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## The generalized Stackelberg equilibrium of the all-pay auction with complete information

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# The generalized Stackelberg equilibrium of the all-pay auction with complete information

Kai A. Konrad\* and Wolfgang Leininger\*\*

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## Abstract

In the equilibrium of the all-pay auction with two groups of individual players who move sequentially, only the player with the lowest effort cost has a positive payoff. This payoff and the overall dissipation crucially depend on group composition.

Keywords: Sequential all-pay auction, complete information, generalized cost, generalized Stackelberg game.

JEL classification code: D72, D74

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## 1 Introduction

The analytics of the all-pay auction with complete information with simultaneous bidding is well understood (see Hillman and Riley 1989, Baye et al. 1996, 2005, Kaplan et al. 2003 and Baye et al. 2005). In this paper we

offer a characterization of the all-pay auction with many bidders who may bid sequentially.

Bidding in all-pay auctions is sometimes sequential both for exogenous reasons and endogenously. For the two-player context, such endogenous timing has been discussed by Deneckere et al. (1992) in a Bertrand framework with consumer loyalty, by Leininger (1991) in an R&D context and by Baik and Shogren (1992) and Leininger (1993) for a Tullock contest; i.e., a specific all-pay auctions with noise. We solve the more general case with many players and discuss the implications of more than two bidders and group composition for the equilibrium payoffs and for endogenous timing.

## 2 Equilibrium

A prize of given size  $V = 1$  is allocated among a set  $N$  of players  $i = 1, \dots, n$  in an all-pay auction with complete information. Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  denote the vector of players' efforts. The players in a subset  $E \subset N$  choose their efforts simultaneously and irreversibly at a point  $e$ (arly) and players from set  $L = N - E$  choose their efforts at point  $l$ (ate), where  $l$  occurs after  $e$ , and players in  $L$  can observe the effort choices made by the players in  $E$ .

Player  $i$ 's payoff is

$$\pi_i(\mathbf{x}) = p_i(\mathbf{x}) \cdot 1 - C_i(x_i). \quad (1)$$

The cost  $C_i(x_i)$  of expending a given effort  $x_i$  is a function of this effort, with  $C_i(0) = 0$ ,  $C_i'(x_i) > 0$ , and  $C_i''(x_i) \geq 0$ . Linear cost,  $C_i(x_i) = c_i x_i$ , and quadratic cost,  $C_i(x_i) = c_i(x_i)^2$  are special cases. We assume that players are asymmetric and can be strictly sorted according to their effort cost. Without loss of generality we consider them numbered such that, for two players  $i$  and  $j$  with  $i < j$ , it holds that  $C_i'(x) < C_j'(x)$  for all  $x \in (0, K]$ . Player 1 has the lowest cost for expending a given effort, player 2 has the second lowest cost, etc. The sorting in  $N$  also induces a similar sorting of players in  $E$  and in  $L$ . We re-number the players in  $E$  and  $L$  according to their cost functions as  $e_{(1)}, e_{(2)}, \dots, e_{(\#E)}$  with  $C_{e_{(i)}}(x) < C_{e_{(j)}}(x)$  and  $l_{(1)}, l_{(2)}, \dots, l_{(\#L)}$  with  $C_{l_{(i)}}(x) < C_{l_{(j)}}(x)$  for all  $i < j$  for all  $x > 0$ . The strict global asymmetry and sorting eliminates some equilibria that may occur otherwise.

The probability that player  $i$  wins the prize is denoted as  $p_i$ . Denote  $\bar{x} \equiv \max_{k \in N} \{x_k\}$ . If  $x_i = \bar{x} > x_j$  for all  $j \in N - \{i\}$ , then  $p_i = 1$  and  $p_j = 0$  for all  $j \neq i$ . If several players have chosen the same, highest effort,  $\bar{x}$ , we assume the following tie-breaking rule. Let  $M$  be the set of players

who choose  $\bar{x}$ . If  $M \subset E$ , or  $M \subset L$ , then each  $i \in M$  wins the prize with the same probability equal to  $1/\#M$ , where  $\#M$  is the cardinality of  $M$ . If  $M \cap E \neq \emptyset$  and  $M \cap L \neq \emptyset$  then the allocation of the prize among the players in  $M$  depends on their cost of effort. If  $C_i(\bar{x}) \geq 1$  for all  $i \in M \cap L$ , then  $p_i = 1/\#(M \cap E)$  for  $i \in M \cap E$  and zero for all other players. Denote  $(M \cap L)^+(\bar{x})$  the subset of  $(M \cap L)$  with players for which  $C_i(\bar{x}) < 1$  holds. If  $(M \cap L)^+(\bar{x}) \neq \emptyset$ , then  $p_i = 1/\#(M \cap L)^+(\bar{x})$  for  $i \in (M \cap L)^+(\bar{x})$  and  $p_i = 0$  for all other players.

