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# Tacit Collusion in an Infinitely Repeated Prisoners' Dilemma

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**Abstract**

In the context of an infinitely repeated Prisoners' Dilemma, we explore how cooperation is initiated when players communicate and coordinate through their actions. There are two types of players - patient and impatient - which are private information. An impatient type is incapable of cooperative play, while if both players are patient types - and this is common knowledge - then they can cooperate with a grim trigger strategy. We find that the longer that players have gone without cooperating, the lower is the probability that they'll cooperate in the next period. While the probability of cooperation emerging is always positive, there is a positive probability that cooperation never occurs.

## 1 Introduction

Antitrust and competition law has recognized that collusion comes in two varieties: explicit and tacit. Explicit collusion involves express communication among the parties regarding the collusive agreement - what outcome is to be supported and how it is to be sustained. Tacit collusion is, essentially, collusion by all other means. A common form of tacit collusion is indirect communication through price signaling. A firm raises its price with the hope that other firms will interpret this move as an invitation to collude and respond by matching the price increase. Of course, such a move is risky in that a firm's rivals may not raise price - either because they fail to properly interpret the price signal or deliberately choose not to collude - in which case the firm that raised price will experience a decline in profit from a loss of demand. In light of that potential cost, a firm desiring of collusion may prefer not to send such a signal and instead wait for a rival to take the initiative by raising price. While waiting means that a firm avoids the possible demand loss from charging a price above that of its rival, waiting could delay the time until a collusive outcome is reached. In this paper, we examine this trade-off towards investigating the dynamics associated with achieving a tacitly collusive outcome.

The setting is an infinitely repeated two-player Prisoners' Dilemma under incomplete information. There are two player types. One type never colludes, perhaps because it is too impatient or it fails to properly read signals. Another type has the capacity to collude and will surely do so once convinced the other player is also willing and able. Collusion requires firms to achieve mutual understanding as to a collusive arrangement. As our approach will deploy the equilibrium framework, we will not be exploring the non-equilibrium process by which firms settle upon a collusive equilibrium; firms will always be playing according to some equilibrium. Tacit collusion in our setting refers to the coordination on *cooperative play* within the context of a particular equilibrium. To be specific, a Markov Perfect Bayesian Equilibrium is characterized in which a cooperative type player randomizes over the cooperative and uncooperative actions, as long as there is uncertainty as to the other player's type. Once one of them chooses the cooperative action - which reveals it is a cooperative type - then the players permanently move to the cooperative outcome (when both are cooperative types) or the uncooperative outcome (when one or both are uncooperative types).

With this simple model, a number of interesting questions can be explored. If players have not yet colluded, is the likelihood of collusion declining over time? If so, does it converge to zero? If it converges to zero, does it occur asymptotically or does it become zero in finite time? That is, does a sufficiently long string of failed attempts to collude (that is, both players having chosen the uncooperative action) result in a cooperative type believing that it is so unlikely the other player is a cooperative type that it gives up trying to collude? Or is collusion assured of eventually occurring?

We find that the probability of collusion emerging in any period is declining over time but is always positive; at no point are beliefs sufficiently pessimistic that cooperative types give up trying to collude. While always positive, the probability of collusion arising in the current period (given it has not yet occurred) asymptotically converges to zero. Furthermore, even if both players are cooperative types, it is generally the case that the probability they never succeed in colluding is positive. Though cooperative type players never give up trying to collude - in the sense that they always choose the cooperative action with positive probability - they may never achieve the collusive outcome.

To our knowledge, there is no previous work which seeks to model the dynamic process by which players coordinate on cooperative play in a game with conflict.[1] However, there are analyses that have some related features. The seminal work of Kreps et al (1982) examines cooperation in a finitely repeated Prisoners' Dilemma with uncertainty as to types. An "irrational" type might be endowed with tit-for-tat or a preference for the cooperative action, while a "rational" type optimizes unconstrained. If it was common knowledge that players were rational then the unique equilibrium has them choose the uncooperative action in every period. However, un-

[1] Coordination within the context of a coordination game is explored in Crawford and Haller (1990).

certainty over the other player's type can support cooperative play for some length of time, at least probabilistically. The pattern in behavior is the reverse of ours in that it can start with cooperative play but must eventually get to uncooperative play when one or both are rational types.

More similar in mathematical structure is Dixit and Shapiro (1985). They consider a repeated Battle of the Sexes game which can be interpreted as two players simultaneously deciding whether or not to enter a market. It is profitable for one and only one firm to enter. The stage game then has two asymmetric pure-strategy equilibria and one symmetric mixed-strategy equilibrium. In the repeated version, the dynamic equilibrium has randomization in each period with, effectively, the game terminating once there is entry. Farrell (1987) considers this structure when players can precede their actions with messages.

In our environment, when play thus far has been uncooperative, a player is uncertain as to whether its rival will choose the cooperative or uncooperative action. In exploring the support of cooperation in a population of randomly matching agents, uncertainty instead occurs once a player faces a deviation from cooperative play. In Kandori (1992), Ellison (1994), and Harrington (1995), players choose the cooperative action in equilibrium but, in response to a partner having chosen the uncooperative action, is supposed to respond with the uncooperative action in its next encounter for the purpose of producing a contagious punishment that spreads through the population and eventually reaches the original deviator. When faced with a deviation, a player is then uncertain whether its partner was the first to deviate - in which case it can expect its next partner to choose the cooperative action - or whether it was responding to having been deviated - in which case it is possible the punishment is widespread and thus the next partner may be likely to select the uncooperative action. There is then uncertainty about play off-of-the equilibrium path which is pertinent to assessing the credibility of the punishment. Our uncertainty is on-the-equilibrium path since players may be cooperative or uncooperative types.

## 2 Model

Consider a two-player Prisoners' Dilemma:

Prisoners' Dilemma

Player 2

| Player 1 | | $C$ | $D$ |
|---|---|---|---|
| | $C$ | $a, a$ | $c, b$ |
| | $D$ | $b, c$ | $d, d$ |

where[2]

$$b > a > d \geq c.$$

[2] It is typical to assume $d > c$ but we will allow $d = c$.

It is further assumed:[3]

$$2a \geq b + c \geq a + d.$$

The first inequality is standard as it means the highest symmetric payoff has both players choosing C rather than taking turns cheating (that is, one player choosing D and the other choosing C).[4] The second inequality is new and will prove critical to our characterization. This assumption can be re-arranged to $b - a \geq d - c$, so that the gain to playing D - when the other player is expected to play C - is at least as great as the gain to playing D - when the other player is expected to play D. This condition holds for the Cournot quantity game with linear demand and constant marginal cost[5] and, loosely speaking, the Bertrand price game with homogeneous goods and constant marginal cost. The Bertrand price game is the special case when:

$$b = 2a, a > d = c = 0.$$

If both set the monopoly price then each earns $a$. Deviation from that outcome involves just undercutting the rival's price which means that the price-cost margin is approximately the same but sales are doubled so that the payoff is $2a$. Given the other firm prices at cost, pricing at cost as well yields a profit of zero (so, $d = 0$) as does pricing at the monopoly price (so, $c = 0$).[6]

Players are infinitely-lived and anticipate interacting in a Prisoners' Dilemma each period. If players have a common discount factor of $\delta$, the grim trigger strategy is a subgame perfect equilibrium iff:

$$\delta > \frac{b-a}{b-d}.$$

To capture uncertainty on the part of a player as to whether the other player is willing to cooperate, it is assumed that a player's discount factor is private information. A player can be of two possible types. A player can be type $L$ (for "long run") which means its discount factor is $\delta$ where $\delta > \frac{b-a}{b-d}$. Or a player can be type $M$ (for "myopic") which means its discount factor is zero (though any value less than $\frac{b-a}{b-d}$ should suffice). Hence, type $M$ players always choose D. A necessary condition for cooperative play to emerge and persist over time is then that both players are type $L$.

[3] Note that we cannot have $d = c$ and $b + c = a + d$ holding simultaneously as it would then imply $b = a$, which violates the assumption that $b > a$.

[4] The condition $2a \geq b + c$ is not necessary for our results but rather is to motivate the focus on players trying to sustain (C,C) in every period.

[5] Assume constant marginal cost $c$ and inverse market demand for firm $i$ is $\beta_0 - \beta_1 q_i - \beta_2 q_j$ where $\beta_0 > 0, \beta_1 \geq \beta_2 > 0$; thus, products can be differentiated. C corresponds to the low quantity $q^l$, and D to the high quantity $q^h$. $b - a > d - c$ is then

$$\begin{aligned} & q^h\left[\beta_0 - \beta_1 q^h - \beta_2 q^l - c\right] - q^l\left[\beta_0 - (\beta_1 + \beta_2) q^l - c\right] > q^h\left[\beta_0 - (\beta_1 + \beta_2) q^h - c\right] - q^l\left[\beta_0 - \beta_1 q^l - \beta_2 q^h - c\right] \\ \Leftrightarrow \quad & q^h > q^l. \end{aligned}$$

[6] The reference to "loosely speaking" is that this interpretation requires three prices - monopoly price, just below the monopoly price, and marginal cost - while the Prisoners' Dilemma has only two actions.

The equilibrium to be characterized will have the property that if both players are type $L$ *and* this is common knowledge then they implement the grim trigger strategy and thus cooperative play occurs. If players' types are common knowledge and one or both are type $M$ then they realize cooperation is infeasible and thereby choose D in every period. Thus, we can think of the game as having a terminal payoff - either $\frac{a}{1-\delta}$ (when both are type $L$) or $\frac{d}{1-\delta}$ (when one or both are type $M$) - when players' types become common knowledge. The focus of our analysis is then on what happens before a state of common knowledge is reached. Towards that end, let $\alpha^t$ denote the probability that a player attaches to the other player being type $L$ in period $t$. In the equilibrium that is to be characterized, $\alpha^t$ will be common to both players. Hence, $\alpha^t$ is not only the probability that player 1 attaches to player 2 being type $L$ but is also player 1's point belief as to the probability that player 2 attaches to player 1 being type $L$, and so forth. $\alpha^1$ is the common prior probability.

Suppose, at the start of period $t$, players' types are not common knowledge, $\alpha^t \in (0,1)$, and a type $L$ player (whether player 1 or 2) chooses C with probability $q^t \in (0,1)$. Future beliefs are described as follows based on the actions chosen in period $t$.

