

Hofmann, Dirk; Qari, Salmai

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Bilateral Search with Horizontal Heterogeneity

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Bilateral Search with Horizontal Heterogeneity

Dirk Hofmann* and Salmal Qari †

Department of Economics, Humboldt-University Berlin

Max Planck Institute for Intellectual Property, Munich

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Abstract

We study a matching model with heterogeneous agents, nontransferable utility and search frictions. Agents differ along a continuous horizontal dimension (e.g. taste) and a discrete vertical dimension, e.g. income. All individuals prefer the highest type along the vertical trait, but have different preferences regarding the horizontal trait. We characterize the equilibrium and study how matching occurs depending on the frictions in the market if income is exogenous. If the level of search frictions is low, assortative mating along both dimensions occurs. For greater levels of search frictions, equilibria with and without income segregation are possible. Finally, we study the impact of a redistributive tax on social welfare if agents decide on their effort levels before they enter the matching market.

Keywords: Matching, Taste, Marriage Markets, Redistributive Taxation

JEL Classification Numbers: D61, H21, J12

*Address: Spandauer Str. 1, 10178 Berlin, e-mail: *dirk.hofmann@staff.hu-berlin.de*

†Address: Max Planck Institute for Intellectual Property, Competition and Tax Law, Department of Public Economics, Marstallplatz 1, 80539 München, Germany, e-mail: *salmal.qari@ip.mpg.de*
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1 Introduction

In this paper we analyze how sorting and matching takes place in an environment with search frictions and nontransferable utilities. We frame the model as the decision problem of an individual on the marriage market, but the framework could also prove useful for similar decision problems, e.g. the labor market. Agents are characterized by two parameters. One parameter captures a discrete vertical dimension, e.g. income and the second parameter represents a horizontal trait, e.g. taste.

If two agents agree to marry each other, both individuals share equally the household income. Both agents additionally enjoy a “non-monetary” utility flow. As utility is nontransferable, agents cannot bargain over split. This implies that all agents prefer the highest type along the vertical dimension. For simplicity we assume that there are only two types, that is poor and rich agents. Along the continuous horizontal dimension agents do not universally agree on who is the best type, but rather prefer someone of similar “taste”. As long as individuals stay single, their utility equals their own income.

When agents are randomly matched to each other, they can perfectly observe each others’ characteristics. Having evaluated the potential partner’s traits, the agent either proposes or declines and waits for the next offer. When declining a potential partner, agents have to optimally balance the disutility from forgone utility flow along the horizontal dimension, the positive or negative utility change induced by sharing the household income and the possibility of receiving a better offer in future periods.

Our framework builds on the existing literature on dynamic bilateral search and matching. The classic model by Becker (1973, 1974) assumes that utility is transferable and that no search frictions are present.¹ Starting from Becker’s observations, a large literature has emerged which interestingly focuses on *vertical* heterogeneity. For example Burdett and Coles (1997); Eeckhout (1999); Smith (2006) analyze vertical heterogeneity in a nontransferable-utility setting, while Bloch and Ryder (2000); Shimer and Smith (2000); Atakan (2006) consider transferable utility. Our paper is also closely related to Konrad and Lommerud (2010), who also introduce a horizontal dimension. However, their model considers only one period in which agents either accept a match or stay single forever. As such it can be interpreted as the case of maximum search frictions.

¹Legros and Newman (2007) and Klumpp (2009) provide recent examples for the case without search frictions.

Our model generates a notion of assortative mating. If the level of search frictions is low, agents expect to receive many offers from their preferred types. Obviously, in equilibrium all agents must have correct expectations. The optimal reservation-utility strategies for low search frictions induce the vertically high agents to reject vertically low agents which we interpret as segregation along income classes. Starting with this result we characterize the equilibrium for both poor and rich agents and derive comparative statics with respect to the level of search frictions. Finally we consider endogenous effort choices on the labor market and analyze the impact of redistributive taxation on social welfare.

We proceed as follows. In the following section we develop the marriage model. Section 3 shows explicitly how a redistributive tax may positively effect welfare by applying type-dependent and endogenous labor-supply decisions. Section 4 concludes. All proofs are relegated to the appendix.