For  $e_{(i)} \in E$ , a pure strategy is an effort choice  $x_{e_{(i)}}$  and the strategy set of player  $e_{(i)}$  is the set of feasible efforts  $[0, K]$ , with  $K$  being a large but finite number.<sup>1</sup> A mixed strategy  $F_{e_{(i)}}$  for player  $e_{(i)} \in E$  is an element of the set of probability distributions over the effort levels from the set  $[0, K]$ . A player  $l_{(i)} \in L$  observes the vector  $(x_{e_{(1)}}, \dots, x_{e_{(\#E)}}) \equiv \mathbf{x}_E$  of effort choices of all players  $e_{(i)} \in E$ . Only  $\bar{x}_E \equiv \max_{e_{(i)} \in E} \{x_{e_{(i)}}\}$  is payoff relevant for the subgame in period  $l$ , so we can denote a pure strategy of  $i \in L$  as a function  $x_i(\bar{x}_E) : [0, K] \rightarrow [0, K]$ , and a mixed strategy of  $i$  as a function  $F_i(\bar{x}_E) : [0, K] \rightarrow \Sigma_{[0, K]}$ , where  $\Sigma_{[0, K]}$  denotes the set of probability distributions over the effort levels from the set  $[0, K]$ .

**Proposition 1** *For any given sets  $L$  and  $E$  there exists a unique subgame perfect equilibrium of the (sequential) all-pay auction. Equilibrium pay-offs are given by:*

$$\pi_j = 0 \text{ for all } j = 2, \dots, n, \text{ and} \quad (2)$$

$$\pi_1 = \begin{cases} 1 & \text{if } L = \{1\} \\ 1 - C_1(\bar{x}_{l_{(2)}}) & \text{if } 1 \in L \text{ and } \#L > 1 \\ 1 - C_1(\bar{x}_2) & \text{if } 1 \in E \end{cases} \quad \begin{cases} \text{with } \bar{x}_{l_{(2)}} \text{ the solution} \\ \text{of } C_{l_{(2)}}(\bar{x}_{l_{(2)}}) = 1 \\ \text{with } \bar{x}_2 \text{ the solution of } C_2(\bar{x}_2) = 1 \end{cases} \quad (3)$$

**Proof.** Consider the following candidate equilibrium of the subgame at stage  $l$ . Let  $L^+(\bar{x}_E)$  be the set of players  $k \in L$  for which  $C_k(\bar{x}_E) < 1$ , and  $L^-(\bar{x}_E)$  the set of the players  $k \in L$  for which  $C_k(\bar{x}_E) \geq 1$ . Then (i) all players  $k \in L^-(\bar{x}_E)$  choose  $x_k(\mathbf{x}_E) = 0$ , (ii) for players  $k \in L^+(\bar{x}_E)$ , if  $\#L^+(\bar{x}_E) = 1$  then  $x_k(\mathbf{x}_E) = \bar{x}_E$ , and if  $\#L^+(\bar{x}_E) > 1$ , then  $x_k(\mathbf{x}_E) = 0$  for all  $k \in L$  with  $k \notin \{l_{(1)}, l_{(2)}\}$ ,

$$F_{l_{(1)}}(\mathbf{x}_E) = \begin{cases} 0 & \text{for } x \in [0, \bar{x}_E] \\ C_{l_{(2)}}(x) & \text{for } x \in [\bar{x}_E, \bar{x}_{l_{(2)}}] \\ 1 & \text{for } x > \bar{x}_{l_{(2)}} \end{cases} \quad (4)$$

<sup>1</sup>For instance, it will be sufficient to define  $K$  as the solution of  $C_1(K) = 1$ .

and

$$F_{l_{(2)}}(\mathbf{x}_E) = \begin{cases} 1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) + C_{l_{(1)}}(\bar{x}_E) & \text{for } x \in [0, \bar{x}_E] \\ 1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) + C_{l_{(1)}}(x) & \text{for } x \in (\bar{x}_E, \bar{x}_{l_{(2)}}] \\ 1 & \text{for } x > \bar{x}_{l_{(2)}}. \end{cases} \quad (5)$$