- If both players chose D in period $t$ then - since both types choose D with positive probability - players remain uncertain as to the other player's type and update their beliefs using Bayes Rule:

$$\alpha^{t+1} = \frac{\alpha^t\left(1-q^t\right)}{1-\alpha^t q^t}. \tag{1}$$

- If both players chose C in period $t$ then - since only a type $L$ player chooses C with positive probability - it is common knowledge they are both type $L$.

- If one player chose C and the other chose D in period $t$ then the former has revealed its type to be type $L$. It is assumed the other player's type becomes known prior to the next period so that, at the start of period $t+1$, players' types are common knowledge.

The assumption that players' types are common knowledge as soon as one player's type is known requires some explanation. Suppose player 1 chose C in the current period and player 2 did not; player 1's type has then been revealed to be type $L$, while uncertainty remains about player 2. What would be natural to expect is that player 1 would choose C in the next period and wait to learn whether player 2 signals it is type $L$ by also playing C. If player 2 did choose C then it would be common knowledge that both players are type $L$, and the players would adopt the grim trigger strategy. If player 2 instead chose D then player 1 would infer that player 2 is type $M$, in which case player 1 (as well as player 2) would play D thereafter. In that case, player 1 earns a low payoff of $c$ for two periods rather than the one period for our specification. By having just one period of loss rather than two periods, expressions are simplified and it would seem to be a reasonable approximation. Furthermore, this approximation does not disturb the main feature of this environment which is that signalling a desire to cooperate is risky, and this risk creates a waiting game between type $L$ players. A type $L$ player wants to learn whether the other player is also type $L$ and thus whether cooperation is feasible. The sooner they learn they are both type $L$, the sooner they are engaged in cooperative play. The necessary step for players to learn that they can cooperate is that at least one of them plays C and thereby signals a capacity to cooperate. However, choosing C is risky in that the other player may choose D - either because the player's type is $M$ or because its type is $L$ and the player has chosen to wait. Each player prefers the other player to take the risk by choosing C but waiting runs the risk of delaying the time until cooperative play is achieved. As a result, in equilibrium, type $L$ players will randomize between playing C and D until one of them chooses C at which time future play is set - both play C (if both are type $L$) or both play D (if one or both are type $M$).

The solution concept to be used is Markov Perfect Bayesian Equilibrium (MPBE); a strategy is Markovian during the phase when players' types are not common knowledge. More specifically, if $\alpha^t \in (0,1)$ then a type $L$ agent's period $t$ play depends only on $\alpha^t$ and no other element of the history. A Markov strategy is then of the form, $q(\cdot) : [0,1] \to [0,1]$. If both players choose D in period $t$ then the next period's beliefs are as specified in (1).

In deriving equilibrium conditions, a player will go through the thought experiment of deviating from $q(\cdot)$. Note, however, that this does not upset the specification of common beliefs. For suppose player 1 deviates in period $t$ by not choosing C with probability $q\left(\alpha^t\right)$. As each player expects the other to have chosen C with probability $q\left(\alpha^t\right)$, each player assigns probability $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$ to the other player being type $L$. While player 1 knows that player 2's beliefs about player 1's type are incorrect, that is irrelevant as all player 1 cares about is player 2's type and player 2's beliefs, both of which are summarized by $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$. Thus, $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$ remains the state variable pertinent to play, even if a player deviates from equilibrium play.

Let $V : [0,1] \to \Re$ denote the value function associated with type $L$ players using some symmetric strategy $q(\cdot)$. By the previous description of play, if player 1 is type $L$ then player 1's continuation payoff, depending on the current period's actions and beliefs $\alpha$ at the start of the period, is:

| Player 1's action | Player 2's action | Player 2's type | Expected continuation payoff |
|---|---|---|---|
| C | C | L | $\frac{a}{1-\delta}$ |
| C | D | L | $\frac{a}{1-\delta}$ |
| C | D | M | $\frac{d}{1-\delta}$ |
| D | C | L | $\frac{a}{1-\delta}$ |
| D | D | L,M | $V\left(\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)}\right)$ |

By examining MPBE, this paper focuses on the dynamics associated with players learning about their capacity to cooperate and the manner in which cooperative play is achieved. There is, however, another equilibrium in which a type $L$ chooses C in the first period and uses a grim punishment; that is, if both do not choose C in the first period then a player chooses D thereafter and otherwise chooses C. This is an equilibrium iff the initial probability that a player is type $L$ is sufficiently large:

$$\begin{aligned}
\alpha\left(\frac{a}{1-\delta}\right)+(1-\alpha)\left(c+\frac{\delta d}{1-\delta}\right) &\geq \alpha b+(1-\alpha)\,d+\frac{\delta d}{1-\delta} \Leftrightarrow \\
\alpha &\geq \frac{d-c}{(d-c)+\delta\left(\frac{a-d}{1-\delta}\right)-(b-a)}.
\end{aligned}$$

The appeal of the equilibrium upon which we focus is that it encompasses the waiting game associated with signalling cooperation and thereby can deliver a richer set of dynamics regarding the emergence of cooperative play.

# 3 Markov Perfect Bayesian Equilibrium

In this section, some properties of a MPBE are provided, while existence is established in the next section. Given the other player chooses C with probability $q$ when she is type $L$, a player's expected payoff from choosing C is

$$W^{C}(\alpha)\equiv\alpha\left[q\left(\frac{a}{1-\delta}\right)+(1-q)\left(c+\frac{\delta a}{1-\delta}\right)\right]+(1-\alpha)\left(c+\frac{\delta d}{1-\delta}\right),$$

which can be simplified to

$$W^{C}(\alpha)=\alpha q\,(a-c)+c+\frac{\alpha\delta\,(a-d)}{1-\delta}+\frac{\delta d}{1-\delta}; \tag{2}$$

and from choosing D is

$$\begin{aligned}
W^{D}(\alpha) \;\equiv\; & \alpha\left[q\left(b+\frac{\delta a}{1-\delta}\right)+(1-q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right)\right] \\
& +(1-\alpha)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right)
\end{aligned}$$

which can be simplified to

$$W^{D}(\alpha)=\alpha q\left(b+\frac{\delta a}{1-\delta}\right)+(1-\alpha q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right). \tag{3}$$

If, in equilibrium, $q\in(0,1)$ then the expressions in (2) and (3) must be the same:

$$\alpha q\,(a-c)+c+\frac{\alpha\delta\,(a-d)}{1-\delta}+\frac{\delta d}{1-\delta}=\alpha q\left(b+\frac{\delta a}{1-\delta}\right)+(1-\alpha q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right).$$

Re-arranging gives us:

$$\alpha q=\frac{\delta\left[\frac{a}{1-\delta}-V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right]-(1-\alpha)\frac{\delta(a-d)}{1-\delta}-(d-c)}{\delta\left(\frac{a}{1-\delta}-V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right)+(b-a)-(d-c)}. \tag{4}$$

Define:

$$\underline{\alpha} \equiv \frac{(1-\delta)(d-c)}{\delta(a-d)} \in [0,1),$$

where $\underline{\alpha} \geq 0$ follows from $d \geq c$ and $a > d$. To show $\underline{\alpha} < 1$, note that

$$\frac{(1-\delta)(d-c)}{\delta(a-d)} < 1 \Leftrightarrow \delta > \frac{d-c}{a-c}.$$

As it is already assumed

$$\delta > \frac{b-a}{b-d},$$

a sufficient condition for $\delta > \frac{d-c}{a-c}$ is

$$\frac{b-a}{b-d} \geq \frac{d-c}{a-c} \Leftrightarrow (b-a)(a-c) \geq (d-c)(b-d) \Leftrightarrow b+c \geq a+d$$

which is true by assumption. Thus, $\underline{\alpha} \in [0,1)$.

Theorem 1 states that a stationary symmetric MPBE has a type $L$ player choose D for sure when the probability that the other player is type $L$ is sufficiently low, $\alpha \leq \underline{\alpha}$. When instead $\alpha > \underline{\alpha}$, a type $L$ player randomizes between playing C and D. In that case, the probability that a player chooses C, $q(\alpha)$, is defined by (4). Proofs are in the appendix.[7]

**Theorem 1** *If $q(\cdot)$ is a stationary symmetric Markov Perfect Bayesian Equilibrium then*

$$q(\alpha)\begin{cases} = 0 & \text{if } \alpha \in [0,\underline{\alpha}] \\ \in (0,1) & \text{if } \alpha \in (\underline{\alpha},1] \end{cases}$$

$$V(\alpha)\begin{cases} = \frac{d}{1-\delta} & \text{if } \alpha \in [0,\underline{\alpha}] \\ \in \left(\frac{d}{1-\delta},\frac{a}{1-\delta}\right) & \text{if } \alpha \in (\underline{\alpha},1] \end{cases}$$

The next result concerns the evolution of beliefs and behavior in response to a failure to cooperate (that is, both players have always chosen D). Recall that if the probability a player assigns to the other player being type $L$ is $\alpha$ then, after observing the other player chose D, the updated probability is $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)}$ where $q(\alpha)$ is the equilibrium probability that a type $L$ player chooses C given beliefs $\alpha$. Further recall that if $\alpha > \underline{\alpha}$ then $q(\alpha) > 0$.

**Theorem 2** *If $q(\cdot)$ is a stationary symmetric Markov Perfect Bayesian Equilibrium then: i) if $\alpha > \underline{\alpha}$ then $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)} > \underline{\alpha}$; ii) if $\alpha^1 > \underline{\alpha}$ then $lim_{t\to\infty}\alpha^t = \underline{\alpha}$ and $q\left(\alpha^t\right) > 0$ for all $t$; iii) if $\underline{\alpha} > 0$ then $lim_{\alpha\downarrow\underline{\alpha}} q(\alpha) = 0$; and iv)* $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right) = 0$.

[7] If $b+c \geq a+d$ does not hold, Theorems 1 and 2 are still true as long as $\delta > \frac{d-c}{a-c}$. However, the results in Section 4 would change.

