

Frahm, Gabriel

**Working Paper**

## Asymptotic distributions of robust shape matrices and scales

Discussion Papers in Statistics and Econometrics, No. 5/07

**Provided in Cooperation with:**

University of Cologne, Institute of Econometrics and Statistics

*Suggested Citation:* Frahm, Gabriel (2008) : Asymptotic distributions of robust shape matrices and scales, Discussion Papers in Statistics and Econometrics, No. 5/07, University of Cologne, Seminar of Economic and Social Statistics, Cologne

This Version is available at:

<https://hdl.handle.net/10419/44943>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# DISCUSSION PAPERS IN STATISTICS AND ECONOMETRICS

SEMINAR OF ECONOMIC AND SOCIAL STATISTICS  
UNIVERSITY OF COLOGNE

No. 5/07

## Asymptotic Distributions of Robust Shape Matrices and Scales

by

Gabriel Frahm

8<sup>th</sup> version

November 19, 2008



## DISKUSSIONSBEITRÄGE ZUR STATISTIK UND ÖKONOMETRIE

SEMINAR FÜR WIRTSCHAFTS- UND SOZIALSTATISTIK  
UNIVERSITÄT ZU KÖLN

Albertus-Magnus-Platz, D-50923 Köln, Deutschland

This page intentionally left blank.

# DISCUSSION PAPERS IN STATISTICS AND ECONOMETRICS

SEMINAR OF ECONOMIC AND SOCIAL STATISTICS  
UNIVERSITY OF COLOGNE

No. 5/07

## Asymptotic Distributions of Robust Shape Matrices and Scales<sup>1</sup>

by

Gabriel Frahm<sup>2</sup>

8<sup>th</sup> version

November 19, 2008

### Abstract

It has been frequently observed in the literature that many multivariate statistical methods require the covariance or dispersion matrix  $\Sigma$  of an elliptical distribution only up to some scaling constant. If the topic of interest is not the scale but only the shape of the elliptical distribution, it is not meaningful to focus on the asymptotic distribution of an estimator for  $\Sigma$  or another matrix  $\Gamma \propto \Sigma$ . In the present work, robust estimators for the shape matrix and the associated scale are investigated. Explicit expressions for their joint asymptotic distributions are derived. It turns out that if the joint asymptotic distribution is normal, the presented estimators are asymptotically independent for one and only one specific choice of the scale function. If it is non-normal (this holds for example if the estimators for the shape matrix and scale are based on the minimum volume ellipsoid estimator) only the presented scale function leads to asymptotically uncorrelated estimators. This is a generalization of a result obtained by Paindaveine (2008) in the context of local asymptotic normality theory.

*Keywords:* local asymptotic normality, M-estimator, R-estimator, robust covariance matrix estimator, scale-invariant function, S-estimator, shape matrix, Tyler's M-estimator.

*AMS Subject Classification:* Primary 62H20, Secondary 62E20.

---

<sup>1</sup>This paper has been previously circulated under the title *Asymptotic covariance matrices of different shape matrix estimators*.

<sup>2</sup>Department of Economic and Social Statistics, Statistics & Econometrics, University of Cologne, Albertus-Magnus-Platz, 50923 Cologne, Germany.

This page intentionally left blank.

# Asymptotic Distributions of Robust Shape Matrices and Scales

Gabriel Frahm<sup>1</sup>

*University of Cologne  
Statistics and Econometrics  
Meister-Ekkehart-Str. 9  
50937 Cologne  
Germany*

---

## Abstract

It has been frequently observed in the literature that many multivariate statistical methods require the covariance or dispersion matrix  $\Sigma$  of an elliptical distribution only up to some scaling constant. If the topic of interest is not the scale but only the shape of the elliptical distribution, it is not meaningful to focus on the asymptotic distribution of an estimator for  $\Sigma$  or another matrix  $\Gamma \propto \Sigma$ . In the present work, robust estimators for the shape matrix and the associated scale are investigated. Explicit expressions for their joint asymptotic distributions are derived. It turns out that if the joint asymptotic distribution is normal, the presented estimators are asymptotically independent for one and only one specific choice of the scale function. If it is non-normal (this holds for example if the estimators for the shape matrix and scale are based on the minimum volume ellipsoid estimator) only the presented scale function leads to asymptotically uncorrelated estimators. This is a generalization of a result obtained by Paindaveine (2008) in the context of local asymptotic normality theory.

*Key words:* local asymptotic normality, M-estimator, R-estimator, robust covariance matrix estimator, scale-invariant function, S-estimator, shape matrix, Tyler's M-estimator.

---

## 1 Motivation

After the seminal paper by Maronna (1976), covariance matrix estimation has become a popular branch of robust statistics. Several techniques have been developed for calculating the asymptotic distributions of robust covariance matrix estimators such as the radial distribution approach of Tyler (1982) and the approach based on

---

<sup>1</sup> Email: frahm@statistik.uni-koeln.de, phone: +49 221 470-4267.

influence functions (Hampel et al., 1986). Moreover, in recent years deep insights have been gained from the viewpoint of local asymptotic normality (LAN) theory (Hallin et al., 2006, Hallin and Paindaveine, 2006a,b).

Let  $X$  be a  $d$ -dimensional random vector possessing an elliptically symmetric distribution, i.e. it can be represented by  $X = \mu + \Lambda \mathcal{R}U$ , where  $U$  is a  $k$ -dimensional random vector, uniformly distributed on the unit hypersphere,  $\mathcal{R}$  is a nonnegative random variable being stochastically independent of  $U$ ,  $\mu \in \mathbb{R}^d$ , and  $\Lambda \in \mathbb{R}^{d \times k}$  (Cambanis et al., 1981, Fang et al., 1990, p. 42). It is assumed that  $\mathcal{R}$  and  $U$  are unobservable quantities. The positive-semidefinite matrix  $\Sigma := \Lambda \Lambda'$  is called the *dispersion matrix* and  $\mathcal{R}$  is the *generating variate* of  $X$ . If  $\mathbb{E}(\mathcal{R}^2) < \infty$ , the covariance matrix of  $X$  is given by  $\mathbb{V}\text{ar}(X) = \mathbb{E}(\mathcal{R}^2)/k \cdot \Sigma$ , whereas in case  $\mathbb{E}(\mathcal{R}^2) = \infty$ , the linear dependence structure of  $X$  can be further described by the dispersion matrix  $\Sigma$  although  $\mathbb{V}\text{ar}(X)$  is not defined.

In general I will assume that  $\Sigma$  is positive-definite, i.e.  $r(\Lambda) = d \leq k$ . In the robust statistics literature (Tyler, 1982, Bilodeau and Brenner, 1999, Ch. 13) and in the context of LAN theory (Hallin and Paindaveine, 2006a, Paindaveine, 2008) it is often supposed that the distribution of  $\mathcal{R}$  is absolutely continuous. Then the density of  $X$  can be written as  $p(x) = \sqrt{\det \Sigma^{-1}} g\{(x - \mu)' \Sigma^{-1} (x - \mu)\}$ , where the so-called *density generator*  $g: \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$  depends on  $x$  only through the quadratic form  $(x - \mu)' \Sigma^{-1} (x - \mu)$ . It can be shown (Frahm, 2004, p. 9) that the density function of  $\mathcal{R}$  is given by  $f(r) \propto r^{d-1} g(r^2)$ .

Tatsuoka and Tyler (2000) wrote that ‘The assumption of an elliptically symmetric distribution is often made simply because of its mathematical tractability’. Nevertheless, the class of elliptically symmetric distributions is a natural extension of the multivariate normal distribution. Moreover, the elliptical distribution assumption is fundamental in multivariate analysis and the results presented in this work generally require that the data are elliptically symmetric distributed. However, there is one exception where the data are only assumed to be *generalized elliptically distributed* (Frahm, 2004, Ch. 3). This will be treated in more detail below.