2 The Model

Each individual i is described by two parameters, income $y_i \in \{y_l, y_h\}$ and a taste parameter $t_i \in [0, 2]$ with $y_h \geq y_l$. The taste parameter t is uniformly distributed on $[0, 2]$. In the matching module two individuals marry if they both agree on mutual proposals. The per-period-utility of individual i marrying individual j is given by the mean of the incomes they have as a couple plus some value for the quality of the emotional match between the spouses. The quality of the emotional match is measured as a function of the distance of the two taste parameters. Geometrically we assume the taste parameters to form a circle, i.e. the distance between two taste parameters is measured as the distance between two points on the border of a circle with circumference 2. As taste is distributed uniformly this is equivalent to the assumption that taste difference between two individuals is a random variable X uniformly distributed on $[0, 1]$. Hence, the per-period-utility of such a couple is assumed to be

$$U_i(\text{individual } i \text{ marrying individual } j) = \frac{y_i + y_j}{2} + 1 - x.$$

To capture the dynamics of the model the life-time-utility of marrying someone with per-period-utility k at period 0 when using discount factor r is

$$\int_0^{\infty} k \cdot e^{-rt} dt = \frac{k}{r}.$$

In the marriage market module the arrival rate of other persons is α . Sometimes we denote by $\Theta := \frac{\alpha}{r}$ the vibrancy of the market, a high Θ denotes higher patience by all participants. By V_i we denote individual i 's expected lifetime utility when staying single. The per-period-utility of a single is just given by his income y_i . Obviously it is optimal to propose to all potential partners which would give a higher lifetime-utility then the expected utility of staying single. A match forms if both spouses agree on the marriage. They leave the market and live forever. They are replaced by clones and there is no divorce to keep things tractable.

In the first section we look at the simple case that there is only one income state. Afterwards we look at a more general case where we have two income classes.

2.1 The Circle

Assume all agents in the marriage market have the same income $y_l = y_h$. When considering a potential partner they only care about the emotional benefit which may arise from a small realization of X , the random variable capturing taste difference.

For the next short time interval Δ an individual's expected discounted lifetime utility can be written as

$$V = \frac{1}{1 + \Delta r} [\Delta y + \Delta \alpha E(\max\{V, \frac{y + 1 - x}{r}\}) + (1 - \Delta \alpha)V] + o(\Delta).$$

The three parts in brackets are the income during the time being single, the utility when meeting another person and considering whether to marry or not, or not meeting anyone in the time period. The small probability events of meeting more than one individual are added by the o -function and do not matter if we let $\Delta \rightarrow 0$. Reformulating this by dividing by Δ , rearranging and then $\Delta \rightarrow 0$ gives

$$rV = y + \Theta E(\max\{0, y + 1 - x - rV\}).$$

The individual makes a proposal if the right term in the maximum-operator is positive. We are looking for symmetric equilibria, i.e. in this model a proposal automatically leads to a marriage. Hence the critical willingness to compromise \bar{x} of an individual is given by $rV = y + 1 - \bar{x}$.

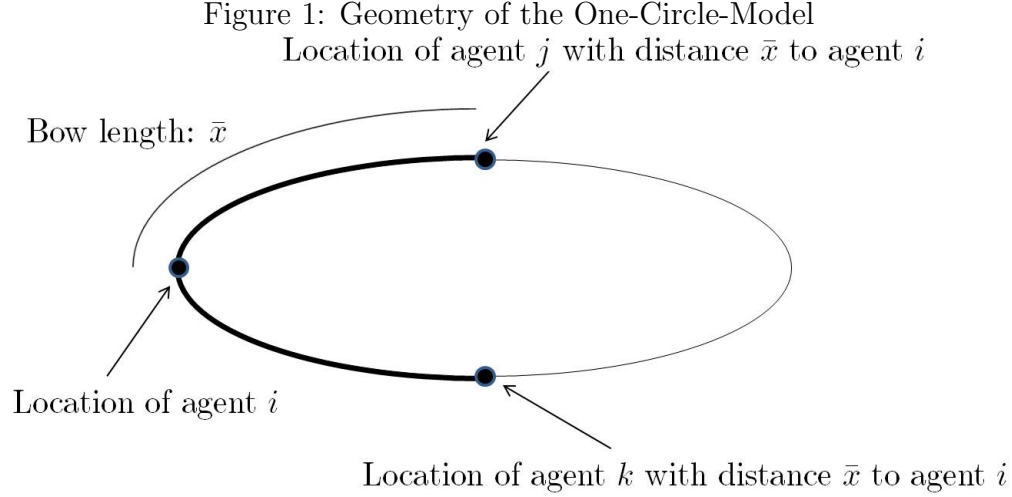


Figure 1 shows the setup for the model of one income level for all agents. As there is only one income level all agents are located on a circle. Agent i is located to the very left and is willing to marry all individuals located on the fat line. He is indifferent between marrying and staying single when meeting agent j or agent k at a certain critical distance \bar{x} which depends on the vibrancy of the marriage institution.