Theorem 2 shows that if $\alpha^1 > \underline{\alpha}$ then $\alpha^t > \underline{\alpha}$ for all $t$ which then implies $q\left(\alpha^t\right) > 0$ for all $t$. Therefore, no matter how long players have failed to cooperate, a type $L$ player will continue to try to initiate cooperation (in the sense of assigning positive probability of choosing C). In other words, beliefs never become so pessimistic about the other player's willingness to cooperate that a player prefers to abandon any prospects of cooperation by playing D for sure.[8] It is also the case, however, that, when $\underline{\alpha} > 0$, the probability of a player initiating cooperation converges to zero over time in response to the probability that the other player is type $L$ converging to $\underline{\alpha}$ after a history of failed cooperation. Note that the probability of a type $L$ player playing C must converge to zero as the probability of a player being type $L$ approaches $\underline{\alpha}\,(> 0)$ from above. If $q(\alpha)$ was instead bounded above zero then a sufficiently long sequence of playing D would have to result in a sufficiently small probability of the player being type $L$, which would contradict this probability being bounded below by $\underline{\alpha}$. Finally, conditional on cooperation not yet having emerged, the probability assigned to a player initiating cooperation is $\alpha^t q\left(\alpha^t\right)$ in which case the probability that cooperation emerges out of period $t$ is $1 - \left(1 - \alpha^t q\left(\alpha^t\right)\right)^2$. While this value is always positive - so collusion is always a possibility - it converges to zero in response to an ever-increasing sequence of failed attempts at collusion, in which case collusion eventually becomes very unlikely to emerge.

# 4 Equilibrium with an Affine Value Function

Given a value function $V(\cdot)$, the resulting symmetric equilibrium strategy, $q^*(\cdot, V(\cdot))$, is defined by:

$$q^*(\alpha, V(\cdot)) = \begin{cases} 0 & \text{if } Y^C(q, \alpha, V(\cdot)) < Y^D(q, \alpha, V(\cdot)),\ \forall q \in [0,1] \\ \widetilde{q}(\alpha, V(\cdot)) & \text{otherwise} \\ 1 & \text{if } Y^C(q, \alpha, V(\cdot)) > Y^D(q, \alpha, V(\cdot)),\ \forall q \in [0,1] \end{cases}$$

where $\widetilde{q}(\alpha, V(\cdot))$ is defined by

$$Y^C(\widetilde{q}(\alpha, V(\cdot)), \alpha, V(\cdot)) = Y^D(\widetilde{q}(\alpha, V(\cdot)), \alpha, V(\cdot)),$$

and the payoffs from choosing C and D, respectively, are:

$$\begin{aligned} Y^C(\widetilde{q}(\alpha, V(\cdot)), \alpha, V(\cdot)) &\equiv \alpha\widetilde{q}(\alpha, V(\cdot))(a-c) + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta} \\ Y^D(\widetilde{q}(\alpha, V(\cdot)), \alpha, V(\cdot)) &\equiv \alpha\widetilde{q}(\alpha, V(\cdot))\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right). \end{aligned}$$

**Definition 3** *An affine Markov Perfect Bayesian Equilibrium (MPBE) is a MPBE in which the value function is affine over* $\alpha \in [\underline{\alpha}, 1]$.

Existence of equilibrium is established by showing that there exists a unique stationary symmetric affine MPBE.

[8] Note that this result is not obvious. If $\underline{\alpha} > 0$ then, in principle, $\alpha^t < \underline{\alpha}$ unless $q(\alpha) \to 0$ sufficiently fast as $\alpha \to \underline{\alpha}$ .

**Theorem 4** *There exists a unique stationary symmetric affine Markov Perfect Bayesian Equilibrium. The value function is*

$$V(\alpha) = \begin{cases} \frac{d}{1-\delta} & \text{if } \alpha \in [0, \underline{\alpha}] \\ x + y\alpha & \text{if } \alpha \in [\underline{\alpha}, 1] \end{cases} \tag{5}$$

*where* $(x, y)$ *is the unique solution to:*

$$x + y\frac{(1-\delta)(d-c)}{\delta(a-d)} = \frac{d}{1-\delta} \tag{6}$$

$$x + y = \frac{2a\delta + (1-\delta)\left[(b-a) - (d-c) - \sqrt{\Omega}\right]}{2\delta(1-\delta)}. \tag{7}$$

*and*

$$\Omega \equiv [(b-a) - (d-c)]^2 + 4\delta(a-c)[(b-a) - (d-c)] + 4\delta(a-c)(d-c). \tag{8}$$

*Furthermore, there is a unique stationary symmetric strategy associated with this equilibrium:*

$$q(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \left[0, \frac{(1-\delta)(d-c)}{\delta(a-d)}\right] \\ \frac{\delta(a-d)-\delta(1-\delta)y}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)} & \text{if } \alpha \in \left(\frac{(1-\delta)(d-c)}{\delta(a-d)}, 1\right] \\ +\left(\frac{1}{\alpha}\right)\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-(1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)}\right] & \end{cases} \tag{9}$$

In the preceding section, we established that $\alpha^t q(\alpha^t)$ converges to zero and thus is eventually decreasing over time. If we focus on equilibria with an affine value function, we can now say that $\alpha^t q(\alpha^t)$ is monotonically declining over time, in which case the probability a player chooses C decreases with the length of time for which cooperative play has not yet occurred. It is also the case that a type $L$ player's equilibrium value is decreasing with the likelihood assigned to players being type $L$.

**Theorem 5** *If* $q(\cdot)$ *is the unique stationary symmetric affine Markov Perfect Bayesian Equilibrium then* $\alpha q(\alpha)$ *is increasing in* $\alpha$ *and* $V(\alpha)$ *is increasing in* $\alpha$.

While $\alpha q(\alpha)$ is increasing in $\alpha$, $q(\alpha)$ need not be increasing in $\alpha$ everywhere, though we know that eventually it must be increasing in $\alpha$ since it converges to zero (when $\underline{\alpha} > 0$). We next show that when $d > c$ then $q(\alpha)$ is decreasing over time as lower probability is attached to players being type $L$ (given only D has been chosen thus far). However, when $d = c$ then $q(\alpha)$ is, interestingly, independent of a player's beliefs as to the other player's type and thus is constant over time. Though it is still the case that $\alpha^t$ is declining, a type $L$ player maintains the same probability of acting cooperatively.

**Theorem 6** *If* $q(\cdot)$ *is the unique stationary symmetric affine Markov Perfect Bayesian Equilibrium then, for* $\alpha > \underline{\alpha}$: *i) if* $d > c$ *then* $q(\alpha)$ *is increasing in* $\alpha$; *and ii) if* $d = c$ *then* $q(\alpha) = q'$ *for some* $q' \in (0, 1)$.

When $d = c$ - so a player is not harmed when choosing the cooperative action - the probability that a type $L$ player chooses C is fixed at some positive value. Thus, if both players are type $L$ then, almost surely, players will eventually achieve the collusive outcome. However, whether cooperative play ultimately emerges is not so clear when $d > c$ as then the probability of cooperation being initiated is declining over time and converges to zero. To examine this issue, define $Q^T$ as the probability that players are still not colluding by the end of period $T$, conditional on both players being type $L$. If $q(\cdot)$ is a stationary symmetric Markov Perfect Bayesian Equilibrium, $Q^T$ is defined by

$$Q^T = \prod_{t=1}^{T} \left(1 - q^t\right)^2$$

where, given $\alpha^1$, $q^t$ is defined recursively by:

$$q^t = q\left(\alpha^t\right), t \geq 1; \ \alpha^t = \frac{\alpha^{t-1}\left(1 - q^{t-1}\right)}{1 - \alpha^{t-1} q^{t-1}}, t \geq 2.$$

The next result shows that, even when both players are type $L$, there is a positive probability that collusion never emerges even though they never give up trying (that is, they always choose C with positive probability).

**Theorem 7** *At the unique stationary symmetric affine Markov Perfect Bayesian Equilibrium, if $d > c$ then* $\lim_{T\to\infty} Q^T > 0$.

If both players are type $L$ then, in any period, there is always a positive probability that one of them will choose the cooperative action and thereby result in the emergence of collusion. This property follows from $\alpha^t > \underline{\alpha}$ for all $t$; regardless of how long the other player has chosen D, a player assigns sufficient probability to its rival being type $L$ that it is optimal to continue to try to cooperate (as reflected in choosing C with positive probability). For $\alpha^t > \underline{\alpha}\,(> 0)$, it must be the case that a long sequence of choosing D is not a sufficiently pessimistic signal that the other player is type $L$ which can only be the case if, as $\alpha^t \to \underline{\alpha}$, the probability that a type $L$ player chooses C converges sufficiently fast to zero, $q^t \to 0$. But, as shown in the previous result, this also has the implication that the probability that two type $L$ players start colluding in period $t$ is going to zero sufficiently fast, which means collusion is not assured. In short, even if both players are willing and able to cooperate, there is a positive probability that they never do so though they never give up trying.

# 5 Examples

In this section, we derive the affine MPBE for some examples. Example 1 is a case in which the probability of a player choosing the cooperative action is independent of $\alpha$ and, therefore, fixed over time. When players are more patient, we show that collusion is more likely to emerge. In Example 2, the probability a type $L$ player chooses the cooperative action is increasing in the likelihood it assigns to the other player also being type $L$. In response to an ever-lengthening sequence of failed cooperation, the probability of cooperation emerging is declining. Furthermore, conditional on both players being type $L$, the probability that collusion never occurs is positive. Finally, Example 3 considers an asymmetric Prisoners' Dilemma in which the collusive outcome does not split the surplus equally. Surprisingly, greater asymmetry makes collusion more likely to emerge.

### 5.1 Example 1: Bertrand Price Game

Assume $b = 2a, d = c = 0$, and normalize so $a = 1$.

Bertrand Price Game

| | | Player 2 | |
|---|---|---|---|
| | | $C$ | $D$ |
| Player 1 | $C$ | $1,1$ | $0,2$ |
| | $D$ | $2,0$ | $0,0$ |

This case approximates the Bertrand price game in which, for example, market demand is perfectly inelastic at two units with a maximum willingness to pay of 1, and firms have zero marginal cost.

