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Default Swaps and Hedging Credit Baskets

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Abstract

We investigate the pricing of basket credit derivatives and their hedging with single name credit default swaps (CDS) based on a model for the joint dynamics of the fair CDS spreads. In the situation of the market flow of information being a pure jump filtration, we present an extremely efficient approach to pricing and study explicit hedging strategies.

Key words: credit default swap, credit basket, hedging

JEL Classification: G13

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1 Introduction

The credit derivatives business has seen a dramatic growth over the last decade. Credit default swaps (CDS) are the dominating plain-vanilla credit derivative product, which serve also as a building block for credit linked notes, credit indices and further synthetic credit investments. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon. A premium, the CDS spread s, is paid on a regular basis (e.g., on a quarterly, act/360 basis) and on a certain notional amount N as an insurance fee against the losses from default of a risky position of notional N, e.g., a bond. The payment of the premium s stops at maturity or at the random time τ of default of the underlying credit, whichever comes first. At the time of default before maturity of the trade the protection buyer receives the payment $N(1-R)$, where R is the recovery rate of the underlying credit risky instrument.

More advanced credit derivative products are linked to several underlying credits $i = 1, \ldots, n$ and the payoff is a function $f(\tau_1, \ldots, \tau_n)$ of the default times τ_i of the involved credits. Examples are basket default swaps, synthetic CDOs or default swaps on certain tranches of losses from a portfolio. What is common to these basket derivative products is that their modelling and pricing requires a model for the dependencies between the underlying credits.

The most important inputs for any credit derivative pricing model are the market observed fair CDS spreads $s^i(0,T)$ for (in principle) all maturities[1] T for credit i. From these spread curves one can back out the market implied (risk neutral) distribution of the default time, $F_i(t) = \mathbf{P}(\tau_i < t), t \geq 0$. Now, by the general no-arbitrage pricing principle, the valuation of a multi-credit derivative with payoff $f(\tau_1, \ldots, \tau_n)$ at time T calls for calculating the risk-neutral expectation

$$\mathbb{E}\left(\exp\left(-\int_0^T r_s \mathrm{d}s\right) f(\tau_1, \ldots, \tau_n)\right), \tag{1}$$

with (r_t) as the riskless short rate. However this requires a model for the joint risk-neutral distribution

$$\mathbf{P}(\tau_1 < t_1, \ldots, \tau_n < t_n)$$

of the default times, where the marginal distributions F_i are "given by the market". A common approach in practice is to link the marginal distribution assuming a certain copula, see e.g. [6],[2]. The calculation of the expectation above is then done either by Monte-Carlo simulation, or, in case of certain low-factor dependencies, by quasi-analytical methods. Recall that by arbitrage pricing theory it is justified to calculate the price via the expectation (1) since it quantifies the cost of dynamic replication (hedging) of the underlying product, provided it can be replicated, which in turn requires a dynamic model. So far, from our review of the standard market approach to the valuation of

[1] CDS are quoted with reasonable liquidity for maturities T of 1, 3, 5, 7, 10 years with highest liquidity normally in 5 years.

multi-credit products, no dynamic model whatsoever has been specified. This is indeed one of the major shortcomings of current market practice in that field. A dynamic model should specify the stochastic dynamics of credit spreads over time.

From both, the theoretical and practical point of view, besides the pricing of a multi-credit derivative, an equally important and related problem is the issue of hedging. Primary hedging instruments in practice are the single name credit default swaps for which a developed and liquid market exists. Practitioners distinguish two sources of risk to hedge against. The first one is the so-called spread risk, which is the risk that the market quoted fair CDS spreads change over time. A change of the spread impacts the distribution of the respective default time and thus the joint distribution and the mark-to-market valuation of the considered basket derivative. The spread risk is thus the risk of changing default probabilities without an actual default of this name having occurred. The second source of risk is the so-called default risk, which is the impact of an actual default on the basket derivative contract. Both sources of risk have to be hedged simultaneously.

In their hedging traders focus primarily on spread risk[2] and calculate hedge positions in single name credit default swaps that immunize the joint position against small changes in the fair market CDS spreads. In addition to that, traders carry out some scenario analysis to analyze how the chosen hedge strategy performs in case of actual defaults.

Analyzing both sources of risk simultaneously requires a model that goes beyond the joint distribution of the default times, which covers just a static snapshot at time $t = 0$. As time t evolves, the flow of information and the stochastic modelling of the actual defaults as well as the stochastic dynamics of the market observables are essential ingredients of the model that determine the hedging strategies. For the dynamics of the CDS spreads one important quantity that measures dependencies is the impact of the default of one of the credits on the spreads of the remaining ones. Given the copula describing the joint distribution, it has been shown in [8], [10] how to determine the conditional distribution $\mathbf{P}(\tau_i > t|\tau_j = t)$ and the fair spread for credit i after the occurrence of the default of credit j.

In this paper we investigate the pricing and hedging of basket credit derivatives starting with a generic model for the joint dynamics of the fair CDS spreads s^i over time for each credit $i = 1, \ldots, n$ in the basket. Our main goal is to analyze the hedging of basket derivatives in terms of single name credit default swaps. In the special case of a pure jump filtration we present an elegant and highly efficient approach to the pricing and to an explicit calculation of hedging strategies. The model we analyze in more detail allows for asymmetric impacts of defaults, i.e., the impact of the default of credit i on credit j measured as spread widening of the spread for j caused by the default of i might be different from the spread widening of i caused by the default of j.

The paper is organized as follows. Section 2 introduces the setup and the basic notation. Section 3 investigates credit default swaps, their fair spreads, the implied distribution of the default time and introduces some useful trading strategies related to credit default

[2]One reason being that most credit default swaps are referring to investment grade credits, i.e., the likelihood of an actual default is considered to be rather small.

swaps. Basket credit default swaps are studied in Section 4. We investigate the hedging of basket products in terms of credit default swaps and related strategies. In case the flow if information is of pure jump type we present a simple numerical approach, which is then illustrated by some examples.

2 No-arbitrage pricing setup and notation

We deal with the pricing of credit derivatives from the point of view of a generic model. We work on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with right continuous filtration (flow of information) $(\mathcal{F}_t)_{t\geq 0}$. Consider non-negative random variables $\tau_1, \ldots, \tau_n$ on this probability space. We interpret τ_i as the random time of default of credit i. We assume that

$$\mathbf{P}(\tau_i = 0) = 0, \ i = 1, \ldots, n \tag{2}$$

$$\mathbf{P}(\tau_1 > t_1, \ldots, \tau_n > t_n) > 0, \ \forall t_i > 0, \ i = 1, \ldots, n. \tag{3}$$

We suppose that $\tau_1, \ldots, \tau_n$ are $(\mathcal{F}_t)_{t\geq 0}$ stopping times but we do not make any particular assumption on the way the default times $\tau_1, \ldots, \tau_n$ are modeled. All basic security price processes in our economy are defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$.

To simplify the exposition and to put a clear focus on the problem of modelling dependent defaults, we *assume that riskless interest rates are zero* in our model. It seems relatively straightforward to extend our analysis to the case of non-vanishing interest rates.