The observations so far lead to the following Proposition:

Proposition 1 *Solution for the Circle*

(a) *The critical willingness to compromise is given by the solution of the following equation system:*

$$(I) \quad rV = y + \frac{1}{2}\Theta\bar{x}^2$$

$$(II) \quad rV = y + 1 - \bar{x}$$

(b) *The critical willingness to compromise is given by $\bar{x} = \frac{1}{\Theta} \cdot (\sqrt{1 + 2\Theta} - 1)$. It holds $0 \leq \bar{x} \leq 1$.*

(c) *The critical willingness to pay is decreasing in Θ and it is*

$$\lim_{\Theta \rightarrow 0} \bar{x} = 1 \quad \text{and} \quad \lim_{\Theta \rightarrow \infty} \bar{x} = 0.$$

The intuition behind these results is as follows. The only additional utility an agent can get by marrying another person is induced by the emotional match as there are no differences in income. That is why agents marry among each other regardless of taste only in the limit in the most adverse market conditions which is Θ close to zero. In these markets there is a very low arrival rate or a very high preference for the present. If people are very patient or respectively the matching institutions work quite effective, i.e. Θ is very high, they are not willing to incur any compromises. In the limit every agent will "wait" (for a time span of zero) for his perfect match. This means in a world where everyone meets anyone in the shortest time interval every agent marries his clone which is an analogy of our two-dimensional base-case to the well-known positive assortative matching results from the literature as e.g. in Becker (1973). In our case ordering is impossible as taste is only a subjective measure but agents are again marrying their clones in the absence of search frictions.

2.2 Two Circles

To expand the framework we now look at the case where agents have two different income levels $y \in \{y_l, y_h\}$; w.l.o.g. we assume that $0 \leq y_l < y_h$ and furthermore $y_h \leq 1$ to treat possible utilities from income and taste similarly and to avoid corner solutions. The different incomes may be due e.g. to some exogenous labor-supply or education decision or by nature (beauty, talent). This decision may depend on a productive type of the agents.² For simplicity we assume that each income type of an agent is equally likely. As in the model with one circle the per-period-utility of being single for an agent i is y_i . The utility of marrying individual j is given by the sum of the mean of the spouses' incomes and the (symmetric) value of the emotional match, i.e. the utility for both spouses is identical and given by

$$U_i(\text{individual } i \text{ marries individual } j) = \frac{y_i + y_j}{2} + 1 - x,$$

where x is the difference in taste between the two spouses. Taking into account the dynamics the lifetime-utility of a marriage is again the per period utility divided by r .

In the following there will be four critical willingness to compromise to discuss. The willingness to compromise for a rich agent among rich agents x_H^R , compromises a rich

²We will come back to endogenous decisions concerning labor-supply in section 3.

is willing to incur in the marriage market with poor agents x_H^P , and the corresponding equivalents for poor agents x_L^R and x_L^P . As obviously the reservation utility of a rich is always at least as high as the reservation utility of a poor the rich are the deciding party in marriages.³ Therefore the outcome of the marriage market will always have $x_H^P = x_L^R$.⁴ Furthermore we are looking for symmetric equilibria. As in the model with one circle a proposal by a rich can be considered as accepted from the corresponding counterpart.