Since $d = c$, previous analysis established that $\underline{\alpha} = 0$ and the probability of a type $L$ player cooperating is independent of $\alpha$ and thus constant over time. The unique stationary symmetric affine Markov Perfect Bayesian Equilibrium is[9]

$$\begin{aligned} q(\alpha) &= \frac{\sqrt{4\delta+1}-1}{\sqrt{4\delta+1}+1} \\ V(\alpha) &= \left(\frac{1+\delta-(1-\delta)\sqrt{4\delta+1}}{2\delta(1-\delta)}\right)\alpha \end{aligned}$$

As one would expect, the probability of choosing C is higher when players are more patient:

$$\frac{\partial q}{\partial \delta} = \frac{4}{\left(\sqrt{4\delta+1}+1\right)^2\sqrt{4\delta+1}} > 0$$

The probability that two type $L$ players are colluding by period $T \geq 2$ is

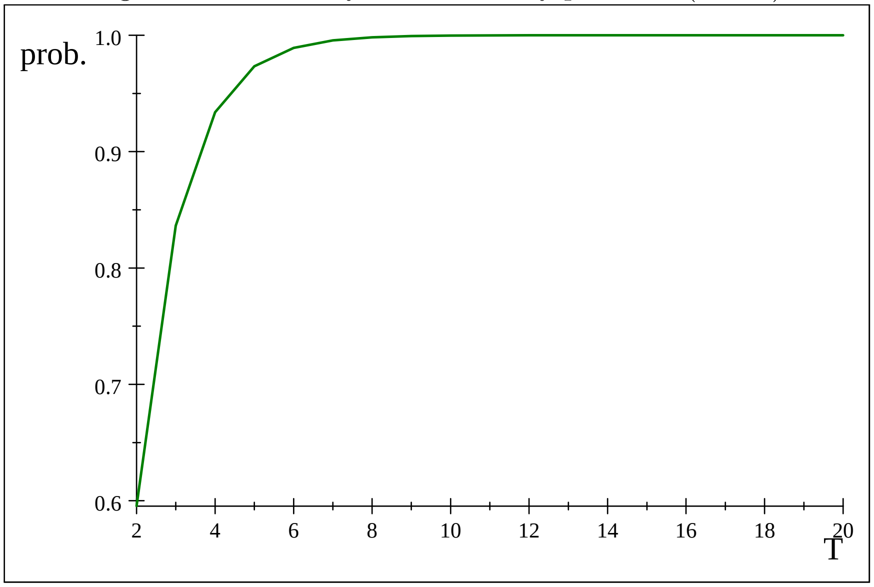
$$1 - \left[1 - \left(\frac{\sqrt{4\delta+1}-1}{\sqrt{4\delta+1}+1}\right)\right]^{2(T-1)}.$$

When $\delta = .9$, Figure 1 shows how the probability of collusion rises rapidly over time,

[9] Derivations for all examples are available on request.

and that it is quite close to one by period 10.

Figure 1: Probability of collusion by period T ($\delta = .9$)




### 5.2 Example 2: Bertrand Price Game with Relative Compensation

Let us modify the Bertrand price game so that managers - not owners - are repeatedly making price decisions and managerial compensation is based on relative performance. Specifically, a manager receives compensation equal to half of firm profit but, in the event that the other firm has higher profit, incurs a penalty equal to one-quarter of the rival firm's profit. The single-period payoff to a manager is then:

$$\text{Payoff of manager } i \text{ in period } t = \begin{cases} (1/2)\,\pi_i^t & \text{if } \pi_i^t \geq \pi_j^t \\ (1/2)\,\pi_i^t - (1/4)\,\pi_j^t & \text{if } \pi_i^t < \pi_j^t \end{cases}$$

where $\pi_i^t$ is the period $t$ profit of firm $i$. If market demand is perfectly inelastic at two units with a maximum willingness to pay of 2 (and zero marginal cost) then the managers' payoff matrix is represented by

Bertrand Price Game
with Relative Compensation
Player 2

| | | $C$ | $D$ |
|---|---|---|---|
| Player 1 | $C$ | $1, 1$ | $-1, 2$ |
| | $D$ | $2, -1$ | $0, 0$ |

The unique stationary symmetric affine Markov Perfect Bayesian Equilibrium is:

$$q(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \left[0, \frac{1-\delta}{\delta}\right] \\ \frac{[\alpha\delta-(1-\delta)]\left(\sqrt{2\delta}-1\right)}{\alpha\sqrt{2\delta}(2\delta-1)} & \text{if } \alpha \in \left(\frac{1-\delta}{\delta}, 1\right] \end{cases}$$

$$V(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \left[0, \frac{1-\delta}{\delta}\right] \\ \frac{[(1+\alpha)\delta-1]\left[\delta-(1-\delta)\sqrt{2\delta}\right]}{\delta(1-\delta)(2\delta-1)} & \text{if } \alpha \in \left(\frac{1-\delta}{\delta}, 1\right] \end{cases}$$

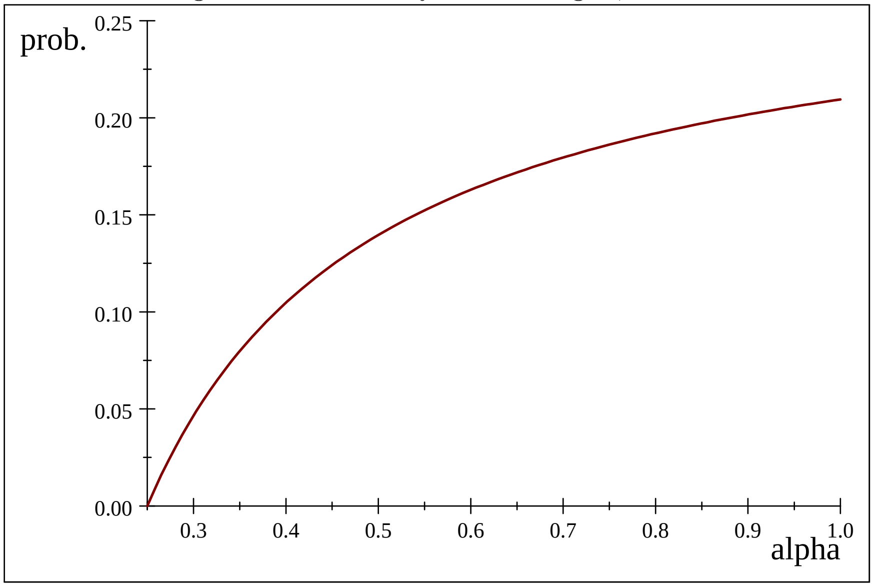
We know from Theorem 6 that, when $\alpha > \frac{1-\delta}{\delta}$ $(= \underline{\alpha})$, $q(\alpha)$ is increasing in $\alpha$.

If $\delta = .8$ then $\underline{\alpha} = .25$ and, for $\alpha > .25$,

$$q(\alpha) \simeq .279 - \frac{.07}{\alpha},$$

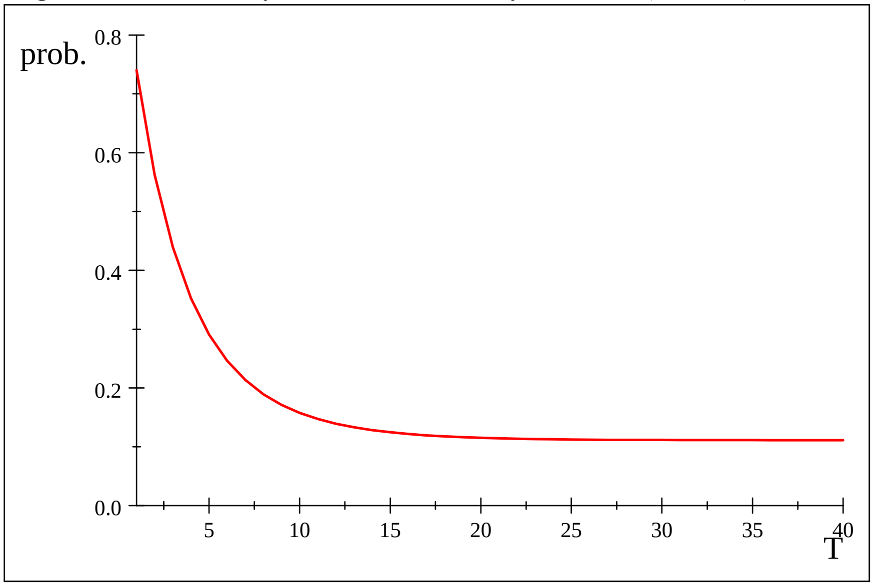
which is plotted in Figure 2. If players have thus far always played D then, in each player updating their beliefs as to the other player's type, $\alpha^t$ will fall over time which then induces type $L$ players to choose C with a lower probability. If a string of (D,D) gets longer and longer, so that $\alpha^t \to \underline{\alpha}$, $q(\alpha) \to 0$ and does so at an increasingly fast rate; note that $q(\alpha)$ is strictly concave in $\alpha$.

Figure 2: Probability of choosing C, $\delta = .8$




When a player initially assigns a 50% chance to its rival being type $L$, the probability that collusion has not been achieved by period $T$ is shown in Figure 3. There is about an 11% chance that collusion is never achieved.

Figure 3: Probability of No Collusion by Period $T$, $\delta = .8$, $\alpha^1 = 0.5$




### 5.3 Example 3: Asymmetric Bertrand Price Game

Consider the following generalization of Example 1 where the collusive outcome is now allowed to be asymmetric and $\gamma \in [1/2, 1)$.[10]

Asymmetric Bertrand Price Game

Player 2

| Player 1 | | *Cooperate* | *Defect* |
|---|---|---|---|
| | *Cooperate* | $\gamma, 1-\gamma$ | $0, 1$ |
| | *Defect* | $1, 0$ | $0, 0$ |

The collusive outcome gives player 1 a market share of $\gamma$ which is at least 1/2. The unique stationary affine MPBE is

$$\begin{aligned}
q_1 &= \frac{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} - \gamma}{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} + \gamma} \\
q_2 &= \frac{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} - (1-\gamma)}{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} + (1-\gamma)} \\
V_2(\alpha_1) &= \left[\frac{(\gamma - \delta + 3\delta(1-\gamma)) - (1-\delta)\sqrt{\gamma(\gamma+4\delta(1-\gamma))}}{2(1-\delta)\delta}\right]\alpha_1 \\
V_1(\alpha_2) &= \left[\frac{(1-\delta-\gamma+3\delta\gamma) - (1-\delta)\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)}}{2(1-\delta)\delta}\right]\alpha_2
\end{aligned}$$

[10] A preliminary analysis suggests that many of the results in Sections 3 and 4 can be extended to when the Prisoners' Dilemma is asymmetric.

As with Example 1, $q_1$ and $q_2$ do not depend on $\alpha$. One can prove that $q_1$ is decreasing in $\gamma$ and increasing in $\delta$, and $q_2$ is increasing in $\gamma$ and $\delta$.