We follow the standard no-arbitrage approach for the pricing of derivatives. Suppose that our market trades as basic instruments default risky zero bonds $(Q^i(t,T))_{t\leq T}$ of all maturities $T > 0$ for all credits $i = 1, \ldots, n$. By definition $Q^i(T,T) = \mathbf{1}_{\{\tau_i > T\}}$. Also there is a risk free money market account (β_t). Since we assume that riskless interest rates are zero, we have $\beta_t = 1$, $t \geq 0$. Also it is natural to extend the definition of $Q^i(t,T)$ beyond the maturity T setting $Q^i(t,T) = Q^i(T,T) = \mathbf{1}_{\{\tau_i > T\}}$ for $t > T$.

There is no arbitrage between the basic securities if all $(Q^i(t,T))_{t\geq 0}$ are $(\mathbf{P}, (\mathcal{F}_t))$-martingales, $i = 1, \ldots, n, \ T > 0$.

Assume from now on that $(Q^i(t,T))_{t\geq 0}$ are $(\mathcal{F}_t)$-martingales, $i = 1, \ldots, n, \ T > 0$ under the measure $\mathbf{P}$ which is then called *risk-neutral* distribution[3].

From the martingale property, we get

$$Q^i(t,T) = \mathbb{E}(\mathbf{1}_{\{\tau_i > T\}} | \mathcal{F}_t). \tag{4}$$

The payoff of a *contingent claim* which is paid at time T is described by an $\mathcal{F}_T$-measurable random variable X. For an integrable claim X we define[4] its value $V_t(X)$ at

[3]Clearly $\mathbf{P}$ might be not uniquely determined with this property, so we just pick one of the measures.

[4]If X is attainable, i.e., hedgeable by an admissible self-financing strategy, this is justified by the no-arbitrage paradigm, otherwise this is a definition.

time $t \leq T$ by

$$V_t(X) = \mathbb{E}(X|\mathcal{F}_t). \tag{5}$$

3 Credit default swaps

A credit default swap (CDS) is currently the most popular type of credit derivative. The market in default swaps is getting increasingly liquid.

A credit default swap offers protection against default of a certain underlying credit over a specified time period. A premium s, the so-called *spread*, is paid on a regular basis (e.g. on a quarterly, act/360 basis) as insurance fee against the losses from default of a default risky position, e.g., a bond. The payment of the premium s stops at maturity or at default of the underlying, whatever comes first. At default at time t before maturity of the trade the protection buyer receives the payment[5] $(1 - R_t)$, where R_t is the recovery rate of the underlying credit risky instrument for default at time t. Although recovery itself is a stochastic quantity, in practice it is often assumed that the recovery rate R_t is deterministic, e.g., $R_t = 40\%$. A reasonable assumption for R can be drawn from historical recovery data as published by various rating agencies. On the other hand, it is easy to verify that the impact of particular recovery assumptions on the pricing of credit default swaps is negligible. So from now on we *assume all recovery rates R_t to be deterministic*, and, to simplify the exposition, independent of time t: $R_t = R$.

3.1 Notation

We start by investigating a CDS on one of our credits $i = 1, \ldots, n$ and write $\tau = \tau_i$ for the random time of default and suppress the index i. Consider a CDS entered into at time $t = 0$ with maturity T and spread s. Since riskless rates are assumed to be vanishing, from the point of view of a protection buyer a credit default swap can be seen as a contingent claim with payoff

$$-s \cdot (\tau \wedge T) + (1 - R)\mathbf{1}_{\{\tau \leq T\}} \tag{6}$$

at time T.
We make use of the following notation:

$$M(t,T) = \mathbb{E}(\tau \wedge T|\mathcal{F}_t) \tag{7}$$

$$B(t,T) = M(t,T) - \tau \wedge t \wedge T \tag{8}$$

$$Q(t,T) = \mathbb{E}(\mathbf{1}_{\{\tau > T\}}|\mathcal{F}_t) \text{ (cf. (4))}. \tag{9}$$

For T fixed, $(M(t,T))_{t\geq 0}$ and $(Q(t,T))_{t\geq 0}$ are martingales. Also $(B(t,T))_{t\geq 0}$ is a non-negative supermartingale vanishing after $\tau \wedge T$. We assume all these processes to be right continuous with left hand limits.

[5] We assume that the default swap protects against losses in a default risky position of a notional amount of one unit.

The value $V(t,T)$ of a CDS with maturity T at time $t \leq T$ can now be written as (cf. (5))

$$\begin{aligned} V(t,T) &= -s \cdot M(t,T) + (1-R)(1-Q(t,T)) \\ &= -s \cdot (\tau \wedge t) - s \cdot B(t,T) + (1-R)(1-Q(t,T)). \end{aligned} \tag{10}$$

In practice the quantity $B(t,T)$ admits an important interpretation: it is the *risky present value of a basis point*; it gives the value at time t of one unit paid for the period of time from time t to $\tau \wedge T$, i.e., up to default or maturity.

Proposition 1 *(i) It holds $B(t,T) > 0$ a.s. on $\{T \wedge \tau > t\}$.*
(ii) We have the following relationship between the risky value of a basis point $B(t,T)$ and risky zero bonds $Q(t,u), u \geq 0$,[6]

$$B(t,T) = \int_t^T Q(t,u)\mathrm{d}u = \int_0^T Q(t,u)\mathrm{d}u - \tau \wedge t,\ t \leq T. \tag{11}$$

Proof: Statement (i) follows from

$$B(t,T) = \mathbb{E}(\tau \wedge T - t|\mathcal{F}_t)$$

a.s. on $\{t < T \wedge \tau\}$.
(ii) It suffices to show that

$$M(t,T) = \mathbb{E}(\tau \wedge T|\mathcal{F}_t) = \int_0^T Q(t,u)\mathrm{d}u.$$

Indeed, with $\alpha_u = \mathbf{P}(\tau \leq u|\mathcal{F}_t)$ increasing right continuous, we can write

$$\begin{aligned} \mathbb{E}(\tau \wedge T|\mathcal{F}_t) &= \int_{(0,T]} u\,\mathbf{P}(\tau \in \mathrm{d}u|\mathcal{F}_t) + T\,\mathbf{P}(\tau > T|\mathcal{F}_t) \\ &= \int_{(0,T]} u\,\mathrm{d}\alpha_u + T(1-\alpha_T) \\ &= T\alpha_T - \int_{(0,T]} \alpha_{u-}\mathrm{d}u + T - T\alpha_T \\ &= \int_{(0,T]} (1-\alpha_{u-})\mathrm{d}u = \int_{(0,T]} (1-\alpha_u)\mathrm{d}u \\ &= \int_{(0,T]} Q(t,u)\mathrm{d}u. \end{aligned}$$

□

[6] We assume a regular version of the conditional distribution $\mathbf{P}(\tau \leq u|\mathcal{F}_t) = 1 - Q(t,u), u \geq 0$.