Note that  $X = \mu + \Lambda \mathcal{R}U = \mu + V S U$  with  $\mathcal{S} := \mathcal{R}/\tau$ ,  $V := \tau \Lambda$ , and  $\tau > 0$ . That means if  $X$  possesses the dispersion matrix  $\Sigma$ , there always exists an equivalent representation of  $X$  with dispersion matrix  $\tau^2 \Sigma$  and so this can be only identified if the distribution of  $\mathcal{R}$  is somehow restricted. However, many multivariate statistical methods like principal components analysis, canonical correlation analysis, linear discriminant analysis, and multivariate regression require the covariance or dispersion matrix only up to some scaling constant. This has been frequently observed in the literature (Croux and Haesbroeck, 1999, Hallin and Paindaveine, 2006a, Oja, 2003, Paindaveine, 2008, Taskinen et al., 2006). If the topic of interest is not the scale but only the *shape* of the distribution of  $X$ , it is not meaningful to focus on the asymptotic covariance matrix (ACM) of an estimator for  $\Sigma$ ,  $\mathbb{V}\text{ar}(X)$  or another matrix  $\Gamma \propto \Sigma$  (i.e.  $\Gamma = \tau^2 \Sigma$ , where  $\tau$  is a *constant* and thus not determined by  $\Sigma$ ).

Therefore I will concentrate on robust estimators for the *shape matrix* of  $X$  (Oja, 2003, Paindaveine, 2008). The associated estimators for the scale are investigated concomitantly. I will derive explicit expressions for their joint asymptotic distributions. The paper is organized as follows. Section 2 introduces the notation and provides some helpful prerequisites about homogeneous functions. The question of how to choose an appropriate scale is investigated in Section 3. This section also contains the main results concerning the joint asymptotic distributions of estimators for the shape matrix and scale. In Section 4 it is shown how to calculate the asymptotic distributions of such estimators on the basis of some well-known robust covariance matrix estimators, namely M-, R-, and S-estimators.

## 2 Prerequisites

### 2.1 Notation

The following notation will be used in the sequel. The  $d^2 \times d^2$  identity matrix is symbolized by  $I_{d^2}$ . Let  $\mathbf{e}_{ij}$  be the  $d \times d$  matrix with 1 in the  $ij$ th position and zeros elsewhere. The  $d^2 \times d^2$  matrix  $J_{d^2}$  is defined as  $J_{d^2} := \sum_{i=1}^d \mathbf{e}_{ii} \otimes \mathbf{e}_{ii}$ , where ‘ $\otimes$ ’ denotes the Kronecker product (Schott, 1997, p. 253). The  $n \times m$  matrix  $A'$  denotes the transpose of an  $m \times n$  matrix  $A$ . In contrast, if  $f$  is an  $\mathbb{R}$ -valued function on an open subset of  $\mathbb{R}$ , then  $f'(x)$  stands for the derivative of  $f$  at  $x \in \mathbb{R}$ . Further, the *commutation matrix*  $K_{d^2}$  is the  $d^2 \times d^2$  matrix given by  $K_{d^2} := \sum_{i,j=1}^d \mathbf{e}_{ij} \otimes \mathbf{e}_{ji}$  (Schott, 1997, p. 277).

For any symmetric  $d \times d$  matrix  $A$ , the  $d^2$ -dimensional vector  $\text{vec } A$  is obtained by stacking the columns of  $A$  on top of each other, whereas  $\text{vech } A$  denotes the  $d(d+1)/2$ -dimensional vector obtained by stacking only the elements of the lower triangular part of  $A$ . Further, the *duplication matrix* is the  $d^2 \times d(d+1)/2$  matrix  $D_d$  such that  $D_d \text{vech } A = \text{vec } A$  (Schott, 1997, p. 283). Then it holds that  $D_d^+ \text{vec } A = \text{vech } A$ , where the  $d(d+1)/2 \times d^2$  matrix  $D_d^+$  is the Moore-Penrose inverse of  $D_d$  (Schott, 1997, p. 284). Let  $I_0$  be defined as the  $\{d(d+1)/2 - 1\} \times d(d+1)/2$  matrix  $I_0 := [0 \ I_{d(d+1)/2-1}]$  and  $N_d := I_0 D_d^+$ , so that  $\text{vech}_0 A := N_d \text{vec } A$  is the *vech* of  $A$  deprived of its first component  $A_{11}$  (Hallin and Paindaveine, 2006a).

I will frequently calculate the differential of an  $\mathbb{R}^m$ -valued function  $f$ , i.e.  $df = \mathcal{J}_f dx$ , where  $\mathcal{J}_f := \partial f(x)/\partial x' \in \mathbb{R}^{m \times n}$  denotes the Jacobi matrix of  $f$  at  $x \in \mathbb{R}^n$ . Suppose that  $x$  represents the *vec* of a symmetric matrix. Then each off-diagonal element in the lower triangular part of that matrix represents an implicit function of the corresponding off-diagonal element in the upper triangular part and vice versa. However, I will not take the symmetry into consideration when calculating the partial derivatives of  $f$ . Otherwise, to adjust for the redundancies caused by the symmetry it would be necessary to apply the operator  $(I_{d^2} + J_{d^2})/2$  on the partial

differentials  $\partial x$  when calculating the total differential  $df$ . Hence, to avoid additional notation and tedious calculations of implicit derivatives, the Jacobi matrix  $\mathcal{J}_f$  is understood to be the matrix of partial derivatives of  $f$  which are obtained by ignoring the symmetry condition. In the present context this poses no problem since  $\mathcal{J}_f$  is always used only in combination with  $\partial x$ .

## 2.2 Homogeneous Functions

Consider a differentiable  $\mathbb{R}^m$ -valued function  $h$  of  $x \in \mathbb{R}^n$ . The function  $h$  is said to be *homogeneous* of degree  $\nu \in \mathbb{R}$  if  $h(\alpha x) = \alpha^\nu h(x)$  for all  $x \in \mathbb{R}^n$  and  $\alpha > 0$ . Due to the Euler relation it holds that  $\mathcal{J}_h x = \nu h(x)$ . A function  $f$  is said to be *scale-invariant* if it is homogeneous of degree 0, i.e.  $f(\alpha x) = f(x)$  for all  $\alpha > 0$ . That means  $\mathcal{J}_f x = 0$  and if  $h$  is homogeneous of degree 1, it holds that  $\mathcal{J}_h x = h(x)$ . In the following a homogeneous function is always understood to be homogeneous of degree 1. Note that the partial derivatives of any homogeneous function are scale-invariant.

Let  $\mathcal{P}^d$  be the set of all symmetric positive-definite  $d \times d$  matrices and  $\varphi: \mathcal{P}^d \rightarrow \mathbb{R}^k$  a scale-invariant function, i.e.  $\varphi(\alpha\Gamma) = \varphi(\Gamma)$  for all  $\alpha > 0$  and  $\Gamma \in \mathcal{P}^d$ . Especially, consider a scale-invariant function  $\Omega(\Gamma) = \Gamma/\sigma^2(\Gamma)$ , where  $\sigma^2: \mathcal{P}^d \rightarrow \mathbb{R}^+$  is an homogeneous function, i.e.  $\sigma^2(\alpha\Gamma) = \alpha\sigma^2(\Gamma) > 0$ . It is supposed that the so-called *scale function*  $\sigma^2$  is differentiable at any point  $\Gamma \in \mathcal{P}^d$  and also that  $\sigma^2(I_d) = 1$ . Then  $\sigma^2(\Gamma)$  is called the *scale* of  $\Gamma$ . The matrix  $\Omega(\Gamma)$  will be called the *shape matrix* (with respect to the scale function  $\sigma^2$ ) belonging to  $\Gamma$ . I will write  $\sigma^2 \equiv \sigma^2(\Gamma)$  and  $\Omega \equiv \Omega(\Gamma)$  whenever these quantities cannot be confounded with the corresponding functions.

