_n \beta_n(P))^2 \le \sigma$$

Alternatively, we can also formulate the model to minimize risk given the level of expected return.

$$\text{Min } \operatorname{var}(P) \quad \text{subject to } E(P) \ge R \tag{20}$$

which can be translated into

$$\min_{w_i} \sum_{n=1}^{N} (D_n)^2 \beta_n(P) - (\sum_{n=1}^{N} D_n \beta_n(P))^2$$

s.t.

$$\overline{lb}_i \le w_i \le \overline{ub}_i$$

$$\sum_{i=1}^{M-1} w_i \le 1 - \overline{lb}_M$$

$$\sum_{i=1}^{M-1} w_i \ge 1 - \overline{ub}_M$$

$$\sum_{n=1}^{N} D_n \beta_n(P) \ge R$$

(2) Models for optimising probability-based risk measures

The probability level-based risk measures, such as value-at-risk (VaR), risk of loss, and shortfall risk, are important for risk management and risk regulation (Gaivoronski and Pflug, 2005). The optimisation models with such constraints can be formulated as,

$$\text{Max } E(P) \quad \text{subject to } prob(P < q) = p \tag{21}$$

Again, using equations (17)-(18), the problem represented by equation (21) can be translated into,

$$\max_{w_i} \sum_{n=1}^{N} D_n \beta_n(P)$$

$$\overline{lb}_i \le w_i \le \overline{ub}_i$$

$$\sum_{i=1}^{M-1} w_i \le 1 - \overline{lb}_M$$

$$\sum_{i=1}^{M-1} w_i \geq 1 - \overline{ub}_M$$

$$\sum_{n=1}^{N} (\beta_n(P) \times I(D_n < q)) \leq p$$

where q is the loss limit, and $I(D_n < q)$ is equal to 1 if $D_n < q$ and 0 otherwise. p is the probability in VaR which is usually set at 5%, risk of loss, or shortfall risk.

The non-linear optimisation can be solved using gradient-based search methods or nonlinear optimization software packages, such as the *fmincon* function in the Optimization Toolbox of Matlab (Coleman et al., 1999).

4. Numerical studies

To illustrate the procedure of using the BRB system to solve portfolio optimisation problems and validate the effectiveness of the techniques under study in this paper, in this section, two numerical studies are conducted.

4.1 Three-asset-class example

Suppose that we select three-asset-class example ($M = 3$)from the RM-WB platform (RiskMetrics, 2004), namely, *US large cap growth*, *US large cap value*, and *US small cap*. The lower bound (*lb*) and upper bound (*ub*) are given in Table 2.

Table 2. Lower and upper bounds of 3 asset classes

	US large cap growth	*US large cap value*	*US small cap*
lb	0	0	0.176852
ub	0.139737	0.683412	1

As discussed in Section 2, we first check whether there is any conflict among the bound constraints, and then select a few referential values from the feasible intervals of *US large cap growth* and *US large cap value*. We set the portfolio weights for the three asset classes as w_1, w_2 and $1 - w_1 - w_2$ respectively.

In this example, here is no conflict among bound constraints. Suppose 5 referential values are uniformly selected for the two asset weights w_1 and w_2, we then have $K = 5 \times 5 = 25$ asset weight combinations or referential belief rule points, and none of them violates the inequality constraints given by equations (7) and (8). It means that we do not need to generate projection rules in this case. The number of buckets used to group the consequential portfolio returns is $N = 1000$.

To generate the portfolio efficient frontier, we select 21 points uniformly from each feasible interval of w_1 and w_2. This leads to $K^* = 21 \times 21 = 441$ sets of new asset weight combinations. Using those points as the input to the BRB model of the problem, we can infer the return distribution for each of the 441 portfolios. To check the accuracy of the BRB interpolation results, we use RM-WB to simulate the distribution for these 441 portfolios. The results are presented in Fig. 5.