Remarks
1. In view of

$$M(t,T) = \int_0^T Q(t,u)\mathrm{d}u \tag{12}$$

the quantity $M(t,T)$ can be interpreted as a "portfolio" of risky zero bonds $Q(t,u)$ with maturities $0 \leq u \leq T$. However, to make this rigorous, one has to extend the usual notation of a portfolio to cover portfolios of infinitely many securities, see, e.g., [1]. The "portfolio" representation above would also justify to consider $M(t,T)$ as a traded security.
However, if we consider the risky zero bonds and credit default swaps as traded instruments, $M(t,T)$ is obviously a traded security since, by (10), it can be represented as a portfolio of a CDS and a risky zero bond.
2. Another consequence of (12) and (10) is the following "hedge representation" of the value $V(t,T)$ of a CDS in terms of risky zero bonds

$$\mathrm{d}V(t,T) = -s\int_0^T (\mathrm{d}Q(t,u))\mathrm{d}u - (1-R)\mathrm{d}Q(t,T).$$

3. For $u \leq t \leq T$ the martingale $\mathbb{E}(B(t,T)|\mathcal{F}_u) = M(u,T) - M(u,t), 0 \leq u \leq t$ is the price at time u of a risky basis point paid for the interval $[t,T]$. This quantity is similar to the so-called swaption numeraire used in the market valuation of options on interest rate swaps. Based on this analogy, one can approach the problem of valuing an option on a credit default swap for the protection time interval from t to T along the same lines as it is standard for interest rate swaptions, see [9]

3.2 The CDS spread

The market quotes CDS by their fair spreads $s(0,T)$, i.e., the price $V(0,T)$ of a CDS at time $t = 0$ is zero by definition if its spread is $s(0,T)$. Given the spread curve $s(0,T)_{T>0}$ and an assumption on the recovery rate R in practice from this information, one derives a curve of implied survival probabilities $Q(0,T)$, which is in turn the basis for all valuations. So the primary market information is the actual spread curve and the trader observes its dynamics over time. This motivates why we devote this section to the spread curve and its relationship to other quantities.

Definition 1 *The fair spread $s(t,T)$ for a CDS entered into at time t and with maturity T is defined as*

$$s(t,T) = \begin{cases} \frac{(1-R)(1-Q(t,T))}{B(t,T)} & : \quad on\ \{\tau > t\}, t < T \\ 0 & : \quad otherwise\ . \end{cases} \tag{13}$$

Observe that from Proposition 1(i) $B(t,T) > 0$ **P**-a.s. on$\{T \wedge \tau > t\}$, so $s(t,T)$ is well defined. For $t \geq \tau$, or $t \geq T$ from its interpretation the CDS spread $s(t,T)$ does in fact not exist and we have chosen to set the spread to zero in these situations.

In practice, at inception of a CDS, its spread is usually set to the fair spread.

Proposition 2 *(i). The fair spread $(s(t,T))_{t\geq 0}$ is a semimartingale on the interval $[0,\tau\wedge T[$, i.e., there exists an increasing sequence (τ_n) of stopping times $\tau_n \leq \tau\wedge T$ with $\lim_n \tau_n = \tau\wedge T$ such that $(s(t\wedge\tau_n,T))_{t\geq 0}$ is a semimartingale.*
(ii). If the limit $s((T\wedge\tau)-,T)$ exists, then $(s(t,T))_{t\geq 0}$ is a semimartingale on the whole axis.

Proof:
(i). For $x\in[0,\infty)$ let $f_n(x) = x\mathbf{1}_{[1/n,\infty)}(x) + g_n(x)\mathbf{1}_{[0,1/n)}(x)$ be a smooth function such that $g_n(x)\geq \frac{1}{2n}$. Define

$$X_t(n) = \frac{(1-R)(1-Q(t,T))}{f_n(B(t,T))},\ t\geq 0.$$

By Ito's formula, $X(n)$ is a semimartingale. Now define

$$\tau_n = \inf\{t\geq 0 : B(t,T)\leq 1/n\}. \tag{14}$$

The sequence (τ_n) is increasing, $\tau_n\leq\tau\wedge T$. We show that $\lim_n\tau_n = \tau\wedge T$. On the set $\bigcap_n\{t>\tau_n\}$, using that $B(t,T)$ is a non-negative supermartingale, we get

$$\begin{aligned} 0 = \lim_n B(t\wedge\tau_n,T) &\geq \lim_n \mathbb{E}\left(B(t,T)|\mathcal{F}_{t\wedge\tau_n}\right) \\ &\geq \mathbb{E}\left(B(t,T)\Big|\bigvee_n \mathcal{F}_{t\wedge\tau_n}\right), \end{aligned}$$

which implies $B(t,T)=0$ a.s. on $\bigcap_n\{t>\tau_n\}$. Now if $\mathbf{P}(\lim_n\tau_n < T\wedge\tau)>0$ this contradicts the statement of Proposition 1(i), namely, that $B(t,T)>0$ a.s. on $\{t<T\wedge\tau\}$.

For $t<\tau_n$ by the definition of $X_t(n)$ it holds that $s(t,T)=X_t(n)$. Finally, to prove that $s(t\wedge\tau_n,T)$ is indeed a semimartingale we write

$$s(t\wedge\tau_n,T) = X_{t\wedge\tau_n}(n) + (s(\tau_n,T) - X_{\tau_n}(n))\mathbf{1}_{[\tau_n,\infty[}(t),$$

where the right hand side is the sum of a semimartingale and a process of bounded variation.
(ii).
By [5], Proposition (5.8), a necessary and sufficient condition for $(s(t,T))$ to be extendable to a semimartingale on the whole axis, is that the left hand limit $s((T\wedge\tau)-,T)$ exists on $B=\bigcap_n\{\tau_n<\tau\wedge T\}$. More precisely, if one defines $\tilde{s}(t,T)$ by

$$\tilde{s}(t,T) = \begin{cases} s(t,T) & on \quad \{t<\tau\wedge T\} \\ 0 = s(\tau\wedge T,T) & on \quad \{t\geq\tau\wedge T\}\cap B^c \\ s((\tau\wedge T)-,T) & on \quad \{t\geq\tau\wedge T\}\cap B \end{cases},$$

then by [5] Proposition (5.8) $(\tilde{s}(t,T))_{t\geq 0}$ is a semimartingale. In view of

$$s(t,T)(\omega) = \tilde{s}(t,T)(\omega) - s((\tau\wedge T)-,T)(\omega)\mathbf{1}_{[\tau\wedge T,\infty)\times B}(t,\omega)$$

the spread process $((s(t,T))_{t\geq 0}$ defined by (13) is a semimartingale too. □

It is not always the case that the left hand limit $s((T\wedge\tau)-,T)$ exists. To illustrate the possible behaviour of the spread $s(t,T)$ when t approaches τ we investigate to extreme examples.

Examples.

1. Denote $F(t) = \mathbf{P}(\tau \leq t)$ and assume that the filtration $(\mathcal{F}_t)$ is generated by the jump process $N_t = \mathbf{1}_{\{\tau\leq t\}}$,

$$\mathcal{F}_t = \mathcal{F}_t^N,\ t \geq 0.$$

Then $Q(t,T) = \mathbf{1}_{\{\tau>t\}}\frac{1-F(T)}{1-F(t)}, t \leq T$, and thus

$$s(t,T) = \frac{(1-R)(F(T)-F(t))}{\int_t^T (1-F(u))\mathrm{d}u}, t<\tau, t<T.$$

Clearly the left hand limit $s(\tau-,T)$ exists on $\{\tau < T\}$ and

$$s(\tau-,T) = \frac{(1-R)(F(T)-F(\tau-))}{\int_\tau^T (1-F(u))\mathrm{d}u}.$$

2. Now assume that τ is a predictable stopping time w.r.t. the filtration $(\mathcal{F}_t)$ and let $(\tau_n), \tau_n \uparrow \tau, \tau_n < \tau$ on $\{\tau > 0\}$ be an announcing sequence. Then $Q(\tau-,T) = \lim_n Q(\tau_n,T) = \mathbf{P}(\tau > T|\mathcal{F}_{\tau-}) = \mathbf{1}_{\{\tau>T\}}$. For the left hand limit of the spread $s(t,T)$ at τ this implies on $\{\tau < T\}$

$$\begin{aligned} s(\tau-,T) &= \lim_n \frac{(1-R)(1-Q(\tau_n,T))}{\int_0^T \mathbf{1}_{\{\tau_n\leq u\}}Q(\tau_n,u)\mathrm{d}u} \\ &= \frac{(1-R)\mathbf{1}_{\{\tau\leq T\}}}{\int_0^T \mathbf{1}_{\{\tau\leq u\}}\mathbf{1}_{\{\tau>u\}}\mathrm{d}u} \\ &= \infty. \end{aligned}$$

The following Proposition shows how the spread curve drives the quantities entering the valuation formula (10) of a CDS.