Note that  $\sigma^2(\Omega) = 1$  and  $\varphi \circ \Omega = \varphi$ , since  $\varphi\{\Omega(\Gamma)\} = \varphi\{\Gamma/\sigma^2(\Gamma)\} = \varphi(\Gamma)$ . For instance, the correlation matrix produced by  $\Gamma$  is scale-invariant and thus it can be derived from any shape matrix  $\Omega$ . Hence, whenever  $\Omega_n$  is an estimator for  $\Omega$ , an estimator for  $\varphi(\Gamma)$  is simply given by  $\varphi(\Omega_n)$ . This is a formal justification of directing one's attention to shape matrices (Frahm and Jaekel, 2007a, Hallin and Paindaveine, 2006a, Oja, 2003, Paindaveine, 2008, Taskinen et al., 2006). General robustness and efficiency properties of scale-invariant functions have been investigated by Tyler (1983).

### 3 Asymptotic Distributions

#### 3.1 The Choice of the Scale Function

In most cases asymptotic normality of robust estimators  $\mu_n$  and  $\Gamma_n$  for the mean vector and covariance matrix can be guaranteed by the usual regularity conditions given in the robust statistics literature. Typically  $\mu_n$  and  $\Gamma_n$  are also asymptotically independent. In the present work it is shown that the asymptotic independence of an estimator  $\Omega_n$  for the shape matrix and an associated estimator  $\sigma_n^2$  for the scale can only be guaranteed for one and only one scale function  $\sigma^2$ . A similar result in the context of LAN theory has been obtained by Paindaveine (2008) (see below).

Let  $\Gamma_n$  be some estimator for  $\Gamma \propto \Sigma$  where  $n$  represents the sample size. The corresponding shape matrix estimator is given by  $\Omega_n := \Gamma_n / \sigma^2(\Gamma_n)$ . At a first glance the choice of the scale function  $\sigma^2$  might be considered as arbitrary and the following variants can be often observed in the literature (Paindaveine, 2008):

- (S1) Frahm (2004, p. 64), Hallin et al. (2006), Hallin and Paindaveine (2006b), Hettmansperger and Randles (2002) as well as Randles (2000) simply choose  $\sigma^2(\Gamma) = \Gamma_{11}$  so that  $\Omega_{11} = 1$ .
- (S2) Dümbgen (1998), Frahm and Jaekel (2007b) as well as Tyler (1987a) take the scale function  $\sigma^2(\Gamma) = (\text{tr } \Gamma) / d$  so that  $\text{tr } \Omega = d$ .
- (S3) Dümbgen and Tyler (2005), Hallin and Paindaveine (2008a,b), Paindaveine (2008), Salibian-Barrera et al. (2006), Taskinen et al. (2006) as well as Tatsuka and Tyler (2000) postulate  $\sigma^2(\Gamma) = (\det \Gamma)^{1/d}$  so that  $\det \Omega = 1$ .

Paindaveine (2008) considers the latter normalization as *canonical* since this is the only one where the Fisher information matrix with respect to the mean vector, shape matrix and scale is block diagonal if the distribution of  $X$  or, more precisely, the corresponding experiment is LAN (van der Vaart, 1998, Ch. 7).

The scale functions defined by **S2** and **S3** correspond to the arithmetic and geometric means of the eigenvalues of  $\Gamma$ , respectively. Hence, another possible scale function is given by the harmonic mean of the eigenvalues of  $\Gamma$ , i.e.

- (S4)  $\sigma^2(\Gamma) = d / (\text{tr } \Gamma^{-1})$  so that  $\text{tr } \Omega^{-1} = d$ .

It is worth to point out that shape matrices are not affine equivariant, since

$$\Omega(V\Gamma V') = \frac{V\Gamma V'}{\sigma^2(V\Gamma V')} = \frac{\sigma^2(\Gamma)}{\sigma^2(V\Gamma V')} \cdot V\Omega(\Gamma)V'$$

for any nonsingular  $d \times d$  matrix  $V$  and generally  $\sigma^2(\Gamma)$  does not correspond to  $\sigma^2(V\Gamma V')$ . This is not surprising because even after an affine-linear transformation

of the data, the shape matrix has to satisfy the scaling condition  $\sigma^2(\Omega) = 1$  and so the equality  $\Omega(V\Gamma V') = V\Omega(\Gamma)V'$  cannot be guaranteed in general. However, a natural requirement is that the equivariance property holds at least for all transformations  $V$  with  $\sigma^2(VV') = 1$ . That means if not the scale but only the shape of the distribution of  $X$  is affected by  $V$ , the shape matrix should remain equivariant.

More generally, it can be required (Tyler, 2002) that

$$\Omega(V\Gamma V') = \frac{V\Omega(\Gamma)V'}{\sigma^2(VV')},$$

i.e.  $\sigma^2(V\Gamma V') = \sigma^2(VV')\sigma^2(\Gamma)$ . Interestingly, from the scale functions considered in **S1–S4** only the canonical one (**S3**) satisfies this kind of affine equivariance property. This is another argument in favor of the determinant-based normalization proposed by Paindaveine (2008).

The previous arguments as well as a thorough discussion in Hallin and Paindaveine (2006a) show that the choice of the scale function must be driven by statistical considerations and should be handled carefully.

**Lemma 1** *Let  $\Omega(\Gamma) = \Gamma/\sigma^2(\Gamma)$  be a  $d \times d$  shape matrix and  $\sigma^2$  a scale function. Then*

$$\mathcal{J}_\Omega := \frac{\partial \text{vec } \Omega(\Gamma)}{\partial (\text{vec } \Gamma)'} = \frac{1}{\sigma^2} \{I_{d^2} - \text{vec } \Omega \mathcal{J}_{\sigma^2}\},$$

where

$$\mathcal{J}_{\sigma^2} := \frac{\partial \sigma^2(\Gamma)}{\partial (\text{vec } \Gamma)'} = \frac{\partial \sigma^2(\Omega)}{\partial (\text{vec } \Omega)'}$$

**Proof.** By the product rule it follows that

$$\mathcal{J}_\Omega = \frac{1}{\sigma^2} \cdot \frac{\partial \text{vec } \Gamma}{\partial (\text{vec } \Gamma)'} - \frac{\text{vec } \Gamma}{\sigma^4} \cdot \mathcal{J}_{\sigma^2} = \frac{1}{\sigma^2} \{I_{d^2} - \text{vec } \Omega \mathcal{J}_{\sigma^2}\}.$$

Since the partial derivatives of an homogeneous function are scale-invariant, it holds that  $\mathcal{J}_{\sigma^2} = \partial \sigma^2(\Omega)/\partial (\text{vec } \Omega)'$ . ■

In the following I will write  $\Psi := I_{d^2} - \text{vec } \Omega \mathcal{J}_{\sigma^2}$  for notational convenience.

### 3.2 Main Results

Let  $\mathcal{Q}$  be a symmetric random  $d \times d$  matrix. A symmetric random  $d \times d$  matrix  $\mathcal{M}$  is said to possess a *radial distribution* if  $\mathcal{O}\mathcal{M}\mathcal{O}' \sim \mathcal{M}$  for any orthogonal  $d \times d$  matrix  $\mathcal{O}$  (Tyler, 1982). In the following let  $\mathcal{N}$  be a symmetric random  $d \times d$  matrix with finite second moments. It is supposed that  $\mathcal{N}$  is of the radial type with respect to a symmetric positive-definite  $d \times d$  matrix  $\Gamma$ . That means  $T\mathcal{N}T'$  has a radial

distribution whenever the  $d \times d$  matrix  $T$  is such that  $T'T = \Gamma^{-1}$ . Further, let  $(\Gamma_n)$  be a sequence of symmetric positive-definite random  $d \times d$  matrices and  $(\sigma_n^2)$  an associated sequence with  $\sigma_n^2 := \sigma^2(\Gamma_n)$ , where  $\sigma^2$  is a scale function. Moreover, consider the sequence  $(\Omega_n)$  of symmetric positive-definite random  $d \times d$  matrices with  $\Omega_n := \Gamma_n/\sigma_n^2$ .