It might be expected that the player with the higher share of collusive profit would play C with a higher probability. However, when the share of collusive profit for player 1 ($\gamma$) is larger, the probability of playing C is actually higher for player 2 and lower for player 1. Since player 1 gains more by achieving cooperative play when $\gamma$ is bigger, player 2 must be more likely to play C if player 1 is to be indifferent between playing C and D; and recall that D is more attractive when the other player is more likely to initiate cooperation. Surprisingly, the player who benefits more from colluding is less likely to take the first move in cooperating.

To explore the effect of asymmetry on the likelihood of collusion, consider the probability that collusion is initiated in any period:

$$1-(1-q_1)(1-q_2)=1-\frac{4}{\left(\sqrt{\frac{\gamma+4\delta(1-\gamma)}{\gamma}}+1\right)\left(\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}}+1\right)}.$$

It is straightforward to show that it is increasing in $\gamma$,

$$\frac{\partial\left[1-(1-q_1)(1-q_2)\right]}{\partial\gamma}>0, \tag{10}$$

so collusion is more likely when the collusive outcome is more skewed to favor one firm.

As the equilibrium condition for the grim trigger strategy is $\delta \geq \gamma$, increasing asymmetry makes collusion more difficult in the sense that the minimum discount factor is higher. However, conditional on the collusive outcome being sustainable, asymmetry reduces the expected time until collusion is achieved, as reflected in (10). In fact, as asymmetry becomes extreme, collusion is achieved immediately.[11]

$$\begin{aligned}
\lim_{\gamma\to 1} q_1(\alpha_1) &= \frac{\sqrt{\gamma(\gamma+4\delta(1-\gamma))}-\gamma}{\sqrt{\gamma(\gamma+4\delta(1-\gamma))}+\gamma}=0\\
\lim_{\gamma\to 1} q_2(\alpha_2) &= \lim_{\gamma\to 1}\frac{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)}-(1-\gamma)}{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)}+(1-\gamma)}=\lim_{\gamma\to 1}\frac{\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}}-1}{\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}}+1}=1
\end{aligned}$$

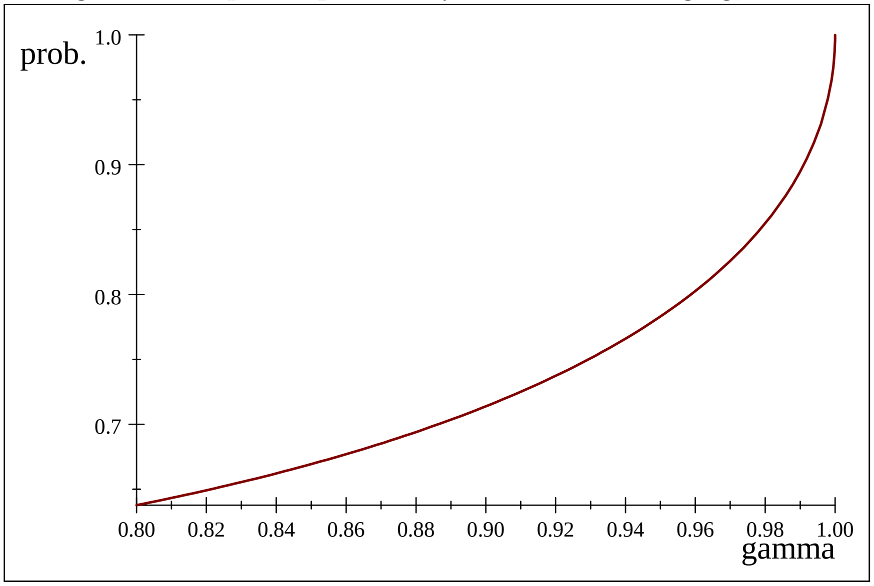
Therefore,

$$\lim_{\gamma\to 1} 1-(1-q_1)(1-q_2)=1.$$

For when $\delta = .8$, Figure 4 depicts the relationship between the asymmetry of the collusive outcome and the probability of collusion emerging, given it has not yet happened.

[11] Keep in mind that as we let $\gamma \to 1$, we must have $\delta \to 1$ so that $\delta \geq \gamma$ is satisfied.

Figure 4: Per period probability of collusion emerging, $\delta = .8$




## 6 Concluding Remarks

The gap between theory and practice on the issue of collusion is probably as great as any in the field of industrial organization. Explicit collusion is illegal and vigorously prosecuted, while tacit collusion is generally legal. Lawyers can meaningfully discuss the distinction between explicit and tacit collusion, but such a discussion is more problematic when it is held among industrial organization theorists. Developing rigorous theoretical notions of explicit and tacit collusion will allow us to characterize those environments conducive to explicit collusion and those conducive to tacit collusion, and also the types of collusive schemes that are implementable using explicit collusion or tacit collusion. Progress on these issues is critical to developing a more sophisticated understanding of collusion that is relevant to the enforcement of competition laws.

In thinking about communication in the context of a formal model, it can manifest itself in two ways - exchange of information and of intentions. Explicit collusion could involve communication on the equilibrium path for the purpose of exchanging information. There is a limited amount of work in oligopoly theory on this topic. In Athey and Bagwell (2001, 2008), firms have private information about their cost and exchange (costless) messages about cost, while in Hanazono and Yang (2007), firms have private signals on demand and seek to share that information. Then there is work in which sales or some other endogenous variable is private information and firms exchange messages for monitoring purposes; see Aoyagi (2002), Chan and Zhang (2009), and Harrington and Skrzypacz (2009).[12] In light of that body of work,

[12] There is also an extensive game theory literature on the issue of private monitoring. See Compte (1998), Kandori and Matsushima (1998), Kandori (2002), Zheng (2008), and Obara (2009)

tacit collusion can be thought of as collusion when firms do not exchange messages. Communication may also be used to resolve strategic uncertainty. Explicit collusion can mean express communication for the purpose of coordinating a move from a non-collusive equilibrium to a collusive equilibrium, while tacit collusion involves more opaque forms of communication. Here, intentions rather than hard information is being communicated.

Within the context of the equilibrium paradigm, the current paper sought to make progress on the tacit signalling of the intention to collude. Our main findings are that if the initial probability that players are capable of colluding is sufficiently high then, in any period, there is always the prospect of collusion emerging. No matter how long is there a history of failed collusion, beliefs as to players being cooperative types remain sufficiently high that it is worthwhile for them to continue to try to cooperate. This does not imply, however, that collusion is assured. For a wide class of situations, there is a positive probability that collusion never emerges. Players never give up trying to collude but they may also never succeed.

While these results are new and, to some extent, surprising, their relevance to understanding tacit collusion among firms in actual markets is not at all clear. While that is our ultimate objective, it was not the goal we set out with this paper. Our goal was simply to start thinking rigorously about tacit collusion in the context of a formal model. By doing so, our hope is that it'll lead to new ways of thinking about the modelling of tacit and explicit collusion, and that those new ways of thinking will shed light on various forms of collusion in actual markets.

In terms of future work, one research direction is to allow a player's type to change over time, rather than remain fixed forever. Recent work by Escobar and Toikka (2009) provides a foundation for such an analysis. When a cooperative type raises price and does not receive a favorable response, it'll infer that its rival is an uncooperative type. In that case, it might be inclined to try again later on the hope that the rival's type has changed. Furthermore, a player who has previously failed to respond in kind may see itself as having the burden in initiating cooperation in the event that its type does change since its rival may be disinclined to try to collude. A deviation would not have different implications as it would be interpreted as a change in a player's type. Assuming persistence in types, the punishment of the deviator would have a certain credibility (beyond simply being an equilibrium) in that the other player believes there is little point in trying to cooperate. Indeed, non-cooperation may be the unique equilibrium. All this could put the burden on the deviator to re-initiate cooperation. Even this cursory analysis suggests that a rich set of behavior could arise from allowing types to evolve stochastically over time.

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# 7 Appendix: Proofs

**Proof of Theorem 1.** Let us first show the properties on $V(\cdot)$ are true, assuming the properties on $q(\cdot)$ hold. First note that, in equilibrium, $V : [0,1] \rightarrow \left[\frac{d}{1-\delta}, \frac{a}{1-\delta}\right]$, as $V(\alpha)$ has a lower bound of $\frac{d}{1-\delta}$ - as a player can assure itself of a payoff of at least $\frac{d}{1-\delta}$ by choosing D every period - and $\frac{a}{1-\delta}$ is an upper bound because the highest average symmetric payoff is $a$. If $q(\alpha) = 0$ then type $L$ players play D for sure in the current period and since $\frac{\alpha(1-q(\alpha))}{1-\alpha q} = \alpha$ then the same is true for all ensuing periods; hence, by stationarity, if $q(\alpha) = 0$ then $V(\alpha) = \frac{d}{1-\delta}$. To show that $V(\alpha) \in \left(\frac{d}{1-\delta}, \frac{a}{1-\delta}\right)$ when $\alpha \in (\underline{\alpha}, 1]$, note that $q(\alpha) \in (0,1)$ implies $V(\alpha) = W^C(\alpha) = W^D(\alpha)$. $\frac{d}{1-\delta}$ is a lower bound on $V(\alpha)$ for all $\alpha$ since at least that value can be achieved by choosing D in every period. Thus, from (3) we have:

$$
\begin{aligned}
V(\alpha) &= \alpha q\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right) \\
&\geq \alpha q\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta \frac{d}{1-\delta}\right) \\
&> \frac{d}{1-\delta} + \alpha q \frac{a-d}{1-\delta} \\
&> \frac{d}{1-\delta}
\end{aligned}
$$

since $b > a > d$. From (2) we have:

$$
\begin{aligned}
V(\alpha) &= \alpha q(a-c) + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta} \\
&= a - (a-c) + \alpha q(a-c) + \frac{\alpha\delta(a-d)}{1-\delta} - \frac{\delta(a-d)}{1-\delta} + \frac{\delta a}{1-\delta} \\
&= \frac{a}{1-\delta} - (1-\alpha q(\alpha))(a-c) - (1-\alpha)\frac{\delta(a-d)}{1-\delta} < \frac{a}{1-\delta}
\end{aligned}
$$

since $a > c, d$. This establishes the properties on $V(\cdot)$.