Proposition 3 *Let $t > 0$ be fixed and all statements are $\mathbf{P}$-a.s. on $\{\tau > t\}$.*
(i) The risky present value of a basis point $B(t,T)$ satisfies the following ordinary differential equation

$$\frac{\partial}{\partial T}B(t,T) + \frac{s(t,T)}{1-R}B(t,T) = 1,\ T > t. \tag{15}$$

(ii) Suppose that the CDS spread curve $(s(t,T))_{T>t}$ is an integrable function in $T>t$ $\mathbf{P}$-a.s. on $\{\tau>t\}$. Then the corresponding term structure of risky zero bonds $(Q(t,T))_{T>t}$ can be inverted from the CDS spread curve $(s(t,T))_{T>t}$ and $Q(t,T)$ is given by

$$Q(t,T)=1-\frac{s(t,T)}{1-R}\int_t^T \exp\left(-\int_v^T \frac{s(t,u)}{1-R}\mathrm{d}u\right)\mathrm{d}v. \tag{16}$$

Moreover, for the risky present value of a basis point we have the relation

$$B(t,T)=\int_t^T \exp\left(-\int_v^T \frac{s(t,u)}{1-R}\mathrm{d}u\right)\mathrm{d}v. \tag{17}$$

Proof: Assertion (i) is an immediate consequence of (13) and (11).
If $(s(t,T))_{T>t}$ is integrable, the solution to (15) is standard and given by (17). The assertion for $Q(t,T)$ then follows in view of $Q(t,T)=\frac{\partial}{\partial T}B(t,T)$. □

Remarks.
1. By their definition it is obvious that $Q(t,T)\geq 0$ and $B(t,T)$ is increasing in $T>t$. For that to be true, in view of (16) and (17), the CDS spread curve has to satisfy some necessary restrictions. Namely, we have $Q(t,T)\geq 0$ if and only if $B(t,T)$ is increasing in $T>t$, which holds, if and only if,

$$1\geq \frac{s(t,T)}{1-R}\int_t^T \exp\left(-\int_v^T \frac{s(t,u)}{1-R}\mathrm{d}u\right)\mathrm{d}v.$$

2. Also, by definition $Q(t,T)$ is a decreasing function in $T>t$ which yields the following further necessary restriction on the CDS spread curve:

$$\frac{s(t,T)}{1-R}\int_t^T \exp\left(-\int_v^T \frac{s(t,u)}{1-R}\mathrm{d}u\right)\mathrm{d}v$$

is an increasing function in $T>t$. Practitioners are well aware of the fact that a too heavily inverted spread curve $(s(0,T))_{T\geq 0}$ could yield "negative" default probabilities for certain time intervals.

3.3 CDS strategies

In this section we investigate simple trading strategies in credit default swaps that generate new securities that in turn can be used as hedging instruments. One motivation for introducing those strategies is that their price dynamics allow for an explicit representation in terms of the dynamics of the spread, see Proposition 5 below. This in turn will be the basis of our pricing approach in Section 4.

We denote by $V(t,u,T)$ the value at time $t\leq T$ of a CDS entered into at time u with (fair) spread $s=s(u,T)$ and maturity T, i.e.,

$$V(t,u,T)=\begin{cases} -s(u,T)\cdot(M(t,T)-\tau\wedge u)+(1-R)(1_{\{\tau>u\}}-Q(t,T)) & : \quad u\leq t\leq T \\ 0 & : \quad t\leq u. \end{cases} \tag{18}$$

Proposition 4 *Consider a CDS with maturity T and with (fair) spread $s(u,T)$ at time $u<T$. At time $T\geq t\geq u$ we have the following equalities for the price $V(t,u,T)$*

$$\begin{aligned} V(t,u,T) &= -s(u,T)\cdot(\tau\wedge t-\tau\wedge u)+(1-R)\mathbf{1}_{\{u<\tau\leq t\}} \qquad (19)\\ &\quad +(s(t,T)-s(u,T))\cdot B(t,T),\\ V(t,u,T) &= -s(u,T)\cdot(\tau\wedge t-\tau\wedge u)-s(u,T)(B(t,T)-B(u,T)) \qquad (20)\\ &\quad (1-R)(Q(u,T)-Q(t,T)). \end{aligned}$$

Proof: For $T\geq t>u$ we have

$$\begin{aligned} V(t,u,T) &= -s(u,T)\mathbb{E}(\tau\wedge T-\tau\wedge u|\mathcal{F}_t)+(1-R)\mathbf{P}(u<\tau\leq T|\mathcal{F}_t)\\ &= -s(u,T)(\tau\wedge t-\tau\wedge u)-s(u,T)B(t,T)\\ &\quad +(1-R)1_{\{u<\tau\leq t\}}+(1-R)\mathbf{P}(t<\tau\leq T|\mathcal{F}_t)\\ &= -s(u,T)(\tau\wedge t-\tau\wedge u)+(1-R)1_{\{u<\tau\leq t\}}+(s(t,T)-s(u,T))B(t,T), \end{aligned}$$

where we have used (13) for the last step.

Finally, from (19) and (13)

$$\begin{aligned} V(t,u,T) &= -s(u,T)(\tau\wedge t-\tau\wedge u)-s(u,T)B(t,T)+(1-R)\mathbf{P}(u<\tau\leq T|\mathcal{F}_t)\\ &= -s(u,T)(\tau\wedge t-\tau\wedge u)-s(u,T)(B(t,T)-B(u,T))\\ &\quad -s(u,T)B(u,T)+(1-R)\mathbf{P}(u<\tau\leq T|\mathcal{F}_u)\\ &\quad +(1-R)[\mathbf{P}(u<\tau\leq T|\mathcal{F}_t)-\mathbf{P}(u<\tau\leq T|\mathcal{F}_u)]\\ &= -s(u,T)(\tau\wedge t-\tau\wedge u)-s(u,T)(B(t,T)-B(u,T))\\ &\quad +(1-R)(Q(u,T)-Q(t,T)). \end{aligned}$$

□

Equality (19) possesses a nice interpretation. The first expression, $-s(u,T)\cdot(\tau\wedge t-\tau\wedge u)$, is the amount of spread payments accrued so far, the so-called called carry; $(1-R)\mathbf{1}_{\{u<\tau\leq t\}}$ is the protection payment if there was already a default up to time t. The last term, $(s(t,T)-s(u,T))\cdot B(t,T)$, is the difference between the spread agreed at inception and the current fair spread for the remaining lifetime times the risky present value of a basis point for the remaining life time. If there has been no default up to time t this last term quantifies the actual mark-to-marked value of the CDS.