**Theorem 1** *Let  $\sigma^2$  be a scale function and  $\Omega \equiv \Omega(\Gamma) = \Gamma/\sigma^2(\Gamma)$  the shape matrix belonging to  $\Gamma$ . Further, let  $(a_n)$  be a sequence of real numbers increasing to infinity such that  $a_n(\text{vec } \Gamma_n - \text{vec } \Gamma) \xrightarrow{d} \text{vec } \mathcal{Q}$  as  $n \rightarrow \infty$  with  $\mathbb{E}(\text{vec } \mathcal{Q}) = 0$  and*

$$\mathbb{V}\text{ar}(\text{vec } \mathcal{Q}) = \gamma_1(I_{d^2} + K_{d^2})(\Gamma \otimes \Gamma) + \gamma_2(\text{vec } \Gamma)(\text{vec } \Gamma)', \quad (1)$$

where  $\gamma_1 \geq 0$  and  $\gamma_2 \geq -2\gamma_1/d$ . Then it follows that

$$a_n \left( \begin{bmatrix} \sigma_n^2 \\ \text{vec } \Omega_n \end{bmatrix} - \begin{bmatrix} \sigma^2 \\ \text{vec } \Omega \end{bmatrix} \right) \xrightarrow{d} \xi, \quad n \rightarrow \infty,$$

where  $\sigma^2 \equiv \sigma^2(\Gamma)$ ,  $\xi$  is a  $(d^2 + 1)$ -dimensional random vector with  $\mathbb{E}(\xi) = 0$ , and

$$\mathbb{V}\text{ar}(\xi) = \begin{bmatrix} \mathcal{V}(\sigma_n^2) & \mathcal{V}(\sigma_n^2, \Omega_n) \\ \mathcal{V}(\sigma_n^2, \Omega_n)' & \mathcal{V}(\Omega_n) \end{bmatrix}.$$

More specifically,

$$\mathcal{V}(\sigma_n^2) = \sigma^4 \{2\gamma_1 \mathcal{J}_{\sigma^2}(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} + \gamma_2\}$$

with  $\mathcal{J}_{\sigma^2} = \partial\sigma^2(\Omega)/\partial(\text{vec } \Omega)'$  and  $\sigma^4 = \{\sigma^2(\Gamma)\}^2$ ,

$$\mathcal{V}(\sigma_n^2, \Omega_n)' = 2\gamma_1 \sigma^2 \Psi(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2},$$

with  $\Psi = I_{d^2} - \text{vec } \Omega \mathcal{J}_{\sigma^2}$ , and

$$\mathcal{V}(\Omega_n) = \gamma_1 \Psi(I_{d^2} + K_{d^2})(\Omega \otimes \Omega) \Psi'.$$

**Proof.** The vector  $\{\sigma^2(\Gamma), \text{vec } \Omega(\Gamma)\}$  is differentiable at  $\text{vec } \Gamma$  and thus

$$a_n \left( \begin{bmatrix} \sigma_n^2 \\ \text{vec } \Omega_n \end{bmatrix} - \begin{bmatrix} \sigma^2 \\ \text{vec } \Omega \end{bmatrix} \right) \xrightarrow{d} \xi := \mathcal{J}_{\sigma^2, \Omega} \text{vec } \mathcal{Q}, \quad n \rightarrow \infty,$$

where  $\mathcal{J}_{\sigma^2, \Omega}$  is defined as  $\partial\{\sigma^2(\Gamma), \text{vec } \Omega(\Gamma)\}/\partial(\text{vec } \Gamma)'$ . From  $\mathbb{E}(\text{vec } \mathcal{Q}) = 0$  it follows that  $\mathbb{E}(\xi) = 0$  and the variance of the first element of  $\xi$  is given by  $\mathcal{V}(\sigma_n^2) = \mathcal{J}_{\sigma^2} \mathbb{V}\text{ar}(\text{vec } \mathcal{Q}) \mathcal{J}'_{\sigma^2}$ . Since  $\sigma^2$  is a homogeneous function it holds that  $\mathcal{J}_{\sigma^2} \text{vec } \Gamma = \sigma^2$ . Note also that  $\mathcal{J}_{\sigma^2}(I_{d^2} + K_{d^2}) = 2\mathcal{J}_{\sigma^2}$  and thus

$$\mathcal{V}(\sigma_n^2) = 2\gamma_1 \mathcal{J}_{\sigma^2}(\Gamma \otimes \Gamma) \mathcal{J}'_{\sigma^2} + \gamma_2 \sigma^4 = \sigma^4 \{2\gamma_1 \mathcal{J}_{\sigma^2}(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} + \gamma_2\}.$$

Similarly, the covariances between the first element of  $\xi$  and its residual elements are given by  $\mathcal{V}(\sigma_n^2, \Omega_n) = \mathcal{J}_{\sigma^2} \text{Var}(\text{vec } \mathcal{Q}) \Psi' / \sigma^2$ . Since  $\Omega$  is a scale-invariant function of  $\Gamma$ , due to Euler's relation it holds that  $(\text{vec } \Gamma)' \Psi' = 0$  and thus

$$\mathcal{V}(\sigma_n^2, \Omega_n) = \gamma_1 \mathcal{J}_{\sigma^2} (I_{d^2} + K_{d^2}) (\Gamma \otimes \Gamma) \Psi' / \sigma^2 = 2\gamma_1 \sigma^2 \mathcal{J}_{\sigma^2} (\Omega \otimes \Omega) \Psi'. \quad (2)$$

The expression for the variances and covariances of the residual elements of  $\xi$ , i.e.  $\mathcal{V}(\Omega_n)$  follows by a straightforward application of the arguments given above. ■

The next proposition ensures that the preceding theorem is applicable to any case where  $\Gamma_n$  represents an affine equivariant covariance matrix estimator and the data stem from an elliptically symmetric distribution.

**Proposition 1** *Let  $\sigma^2$  be a scale function and  $\Omega \equiv \Omega(\Gamma) = \Gamma / \sigma^2(\Gamma)$  the shape matrix belonging to  $\Gamma$ . Further, let  $(a_n)$  be a sequence of real numbers increasing to infinity such that  $a_n(\text{vec } \Gamma_n - \text{vec } \Gamma) \rightarrow_d \text{vec } \mathcal{N}$  as  $n \rightarrow \infty$ . Here  $\mathbb{E}(\text{vec } \mathcal{N}) = 0$  and  $\mathcal{N}$  is of the radial type with respect to the matrix  $\Gamma$ . Then the conditions of Theorem 1 are satisfied.*

**Proof.** It is only necessary to show that the second moment condition (1) is satisfied. Since  $\mathcal{N}$  is of the radial type, this follows immediately from Corollary 1 of Tyler (1982). ■

In the following  $\Gamma_n$  can be interpreted as a covariance matrix estimator. Due to the central limit theorem, in most practical situations it can be found that  $a_n = \sqrt{n}$  and the random vector  $\text{vec } \mathcal{N}$  is multivariate normally distributed. A well-known exception is the *minimum volume ellipsoid* (MVE) estimator (Rousseeuw, 1985). This is only  $\sqrt[3]{n}$ -consistent and its asymptotic distribution is non-normal (Davies, 1992). Nonetheless, whenever  $\Gamma_n$  is affine equivariant and the data stem from an elliptically symmetric distribution, the limiting random matrix  $\mathcal{N}$  is of the radial type (Tyler, 1982). Hence, Proposition 1 is applicable to a wide range of covariance matrix estimators.