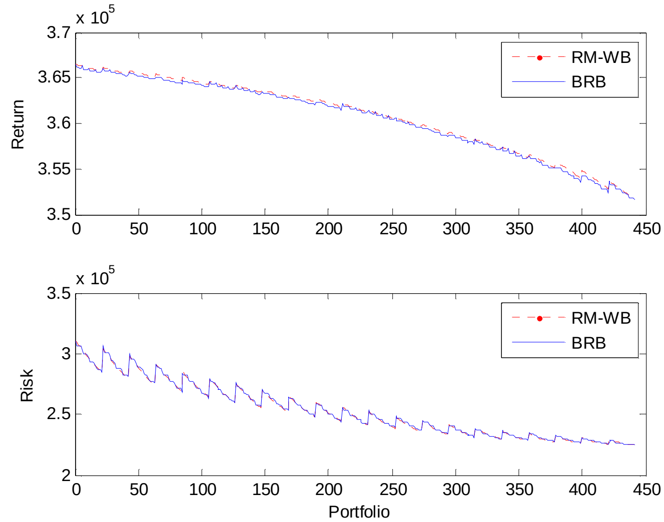

Fig. 5 Comparison between RM-WB and BRB with 3-asset-class portfolios

As shown in Fig. 5, the BRB system can closely replicate the non-linear relationship between asset weight combination and the mean and risk of portfolio returns. The maximum absolute deviation between RM-WB and BRB outputs is less than 0.2%. Using these interpolation points, we can get the approximated efficient frontier as shown in Fig. 6.

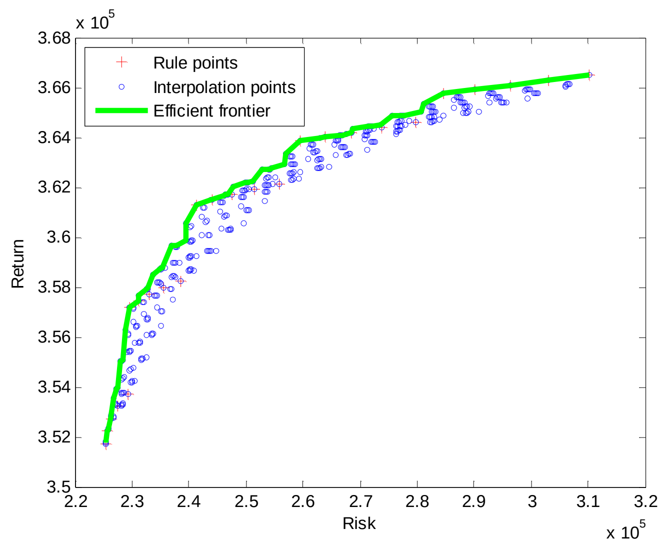

Fig. 6 BRB Efficient frontier for 3-asse-class example

Along the efficient frontier, the optimal portfolio can easily be found with a given level of expected return or risk. In this example, the computational time (seconds) is given in Table 3.

Table 3. Computational time for 3-asset-class (M=3) example

RM-WB Simulation ($K^* = 441$)	Portfolio optimisation using BRB		
	Rule base generation using RM-WB ($K = 25$)	BRB interpolation ($K^* = 441$)	BRB in total
17 minutes	60 seconds	15 seconds	75 seconds

It is obvious that the BRB system studied in this paper is quite efficient for generating the efficient frontier.

4.2 Nine-asset-class application

This section solves 9-asset-class set which includes *World equity (ex US)*, *US large cap growth*, *US large cap value*, *US midcap*, *US smallcap*, *Cash*, *US bonds*, *US muni bonds*, *World bonds (ex US)*. The bounds are given in Table 4.

Table 4. Lower and upper bounds of 9 asset classes

	World equity (ex US)	US large cap growth	US large cap value	US mid cap	US small cap	Cash	US bonds	US muni bonds	World bonds (ex US)
lb	0	0.15	0	0	0.05	0	0.1	0	0.15
ub	0.2	0.45	0.21	0.17	0.25	0.9	0.25	0.2	0.45

In this case, the upper bound of *Cash* is updated to 0.55 at first according to the equations (5) and (6). The numbers of referential values for the first $M-1=8$ asset classes are 2, 3, 2, 2, 2, 5, 2 and 2 respectively. Referential values are positioned evenly in the feasible interval of each asset weight. This leads to 960 initial weight combinations. As there are 874 points in infeasible space, 418 projected rule points are generated to construct an initial belief rule base with $K=960-874+418=504$ referential rule points in total. The number of buckets used to group the consequential returns is $N=100$.