Let $0=t_0<t_1<t_2<\cdots<t_N=T$ be a partition P of time and consider the following strategy. At time t_0 we enter into a fair CDS with maturity T, at time t_1 this CDS is unwound at the then prevailing market value and we enter into a new fair CDS, starting at time t_1 with maturity T and so on.

The value process $D^P(t,T)$ of this strategy is obviously

$$\begin{aligned} D^P(t,T) &= \sum_{j=1}^{N} \Big\{ -s(t_{j-1},T)\,[t_j \wedge t \wedge \tau - t_{j-1} \wedge t \wedge \tau] + \\ &\qquad [s(t_j \wedge t,T) - s(t_{j-1} \wedge t,T)]\,B(t_j \wedge t,T) \Big\} + (1-R)\mathbf{1}_{\{\tau \le t\}} \\ &= \sum_{j=1}^{N} V(t_j \wedge t, t_{j-1} \wedge t, T). \end{aligned} \tag{21}$$

Now, using (20) and (13) this value process can also be written as

$$\begin{aligned} D^P(t,T) &= -\int_{(0t\wedge\tau]} \sum_{j=1}^{N} s(t_{j-1},T) 1_{(t_{j-1},t_j]}(u)\mathrm{d}u \\ &\quad - \int_{(0,t\wedge\tau]} \sum_{j=1}^{N} s(t_{j-1},T) 1_{(t_{j-1},t_j]}(u)\mathrm{d}B(u,T) + (1-R)(Q(0,T) - Q(t,T)) \\ &= -\int_{(0,t\wedge\tau]} \sum_{j=1}^{N} s(t_{j-1},T) 1_{(t_{j-1},t_j]}(u)(\mathrm{d}u + \mathrm{d}B(u,T)) \\ &\quad + (1-R)(Q(0,T) - Q(t,T)). \end{aligned}$$

Lemma 1 *On the stochastic interval* $[0, \tau \wedge T[$ *the limit of* $D^P(t,T)$ *as* $\Delta_P = \max_i(t_j - t_{j-1}) \to 0$ $\mathbf{P}$*-a.s. exists and is given by*

$$\lim_{\Delta_P \to 0} D^P(t,T) = -\int_{(0,t]} s(u-,T)(\mathrm{d}u + \mathrm{d}B(u,T)) + (1-R)(Q(0,T) - Q(t,T)).$$

If the limit $s((\tau \wedge T)-,T)$ *exists, then the process* $(D^P(t,T))_{t\ge 0}$ *converges to*

$$-\int_{(0,t]} s(u-,T)(\mathrm{d}u + \mathrm{d}B(u,T)) + (1-R)(Q(0,T) - Q(t,T), t \ge 0,$$

on the whole time axis.

Proof: The assertion follows from [7], Theorem II.21, where, in case the limit $s((\tau \wedge T)-,T)$ does not exist, we have to localize by the sequence $\tau_n \uparrow \tau \wedge T$, defined by (14).□

Definition 2 *If the limit* $s((\tau \wedge T)-,T)$ *exists we denote*

$$C(t,T) = -\int_{(0,t]} s(u-,T)(\mathrm{d}u + \mathrm{d}B(u,T)) + (1-R)(Q(0,T) - Q(t,T)),\ t \le T. \tag{22}$$

We interpret $C(t,T)$ *as the price at time* t *of a strategy in credit default swaps which consists in continuously resettling into a fair credit default swap with maturity* T*.*

Remark. It might be possible to extend some results of the remainder of this paper also to the situation where the limit $s((\tau \wedge T)-, T)$ does not exist. However, this seems to be quite technical and not straightforward to accomplish. More importantly, in Section 4 in our main application of the results, we deal with models where the limit $s((\tau \wedge T)-, T)$ exists by definition.

So from now on *we assume that the limit* $s((\tau \wedge T)-, T)$ *exists.* The process $(C(t,T))_{t \leq T}$ is then well-defined and by Proposition 2, $(s(t,T))_{t \geq 0}$ is a semimartingale.

Proposition 5 *If the continuous martingale part of the semimartingale* $(s(t,T))_{t \geq 0}$ *vanishes, then*

$$C(t,T) = -\int_0^t s(u,T)\mathrm{d}u + \int_0^t B(u,T)\mathrm{d}s(u,T),\ t < \tau \wedge T. \tag{23}$$

Proof: By assumption $(s(t,T))$ is locally of bounded variation, so the integral w.r.t. $(s(t,T))$ is in fact a Lebesgue-Stieltjes integral. First integrating by parts and then rearranging and simplifying we get

$$\begin{aligned}
C(t,T) &= -\int_0^t s(u-,T)\mathrm{d}u - B(t,T)s(t,T) + B(0,T)s(0,T) + \int_0^t B(u-,T)\mathrm{d}s(u,T) \\
&\quad + [B(.,T), s(.,T)]_t + (1-R)(Q(0,T) - Q(t,T)) \\
&= -\int_0^t s(u-,T)\mathrm{d}u + \int_0^t B(u,T)\mathrm{d}s(u,T) \\
&\quad - \sum_{u \leq t} \Delta B(u,T)\Delta s(u,T) + [B(.,T), s(.,T)]_t \\
&\quad + B(0,T)s(0,T) - (1-R)(1-Q(0,T)) \\
&\quad - B(t,T)s(t,T) + (1-R)(1-Q(t,T)) \\
&= -\int_0^t s(u-,T)\mathrm{d}u + \int_0^t B(u,T)\mathrm{d}s(u,T),
\end{aligned}$$

where we have used (13) for the last step. □

Equation (23) has an intuitive interpretation. The first term quantifies the accrued premiums from the CDS positions up to time t. The second integral $\int_0^t B(u,T)\mathrm{d}s(u,T)$ expresses the cumulative costs of resettling the CDS positions to be fair: over the "time interval" $\mathrm{d}u$ the mark-to-market value of our CDS position from the beginning of this interval is just the change in fair spread $\mathrm{d}s(u,T)$ times the present value of a basis point $B(u,T)$ at the end of the interval and for the remaining time to maturity T.

The result of Proposition 5 is of critical importance for our pricing and hedging approach for basket derivatives in Section 4. In particular, we will make use of the fact that, by (23) and (17), the price dynamics of the strategy $C(t,T)$ are explicitly represented in terms of the dynamics of the spread $s(t,T)$.

4 Credit basket derivatives

Consider n credits with default times $\tau_1, \ldots, \tau_n$. A *credit basket derivative* with maturity T is a contingent claim with payoff V_T at time T where

$$V_T = f(T, \tau_1, \ldots, \tau_n)$$

with a measurable function f such that $f(T, \tau_1, \ldots, \tau_n)$ is a $\mathcal{F}_T$ measurable random variable.
Important examples are basket default swaps to be discussed in more detail below.

Assume from now on that

$$\mathbf{P}(\tau_i = \tau_j) = 0, \forall i \neq j. \tag{24}$$

The default times $\tau_1, \ldots, \tau_n$ can then be uniquely ordered and we denote by $\tau_{[k]}$ the time of the kth default, i.e., $\tau_{[k]} \in \{\tau_1, \ldots, \tau_n\}$ and

$$\tau_{[1]} < \tau_{[2]} < \cdots < \tau_{[n]}.$$

A kth to default swap (basket CDS) with maturity T and premium s is like a credit default swap where the event to protect is the occurrence of the kth default before maturity T. It is a credit basket derivative with payoff

$$V_T^{k\text{th}} = -s \cdot (\tau_{[k]} \wedge T) + \sum_{i=1}^{n} P_i(\tau_i) 1_{\{\tau_{[k]} \leq T, \tau_{[k]} = \tau_i\}}, \tag{25}$$

where $P_i(\tau_i)$ is an insurance premium paid if credit i is the kth defaulting. In practice one usually has $P_i(\tau_i) = 1 - R_i$, where R_i is the recovery rate for credit i.