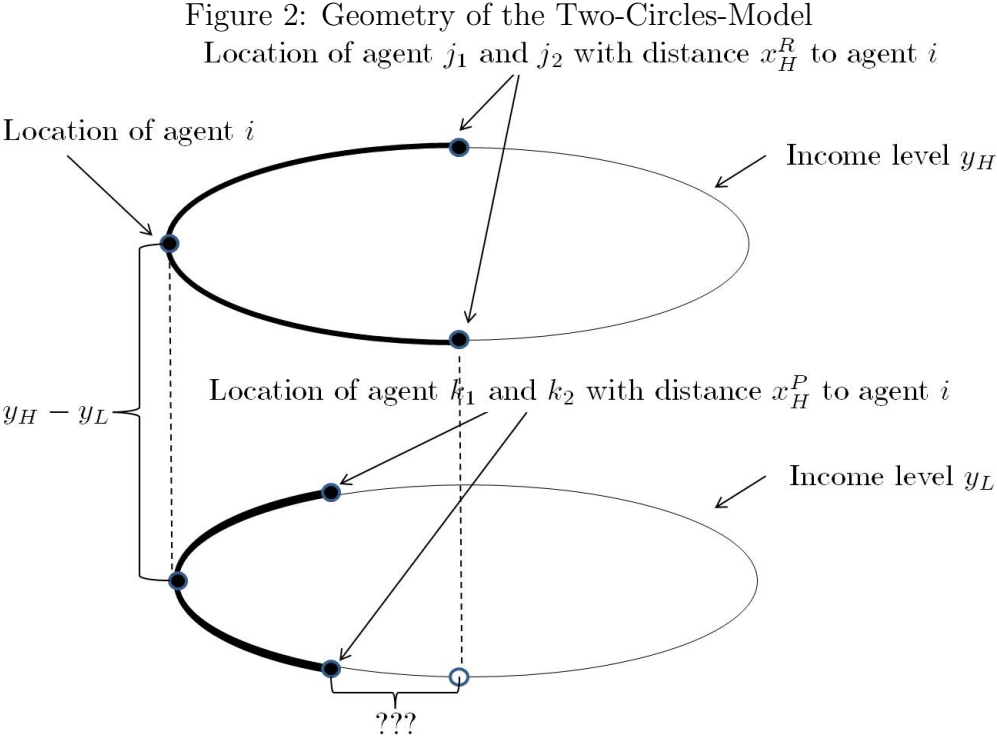


Figure 2 shows the basic geometry for the model with two circles. All agents are located on one of the two circles having high or low income. Furthermore they have a specific taste parameter which gives their exact position on the circle. In the figure the willingness to compromise for a high-income agent is adumbrated. There is a higher willingness to compromise on the rich circle than on the poor circle. An elementary

³The alternative to a marriage of staying single is always less attractive for people from a lower income class as a person's income in a household is the mean of the two spouses. Therefore the one-sided willingness to compromise for marriages in the other income class is always higher for agents with low incomes. Therefore, the upper income class is the decisive part in a marriage.

⁴Note, that we directly solve for the willingness to compromise in the effective outcome of the game. Poor agents have a higher acceptance in taste for rich partners than x_L^R but their proposals are rejected.

consideration shows that the difference in the length of the bows (the length which is signed with the question marks) is $\frac{y_H - y_L}{2}$. A rich agent must be indifferent between the emotionally most different rich agent he is willing to accept and the corresponding poor agent. Hence, the low agent has to reimburse the rich with an emotional match equalling the income loss of $\frac{y_H - y_L}{2}$ compared to the rich agent in distance x_H^R . We will be more detailed on this below.

To start with the analysis one sees that above a certain threshold value $\tilde{\Theta}$ rich people only marry among each other. For these values of $\Theta \geq \tilde{\Theta}$ even the poor emotional equivalent of a rich person does not give enough lifetime-utility to compensate the rich agent for the redistributive loss in income. This threshold can be expressed in Θ as a higher Θ indicates higher patience or better marriage institutions.

If Θ is above this critical value $\tilde{\Theta}$ the results from the circle with only one income can be applied. In the result for the critical willingness to compromise one has to replace Θ by $\frac{\Theta}{2}$ as the probabilities to meet an agent of a certain type are reduced to a half. Hence, for high values of Θ the willingness to compromise is given by

$$x_H^R = x_L^P = \frac{2}{\Theta}(\sqrt{1 + \Theta} - 1), \quad x_H^P = x_L^R = 0.$$

To get an expression for the critical vibrancy parameter $\tilde{\Theta}$ we do the following consideration. A rich agent will be willing to marry his emotional match from the poor circle as soon as he is willing to marry the rich agent which gives him exactly the same payoff. This is the case if

$$y_h + 1 - x_H^R = \frac{y_l + y_h}{2} + 1 \Leftrightarrow x_H^R = \frac{y_h - y_l}{2}.$$

This is intuitive: The willingness to compromise in taste on the rich circle is equal to the income reduction induced by the marriage to the agent's emotional match from the poor circle.