Let us now establish the stated properties on $q(\cdot)$. A player strictly prefers D to C iff:

$$
\alpha q\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right) > \alpha q(a-c) + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta}. \tag{11}
$$

Since $V\left(\frac{\alpha(1-q)}{1-\alpha q}\right) \geq \frac{d}{1-\delta}$, a sufficient condition for (11) involves substituting $\frac{d}{1-\delta}$ for $V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)$:

$$
\alpha q\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(\frac{d}{1-\delta}\right) > \alpha q(a-c) + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta} \tag{12}
$$

or

$$
\alpha q\left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(\frac{d}{1-\delta}\right) - \alpha q(a-c) - c - \frac{\alpha\delta(a-d)}{1-\delta} - \frac{\delta d}{1-\delta} > 0. \tag{13}
$$

Take the derivative of the LHS of (13) with respect to $q$:

$$\alpha\left(b+\frac{\delta a}{1-\delta}\right)-\alpha\left(\frac{d}{1-\delta}\right)-\alpha\left(a-c\right)=\alpha\left[\left(b-a\right)-\left(d-c\right)\right]+\alpha\delta\left(\frac{a-d}{1-\delta}\right)>0,$$

since $b-a\geq d-c$ and $a-d>0$. Hence, the difference between the payoff to D and the payoff to C is minimized when $q=0$. Thus, D is surely strictly preferred to C if (12) holds when $q=0$:

$$\frac{d}{1-\delta}>c+\frac{\alpha\delta\left(a-d\right)}{1-\delta}+\frac{\delta d}{1-\delta}, \tag{14}$$

which is equivalent to

$$d>c+\frac{\alpha\delta\left(a-d\right)}{1-\delta}\Leftrightarrow\left(1-\delta\right)\left(d-c\right)>\alpha\delta\left(a-d\right)\Leftrightarrow\left(\underline{\alpha}\equiv\right)\frac{\left(1-\delta\right)\left(d-c\right)}{\delta\left(a-d\right)}>\alpha.$$

Thus, if $\alpha<\underline{\alpha}$ then, in equilibrium, $q\left(\alpha\right)=0$.

To prove that $q\left(\underline{\alpha}\right)=0$, suppose not. It follows from $q\left(\underline{\alpha}\right)>0$ that

$$\frac{\underline{\alpha}\left(1-q\left(\underline{\alpha}\right)\right)}{1-\underline{\alpha}q\left(\underline{\alpha}\right)}<\underline{\alpha}$$

and, by the preceding analysis,

$$V\left(\frac{\underline{\alpha}\left(1-q\left(\underline{\alpha}\right)\right)}{1-\underline{\alpha}q\left(\underline{\alpha}\right)}\right)=\frac{d}{1-\delta}. \tag{15}$$

For $q\left(\underline{\alpha}\right)>0$, the expected payoff from choosing D must equal that from choosing C for some $q>0$:

$$\alpha q\left(b+\frac{\delta a}{1-\delta}\right)+\left(1-\alpha q\right)\left(\frac{d}{1-\delta}\right)=\alpha q\left(a-c\right)+c+\frac{\alpha\delta\left(a-d\right)}{1-\delta}+\frac{\delta d}{1-\delta}, \tag{16}$$

where we used (15). However, notice that the LHS and RHS of (16) are exactly the same as in (12). By the previous analysis, if $\alpha=\underline{\alpha}$ then the payoff to D and C are equal when $q=0$ and the payoff to D exceeds that from C when $q>0$ in which case (16) cannot hold. We conclude that $q\left(\underline{\alpha}\right)=0$.

Finally, let us prove that if $\alpha\in\left(\underline{\alpha},1\right]$ then $q\left(\alpha\right)\in\left(0,1\right)$. To show that $q\left(\alpha\right)>0$, suppose not so $\exists\alpha'>\underline{\alpha}$ such that $q\left(\alpha'\right)=0$. By the preceding logic, $V\left(\alpha'\right)=\frac{d}{1-\delta}$. In that case, the payoff to D is at least as great as that from C iff (14) holds with a weak inequality, but the previous analysis showed that is the case iff $\alpha\leq\underline{\alpha}$. Therefore, if $\alpha\in\left(\underline{\alpha},1\right]$ then $q\left(\alpha\right)>0$. To show that $q\left(\alpha\right)<1$, consider the payoffs from C and D when the other player (if type $L$) chooses C for sure:

$$\begin{aligned}
\text{Play C} &: \quad \alpha\left(a+\frac{\delta a}{1-\delta}\right)+\left(1-\alpha\right)\left(c+\delta\left(\frac{d}{1-\delta}\right)\right)\\
\text{Play D} &: \quad \alpha\left(b+\frac{\delta a}{1-\delta}\right)+\left(1-\alpha\right)\left(d+\delta\left(\frac{d}{1-\delta}\right)\right)
\end{aligned}$$

Since choosing D yields a strictly higher payoff, it cannot be the case that $q(\alpha) = 1$. Therefore, $q(\alpha) < 1$. ■

**Proof of Theorem 2.** To show that $\alpha > \underline{\alpha}$ implies $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)} > \underline{\alpha}$, suppose not so that $\exists \alpha' > \underline{\alpha}$ such that $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')} \leq \underline{\alpha}$. By the proof of Theorem 1, $V\left(\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')}\right) = \frac{d}{1-\delta}$ and, from (4), we have:

$$\alpha' q(\alpha') = \frac{\delta\left(\frac{a-d}{1-\delta}\right) - (d-c) - (1-\alpha')\frac{\delta(a-d)}{1-\delta}}{\delta\left(\frac{a-d}{1-\delta}\right) - (d-c) + (b-a)} = \frac{\alpha'\delta\left(\frac{a-d}{1-\delta}\right) - (d-c)}{\delta\left(\frac{a-d}{1-\delta}\right) - (d-c) + (b-a)}. \quad (17)$$

We've made the supposition

$$\underline{\alpha} \geq \frac{\alpha'(1-q(\alpha'))}{1-\alpha' q^*(\alpha')}$$

which is equivalent to

$$\alpha' q(\alpha') \geq \frac{\alpha' - \underline{\alpha}}{1-\underline{\alpha}}. \quad (18)$$

Substituting (17) into (18):

$$\begin{aligned}
\frac{\alpha'\delta\left(\frac{a-d}{1-\delta}\right) - (d-c)}{\delta\left(\frac{a-d}{1-\delta}\right) - (d-c) + (b-a)} &\geq \frac{\alpha' - \underline{\alpha}}{1-\underline{\alpha}} \\
\frac{\alpha'\delta\left(\frac{a-d}{1-\delta}\right) - (d-c)}{\delta\left(\frac{a-d}{1-\delta}\right) - (d-c) + (b-a)} &\geq \frac{\alpha' - \frac{(1-\delta)(d-c)}{\delta(a-d)}}{1 - \frac{(1-\delta)(d-c)}{\delta(a-d)}} \\
\frac{\delta(a-d)\alpha' - (1-\delta)(d-c)}{\delta(a-d) - (1-\delta)(d-c) + (1-\delta)(b-a)} &\geq \frac{\delta(a-d)\alpha' - (1-\delta)(d-c)}{\delta(a-d) - (1-\delta)(d-c)} \\
\delta(a-d) - (1-\delta)(d-c) &\geq \delta(a-d) - (1-\delta)(d-c) + (1-\delta)(b-a) \\
0 &\geq (1-\delta)(b-a),
\end{aligned}$$

which is not true. Hence, $\nexists \alpha' > \underline{\alpha}$ such that $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')} \leq \underline{\alpha}$ which means if $\alpha' > \underline{\alpha}$ then $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')} > \underline{\alpha}$.

Next consider: if $\alpha^1 > \underline{\alpha}$ then $lim_{t\to\infty}\alpha^t = \underline{\alpha}$. By Bayes rule,

$$\alpha^{t+1} = \alpha^t\left(\frac{1-q^t}{1-\alpha^t q^t}\right) \Rightarrow \alpha^{t+1} < \alpha^t.$$

By part (i) of this theorem, if $\alpha^1 > \underline{\alpha}$ then $\underline{\alpha}$ is a lower bound of the sequence $\{\alpha^t\}$. Hence, $\{\alpha^t\}$ has a limit and it is sufficient to show that $\underline{\alpha}$ is the infimum of $\{\alpha^t\}$. Suppose not, and let $\alpha' > \underline{\alpha}$ be the infimum of $\{\alpha^t\}$. Then as $\alpha^t \to \alpha'$, $\alpha^{t+1} \to \alpha^t$, which indicates $q^t \to 0$. As $q^t \to 0$, $V(\alpha^{t+1}) \to \frac{d}{1-\delta}$. But we know from the proof of Theorem 1 that the payoff to D is the same as the payoff from C iff $\alpha^t \to \underline{\alpha}$, which contradicts $\alpha^t \to \alpha'$ and $\alpha' > \underline{\alpha}$. Therefore, $lim_{t\to\infty}\alpha^t = \underline{\alpha}$, for $\alpha^1 > \underline{\alpha}$.

That $\alpha^1 > \underline{\alpha}$ implies $q\left(\alpha^t\right) > 0\ \forall t$ immediately follows from $\alpha^t > \underline{\alpha}\ \forall t$ and Theorem 1.

Next let us show that $lim_{\alpha\downarrow\underline{\alpha}} q(\alpha) = 0$ when $\underline{\alpha} > 0$. It has already been proven: if $\alpha^1 > \underline{\alpha}$ then $lim_{t\to\infty}\alpha^t = \underline{\alpha}$. Therefore,

$$\lim_{\alpha\downarrow\underline{\alpha}} \frac{\alpha\left(1-q(\alpha)\right)}{1-\alpha q(\alpha)} = \underline{\alpha}\,(>0)\,,$$

which implies $lim_{\alpha\downarrow\underline{\alpha}} q(\alpha) = 0$.