Further, we uniformly select 3, 3, 3, 2, 2, 7, 2 and 3 points from the feasible intervals of asset classes. Using the similar process of constructing rule points above, $K^*=1775$ new portfolios can be generated for the purpose of constructing the efficient frontier. Fig. 7 compares the mean and standard deviation associated with the 1775 portfolios produced from RM-WB simulation and BRB interpolation.

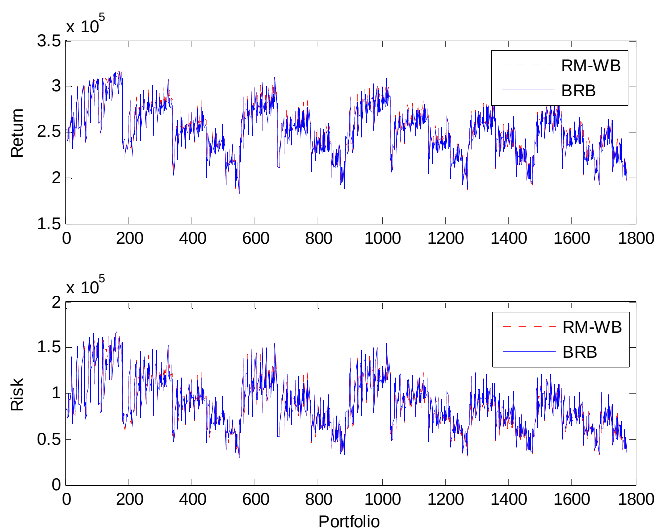

Fig. 7 Comparison between RM-WB and BRB with 9-asset-class portfolios

In Fig. 7, the maximum absolute deviation between RM-WB and BRB outputs is less than 7%. The corresponding efficient frontier is shown in Fig. 8.

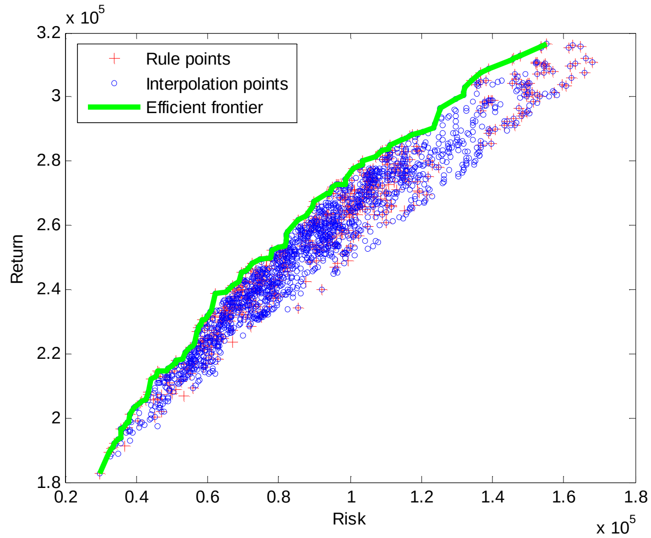

Fig. 8 BRB efficient frontier for 9-asset-class application

Along the efficient frontier, we can search for the optimal portfolios under given return or risk levels. Table 5 lists the computational time for RM-WB simulation and BRB interpolation for this 9-asset-class application.

Table 5. Computational time for 9-asset-class (M=9) application

RM-WB Simulation ($K^* = 1775$)	Portfolio optimisation using BRB		
	Rule base generation using RM-WB ($K = 504$)	BRB interpolation ($K^* = 1775$)	BRB in total
59 minutes	1010 seconds	80 seconds	18 minutes

It demonstrates that the BRB system is efficient enough for solving the 9-asset-class investment problem.

4.3 Optimal portfolio weights

For a given level of risk, we can locate the portfolio that generated the highest amount of return from the set of 1775 portfolios produced in Section 4.2. Alternatively, we can also use equation (19) in Section 3.2 together with a nonlinear optimiser to find the optimum weights. The initial condition can be randomly generated or taken from the optimum portfolio from Section 4.2. Fig. 9 shows the found optimal portfolios under different risk levels.