We can now solve

$$\frac{y_h - y_l}{2} = \frac{2}{\tilde{\Theta}}(\sqrt{1 + \tilde{\Theta}} - 1)$$

for the unknown value of $\tilde{\Theta}$. Some reformulations finally give that

$$\tilde{\Theta} = 8 \frac{2 - (y_h - y_l)}{(y_h - y_l)^2}.$$

This shows that the higher the difference in income the lower the critical market parameter. Intuitively, the higher the potential loss by the redistribution of income in a marriage the worse the marriage institution has to be to provide some mixed marriages.⁵ Furthermore, $\tilde{\Theta}$ is decreasing in the scale of wealth, i.e. if one replaces all y_i by $k \cdot y_i$ then one gets $\tilde{\Theta} = 8 \frac{2-(y_h-y_l)}{k(y_h-y_l)^2}$. An up-scaled income increases the income difference and therefore reduces the willingness to compromise to the lower income level. The marriage institution has to be worse that agents start to incur mixed marriages.⁶

For values of the market parameter $\Theta \leq \tilde{\Theta}$ there are interior solutions for the respective functions $x_H^R, x_H^P = x_L^R$ and x_L^P , i.e. the respective values are somewhere in the open interval $(0, 1)$. For these market parameters the lifetime-utility of a rich agent is

$$\begin{aligned} V_H = & \Delta y_h + \frac{1}{1 + \Delta r} \left[\frac{1}{2} \alpha \Delta E_H(\max\{V_H, \frac{y_h + 1 - x}{r}\}) \right. \\ & \left. + \frac{1}{2} \alpha \Delta E_H(\max\{V_H, \frac{\frac{y_h + y_l}{2} + 1 - x}{r}\}) + (1 - \Delta \alpha) V_H \right]. \end{aligned}$$

Similar as in the case of the circle this can be reformulated as

$$rV_H = y_h + \frac{1}{2} \Theta [E_H(\max\{0, y_h + 1 - x - rV_H\}) + E_H(\max\{0, \frac{y_h + y_l}{2} + 1 - x - rV_H\})].$$

The left summand in brackets is the decision whether to marry a rich and the right summand is whether to marry a poor with which the agent is faced randomly. Hence, the critical values are given by

$$rV_H = y_h + 1 - x_H^R \text{ and } rV_H = \frac{y_h + y_l}{2} + 1 - x_H^P.$$

These two equations imply that the difference between the critical thresholds x_H^R and x_H^P is $\frac{y_h - y_l}{2}$. Intuitively the two potential partners on the border of being indifferent between marriage and staying single have to be identical in utility for the proposer. The additional loss by marrying a poor is compensated by a better emotional match in

⁵Remember that we reduced ourselves to income values between zero and one to avoid corner solutions in a sense that the rich marry among each other regardless of taste before they start to mix with the poor.

⁶This observation indicates that a redistributive tax t - agents get $(1-t)y_i$ as net income - increases the tendency to incur mixed marriages. These mixed marriages have by construction of the model a welfare increasing effect while the redistribution of income is welfare neutral. In the next section we present an example with a welfare distortion induced by an income tax and show that under some circumstances the match quality can outweigh the negative effect of this distortion.

the utility value of the loss by income redistribution. Solving these two equations for rV_H and plugging the respective expressions in the formula for the life-time-utility of a rich agent and finally evaluating the expectation operators gives

$$rV_H = y_h + \frac{1}{2}\Theta \left[\frac{1}{2}(x_H^R)^2 + \frac{1}{2}(x_H^P)^2 \right].$$

Corollary 1 *Characterization of the solution for two Circles and rich agents*

For $\Theta \leq \tilde{\Theta}$ the critical willingness to compromise for the rich is given by the solution of the following equation system:

$$\begin{aligned} (I) \quad rV_H &= y_H + \frac{1}{2}\Theta \left[\frac{1}{2}(x_H^R)^2 + \frac{1}{2}(x_H^P)^2 \right] \\ (II) \quad rV_H &= y_H + 1 - x_H^R \\ (III) \quad rV_H &= \frac{y_h + y_l}{2} + 1 - x_H^P \end{aligned}$$

Obviously, only in the limit, i.e. a dried-up marriage market, the willingness to compromise among the rich is 1.⁷ This is intuitive as marrying someone with the same income and opposite taste exactly gives back the utility of staying single. Consequently, the upper limit for $\Theta \rightarrow 0$ of x_H^P is $1 - \frac{y_h - y_l}{2}$. Hence, in a marriage market where agents have a type dependent utility of staying single in a component which falls victim to redistribution even with the worst marriage institution some matches will never form.