An important consequence of Theorem 1 is that the asymptotic distribution of  $\Omega_n$  is only driven by the number  $\gamma_1$ . That means  $\gamma_2$  has no impact on the asymptotic distribution of  $\Omega_n$ . Hence, the asymptotic relative efficiency of some shape matrix estimator  $\Omega_{1n}$  compared to another shape matrix estimator  $\Omega_{2n}$  (i.e. both estimators are based on the *same* scale function  $\sigma^2$  but different covariance matrix estimators) can be simply calculated by the ratio  $\gamma_{12}/\gamma_{11}$ , where  $\gamma_{11}$  is the  $\gamma_1$  of  $\Omega_{1n}$  and  $\gamma_{12}$  is the  $\gamma_1$  of  $\Omega_{2n}$  (Tyler, 1983).

**Corollary 1** *Suppose that the conditions of Theorem 1 are satisfied and  $\sigma^2$  corresponds to the scale function given by S3. Then it holds that*

$$\mathcal{V}(\sigma_n^2) = \sigma^4 \left( \frac{2\gamma_1}{d} + \gamma_2 \right), \quad \mathcal{V}(\sigma_n^2, \Omega_n)' = 0,$$

and

$$\mathcal{V}(\Omega_n) = \gamma_1 (I_{d^2} + K_{d^2})(\Omega \otimes \Omega) - \frac{2\gamma_1}{d} \cdot (\text{vec } \Omega)(\text{vec } \Omega)'. \quad (3)$$

In particular, if  $\text{vec } \mathcal{Q}$  is multivariate normally distributed, the quantities  $\sigma_n^2$  and  $\Omega_n$  are asymptotically independent.

**Proof.** Note that

$$\mathcal{J}_{\sigma^2} = \frac{\sigma^2}{d \det \Gamma} \cdot \frac{\partial \det \Gamma}{\partial (\text{vec } \Gamma)'} = \frac{\sigma^2}{d} \cdot (\text{vec } \Gamma^{-1})' = (\text{vec } \Omega^{-1})'/d.$$

Due to Theorem 1 the asymptotic variance  $\mathcal{V}(\sigma_n^2)$  is given by

$$\mathcal{V}(\sigma_n^2) = \sigma^4 \left\{ 2\gamma_1 \mathcal{J}_{\sigma^2}(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} + \gamma_2 \right\}$$

and note that  $(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} = \text{vec } \Omega/d$ . Moreover,  $\mathcal{J}_{\sigma^2} \text{vec } \Omega = 1$ , which means that  $\mathcal{V}(\sigma_n^2) = \sigma^4(2\gamma_1/d + \gamma_2)$ . Further,

$$\mathcal{V}(\sigma_n^2, \Omega_n)' = 2\gamma_1 \sigma^2 \Psi(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} = 2\gamma_1 \sigma^2 \Psi \text{vec } \Omega/d.$$

Due to Euler's relation it holds that  $\Psi \text{vec } \Omega = 0$  and thus  $\mathcal{V}(\sigma_n^2, \Omega_n)' = 0$ . That means  $\sigma_n^2$  and  $\Omega_n$  are asymptotically uncorrelated or even independent if  $\text{vec } \mathcal{Q}$  is multivariate normally distributed. Finally, the expression for  $\mathcal{V}(\Omega_n)$  follows by a straightforward calculation after noting that  $\mathcal{J}_{\sigma^2}(\Omega \otimes \Omega) \mathcal{J}'_{\sigma^2} = 1/d$ . ■

**Theorem 2** *Suppose that the conditions of Theorem 1 are satisfied with  $\gamma_1 > 0$ . Then the scale function given by **S3** is the only one where  $\sigma_n^2$  and  $\Omega_n$  are asymptotically uncorrelated.*

**Proof.** Paindaveine (2008) shows that the determinant-based scale function given by **S3** is the only one where the Fisher information  $\mathcal{I}_{\sigma^2, \Omega}$  is a block diagonal matrix if the considered family of elliptically symmetric distributions is LAN. Suppose that the data are multivariate normally distributed. Then Theorem 1 applies to the sample covariance matrix and it is clear that the family of multivariate normal distributions is LAN. The Fisher information is the inverse of the ACM of  $\sigma_n^2$  and  $\Omega_n$  (which can be obtained after re-shaping  $\Omega_n$  to avoid singularity (Hallin and Paindaveine, 2006a,b)). Hence, there is no other scale function such that (2) vanishes. Since the latter is only an algebraic statement, the same must hold for any other distribution under the conditions of Theorem 2. ■

Theorem 2 extends the main result of Paindaveine (2008) which has been obtained in the context of LAN theory. Similarly, it can be shown that the canonical scale function is the only one which admits the simple representation of the ACM of a shape matrix estimator given by Eq. 3. In fact, this ACM exhibits the same desirable form as the ACM of any affine equivariant covariance matrix estimator according to Theorem 2 and Eq. 1. The operators  $\Psi$  and  $\mathcal{J}_{\sigma^2}$  corresponding to the remaining

scale functions defined by **S1**, **S2**, and **S4** are now given for convenience without an explicit derivation.

**ad S1.**  $\mathcal{J}_{\sigma^2} = \mathbf{e}'_1$ , where  $\mathbf{e}_1$  is the  $d^2 \times 1$  vector with 1 in the first position and zeros elsewhere, so that  $\Psi = I_{d^2} - \text{vec } \Omega \mathbf{e}'_1$ .

**ad S2.**  $\mathcal{J}_{\sigma^2} = (\text{vec } I_d)' / d$  and thus  $\Psi = I_{d^2} - (\text{vec } \Omega)(\text{vec } I_d)' / d$  (see also Theorem 5 in Sirkiä et al., 2007).

**ad S4.** It can be shown that  $\mathcal{J}_{\sigma^2} = d / (\text{tr } \Gamma^{-1})^2 \cdot (\text{vec } \Gamma^{-2})' = (\text{vec } \Omega^{-2})' / d$ , where  $\Gamma^{-2} := \Gamma^{-1} \Gamma^{-1}$  and  $\Omega^{-2} := \Omega^{-1} \Omega^{-1}$ , i.e.  $\Psi = I_{d^2} - (\text{vec } \Omega)(\text{vec } \Omega^{-2})' / d$ .

If a shape matrix estimator  $\Omega_{1n}$  defined via a scale function  $\sigma_1^2$  is *re-normalized* by applying some other scale function  $\sigma_2^2$  to  $\Omega_{1n}$ , its ACM simply corresponds to

$$\mathcal{V}(\Omega_{2n}) = \gamma_1 \Psi_2 (I_{d^2} + K_{d^2}) (\Omega_2 \otimes \Omega_2) \Psi_2', \quad (4)$$

where  $\Psi_2 = I_{d^2} - \text{vec } \Omega_2 \mathcal{J}_{\sigma_2^2}$  and  $\Omega_2$  is the shape matrix belonging to  $\Gamma$  with respect to the scale function  $\sigma_2^2$ . That means the first normalization has no impact on the asymptotic distribution of  $\Omega_{2n}$ .

## 4 Robust Covariance Matrix Estimation

In the following I will present some well-known robust covariance matrix estimators (i.e. M-, R-, and S-estimators) which satisfy the aforementioned conditions and calculate the joint asymptotic distributions of the corresponding estimators for the shape matrix and scale. It is neither possible nor reasonable to study here all existing robust covariance matrix estimators (for some contemporary overviews see, e.g., Zuo, 2006, Maronna et al., 2006, Ch. 6), but the essential concept might become clear from the subsequent discussion.

Let  $\Gamma_n$  be an affine equivariant estimator which is consistent for  $\Gamma$ . Due to the general result of Tyler (1982), in most practical situations  $\Gamma_n$  is asymptotically normally distributed with ACM  $\mathcal{V}(\Gamma_n) = \gamma_1 (I_{d^2} + K_{d^2}) (\Gamma \otimes \Gamma) + \gamma_2 (\text{vec } \Gamma)(\text{vec } \Gamma)'$ , where  $\gamma_1 \geq 0$  and  $\gamma_2 \geq -2\gamma_1/d$  usually depend on the generating variate  $\mathcal{R}$ . In the following I will only present the numbers  $\gamma_1$  and  $\gamma_2$ . The  $\sqrt{n}$ -convergence to the normal law is implicitly assumed. Hence, Theorem 2 implies that the canonical scale function is the only one where the estimators for the shape matrix and scale are asymptotically independent. As a counterexample consider the MVE-estimator. This is not  $\sqrt{n}$ -consistent and asymptotically normally distributed (Davies, 1992). However, since the MVE-estimator is affine equivariant and the rate of convergence does not matter, the corresponding MVE-estimators for the shape matrix and scale remain asymptotically uncorrelated (under the elliptical distribution assumption).