These strategies are mutually optimal replies. (i) For given  $\bar{x}_E$ ,  $x_k = 0$  uniquely maximizes the payoff of  $k \in L^-(\bar{x}_E)$ , independent of the choices of other players in  $L$ . (ii) Consider  $k \in L^+(x_E)$  with  $k \notin \{l_{(1)}, l_{(2)}\}$ . The payoff in the candidate equilibrium is  $\pi_k(0) = 0$  and, given (??),  $\pi_k(x_k) < 0$  for any  $x_k > 0$ . Thus  $x_k = 0$  is the unique best reply. Consider next  $l_{(1)}$ . The payoff of  $l_{(1)}$  given the candidate equilibrium strategies of other players is  $\pi_{l_{(1)}}(0) = 0$ ,  $\pi_{l_{(1)}}(x) = -C_{l_{(1)}}(x)$  for  $x \in (0, \bar{x}_E)$  and

$$\pi_{l_{(1)}} = F_{l_{(2)}}(x) \cdot 1 - C_{l_{(1)}}(x) \quad (6)$$

for  $x_{l_{(1)}} \geq \bar{x}_E$ . By (??) this payoff is equal to  $1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) > 0$  for all  $x_{l_{(1)}} \in [\bar{x}_E, \bar{x}_{l_{(2)}}]$ , and smaller than this for all  $x$  outside this interval. Consider finally the payoff of  $l_{(2)}$ . This payoff is  $\pi_{l_{(2)}} = 0$  for  $x_{l_{(2)}} = 0$ ,  $\pi_{l_{(2)}}(x) = -C_{l_{(2)}}(x)$  for  $x \in (0, \bar{x}_E)$ , and

$$\pi_{l_{(2)}} = F_{l_{(1)}}(x) \cdot 1 - C_{l_{(2)}}(x) \quad (7)$$

for  $x_{l_{(2)}} > \bar{x}_E$ . This payoff (??) is equal to zero for all  $x_{l_{(2)}} \in (\bar{x}_E, \bar{x}_{l_{(2)}}]$ , because of (??), and negative for all  $x$  outside this interval. Accordingly, any  $x \in \{0\} \cup (\bar{x}_E, \bar{x}_{l_{(2)}}]$  is an optimal reply for  $l_{(2)}$ .

The equilibrium in the subgame among players from set  $L$  is the equilibrium of a simultaneous all-pay auction with a minimum bid of  $\bar{x}_E$ . The uniqueness of this equilibrium can be shown following the line of arguments in Baye, Kovenock and de Vries (1996).

Consider stage  $e$ . Define  $\max_{l_{(i)} \in L} \{\bar{x}_{l_{(i)}}\} \equiv \bar{x}_L$ . A player  $i \in E$  who made the highest bid  $\bar{x}_E = \max_{e_{(i)} \in E} \{x_{e_{(i)}}\}$  wins the prize *if and only if*  $\bar{x}_E > \bar{x}_L$ . In the characterization of the equilibrium strategies of players from the set  $E$ , this  $\bar{x}_L$  plays a similar role for players from the set  $E$  when they choose their efforts, as  $\bar{x}_E$  does in the subgame at  $l$ . This  $\bar{x}_L$ , unlike  $\bar{x}_E$ , is not determined by the actual effort choices of players, but is uniquely determined by the cost structure of the players in  $L$ . Hence, the game among players at stage  $e$ , too, has the structure of a simultaneous all-pay auction with a minimum bid, which now is given by  $\bar{x}_L$ .  $\bar{x}_L$  results from *foreseeing* the optimal behaviour of players moving at  $l$ . The formal reasoning for the equilibrium choices of players in  $E$  given  $\bar{x}_L$  is analogous to the reasoning

for  $L$  given  $\bar{x}_E$  and not repeated here, leading to the following equilibrium:<sup>2</sup>  
(i) All players  $k \in E^-(\bar{x}_L)$  choose  $x_k = 0$ . (ii) If  $\#E^+(\bar{x}_L) = 1$ , then this player  $k \in E^+(\bar{x}_L)$  chooses  $x_k = \bar{x}_L$ . (iii) If  $\#E^+(\bar{x}_L) > 1$ , then all players  $e_{(k)} \in E^+(\bar{x}_L)$  with  $k > 2$  choose  $x_{e_{(k)}} = 0$ , player  $k = e_{(1)}$  chooses a mixed strategy that is described by the cumulative distribution function  $F_{e_{(j)}}(\bar{x}_L)$  that are obtained from (??) and (??) by replacing  $l$  by  $e$  and  $E$  by  $L$  throughout. In this equilibrium,  $\pi_k = 0$  for all  $k \neq 1$ . Consider  $\pi_1$ . If  $1 \in L$ , then  $x_k = 0$  for all  $k \in E$ . Hence,  $\bar{x}_E = 0$ . If  $L = \{1\}$ , then 1 wins with no effort and has payoff  $\pi_1 = 1$ . If  $\#L > 1$ , then  $\pi_1 = 1 - C_1(\bar{x}_{l_{(2)}})$ . If  $1 \in E$ , it has been shown that  $x_1 = \bar{x}_2$  is within player 1's equilibrium support and yields  $\pi_1 = 1 - C_1(\bar{x}_2)$ . ■

According to Proposition 1, for any partition of players into the sets  $E$  and  $L$ , there is a *unique* subgame perfect equilibrium. The number of different partitions of  $n$  players into two sets is  $2^n$ . Hence, we have characterized  $2^n$  possible equilibrium configurations, which, however, result in only  $n$  different equilibrium payoff vectors  $(\pi_1, 0, \dots, 0)$ , where  $\pi_1$  can assume the values 1 and  $1 - C_1(\bar{x}_i), i = 2, \dots, n$ .