Still to determine is the willingness to compromise of the poor among their own people if $\Theta \leq \tilde{\Theta}$. For such values of Θ there is a positive probability that a poor gets a proposal by a rich (which he accepts). The determining parameter for the probability and the emotional match of this event is x_H^P the willingness to compromise of a rich agent on the circle of poor agents. In case that such a marriage takes place the expected utility of the poor concerning the emotional match quality (taste distance) is $x_H^P/2$. Again, all agents play a symmetric strategy, meaning that proposals among the poor lead to marriage with certainty.

⁷This can be seen analytically when equating (I) and (II) from the equation system with $x_H^R = 1$.

The present lifetime-utility of a poor agent is

$$V_L = \frac{1}{1 + \Delta r} \left[\Delta y_l + \frac{1}{2} \alpha \Delta x_H^P \frac{\frac{y_h + y_l}{2} + 1 - \frac{x_H^P}{2}}{r} + \frac{1}{2} \alpha \Delta E_L(\max\{V_L, \frac{y_l + 1 - x}{r}\}) + (\frac{1}{2} \alpha \Delta (1 - x_H^P) + 1 - \alpha \Delta) V_L \right].$$

Reformulating this in the usual manner gives

$$rV_L = y_L + \frac{1}{2} \Theta x_H^P \left[\frac{y_h + y_l}{2} + 1 - \frac{x_H^P}{2} - rV_L \right] + \frac{1}{2} \Theta E_L(\max\{0, y_l + 1 - x - rV_L\}).$$

In this case the poor agent can only decide on the marriage with another poor (by symmetry) which is the last summand. Hence, a poor agent's reservation utility is described by $rV_L = y_L + 1 - x_L^P$. Plugging this into the right hand side of the equation given above and evaluating the expectation operator one gets

$$rV_L = y_l + \frac{1}{2} \Theta x_H^P \left[\frac{y_h - y_l}{2} + x_L^P - \frac{x_H^P}{2} \right] + \frac{1}{4} \Theta (x_L^P)^2.$$

This leads to the following Proposition:

Proposition 2 *Solution for two Circles*

For $\Theta \leq \tilde{\Theta}$ the critical willingness to compromise are given by the solution of the following equation system:

$$\begin{aligned} (I) \quad rV_H &= y_H + \frac{1}{2} \Theta \left[\frac{1}{2} (x_H^R)^2 + \frac{1}{2} (x_H^P)^2 \right] \\ (II) \quad rV_H &= y_H + 1 - x_H^R \\ (III) \quad rV_H &= \frac{y_h + y_l}{2} + 1 - x_H^P \\ (IV) \quad rV_L &= y_l + \frac{1}{2} \Theta x_H^P \left[\frac{y_h - y_l}{2} + x_L^P - \frac{x_H^P}{2} \right] + \frac{1}{4} \Theta (x_L^P)^2 \\ (V) \quad rV_L &= y_l + 1 - x_L^P \end{aligned}$$

2.2.1 Comparative Statics of the General Solution

The solution of the equation system in Proposition 2 can be determined without any general difficulties. By the construction of the model there remain some interesting

features of the functions solving the model which are summarized in the following Proposition. An intuition for these results will be given below.

Proposition 3 *Comparative Statics of the General Solution*

Define $d := y_H - y_L$.

(a₁) *The function characterizing the willingness to compromise among the rich is*

$$x_H^R(\Theta) = \begin{cases} \frac{d}{4} + \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta}, & \text{if } 0 \leq \Theta < \tilde{\Theta} \\ \frac{2}{\tilde{\Theta}}(\sqrt{1 + \tilde{\Theta}} - 1), & \text{if } \tilde{\Theta} \leq \Theta \end{cases}.$$

(a₂) x_H^R is continuous for all $\Theta \in \mathbb{R}_+$.

(a₃) x_H^R is differentiable for all $\Theta \in \mathbb{R}_+$. x_H^R is strictly decreasing in \mathbb{R}_+ .