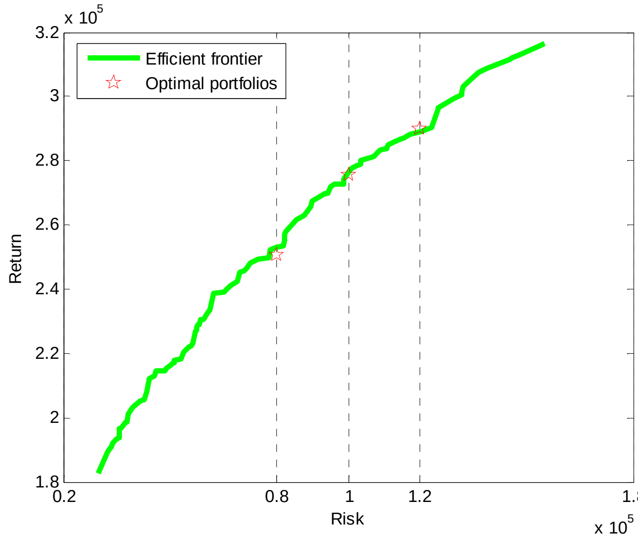

Fig. 9 Optimal portfolios under different risk levels

Under the risk level 1.0×10^5, the optimal portfolio is (0.2, 0.15, 0, 0.17, 0.0647, 0, 0.25, 0, 0.1653). The time spent on the nonlinear optimisation is about 135 seconds.

Since the relationship between the portfolio return and asset weight combination is highly non-linear along the efficient frontier, the local linearization and perturbation methods have been used to approximate the optimal portfolio (Speranza, 1993; Judd, 1996; RiskMetrics, 2004). Here, we pick up 10 portfolios from the interpolated efficient frontier at around the risk level 1.0×10^5. The weights of asset classes and the return and risk of RM-WB and BRB outputs are shown in Table 6.

Table 6. 10 portfolios nearest to the optimal solution

Asset class									RM-WB		BRB	
World equity (ex US)	US large cap growth	US large cap value	US midcap	US small cap	Cash	US bonds	US muni bonds	World bonds (ex US)	Return	Risk	Return	Risk
0.2	0.15	0	0.17	0.05	0	0.1	0	0.33	273912	100524	273912	100605
0.2	0.15	0.21	0	0.05	0	0.24	0	0.15	274320	98634	274320	98715
0.2	0.15	0.03	0.17	0.05	0	0.25	0	0.15	277485	101024	277485	101099
0.1	0.15	0.13	0.17	0.05	0	0.25	0	0.15	279392	104636	271595	99824
0.095	0.15	0.105	0	0.25	0	0.25	0	0.15	282176	105308	271415	98341
0.2	0.15	0	0	0.25	0.092	0.158	0	0.15	272527	99942	272280	100359
0.2	0.15	0.21	0	0.05	0	0.1	0.1	0.19	273350	99157	272462	98081
0.2	0.15	0	0	0.25	0.05	0.1	0.1	0.15	274620	100901	274871	101586
0.2	0.15	0	0	0.25	0.092	0.1	0.058	0.15	271747	99603	271918	100176
0.2	0.15	0.21	0	0.05	0	0.14	0.1	0.15	272992	97867	272845	98578

The stacked area graph in Fig. 10 shows that the 10 portfolios are quite different although they are close on the expected return and risk.

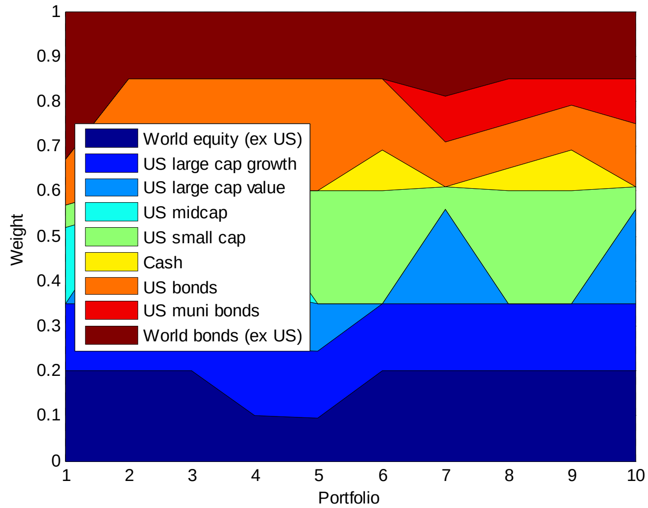

Fig. 10 Investment weights of 10 portfolios closest to the optimal solution

Given that the asset weights surrounding the optimal solution are highly nonlinear, the BRB search routine might be a better method that the local linearization method commonly used in practice.