Throughout this section it is supposed that the unknown location vector  $\mu \in \mathbb{R}^d$

can be substituted by some  $\sqrt{n}$ -consistent estimate (here, too, it has been already demonstrated by Rousseeuw (1985) that the MVE-estimator for the location is only  $\sqrt[3]{n}$ -consistent and its asymptotic distribution is non-normal). In most cases – under mild regularity conditions concerning the distribution of  $X$  (see, e.g., Hallin and Paindaveine, 2006b, Tyler, 1987a, Bilodeau and Brenner, 1999, Ch. 13) – it can be shown that the resulting covariance matrix estimator is asymptotically normally distributed possessing an ACM of that form which is required in Theorem 1. Hence, in the following  $X_1, \dots, X_n$  will represent *centered* i.i.d. random vectors for the sake of simplicity and without loss of generality.

#### 4.1 M-Estimation

An *M-estimator* for  $\Gamma$  (Maronna, 1976) is defined as a solution of

$$\Gamma_n = \frac{1}{n} \sum_{t=1}^n w(X_t' \Gamma_n^{-1} X_t) X_t X_t',$$

where  $w: \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$  satisfies a set of general conditions (Maronna, 1976, Bilodeau and Brenner, 1999, Section 13.4.1). The estimator  $\Gamma_n$  is strongly consistent for the matrix  $\Gamma = \mathbb{E}\{w(X' \Gamma^{-1} X) X X'\}$  which is related to the dispersion matrix of  $X$  by  $\Gamma = \tau^2 \Sigma$ , where  $\tau > 0$  is such that  $\mathbb{E}\{\psi(\mathcal{R}^2/\tau^2)\} = d$  with  $\psi(t) := tw(t)$ . The numbers  $\gamma_1$  and  $\gamma_2$  can be calculated by  $\gamma_1 = (d+2)^2 \psi_1 / (d+2\psi_2)^2$  and

$$\gamma_2 = \frac{(\psi_1 - 1) - 2(\psi_2 - 1) \psi_1 \{d + (d+4)\psi_2\} / (d+2\psi_2)^2}{\psi_2^2},$$

where  $\psi_1 := \mathbb{E}\{\psi^2(\mathcal{R}^2/\tau^2)\} / \{d(d+2)\}$  and  $\psi_2 := \mathbb{E}\{\psi'(\mathcal{R}^2/\tau^2) \mathcal{R}^2\} / (d\tau^2)$  (Tyler, 1982, Bilodeau and Brenner, 1999, p. 223).

If  $X$  possesses a continuous elliptical distribution and  $\Sigma_n$  is the corresponding *ML-estimator* for the dispersion matrix  $\Sigma$ , it holds that  $\gamma_1 = \{d(d+2)/4\} / \mathbb{E}\{h^2(\mathcal{R}^2)\}$  and  $\gamma_2 = -2\gamma_1(1-\gamma_1) / \{2+d(1-\gamma_1)\}$ , where  $h(t) := t \partial \log g(t) / \partial t$ . If  $X \sim \mathcal{N}_d(0, \Sigma)$  and  $\Sigma_n$  represents the sample covariance matrix, it holds that  $\gamma_1 = 1$  and  $\gamma_2 = 0$ . Otherwise the sample covariance matrix is an M-estimator where  $\psi(t) = t$ . That means  $\mathbb{E}(\mathcal{R}^2/\tau^2) = \mathbb{E}\{\psi(\mathcal{R}^2/\tau^2)\} = d$ ,  $\psi_1 = d/(d+2) \cdot \mathbb{E}(\mathcal{R}^4) / \mathbb{E}^2(\mathcal{R}^2)$ , and  $\psi_2 = 1$  so that  $\gamma_1 = \psi_1$  and  $\gamma_2 = \gamma_1 - 1$  if  $\mathcal{R}$  has a finite fourth moment.

Now special attention is devoted to Tyler's M-estimator (Tyler, 1983, 1987a)

$$T_n = \frac{d}{n} \sum_{t=1}^n \frac{X_t X_t'}{X_t' T_n^{-1} X_t} = \frac{d}{n} \sum_{t=1}^n \frac{S_t S_t'}{S_t' T_n^{-1} S_t}, \quad (5)$$

where  $S_t := X_t / \|X_t\|$ ,  $\|\cdot\|$  denotes the Euclidean norm, and it is only supposed that  $\mathbb{P}(\mathcal{R} > 0) = 1$ . Note that  $T_n$  is not affected by the realizations of the generating variate  $\mathcal{R}$ , since  $S = X / \|X\| = \mathcal{R} \Lambda U / \|\mathcal{R} \Lambda U\| = \Lambda U / \|\Lambda U\|$  (a.s.).

That means Tyler's M-estimator is *distribution-free* in the context of elliptically symmetric distributions. This has been already observed by Tyler (1987b). Frahm and Jaekel (2007a,b) pointed out that the distribution-free property even holds within the class of generalized elliptical distributions. A random vector is said to be generalized elliptically distributed if its generating variate  $\mathcal{R}$  can be negative and might depend on  $U$  (Frahm, 2004, p. 46). This feature allows for the modeling of various kinds of asymmetries (Kring et al., 2007, Frahm, 2004, Section 3.4). For instance it can be shown that any *skew-elliptical distribution* (Liu and Dey, 2004) belongs to the class of generalized elliptical distributions (Frahm, 2004, p. 47).

Tyler's M-estimator (5) is unique up to a scaling constant. Hence, in fact  $T_n$  is a genuine *shape matrix* estimator since it can be only calculated with some suitable scale function  $\sigma^2$  such that  $\sigma^2(T_n) = 1$ . Originally, Tyler (1987a,b) applied the trace-based scale function given by **S2**, whereas in Tatsuoka and Tyler (2000) the authors prefer to use the canonical normalization **S3**. For the purpose of calculating the asymptotic distribution, Tyler (1987a,b) focuses on  $\bar{T}_n := d/(\text{tr } \Sigma^{-1} T_n) \cdot T_n$ , that means he defines the scale of  $T_n$  via  $\Sigma$  by  $\sigma^2(T_n) = \text{tr } \Sigma^{-1} T_n / d$ . This leads to  $\sigma^2(\bar{T}_n) = \sigma^2(\Sigma) = 1$  for any positive-definite  $d \times d$  matrix  $\Sigma$ .

Note that in contrast to some normalization according to **S1–S4**, the shape matrix estimator  $\bar{T}_n$  indeed is affine equivariant and consequently its ACM (Tyler, 1987b) exhibits the simple structure suggested by Eq. 1, viz

$$\mathcal{V}(\bar{T}_n) = \frac{d+2}{d} \cdot (I_{d^2} + K_{d^2})(\Sigma \otimes \Sigma) - \frac{2(d+2)}{d^2} \cdot (\text{vec } \Sigma)(\text{vec } \Sigma)'. \quad (6)$$

Since  $\Sigma$  represents a shape matrix with respect to Tyler's scale function, this ACM in fact corresponds to the ACM given by Eq. 3 with  $\gamma_1 = (d+2)/d$ . Furthermore, the Jacobian of Tyler's scale function is given by  $\mathcal{J}_{\sigma^2} = (\text{vec } \Sigma^{-1})'/d$  and this actually corresponds to the Jacobian of the *canonical* scale function (see the proof of Corollary 1). That means by using Tyler's scale function in association with some other affine equivariant covariance matrix estimator, the corresponding estimators for the shape matrix and scale become asymptotically uncorrelated. This seems to contradict Theorem 2. However, note that Tyler's  $\sigma^2$  in general does not meet the natural requirement  $\sigma^2(I_d) = 1$  and unfortunately  $\bar{T}_n$  cannot be applied in practical situations, since  $\sigma^2$  is determined by the unknown parameter  $\Sigma$ .