(a₄) It holds $\lim_{\Theta \rightarrow 0} x_H^R(\Theta) = 1$ and $\lim_{\Theta \rightarrow \infty} x_H^R(\Theta) = 0$.

(b₁) *The function for the willingness to compromise among the poor is*

$$x_L^P(\Theta) = \begin{cases} \frac{1}{4\Theta} \left(-4 + d\Theta - \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2} \right. \\ \left. + 2\sqrt{16 + \Theta(32 + d^2\Theta - 2d\sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2})} \right), & \text{if } 0 \leq \Theta < \tilde{\Theta} \\ \frac{2}{\tilde{\Theta}}(\sqrt{1 + \tilde{\Theta}} - 1), & \text{if } \tilde{\Theta} \leq \Theta \end{cases}.$$

(b₂) x_L^P is continuous for all $\Theta \in \mathbb{R}_+$.

(b₃) x_L^P is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$. There exists a threshold value Θ^* s.t. x_L^P is strictly decreasing in $\mathbb{R}_+ \setminus (\Theta^*, \tilde{\Theta}]$ and is increasing in $(\Theta^*, \tilde{\Theta}]$.

(b₄) It holds $\lim_{\Theta \rightarrow 0} x_L^P(\Theta) = 1$ and $\lim_{\Theta \rightarrow \infty} x_L^P(\Theta) = 0$.

(c₁) *The function for the willingness to compromise among the poor for a rich is*

$$x_H^P(\Theta) = x_L^R(\Theta) = \begin{cases} -\frac{d}{4} + \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta}, & \text{if } 0 \leq \Theta < \tilde{\Theta} \\ 0, & \text{if } \tilde{\Theta} \leq \Theta \end{cases}.$$

(c₂) x_H^P is continuous for all $\Theta \in \mathbb{R}_+$.

(c₃) x_H^P is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$. x_H^P is strictly decreasing in \mathbb{R}_+ .

(c₄) It holds $0 \leq \lim_{\Theta \rightarrow 0} x_H^P(\Theta) = 1 - \frac{d}{2} \leq 1$ and $\lim_{\Theta \rightarrow \infty} x_H^P(\Theta) = 0$.

(d₁) *The discounted life-time-utilities for the agents are*

$$V_H(\Theta) = \frac{y_H + 1 - x_H^R(\Theta)}{r} \text{ and } V_L(\Theta) = \frac{y_L + 1 - x_L^P(\Theta)}{r}.$$

(d₂) *The functions for the discounted expected lifetime-utility of the agents V_H and V_L*

are continuous and differentiable for all $\Theta \in \mathbb{R}_+$.

(d₃) $V_H(\Theta)$ is differentiable for all $\Theta \in \mathbb{R}_+$. $V_L(\Theta)$ is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$.

(d₄) $(V_H + V_L)(\Theta)$ is a monotonically increasing function in \mathbb{R}_+ . It is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$.

In the following we want to discuss the properties of the functions solving the problem of Proposition 2 and give some intuition for the findings.⁸ Looking at the willingness to compromise among the rich the differentiability of x_H^R is intuitive. Reaching a vibrancy of $\tilde{\Theta}$ the rich start to make proposals (which succeed) to the poor. But as meeting the perfect emotional match from the poor side starts out as a probability zero event, the differentiability of x_H^R is not affected and the function behaves smooth to a certain degree. Clearly under the worst matching institutions $\Theta = 0$ a rich is indifferent between staying single and marrying his emotional complement as the per-period-utility of doing this remains the same. Having perfect matching institutions, i.e. $\Theta \rightarrow \infty$, all rich agents marry their own clone and the willingness to compromise decreases to zero.

A similar logic applies for the willingness to compromise among the poor x_L^P . The most important difference appears by the missing differentiability of the corresponding solution. An intuitive consideration for this finding shows that for the vibrancy becoming lower than $\tilde{\Theta}$ the rich start to marry some well-emotionally matched poor people. But the rich are the decisive part in such marriages. Hence, starting out as a probability zero event when decreasing Θ even further more and more rich agents are available for a poor agent. These rich agents give a higher utility than the most extreme person this poor would be willing to marry. So, x_L^P is not only not differentiable but is for a short interval even an increasing function in the vibrancy.⁹ The intuition for the limiting cases is as in the case of the willingness to compromise among the rich as there are no re-distributional concerns when marrying within your own income class.