4.1 Hedging basket claims

In the following we investigate the problem of hedging a basket credit derivative with primary securities such as credit default swaps $V^i(t,T)$ on credit i or strategies $D^{i,P}(t,T)$, $C^i(t,T)$ as defined in Section 3.3. The superscript i indicates that the respective security refers to credit i. We refer to the Definition 2 of $(C^i(t,T))_{t\geq 0}$ and recall that we always assume that the limit $s^i((\tau_i \wedge T)-, T)$ exists.

Definition 3 *The basket credit derivative $V_T = f(T, \tau_1, \ldots, \tau_n)$ is called hedgeable in the hedge instruments $H^i(., S) \in \{V^i(., S), D^{i,P}(., S), C^i(., S)\}$ with $S \in M_i$ and M_i a finite set of maturities, if*

$$V_T = K + \sum_{i=1}^{n} \sum_{S \in M_i} \int_0^T n^{i,S}(u) \mathrm{d}H^i(u, S), \tag{26}$$

with some constant K and predictable integrands $n^{i,S}$ such that the integrals are well-defined. The integrands $\{n^{i,S}, S \in M_i, i = 1, \ldots, n\}$ are called a hedging strategy in the hedge securities $\{H^i(., S), S \in M_i, i = 1, \ldots, n\}$.

Following our definition (5) for the value of a claim we get

Lemma 2 *Suppose* $V_T = f(T, \tau_1, \ldots, \tau_n)$ *is hedgeable in the instruments* $H^i(., S)$ *with strategy* $\{n^{i,S}, S \in M_i, i = 1, \ldots, n\}$ *such that* $\int_0^t n^{i,S}(u)\mathrm{d}H^i(u, S), t \leq T$, *is a* $\mathbf{P}$*-martingale for every* i, S. *Then the price* V_0 *is*

$$V_0 = \mathbb{E}(V_T) = K.$$

Remark: The strategy $\{n^{i,S}, S \in M_i, i = 1, \ldots, n\}$ can be extended to a self-financing strategy in the securities $\{H^i(., S), S \in M_i, i = 1, \ldots, n\} \cup \{\beta\}$ putting a respective amount n^β into the risk free savings account $\beta_t = 1$,

$$n^\beta(t) = \sum_{i=1}^{n} \sum_{S \in M_i} n^{i,S}(t) H^i(t, S) - \left(K + \sum_{i=1}^{n} \sum_{S \in M_i} \int_0^t n^{i,S}(u)\mathrm{d}H^i(u, S) \right), t \leq T.$$

4.2 Hedging in the pure jump case

Denote by N^i the jump process associated with the default time τ_i

$$N_t^i = 1_{\{\tau_i \leq t\}}, t \geq 0.$$

In this section we assume that the underlying filtration $(\mathcal{F}_t)$ is

$$\mathcal{F}_t = \mathcal{F}_t^{N^1, \ldots, N^n}, t \geq 0, \tag{27}$$

i.e., the filtration is generated by the pure jump processes $N^1, \ldots, N^n$. In other words, defaults are the only observable information in the market.

Proposition 6 *Consider a basket credit derivative with integrable payoff*

$$V_T = f(T, \tau_1, \ldots, \tau_n)$$

at time T. *Denote* $V_t = \mathbb{E}(V_T | \mathcal{F}_t), t \geq 0$. *Then* V_T *is hedgeable in the hedge instruments* $H^i(., S) \in \{V^i(., S), D^{i,P}(., S), C^i(., S)$ *with* $S \in M_i$ *and* M_i *a finite set of maturities, if and only if*

$$\Delta V_{\tau_j} = \sum_{i=1}^{n} \sum_{S \in M_i} n^{i,S}(\tau_j) \Delta H^i(\tau_j, S), j = 1, \ldots, n.$$

Proof: Because of assumption (27) the processes V and $H^i(., S)$ are pure jump martingales, i.e., compensated sums of jumps. The jumps are exhausted by $\tau_1, \ldots, \tau_n$. The assertion follows since pure jump martingales coincide if and only if their jumps coincide (see, e.g,. [4], Chapter VIII). □

It well-known that in the pure jump case the flow of information $(\mathcal{F}_t)$ possesses a very simple and explicit form, which will be the key to our further analysis of hedging and pricing basket derivatives.

Denote by z_k the random variable indicating the identity of the kth default:

$$z_k = \sum_{i=1}^{n} i\, 1_{\{\tau_{[k]}=\tau_i\}}, k = 1, \ldots, n. \tag{28}$$

Lemma 3 *Let (X_t) be $(\mathcal{F}_t)$-adapted, then X_t admits a representation*

$$\begin{aligned} X_t = {} & f_0(t)1_{[0,\tau_{[1]})}(t) + f_1(\tau_{[1]}, z_1, t)1_{[\tau_{[1]},\tau_{[2]})}(t) + \ldots \\ & + f_{n-1}(\tau_{[1]}, \ldots, \tau_{[n-1]}, z_1, \ldots, z_{n-1}, t)1_{[\tau_{[n-1]},\tau_{[n]})}(t) \\ & + f_n(\tau_{[1]}, \ldots, \tau_{[n]}, z_1, \ldots, z_n, t)1_{[\tau_{[n]},\infty)}(t), \end{aligned} \tag{29}$$

with deterministic functions $f_k(t_1, \ldots, t_k, i_1, \ldots, i_k, t), t_l \in \mathbb{R}_+, i_l \in \{1, \ldots, n\}$ that are measurable in $(t_1, \ldots, t_k)$.

Proof. As a consequence of [3], Theorem A2.32, for each t and k there exists an $\mathcal{F}_{\tau_{[k]}}$-measurable variable $V(k,t)$ such that the $\mathcal{F}_t$ measurable random variable X_t is of the form

$$X_t \mathbf{1}_{\{\tau_{[k]} \le t < \tau_{k+1]}\}} = V(k,t)\mathbf{1}_{\{\tau_{[k]} \le t < \tau_{k+1]}\}}.$$

Finally, from [3], Theorem A2.30, we get

$$V(k,t) = f_k(\tau_{[1]}, \ldots, \tau_{[k]}, z_1, \ldots, z_k, t)$$

with a function f_k with the stated properties. □

4.3 Pricing and hedging a basket CDS in the pure jump case

In view of (29) the CDS spreads $s^i(t,T)$ can be written in the form

$$s^i(t,T) = \tag{30}$$
$$\begin{cases} a^i(t,T) & : \quad t < \tau_{[1]} \\ a^i(t,T) + b^i(\tau_{[1]}, z_1, t, T) & : \quad \tau_{[1]} \le t < \tau_{[2]}, z_1 \ne i \\ a^i(t,T) + b^i(\tau_{[1]}, z_1, t, T) + b^i(\tau_{[1]}, \tau_{[2]}, z_1, z_2, t, T) & : \quad \tau_{[2]} \le t < \tau_{[3]}, z_1 \ne i, z_2 \ne i \\ \ldots & : \quad \ldots \end{cases}.$$

The function $a^i(t,T)$ is the deterministic base CDS spread up and until the time of first default, the function $b^i(\tau_{[1]}, z_1, t, T)$ is the spread widening relative to the base spread $a^i(t,T)$ which is caused by the occurrence of the first default and the first default being credit $z_1 \ne i$ etc.