5. Concluding remarks

The study is dedicated to apply a BRB system to model and solve the portfolio optimisation problem. The procedures of implementing the BRB system to portfolio optimisation are discussed in details, and two methods are proposed to locate the optimal portfolios under different constraints. The advantages of the BRB system to portfolio optimisation are that it can:

- Represent the non-linear relationship between a multi-asset class portfolio (represented by asset weights) and the mean and risk (indicated by standard deviation) of portfolio returns;
- Construct efficient frontier for long-term investment using BRB interpolation; and
- Search for the optimal portfolios with analytical mathematical model;

Acknowledgements

This research is partly funded by the European Community's Seventh Framework Programme FP7-PEOPLE-ITN-2008 under grant agreement number PINN-GA-2009-237984 and the UK Engineering and Physical Science Research Council (EPSRC) under grant number EP/F024606/1. The EC funding and the help from our industry partner, RiskMetrics Group, are gratefully acknowledged. We would like to thank, in particular, Phil Jacob from RiskMetrics for suggesting this research problem and the valuable advice he has given.

References

Bawa, V.S. Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2(1) (1975), 95–121.

Bekkers N., R.Q. Doeswijk, T.W. Lam. Strategic Asset Allocation: Determining the Optimal Portfolio with Ten Asset Classes. *Journal of Wealth Management*, 12(3) (2009), 61–77.

Black, F., and R. Litterman. Global Portfolio Optimization. *Financial Analysts Journal*, 48(5) (1992), 28–43.

Brandt, M.W., A. Goyal, P. Santa-Clara, J.R. Stroud. A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning About Return Predictability. *Review of Financial Studies*, 18(3) (2005), 831–873.

Brinson, G.P., L.R. Hood, and G.L. Beebower. Determinants of Portfolio Performance. *Financial Analysts Journal*, 42(4) (1986), 39–48.

Campbell, J.Y., L.M. Viceira. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press, Oxford, UK. 2002.

Çelikyurta and S. Özekici. Multiperiod portfolio optimization models in stochastic markets using the mean–variance approach. *European Journal of Operational Research*, 179(1) (2007), 186–202.

Chen Y.W., Yang Jian-Bo, Dong-Ling Xu, Zhi-Jie Zhou, and Da-Wei Tang (in review). Inference analysis and adaptive training for belief rule based systems. *Expert Systems with Applications*, 2010.

Coleman T., Branch M.A., Grace A. *Optimization Toolbox – For Use With MATLAB*. Natick, MA: The Mathworks Inc. 1999.

Dempster A.P. A generalization of Bayesian inference. *Journal of the Royal Statistical Society*, Series B 30(2) (1968), 205–247.

Ehrgott, M., K. Klamroth, C. Schwehm. An MCDM approach to portfolio optimization. *European Journal of Operational Research* 155 (2004), 752–770.

Gaivoronski, A.A., G. Pflug. Value at Risk in Portfolio Optimization: Properties and Computational Approach. *Journal of Risk*, 7(2) (2005), 1–31.

Giovanni P., Lurdes I. *Decision Theory: Principles and Approaches*. John Wiley & Sons. 2009.

Glasserman, P. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York. 2004.

Gratcheva, E.M., and Falk J.E. Optimal deviations from an asset allocation. *Computers & Operations Research*, 30 (2003), 1643–1659.

Greer, R.J. What is an asset class, anyway? *The Journal of Portfolio Management*, 23(2) (1997), 86–91.

Hensel, C.R., D.D. Ezra, J.H. Ilkiw. The Importance of the Asset Allocation Decision. *Financial Analysts Journal*, 47(4) (1991), 65–72.