Finally, it is easy to prove $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right) = 0$. If $\alpha^1 \leq \underline{\alpha}$ then $q\left(\alpha^t\right) = 0 \forall t$ and therefore $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right) = 0$. If $\alpha^1 > \underline{\alpha} > 0$ then, by the other results of Theorem 2, $\lim_{t\to\infty}\alpha^t = \underline{\alpha}$ and $\lim_{\alpha\downarrow\underline{\alpha}} q(\alpha) = 0$ which implies $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right) = 0$. If $\alpha^1 > \underline{\alpha} = 0$ then $\lim_{t\to\infty}\alpha^t = 0$ which implies $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right) = 0$. ■

**Proof of Theorem 4.** Re-arranging (4), an equilibrium $q(\cdot)$ is defined by

$$\alpha q\left[(b-a)-(d-c)\right]+(1-\alpha)\frac{\delta(a-d)}{1-\delta}+(d-c) = \delta(1-\alpha q)\left[\frac{a}{1-\delta} - V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right] \tag{19}$$

Conjecturing that the value function is linear in $\alpha$,

$$V(\alpha) = x + y\alpha, \tag{20}$$

substitute (20) into (19).

$$\begin{aligned} & \alpha q\left[(b-a)-(d-c)\right]+(1-\alpha)\frac{\delta(a-d)}{1-\delta}+(d-c) \\ = \; & \delta(1-\alpha q)\left[\frac{a}{1-\delta} - x - y\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right] \end{aligned} \tag{21}$$

$$\alpha q\left[(b-a)-(d-c)\right]+(1-\alpha)\frac{\delta(a-d)}{1-\delta}+(d-c) = \delta(1-\alpha q)\frac{a}{1-\delta} - \delta(1-\alpha q)x - \delta y\alpha(1-q)$$

$$\alpha q\left\{\left[(b-a)-(d-c)\right]+\frac{\delta a}{1-\delta} - \delta x - \delta y\right\} = \frac{\delta a}{1-\delta} - \delta x - \delta y\alpha - (1-\alpha)\frac{\delta(a-d)}{1-\delta} - (d-c)$$

$$\begin{aligned} \alpha q &= \frac{\frac{\delta a}{1-\delta} - \delta x - \delta y\alpha - (1-\alpha)\frac{\delta(a-d)}{1-\delta} - (d-c)}{\left[(b-a)-(d-c)\right]+\frac{\delta a}{1-\delta} - \delta x - \delta y} \\ \alpha q &= \frac{\delta a - \delta(1-\delta)(x+y\alpha) - (1-\alpha)\delta(a-d) - (1-\delta)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a - \delta(1-\delta)(x+y)} \end{aligned}$$

$$\begin{aligned} \alpha q &= \alpha\left[\frac{\delta(a-d)-\delta(1-\delta)y}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right] \\ &\quad + \frac{\delta a - \delta(1-\delta)x - \delta(a-d) - (1-\delta)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a - \delta(1-\delta)(x+y)} \end{aligned} \tag{22}$$

Thus, $\alpha q$ is affine in $\alpha$ if the value function is affine in $\alpha$. As a player is indifferent between playing C and D, the value can be given by the payoff to choosing C for sure:

$$V(\alpha) = \alpha q (a-c) + \frac{\alpha\delta(a-d)}{1-\delta} + c + \frac{\delta d}{1-\delta}.$$

The value function is affine in $\alpha q$ and, since $\alpha q$ is affine in $\alpha$, $V(\alpha)$ is affine in $\alpha$.

The next step is to show that there exist unique values for $x$ and $y$. Using the payoff to playing C, in equilibrium the value function equals:

$$\begin{aligned}
V(\alpha) &= \alpha q (a-c) + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta} \\
&= \alpha \left[\frac{\delta(a-d) - \delta(1-\delta)y}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)}\right](a-c) \\
&\quad + \left[\frac{\delta a - \delta(1-\delta)x - \delta(a-d) - (1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)}\right](a-c) \\
&\quad + c + \frac{\alpha\delta(a-d)}{1-\delta} + \frac{\delta d}{1-\delta}
\end{aligned}$$

$$\begin{aligned}
= \ & \alpha\left[\frac{\delta(a-c)[(a-d)-(1-\delta)y]}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)} + \frac{\delta(a-d)}{1-\delta}\right] \\
& + (a-c)\left[\frac{\delta a - \delta(1-\delta)x - \delta(a-d) - (1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)}\right] \\
& + c + \frac{\delta d}{1-\delta}
\end{aligned} \tag{23}$$

Equating coefficients between (20) and (23), we have

$$\begin{aligned}
x &= (a-c)\left[\frac{\delta d - \delta(1-\delta)x - (1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)}\right] \\
&\quad + c + \frac{\delta d}{1-\delta}
\end{aligned} \tag{24}$$

$$y = \frac{\delta(a-c)[(a-d)-(1-\delta)y]}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta(1-\delta)(x+y)} + \frac{\delta(a-d)}{1-\delta} \tag{25}$$

To show that there is a unique solution to (24)-(25), define $z \equiv x+y$ and note that:

$$z = x + y = V(1) = W^C(1) = Q(a-c) + \frac{\delta(a-d)}{1-\delta} + c + \frac{\delta d}{1-\delta},$$

where $Q = q(1)$. Simplifying the preceding equation gives:

$$z = Q(a-c) + \frac{\delta a}{1-\delta} + c. \tag{26}$$

If we can show that there exists a unique $Q \in (0,1)$ satisfying the equilibrium condition (21) when $\alpha = 1$, then $z = x + y = V(1)$ is unique.

Evaluating (21) at $\alpha = 1$, we have:

$$\begin{aligned}
Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta\left(1-Q\right)\left[\frac{a}{1-\delta}-x-y\left(\frac{1-Q}{1-Q}\right)\right] \\
Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta\left(1-Q\right)\left[\frac{a}{1-\delta}-z\right] \\
Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta\left(1-Q\right)\left[\frac{a}{1-\delta}-\left(Q(a-c)+\frac{\delta a}{1-\delta}+c\right)\right] \\
Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta\left(1-Q\right)^{2}\left(a-c\right),
\end{aligned}$$

and re-arranging gives us

$$\delta\left(a-c\right)Q^{2}-\left[2\delta\left(a-c\right)+(b-a)-(d-c)\right]Q+\left[\delta\left(a-c\right)-(d-c)\right]=0.$$

This quadratic has two solutions:

$$Q=\frac{2\delta\left(a-c\right)+(b-a)-(d-c)\pm\sqrt{\Omega}}{2\delta\left(a-c\right)},$$

where

$$\Omega \equiv \left[2\delta\left(a-c\right)+(b-a)-(d-c)\right]^{2}-4\delta\left(a-c\right)\left[\delta\left(a-c\right)-(d-c)\right] \tag{27}$$

$$\begin{aligned}
&= 4\delta^{2}\left(a-c\right)^{2}+\left[(b-a)-(d-c)\right]^{2}+4\delta\left(a-c\right)\left[(b-a)-(d-c)\right] \\
&\quad -4\delta^{2}\left(a-c\right)^{2}+4\delta\left(a-c\right)\left(d-c\right) \\
&= \left[(b-a)-(d-c)\right]^{2}+4\delta\left(a-c\right)\left[(b-a)-(d-c)\right]+4\delta\left(a-c\right)\left(d-c\right) \\
&> 0
\end{aligned}$$

since $a > c$, $d \geq c$ and $b + c \geq a + d$; and recall that the assumption $b > a$ implies $d = c$ and $b + c = a + d$ cannot both hold. Hence, the two solutions are real. Next note that the bigger root exceeds one:

$$Q^{b}=1+\frac{(b-a)-(d-c)+\sqrt{\Omega}}{2\delta\left(a-c\right)}>1.$$

Thus, we only need to show that the smaller root falls in $(0, 1)$.

$$Q^{s}=1+\frac{(b-a)-(d-c)-\sqrt{\Omega}}{2\delta\left(a-c\right)}<1$$

if and only if

$$\begin{aligned}
(b-a)-(d-c) &< \sqrt{\Omega}\Leftrightarrow\left[(b-a)-(d-c)\right]^{2}<\Omega\Leftrightarrow \\
\left[(b-a)-(d-c)\right]^{2} &< \left[(b-a)-(d-c)\right]^{2}+4\delta\left(a-c\right)\left[(b-a)-(d-c)\right]+4\delta\left(a-c\right)\left(d-c\right),
\end{aligned}$$

which is equivalent to

$$4\delta\left(a-c\right)\left[(b-a)-(d-c)\right]+4\delta\left(a-c\right)\left(d-c\right)>0,$$

and, therefore, $Q^s < 1$. $Q^s > 0$ if and only if

$$\begin{aligned} 2\delta (a-c) + (b-a) - (d-c) &> \sqrt{\Omega} \\ [2\delta (a-c) + (b-a) - (d-c)]^2 &> \Omega. \end{aligned}$$

From (27), the preceding condition is equivalent to

$$4\delta (a-c) [\delta (a-c) - (d-c)] > 0,$$

which holds since

$$\delta (a-c) - (d-c) > 0 \Leftrightarrow \delta > \frac{d-c}{a-c}.$$

The last property follows from $\delta > \frac{b-a}{b-d} \geq \frac{d-c}{a-c}$.

There then exists a unique $Q \in (0,1)$, and $z = x + y = V(1)$ is unique since it is linear in $Q$. In addition, plugging $Q^s$ in (26) gives

$$\begin{aligned} z &= \frac{2\delta (a-c) + (b-a) - (d-c) - \sqrt{\Omega}}{2\delta} + \frac{\delta a}{1-\delta} + c \\ &= \frac{a}{1-\delta} + \frac{(b-a) - (d-c) - \sqrt{\Omega}}{2\delta} \\ &= \frac{2a\delta + (1-\delta)\left[(b-a) - (d-c) - \sqrt{\Omega}\right]}{2\delta (1-\delta)}. \end{aligned}$$

To close the model, use the initial condition

$$V(\underline{\alpha}) = \frac{d}{1-\delta},$$

which takes the form:

$$x = \frac{d}{1-\delta} - y\frac{(1-\delta)(d-c)}{\delta (a-d)}.$$

$x^*$ is then the unique solution to

$$x^* = \frac{d}{1-\delta} - (z - x^*) \frac{(1-\delta)(d-c)}{\delta (a-d)},$$

and $y^*$ is the unique solution to: $y^* = z - x^*$. This completes the proof that there is a unique stationary symmetric affine MPBE. Finally, solving for $q$ from (22) gives us (9). ■

**Proof of Theorem 5.** Since the equilibrium probability of choosing C is

$$\begin{aligned} \alpha q(\alpha) &= \alpha \left[ \frac{\delta (a-d) - \delta (1-\delta) y}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta (1-\delta)(x+y)} \right] \\ &\quad + \left[ \frac{\delta a - \delta (1-\delta) x - \delta (a-d) - (1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)] + \delta a - \delta (1-\delta)(x+y)} \right] \end{aligned}$$

then $\alpha q(\alpha)$ is increasing in $\alpha$ iff

$$\frac{\delta(a-d)-\delta(1-\delta)y}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)}>0. \tag{28}$$

By assumption

$$(b-a)-(d-c)\geq 0,$$

and $V(1)<\frac{a}{1-\delta}$ implies

$$\frac{a}{1-\delta}-(x+y)>0. \tag{29}$$

Thus, (28) is true iff the numerator is positive:

$$(a-d)-(1-\delta)y>0. \tag{30}$$

Suppose (30) was not true. From (25), we have

$$y=\frac{\delta(a-c)[(a-d)-(1-\delta)y]}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)}+\frac{\delta(a-d)}{1-\delta}. \tag{31}$$

If (30) is not true then the first term of (31) is non-positive, but then (31) implies

$$y\leq\frac{\delta(a-d)}{1-\delta}$$

which contradicts the supposition that (30) is not true. From this contradiction, we conclude (30) and thus $\alpha q(\alpha)$ is increasing in $\alpha$.