Since $s^i(t,T)$ is a semimartingale we will assume that the functions

$$a^i(t,T), b^i(t_1, \ldots, t_k, i_1, \ldots, i_k, t, T)$$

are of finite variation and càdlag in the variable $t \in [0,T]$.

In addition, to be able to use the result of Proposition 3, the functions $a^i(t,T), b^i(u,j,t,T)$ are supposed to be integrable in the variable T. In view of the remarks following Proposition 3, the functions

$$a^i(t,T), b^i(t_1,\dots,t_k,i_1,\dots,i_k,t,T)$$

have to satisfy some additional conditions, which we assume to be fulfilled.

By [3], Theorem A2.34, the predictable hedge strategy $n(t)$ in any of our hedge instruments is necessarily of the form

$$n(t) = \begin{cases} n_0(t) & : \quad 0 \le t \le \tau_{[1]} \\ n_1(\tau_{[1]}, z_1, t) & : \quad \tau_{[1]} < t \le \tau_{[2]}, \\ n_2(\tau_{[1]}, \tau_{[2]}, z_1, z_2, t) & : \quad \tau_{[2]} < t \le \tau_{[3]}, \\ \dots & : \quad \dots \end{cases} . \tag{31}$$

with measurable functions n_k.

4.3.1 Hedging and pricing a first-to-default swap

Now, it turns out that the hedging and pricing of a first to default swap can be made quite explicit.

We need a purely technical condition on the support of the measure $\mathbf{P}(\tau_i \in \mathrm{d}t, \tau_{[1]} = \tau_i)$

$$\mathrm{supp}(\mathbf{P}(\tau_i \in \mathrm{d}t, \tau_{[1]} = \tau_i \le T)) = [0,T]. \tag{32}$$

Proposition 7 *Consider a first-to-default (FTD) swap with maturity T and payoff V_T^{first} as in (25) for $k=1$,*

$$V_T^{first} = -s \cdot (\tau_{[1]} \wedge T) + \sum_{i=1}^n P_i(\tau_i) 1_{\{\tau_{[1]} \le T, \tau_{[1]} = \tau_i\}}.$$

Chose as hedge instruments CDS strategies $C^i(t,T)$ (see (22)) for the underlying credits $i = 1,\dots,n$. The FTD swap is hedgeable in the instruments $C^i(t,T)$ with strategies $n^i(t)$ as in (31), i.e.,

$$V_T^{first} = K + \sum_{i=1}^n \int_0^T n^i(s) \mathrm{d}C^i(s,T),$$

if and only if the vector function $\mathbf{n}_0(t) = (n_0^1(t),\dots,n_0^n(t))^T$ satisfies the following system of ordinary integral equations

$$-s \cdot t \cdot \mathbf{1} + \mathbf{P}(t) = K \cdot \mathbf{1} + \int_{(0,t)} \mathrm{d}\mathbf{F}(u,T) \cdot \mathbf{n_0}(u) \cdot \mathbf{1} + \mathbf{E}(t,T) \cdot \mathbf{n}_0(t), \ t \le T \tag{33}$$

$$-s \cdot T = K + \int_0^T \mathrm{d}\mathbf{F}(u,T) \cdot \mathbf{n_0}(u), \tag{34}$$

with the notation[7]

$$
\begin{aligned}
\mathbf{P}(t) &= (P_1(t), \dots, P_n(t))^T \\
\mathbf{1} &= (1, \dots, 1)^T \\
\mathbf{E}(t,T) &= (E_{i,j}(t,T))_{i,j=1,\dots,n} \\
E_{i,j}(t,T) &= \begin{cases} 1 - R_i & : \quad i = j \\ (b^i(t,j,t,T) + \Delta a^i(t,T)) \int_t^T \exp\left(- \int_v^T \frac{a^i(t,u)+b^i(t,j,t,u)}{1-R_i} \mathrm{d}u \right) \mathrm{d}v & : \quad i \neq j \end{cases} \\
\mathrm{d}\mathbf{F}(u,T) &= (-a^1(u,T)\mathrm{d}u + B^1(u,T)\mathrm{d}a^1(u,T), \dots, -a^n(u,T)\mathrm{d}u + B^n(u,T)\mathrm{d}a^n(u,T)) \\
B^i(u,T) &= \int_t^T \exp\left(- \int_v^T \frac{a^i(t,u)}{1-R_i} \mathrm{d}u \right) \mathrm{d}v
\end{aligned}
$$

and functions a^i, b^i *from* (30).

Proof: Using Proposition 5 the FTD swap is hedgeable in the instruments $C^i(t,T)$ with strategies $n^i(t)$ as in (31) if and only if

$$
\begin{aligned}
& -s \cdot (\tau_{[1]} \wedge T) + \sum_{i=1}^n P_i(\tau_i) 1_{\{\tau_{[1]} \leq T, \tau_{[1]} = \tau_i\}} \\
& = K + \sum_{i=1}^n \left[- \int_0^{\tau_{[1]} \wedge T} n_0^i(u) s^i(u,T) \mathrm{d}u + \int_0^{\tau_{[1]} \wedge T} n_0^i(u) B^i(u,T) \mathrm{d}s^i(u,T) \right. \\
& \qquad \left. + n_0^i(\tau_{[1]})(1 - R_i) \mathbf{1}_{\{\tau_{[1]} = \tau_i \leq T\}} \right].
\end{aligned}
$$

On the set $\{\tau_{[1]} = \tau_j = t \leq T\}$ using (17) and (30) this can be written as

$$
\begin{aligned}
& -s \cdot t + P_j(t) = \\
& K + \sum_{i=1}^n \left[- \int_0^t n_0^i(u) a^i(u,T) \mathrm{d}u + \int_{(0,t)} n_0^i(u) \int_u^T \exp\left(- \int_v^T \frac{a^i(t,w)}{1-R_i} \mathrm{d}w \right) \mathrm{d}v \, \mathrm{d}a^i(u,T) \right] \\
& + \sum_{i \neq j} n_0^i(t) (b^i(t,j,t,T) + \Delta a^i(t,T)) \int_t^T \exp\left(- \int_v^T \frac{a^i(t,u) + b^i(t,j,t,u)}{1-R_i} \mathrm{d}u \right) \mathrm{d}v \\
& + n_0^j(t)(1 - R_j).
\end{aligned}
$$

In vector notation this is just equation (33). In the same way, on the set $\{\tau_{[1]} > T\}$ we obtain (34). In view of condition (32) this proves the assertion. □

Corollary 1 *Suppose that for every* s, K *equation* (33) *possesses a solution*

$$
\mathbf{n}_0^{s,K}(t), t \leq T.
$$

[7] $\Delta a^i(t,T) = a^i(t,T) - a^i(t-,T)$.