An alternative way for obtaining the desired ACM of Tyler's M-estimator is given as follows. Note that  $T_n$  is simply an M-estimator with  $\psi(t) = d$ . That means  $\psi_1 = d/(d+2)$  and  $\psi_2 = 0$  so that  $\gamma_1 = (d+2)/d$  and  $\gamma_2$  is not defined (since  $\sigma^2$  cannot be estimated by  $T_n$ ). Hence, due to Theorem 1, the ACM of  $T_n$  generally corresponds to  $\mathcal{V}(T_n) = (d+2)/d \cdot \Psi(I_{d^2} + K_{d^2})(\Omega \otimes \Omega)\Psi'$ . Moreover, due to Corollary 1 the ACM of Tyler's M-estimator, based on the *canonical* scale function, corresponds to (6) where  $\Sigma$  has to be substituted by  $\Omega$ .

## 4.2 R-Estimation

The R-estimator for the shape matrix has been introduced by Hallin et al. (2006). Consider Tyler's M-estimator  $T_n$  which is normalized according to **S1**, i.e. the upper left element corresponds to 1. The R-estimator is based on a *discretized version* of  $T_n$ . Suppose that  $x$  is an element of  $T_n$ . Then the discretization can be made by  $x^\# := \text{sgn } x / \sqrt{n} \lceil \sqrt{n} |x| \rceil$  (Hallin et al., 2006), where  $\lceil y \rceil$  denotes the smallest integer not smaller than  $y \in \mathbb{R}$ . The corresponding discretized version of Tyler's M-estimator is denoted by  $T_n^\#$ . Hallin and Paindaveine (2006b) also define  $U_t := (T_n^\#)^{-1/2} X_t / \|(T_n^\#)^{-1/2} X_t\|$ . Here  $A^{-1/2}$  denotes a positive-definite  $d \times d$  matrix such that  $A^{-1/2} A^{-1/2'} = A^{-1}$ , where  $A^{-1}$  is the inverse of a symmetric positive-definite  $d \times d$  matrix  $A$ . Further,  $R_t$  represents the rank of  $\|(T_n^\#)^{-1/2} X_t\|$  with respect to the sample  $X_1, \dots, X_n$ .

Let  $f_S: \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$  be the density function of some imaginary generating variate  $\mathcal{S}$ , whereas  $f_{\mathcal{R}}$  refers to the true generating variate  $\mathcal{R}$ . Consider the cumulative distribution function  $F_S(x) = \int_0^x f_S(r) dr$  and  $F_{\mathcal{R}}$  respectively. Here both  $\mathcal{R}$  and  $\mathcal{S}$  are absolutely continuous and satisfy some weak regularity conditions which guarantee local asymptotic normality (Hallin and Paindaveine, 2006b). As already mentioned before, the density function of  $\mathcal{S}$  is given by  $f_S(r) \propto r^{d-1} g_S(r^2)$ , where  $g_S$  is the density generator of  $\mathcal{S}$ . However, in the following consider the function  $f_S^*(r) := r^{-(d-1)} f_S(r) = g_S(r^2)$  and for  $0 < p < 1$  define  $K_S(p) := \psi_S\{F_S^{-1}(p)\} F_S^{-1}(p)$ , where  $F_S^{-1}$  is the quantile function of  $\mathcal{S}$  and  $\psi_S(x) := -f_S^{*'}(x) / f_S^*(x)$ . Now, the so-called *cross-information coefficient* (Hallin et al., 2006) is given by

$$\mathcal{I}_{\mathcal{R}, \mathcal{S}} := \int_0^1 K_{\mathcal{R}}(p) K_{\mathcal{S}}(p) dp. \quad (7)$$

Also define

$$\Delta_n := M_d \left( T_n^\# \otimes T_n^\# \right)^{-1/2} \sum_{t=1}^n \left\{ K_S \left( \frac{R_t}{n+1} \right) \text{vec} \left( U_t U_t' \right) - \frac{\overline{K}_S}{d} \cdot \text{vec} I_d \right\}$$

with  $\overline{K}_S := 1/n \sum_{t=1}^n K_S(t/(n+1))$ . The  $\{d(d+1)/2 - 1\} \times d^2$  matrix  $M_d$  symbolizes the Moore-Penrose inverse of  $N_d'$  (where  $N_d$  is such that  $N_d \text{vec} A = \text{vech}_0 A$ ). Further, let  $\Psi_n := I_{d^2} - \text{vec} T_n^\# \mathbf{e}_1'$  and  $Q_n := N_d \Psi_n (I_{d^2} + K_{d^2}) (T_n^\# \otimes T_n^\#) \Psi_n' N_d'$ . Now the R-estimator  $\Omega_n$  is defined in terms of the  $\text{vech}_0$  operator, viz

$$\text{vech}_0 \Omega_n = \text{vech}_0 T_n^\# + \frac{d(d+2)}{2n} \cdot \widehat{\mathcal{I}}_{\mathcal{R}, \mathcal{S}, n}^{-1} Q_n \Delta_n,$$

where  $\widehat{\mathcal{I}}_{\mathcal{R}, \mathcal{S}, n}$  represents some consistent estimator for the cross-information coefficient (7) (Hallin et al., 2006). The upper left element of  $\Omega_n$  is set to 1.

Thereafter, following the arguments of Hallin and Paindaveine (2006a) and Paindaveine (2008), one can apply a re-normalization by using the canonical scale

function and the ACM of the resulting R-estimator readily follows by applying Eq. 4 with  $\gamma_1 = d(d+2)\mathcal{I}_{\mathcal{S},\mathcal{S}}/\mathcal{I}_{\mathcal{S},\mathcal{R}}^2$ . Especially, if  $\mathcal{S} \sim \mathcal{R}$  it holds that  $\gamma_1 = d(d+2)/\mathcal{I}_{\mathcal{R},\mathcal{R}}$  with  $\mathcal{I}_{\mathcal{R},\mathcal{R}} = \int_0^1 K_{\mathcal{R}}^2(p) dp = \mathbb{E}(\psi_{\mathcal{R}}^2(\mathcal{R}) \mathcal{R}^2)$ . From  $\psi_{\mathcal{R}}(r) r = -2r^2 g'(r^2)/g(r^2)$  it follows that  $\psi_{\mathcal{R}}^2(r) r^2 = 4h^2(r^2)$ , where  $h$  has been already defined in Section 4.1. Recall that the function  $h$  is used for calculating the ACM of an ML-estimator. That means if  $\mathcal{S} \sim \mathcal{R}$ , the R-estimator has the same limiting distribution as the corresponding ML-estimator and thus it becomes asymptotically efficient.

### 4.3 S-Estimation

The S-estimator for the dispersion matrix (Davies, 1987) can be defined as  $\Gamma_n = \arg \min_{\Upsilon \in \mathcal{P}^d} \det \Upsilon$  subject to

$$\frac{1}{n} \sum_{t=1}^n \rho\left(\sqrt{X_t' \Upsilon^{-1} X_t}\right) = \alpha \rho(\infty),$$

where  $0 < \alpha < 1$  and  $\rho: \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$  has to be bounded, increasing, and sufficiently smooth (Croux and Haesbroeck, 1999, Tyler, 2002, Bilodeau and Brenner, 1999, Section 13.4.2). The chosen constraint guarantees that  $\Gamma_n$  is consistent for  $\Gamma = \tau^2 \Sigma$ , where  $\tau > 0$  is such that  $\mathbb{E}\{\rho(\mathcal{R}/\tau)\} = \alpha \rho(\infty)$ .