To show that $V(\alpha)$ is increasing in $\alpha$, recall that

$$V(\alpha)=\alpha q(\alpha)(a-c)+\frac{\alpha\delta(a-d)}{1-\delta}+c+\frac{\delta d}{1-\delta}.$$

That $\alpha q(\alpha)$ is increasing in $\alpha$ delivers the result. ■

**Proof of Theorem 6.** For $\alpha\leq\underline{\alpha}$, $q(\alpha)=0$, so it is non-decreasing in $\alpha$ for $\alpha\in[0,\underline{\alpha}]$. From hereon, suppose $\alpha>\underline{\alpha}$ so that

$$\begin{aligned} q(\alpha) &= \frac{\delta(a-d)-\delta(1-\delta)y}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)} \\ &\quad+\left(\frac{1}{\alpha}\right)\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-(1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)}\right]\end{aligned}$$

Thus, $q(\alpha)$ is increasing in $\alpha$ iff

$$\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-(1-\delta)(d-c)}{(1-\delta)[(b-a)-(d-c)]+\delta a-\delta(1-\delta)(x+y)}\right]<0 \tag{32}$$

The denominator of the LHS of (32) is positive because $b-a\geq d-c$ by assumption and

$$\frac{a}{1-\delta}>x+y$$

as shown in (29). Thus, (32) is true iff the numerator is negative:

$$\delta a - \delta\left(1-\delta\right)x - \delta\left(a-d\right) - \left(1-\delta\right)\left(d-c\right) < 0. \tag{33}$$

Suppose (33) was not true. From (24), we would then have

$$x \geq c + \frac{\delta d}{1-\delta}$$

which implies

$$\delta a - \delta\left(1-\delta\right)x - \delta\left(a-d\right) - \left(1-\delta\right)\left(d-c\right) \leq \delta a - \delta\left(1-\delta\right)\left(c + \frac{\delta d}{1-\delta}\right) - \delta\left(a-d\right) - \left(1-\delta\right)\left(d-c\right) \tag{34}$$

By rearranging terms, the RHS of (34) is equivalent to

$$-(1-\delta)^2(d-c) \tag{35}$$

which is negative iff $d > c$. Hence, the LHS of (33) is negative for $d > c$, which contradicts the supposition that (33) is not true. From this contradiction, we conclude (33) is true for $d > c$. Namely, $q\left(\alpha\right)$ is increasing in $\alpha$ for $d > c$.

If $d = c$, (35) implies

$$\delta a - \delta\left(1-\delta\right)x - \delta\left(a-d\right) - \left(1-\delta\right)\left(d-c\right) = 0 \Rightarrow \frac{\partial q\left(\alpha\right)}{\partial \alpha} = 0.$$

■

**Proof of Theorem 7.** First note that if $\alpha^1 \leq \underline{\alpha}$ then $q^t = 0$ $\forall t$ in which case $Q^T = 1$. From hereon, assume $\alpha^1 \in \left(\underline{\alpha}, 1\right)$. If $d > c$ then, with the affine MPBE,

$$q\left(\alpha\right) = A + B\left(\frac{1}{\alpha}\right)$$

for some $A$ and $B$ where $B < 0$ and $A + B < 1$. Then

$$\alpha^t = \frac{\alpha^{t-1}\left(1-q^{t-1}\right)}{1-\alpha^{t-1}q^{t-1}} = \frac{\alpha^{t-1}\left(1-A-\frac{B}{\alpha^{t-1}}\right)}{1-\alpha^{t-1}\left(A+\frac{B}{\alpha^{t-1}}\right)}$$

$$q^t = A + B\left(\frac{1}{\alpha^t}\right) = A + B\left(\frac{1-\alpha^{t-1}\left(A+\frac{B}{\alpha^{t-1}}\right)}{\alpha^{t-1}\left(1-A-\frac{B}{\alpha^{t-1}}\right)}\right) \tag{36}$$

Since $B \neq 0$, we can invert

$$q^{t-1} = A + B\left(\frac{1}{\alpha^{t-1}}\right)$$

to derive

$$\alpha^{t-1} = \frac{B}{q^{t-1}-A}.$$

Insert this expression in (36),

$$
\begin{aligned}
q^t &= A + B\left(\frac{1-\alpha^{t-1}\left(A+\frac{B}{\alpha^{t-1}}\right)}{\alpha^{t-1}\left(1-A-\frac{B}{\alpha^{t-1}}\right)}\right) \\
q^t &= A + B\left(\frac{1-\left(\frac{B}{q^{t-1}-A}\right)\left(A+\frac{B}{\frac{B}{q^{t-1}-A}}\right)}{\left(\frac{B}{q^{t-1}-A}\right)\left(1-A-\frac{B}{\frac{B}{q^{t-1}-A}}\right)}\right) \\
q^t &= A + B\left(\frac{1-\left(\frac{B}{q^{t-1}-A}\right)\left(A+q^{t-1}-A\right)}{\left(\frac{B}{q^{t-1}-A}\right)\left(1-A-q^{t-1}+A\right)}\right) \\
q^t &= A + B\left(\frac{\frac{q^{t-1}-A-Bq^{t-1}}{q^{t-1}-A}}{\frac{B(1-q^{t-1})}{q^{t-1}-A}}\right) = A+\left(\frac{q^{t-1}-A-Bq^{t-1}}{1-q^{t-1}}\right)
\end{aligned}
$$

$$
q^t = A+\left(\frac{q^{t-1}-A-Bq^{t-1}}{1-q^{t-1}}\right) = \frac{A\left(1-q^{t-1}\right)+q^{t-1}-A-Bq^{t-1}}{1-q^{t-1}} = q^{t-1}\left(\frac{1-A-B}{1-q^{t-1}}\right)
$$

$$
q^t = q^{t-1}\left(\frac{1-A-B}{1-q^{t-1}}\right) \tag{37}
$$

By $B<0$ and $\alpha^t<1$, we have

$$
A+B > A+B\left(\frac{1}{\alpha^t}\right) = q^t,\ \forall\ t.
$$

By $B<0$ and that $\alpha^t$ decreasing over time, we have that $q^t$ decreasing over time. Hence,

$$
1-q^1 \leq 1-q^{t-1},\ \forall\ t>2,
$$

Therefore,

$$
q^t \leq \left(\frac{1-A-B}{1-q^1}\right)q^{t-1}.
$$

As this holds for all $t$, it implies

$$
q^t \leq \left(\frac{1-A-B}{1-q^1}\right)^{t-1} q^1 = \nu^{t-1}q,
$$

where $\nu \equiv \left(\frac{1-A-B}{1-q^1}\right) \in (0,1)$. Hence,

$$
\prod_{t=1}^{T}\left(1-q^t\right)^2 > \left[\prod_{t=1}^{T}\left(1-\nu^{t-1}q\right)\right]^2.
$$

To prove this theorem, it is then sufficient to show

$$
\lim_{T\to\infty}\prod_{t=1}^{T}\left(1-\nu^{t-1}q\right) > 0,
$$

which is equivalent to

$$\lim_{T\to\infty}\prod_{t=1}^{T}\left(1-\nu^{t}q\right)>0,$$

which, because $q\in(0,1)$, is true if

$$\lim_{T\to\infty}\prod_{t=1}^{T}\left(1-\nu^{t}\right)>0,$$

which is equivalent to

$$\sum_{t=1}^{\infty}\log\left(1-\nu^{t}\right)>-\infty.$$

Since $\nu\in(0,1)$ then

$$\begin{aligned}
&\sum_{t=1}^{\infty}\log\left(1-\nu^{t}\right)\\
= & -\left[\left(\nu+\frac{\nu^2}{2}+\frac{\nu^3}{3}+...\right)+\left(\nu^2+\frac{\nu^4}{2}+\frac{\nu^6}{3}+...\right)+...+\left(\nu^t+\frac{\nu^{2t}}{2}+\frac{\nu^{3t}}{3}+...\right)+...\right]
\end{aligned}$$

Some manipulation yields the desired result:

$$\begin{aligned}
& -\left[\left(\nu+\frac{\nu^2}{2}+\frac{\nu^3}{3}+...\right)+\left(\nu^2+\frac{\nu^4}{2}+\frac{\nu^6}{3}+...\right)+...+\left(\nu^t+\frac{\nu^{2t}}{2}+\frac{\nu^{3t}}{3}+...\right)+...\right]\\
= & -\left[\left(\nu+\nu^2+\nu^3+...\right)+\frac{1}{2}\left(\nu^2+\nu^4+\nu^6+...\right)+\frac{1}{3}\left(\nu^3+\nu^6+\nu^9+...\right)+...\right]\\
= & -\left(\frac{\nu}{1-\nu}+\frac{1}{2}\frac{\nu^2}{1-\nu^2}+\frac{1}{3}\frac{\nu^3}{1-\nu^3}+...\right)=-\frac{\nu}{1-\nu}\left(1+\frac{1}{2}\frac{\nu}{1+\nu}+\frac{1}{3}\frac{\nu^2}{1+\nu+\nu^2}+...\right)\\
\geq & -\frac{\nu}{1-\nu}\left(1+\nu+\nu^2+...\right)=-\frac{\nu}{\left(1-\nu\right)^2}>-\infty
\end{aligned}$$

■