Adding another player  $(n + 1)$  doubles the number of feasible partitions (as this player could "join" any previous partition in either  $E$  or  $L$ ), but add only one further equilibrium payoff vector  $\boldsymbol{\pi} = (1 - C_1(\bar{x}_{n+1}), 0, \dots, 0)$  with a new positive payoff for player 1. If the new player is not the one with the highest cost of effort (and hence change our labelling of players), the *number* of different equilibrium payoffs for player 1 only increases by one to  $(n + 1)$ . Moreover,  $n$  of those are identical to the previous ones unless the new player happens to be the -new- strongest one with the lowest cost of effort. In this latter case all  $n$  feasible equilibrium payoffs for player 1, which are smaller than 1, can be different from the previous feasible equilibrium values with only  $n$  players. With just 2 players the equilibrium payoffs for three player partitions  $(E_1, L_1) = (\{1\}, \{2\}), (E_2, L_2) = (\{1, 2\}, \{\emptyset\})$  and  $(E_3, L_3) = (\{\emptyset\}, \{1, 2\})$  are identically equal to  $(1 - C_1(\bar{x}_2), 0)$ , whereas  $(E_4, L_4) = (\{2\}, \{1\})$  yields the payoffs  $(1, 0)$ .

The equilibrium with  $n$  players has similar properties as the solution with two players. Only the player with the strongest cost advantage has a positive payoff. This payoff is larger if this player chooses his effort last. An interesting and new aspect is that the all-pay auction that takes place at stage  $l$  and the competition it involves for those players, does not benefit

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<sup>2</sup>Uniqueness then again follows from reasoning as in Baye et al. (1996) for any set  $E$  and any  $\bar{x}_L$ . For a more detailed exposition see the working paper version, Konrad and Leininger (2005).

the players who choose at stage  $e$  compared to the situation in which there is only one player who chooses at  $l$ . The reason is that  $\bar{x}_L$ , the minimal bid for a player in  $E$  in order to win, does not depend on actual behavior of players in the competition at  $L$ , but their *potential* limit behavior in rational play. This largest possible rational counter bid, however, is determined by exogenous data, namely the cost parameters of the players moving at  $l$ .

Also similar to the two-player case, the player with the lowest cost is best off if he moves later than all other players. For all other players, timing does not affect their payoff. They all cannot expect to gain something from participating in the contest and from making positive bids. Turning to the case in which players can choose whether they belong to group  $E$  or group  $L$ , by the nature of the equilibrium, all players  $i \geq 2$  are indifferent as to which group they would join as their equilibrium payoff is zero. Player 1's payoff depends on his own and the other players' choice, but  $1 \in L$  is an almost strictly dominant choice: only when player 2 chooses  $l$  is player 1 indifferent between bidding at  $e$  (and preempting player 2) or  $l$  (and entering a simultaneous all-pay auction with player 2). Still, any offer of a small "premium" payment for moving at  $e$  - either by the rules of the game or by player 1 - would select the efficient equilibrium as the unique one. This essentially characterizes the possible endogenous equilibrium partitions of players into the groups  $E$  and  $L$ .

These results do not depend in an essential way on our choice of tie-breaking rule. This rule was chosen in order to arrive at *strict* subgame perfect equilibria. Choice of other tie-breaking rules; i.e. the probably more "obvious" one, which stipulates the winning probability of a player from  $M$  as always equal to  $\frac{1}{\#M}$  *regardless* of his membership in  $E$  or  $L$ , would only produce  $\varepsilon$ -equilibria. Moreover, these  $\varepsilon$ -equilibria would all lie in  $\varepsilon$ -neighborhoods of our strict equilibria.

### 3 Conclusions

The sequential structure of bids in an all-pay auction with complete information and many players favors the strongest player, particularly if some of his strongest competitors have to make their bids prior to him. For an appropriate partition of players the prize is efficiently allocated to the strongest bidder, with aggregate bid cost in this equilibrium of zero. Moreover, this partition of players into groups can emerge as an equilibrium outcome if all players choose their timing of bids simultaneously in a stage prior to the actual bidding stages. This result has important consequences



for the theory of rent-seeking and patent competition among heterogeneous agents. For instance, Konrad and Leininger (2006) show that the problem of efficient provision of collective effort in contests between groups of heterogeneous agents, which have to cope with the possibility of internal conflict over the returns of the collective effort, becomes tractable by implementing the equivalent of the efficient equilibrium in the internal conflict.

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