Let  $\psi$  be the first and  $\psi'$  the second derivative of  $\rho$ . It is assumed that

$$\mathbb{E}\{\psi'(\mathcal{R}/\tau)\} > 0 \quad \text{and} \quad \mathbb{E}\{\psi'(\mathcal{R}/\tau) \mathcal{R}^2/\tau + (d+1) \psi(\mathcal{R}/\tau) \mathcal{R}\} > 0.$$

Then the numbers  $\gamma_1$  and  $\gamma_2$  are given by

$$\gamma_1 = \frac{d(d+2) \mathbb{E}\{\psi^2(\mathcal{R}/\tau) \mathcal{R}^2\}}{\mathbb{E}^2\{\psi'(\mathcal{R}/\tau) \mathcal{R}^2/\tau + (d+1) \psi(\mathcal{R}/\tau) \mathcal{R}\}}$$

and

$$\gamma_2 = \frac{4\tau^2 \text{Var}\{\rho(\mathcal{R}/\tau)\}}{\mathbb{E}^2\{\psi(\mathcal{R}/\tau) \mathcal{R}\}} - \frac{2\gamma_1}{d}$$

(Davies, 1987, Lopuhaä, 1989, Bilodeau and Brenner, 1999, p. 225).

### Acknowledgements

The author would like to thank Karl Mosler for his valuable comments as well as two anonymous referees for their important suggestions and valuable contributions.

## References

- Bilodeau, M., Brenner, D., 1999. *Theory of Multivariate Statistics*. Springer.
- Cambanis, S., Huang, S., Simons, G., 1981. On the theory of elliptically contoured distributions. *Journal of Multivariate Analysis* 11, 368–385.
- Croux, C., Haesbroeck, G., 1999. Influence function and efficiency of the minimum covariance determinant scatter matrix estimator. *Journal of Multivariate Analysis* 71, 161–190.
- Davies, P., 1987. Asymptotic behaviour of S-estimates of multivariate location parameters and dispersion matrices. *Annals of Statistics* 15, 1269–1292.
- Davies, P., 1992. The asymptotics of Rousseeuw’s minimum volume ellipsoid estimator. *Annals of Statistics* 20, 1828–1843.
- Dümbgen, L., 1998. On Tyler’s M-functional of scatter in high dimension. *Annals of the Institute of Statistical Mathematics* 50, 471–491.
- Dümbgen, L., Tyler, D., 2005. On the breakdown properties of some multivariate M-functionals. *Scandinavian Journal of Statistics* 32, 247–264.
- Fang, K., Kotz, S., Ng, K., 1990. *Symmetric Multivariate and Related Distributions*. Chapman & Hall.
- Frahm, G., 2004. *Generalized Elliptical Distributions: Theory and Applications*. Ph.D. thesis, University of Cologne, Department of Economic and Social Statistics, Germany.
- Frahm, G., Jaekel, U., 2007a. Distribution-free shape matrix estimation for incomplete data. Discussion paper, University of Cologne, Department of Economic and Social Statistics, Germany.
- Frahm, G., Jaekel, U., 2007b. Tyler’s M-estimator, random matrix theory, and generalized elliptical distributions with applications to finance. Discussion paper, University of Cologne, Department of Economic and Social Statistics, Germany.
- Hallin, M., Oja, H., Paindaveine, D., 2006. Semiparametrically efficient rank-based inference for shape. II. Optimal R-estimation of shape. *Annals of Statistics* 34, 2757–2789.
- Hallin, M., Paindaveine, D., 2006a. Parametric and semiparametric inference for shape: the role of the scale functional. *Statistics and Decisions* 24, 327–350.
- Hallin, M., Paindaveine, D., 2006b. Semiparametrically efficient rank-based inference for shape. I. Optimal rank-based tests for sphericity. *Annals of Statistics* 34, 2707–2756.
- Hallin, M., Paindaveine, D., 2008a. Optimal rank-based tests for homogeneity of scatter. *Annals of Statistics* 36, 1261–1298.
- Hallin, M., Paindaveine, D., 2008b. Optimal tests for homogeneity of covariance, scale, and shape. *Journal of Multivariate Analysis*, in press (DOI: 10.1016/j.jmva.2008.05.010).
- Hampel, F., Ronchetti, E., Rousseeuw, P., Stahel, W., 1986. *Robust Statistics*. John Wiley.
- Hettmansperger, T., Randles, R., 2002. A practical affine equivariant multivariate median. *Biometrika* 89, 851–860.
- Kring, S., Rachev, S., Höchstötter, M., Fabozzi, F., 2007. Multi-tail elliptical distri-

- butions. Preprint, University of Karlsruhe, Department of Econometrics, Statistics and Mathematical Finance, Germany.
- Liu, J., Dey, D., 2004. Skew-elliptical distributions. In: Genton, M. (Ed.), *Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality*. Chapman & Hall, Ch. 3.
- Lopuhaä, H., 1989. On the relation between S-estimators and M-estimators of multivariate location and covariance. *Annals of Statistics* 17, 1662–1683.
- Maronna, R., 1976. Robust M-estimators of multivariate location and scatter. *Annals of Statistics* 4, 51–67.
- Maronna, R., Martin, D., Yohai, V., 2006. *Robust Statistics*. John Wiley.
- Oja, H., 2003. Multivariate M-estimates of location and shape. In: Höglund, R., Jäntti, M., Rosenqvist, G. (Eds.), *Statistics, Econometrics and Society. Essays in Honor of Leif Nordberg*. Statistics Finland.
- Paindaveine, D., 2008. A canonical definition of shape. *Statistics and Probability Letters* 78, 2240–2247.
- Randles, R., 2000. A simpler, affine-invariant, multivariate, distribution-free sign test. *Journal of the American Statistical Association* 95, 1263–1268.
- Rousseeuw, P., 1985. Multivariate estimation with high breakdown point. In: Grossmann, W., Pflug, G., Vincze, I., Wertz, W. (Eds.), *Mathematical Statistics and Applications*. Reidel, Dordrecht, pp. 283–297.
- Salibian-Barrera, M., van Aelst, S., Willems, G., 2006. Principal components analysis based on multivariate MM-estimators with fast and robust bootstrap. *Journal of the American Statistical Association* 101, 1198–1211.
- Schott, J., 1997. *Matrix Analysis for Statistics*. John Wiley.
- Sirkiä, S., Taskinen, S., Oja, H., 2007. Symmetrized M-estimators of multivariate scatter. *Journal of Multivariate Analysis* 98, 1611–1629.
- Taskinen, S., Croux, C., Kankainen, A., Ollila, E., Oja, H., 2006. Influence functions and efficiencies of the canonical correlation and vector estimates based on scatter and shape matrices. *Journal of Multivariate Analysis* 97, 1219–1243.
- Tatsuoka, K., Tyler, D., 2000. On the uniqueness of S-functionals and M-functionals under nonelliptical distributions. *Annals of Statistics* 28, 1219–1243.
- Tyler, D., 1982. Radial estimates and the test for sphericity. *Biometrika* 69, 429–436.
- Tyler, D., 1983. Robustness and efficiency properties of scatter matrices. *Biometrika* 70, 411–420.
- Tyler, D., 1987a. A distribution-free M-estimator of multivariate scatter. *Annals of Statistics* 15, 234–251.
- Tyler, D., 1987b. Statistical analysis for the angular central Gaussian distribution on the sphere. *Biometrika* 74, 579–589.
- Tyler, D., 2002. High breakdown point multivariate M-estimation. *Estadística* 54, 213–247.
- van der Vaart, A., 1998. *Asymptotic Statistics*. Cambridge University Press.
- Zuo, Y., 2006. Robust location and scatter estimators in multivariate analysis. In: Fan, J., Koul, H. (Eds.), *Frontiers of Statistics (in honor of Professor P.J. Bickel's 65th birthday)*. Imperial College, pp. 467–490.