Ibbotson, R.G., P.D. Kaplan. Does Asset Allocation Policy Explain 40, 90, or 100 Percent of Performance? *Financial Analysts Journal*, 56(1) (2000), 26–33.

Idzorek, T. *Developing Robust Asset Allocations*. Working Paper, Ibbotson Associates, Chicago, 2006.

Jacobs B.I., K.N. Levy, H.M. Markowitz. Portfolio Optimization with Factors, Scenarios, and Realistic Short Positions. *Operations Research*, 53(4) (2005), 586–599.

Jorion, P. Bayesian and CAPM estimators of the means: Implications for portfolio selection. *Journal of Banking & Finance*, 15(3) (1991), 717–727.

Judd, K.L. Approximation, perturbation, and projection methods in economic analysis. In: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), *Handbook of Computational Economics*. North-Holland, Amsterdam, pp. 511–585 (Chapter 12). 1996.

Kan, R., G. Zhou. Optimal Portfolio Choice with Parameter Uncertainty. *Journal of Financial and Quantitative Analysi*s, 42(3) (2007), 621–656.

Kim J.S., Y.C. Kim, K.Y. Shin. An Algorithm for Portfolio Optimization Problem, *Informatica*, 16(1) (2005) 93–106.

Koskosidis Y., Y.A., A.M. Duarte. A Scenario-Based Approach to Active Asset Allocation. *The Journal of Portfolio Management*, 23(2) (1997), 74–85.

Kritzman, M. Toward Defining an Asset Class. *The Journal of Alternative Investments*, 2(1) (1999), 79–82.

Markowitz, H., Portfolio Selection. *The Journal of Finance,* 7(1) (1952), 77–91.

Markowitz, H.M., A.F. Perold. Portfolio Analysis with Factors and Scenarios. *The Journal of Finance*, 36(4) (1981), 871–877.

Palmquist, J., P. Krokhmal, S. Uryasev. Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of Risk*, 4(2) (2001), 21–41.

RiskMetrics, *Risk Management: A Practical Guide*. RiskMetrics Group. August 1999.

RiskMetrics, *WealthBench Technical Document*, Preliminary Client Version, RiskMetrics Group. January, 2004.

Shafer G. *Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press. 1976.

Speranza, M.G. Linear programming models for portfolio optimization, *Finance*, 14 (1993), 107–123.

Tütüncü, R.H., M. Koenig. Robust asset allocation. *Annals of Operations Research*, 132 (2004), 157–187.

Wang Y.M., Yang J.B., Xu D.L. Environmental Impact Assessment Using the Evidential Reasoning Approach. *European Journal of Operational Research*, 174(3) (2006), 1885–1913.

Xu D.L., Liu J., Yang J.B., Liu G.P., Wang J., Jenkinson I., Ren J. Inference and learning methodology of belief-rule-based expert system for pipeline leak detection. *Expert Systems with Applications* 32(1) (2007), 103–113.

Yang J.B., Singh M.G. An evidential reasoning approach for multiple attribute decision making with uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics* 24(1) (1994), 1–18.

Yang J.B., Liu J., Wang J., Sii H.S., Wang H.W. A belief rule-base inference methodology using the evidential reasoning approach – RIMER. *IEEE Transactions on Systems, Man, and Cybernetics* – Part A, 36(2) (2006), 266–285.

Yang J.B., Liu J., Xu D.L., Wang J., Wang H.W. Optimization models for training belief rule based systems. *IEEE Transactions on Systems, Man, and Cybernetics – Part A* 37(4) (2007), 569–585.

Yang J.B., Xu D.L. On the evidential reasoning algorithm for multiattribute decision analysis under uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans* 32(3) (2002), 289–304.

Yoon K., Hwang C.L. *Multiple Attribute Decision Making: An Introduction*. Thousand Oaks, CA: Sage Publications. 1995.

Zadeh L.Z. Fuzzy sets. *Information Control* 8(3) (1965), 338–353.

Zhou Z.J., Hu C.H., Yang J.B., Xu D.L., Zhou D.H. Online updating belief-rule-based system for pipeline leak detection under expert intervention. *Expert Systems with Applications* 36 (2009), 7700–7709.