(i) For given spread s the price K of the FTD swap is a solution of the equation

$$K = -s \cdot T - \int_0^T \mathrm{d}\mathbf{F}(u,T) \cdot \mathbf{n}_{\mathbf{0}}^{\mathbf{s},\mathbf{K}}(u).$$

(ii) The fair spread s^{FTD} of the FTD swap is a solution of the equation

$$s^{FTD} = \frac{-\int_0^T \mathrm{d}\mathbf{F}(u,T) \cdot \mathbf{n}_{\mathbf{0}}^{\mathbf{s}^{FTD},\mathbf{0}}(u)}{T}.$$

Remark. Even in case of a model that is not of pure jump type, the results above seem to be applicable to a certain extend, at least for the purpose of pricing. Start with some model for $\tau_1, \ldots, \tau_n$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with risk-neutral distribution $\mathbf{P}$. Suppose we get a hand on the joint distribution of the default times and their copula

$$\mathbf{P}(\tau_1 < t_1, \ldots, \tau_n < t_n) = C(F_1(t_1), \ldots, F_n(t_n)).$$

Now construct a model with pure jump filtration possessing exactly the same joint distribution of default times. For this purpose, the default implied spread widenings can be calculated as shown in [8],[10] and the setup (30) can be made explicit from the copula. Prices and fair spreads calculated as above in the framework of a pure jump model then coincide with the prices and fair spreads in the original model since they depend solely on the joint distribution.

4.3.2 Numerical examples

Contrary to most pricing models for FTD swap in practice, which are based on a time consuming Monte-Carlo simulation of a copula model or which use somewhat restricting factor dependency structures, the approach above allows for an extremely fast and efficient implementation of the pricing of an FTD swap, producing at the same time explicit hedging strategies in case of a pure jump model.

We start with a model setup as in (30) assuming for simplicity of exposition that all functions $a^i(t,T), b^i(u,j,t,T), \ldots$ are constant over time, i.e.

$$a^i(t,T) = a^i, b^i(u,j,t,T) = b^i(j), \ldots$$

For the premiums $P_i(t)$ of the FTD swap we assume, as it is common in practice, that

$$P_i(t) = 1 - R_i.$$

In this case equation (33) simplifies, for $j = 1, \ldots, n$ and $0 \leq t \leq T$

$$\begin{aligned} -s\ t + (1 - R_j) =& K - \sum_{i=1}^{n} a^i \int_0^t n_0^j(u)\mathrm{d}u \\ &+ \sum_{i\neq j} n_0^i(t) b^i(j) \frac{1-R_i}{a^i + b^i(j)} \left[1 - \exp\left(-\frac{a^i + b^i(j)}{1-R_i}(T-t)\right)\right] \\ &+ (1 - R_j) n_0^j(t). \end{aligned}$$

This system of equations possesses a unique solution for every s, K, which can be even made explicit. However, we prefer a numerical solution based on a discretisation of time.

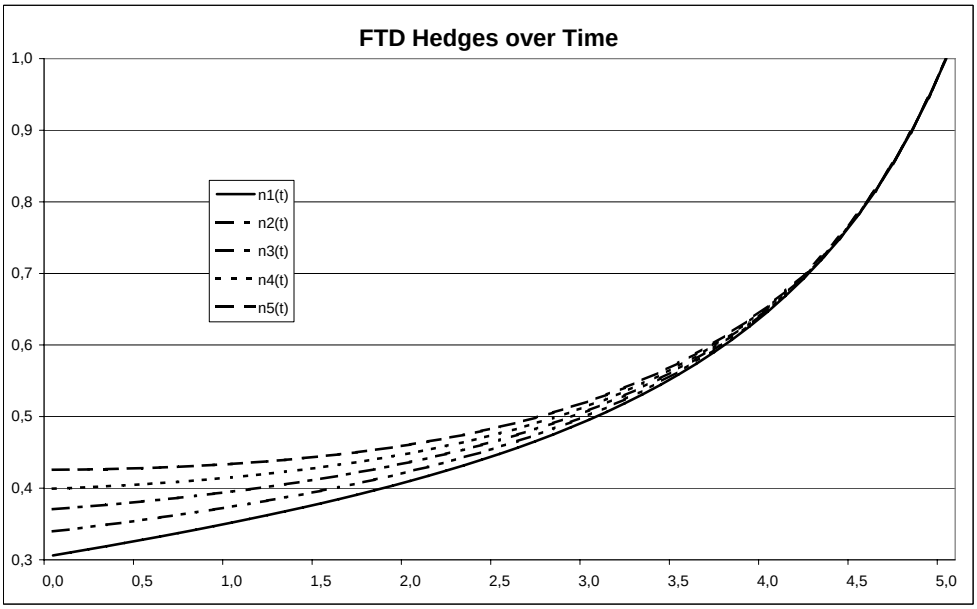
As an example, consider $n = 5$ names with base spreads $a^1 = 0,80\%, a^2 = 0,90\%, a^3 = 1,00\%, a^4 = 1,10\%, a^5 = 1,20\%$ and assume recoveries $R_i = 20\%$ throughout. The following table shows the fair FTD spread s^{FTD} for maturities $T = 1, \ldots, 5$ and for constant default implied spread jumps $b^i(j) = 1\%, 5\%, 10\%$.

$b^i(j)/T$	1	2	3	4	5
1%	4,878%	4,764%	4,657%	4,555%	4,459%
5%	4,467%	4,073%	3,765%	3,516%	3,310%
10%	4,074%	3,523%	3,148%	2,872%	2,660%

Now consider the same example as above with maturity $T = 5$ and with default implied spread jumps $b^i(j) = 5\%$ throughout. Here are the fair prices K of the FTD swap for different given spreads s.

spread s	2,00%	2,50%	3,00%	3,31%	3,50%	4,00%	4,50%
K in basis points	598,16	369,85	141,54	0,00	-86,75	-315,06	-543,37

The final example takes a rather extreme situation. We consider a basket with $n = 5$ names and with base spreads $a^1 = 1,00\%, a^2 = 2,00\%, a^3 = 3,00\%, a^4 = 4,00\%, a^5 = 5,00\%$, recoveries $R_i = 20\%$, a maturity of $T = 5$ and equal default implied spread jumps $b^i(j) = 10\%$. The fair spread is then $s^{\text{FTD}} = 7,992\%$. The following picture shows the hedges notionals $n^i(t)$ over time for a fair FTD swap.



The hedges start at time $t = 0$ with relatively small hedge amounts $n^i(t) \ll 100\%$ that increase over time and tend to 100% as t approaches the maturity T. This admits an intuitive explanation. Let us think of a bank that bought FTD protection from a client and hedges by selling protection in single name CDS in the underlying names. In case the first credit defaulting before maturity is i, the bank receives the amount $(1 - R_i)$ from the client. This amount is, according to the shown hedge ratios of $n^k(t) < 1$, higher than what the bank has to pay in the hedge, namely, the amount $n^i(t)(1 - R_i)$. The difference, $D_i(t) = (1 - n^i(t))(1 - R_i)$, between the two is, of course, not a windfall profit for the bank. Due to the spread widening in the remaining names $k \neq i$ unwinding the now redundant hedges in these names is costly, the cost becoming higher the higher the remaining life time $T - t$ of the CDS. On the other hand, there might be a (so-called carry-) mismatch between the premium the bank has paid to the client and what was earned from the hedges; up to time t the carry is

$$-s^{\mathrm{FTD}} \cdot t + \sum_k \int_0^t n^k(u) s^k(u, T) \mathrm{d}u.$$

Initially, for small t the carry will be negative as the hedge amounts $n^k(t)$ are small. In case of default any negative carry has to be be covered as well by the difference $D_i(t)$. For t getting closer to maturity T, the hedge ratios $n^k(t)$ approach 100%, generating a locally positive carry, which, in case there is no default at all, will balance the final overall carry to be zero at maturity T.

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