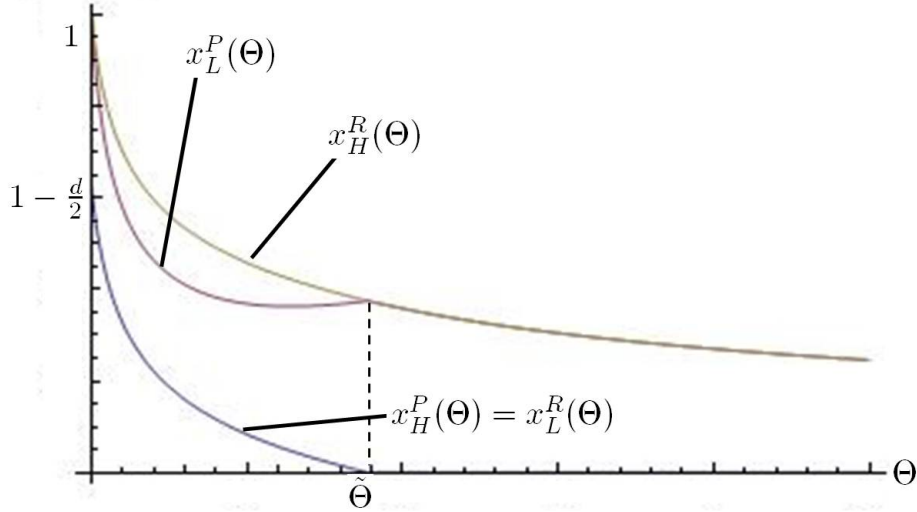
By construction of the model and as already mentioned the difference between x_H^R and x_H^P is always $\frac{d}{2}$ as long as the latter function is positive. This also explains the missing differentiability in $\tilde{\Theta}$. The behavior for very bad matching institutions, i.e. $\Theta \rightarrow 0$, is due to the fact that the decisive part in a marriage is the rich partner. In the limit he is indifferent of staying single or marrying a person which compensates for the

⁸Most parts of the argumentation are based on the solutions considered as functions of the market's vibrancy Θ . The higher the vibrancy the more effective are the marriage institutions.

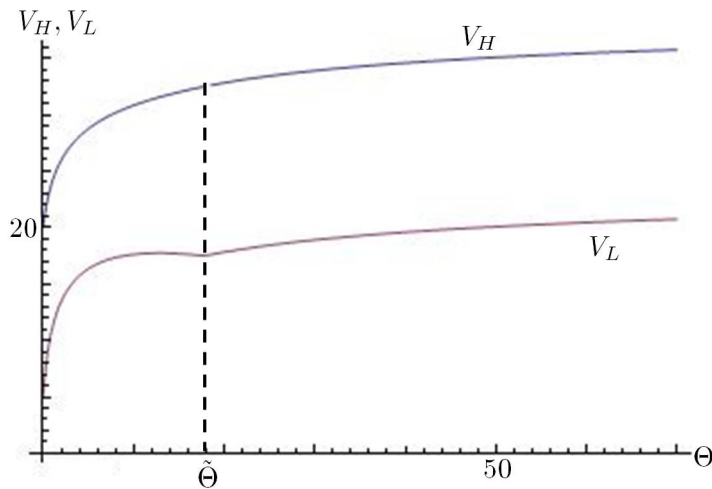
⁹Calculate the interval in which this is the case.

redistributive loss in income $\frac{y_H - y_L}{2} = \frac{d}{2}$ by the additional gain induced by the emotional match, i.e. $1 - x \stackrel{!}{=} \frac{d}{2} \Leftrightarrow x = 1 - \frac{d}{2}$.

$$x_H^R(\Theta), x_H^P(\Theta), x_L^P(\Theta)$$



The figure shows the willingness to compromise from parts a) to c) of the Proposition given above for parameter value $d = 0.75$ and a corresponding $\tilde{\Theta} = \frac{160}{9}$. While for values of the vibrancy above $\tilde{\Theta}$ the willingness to compromise among the rich and the willingness among the poor coincide there are no marriages between different income classes. In the left part one can see the proofed limiting behavior as well as the constant distance between x_H^R and x_H^P and the interval where the willingness to compromise among the poor increases when making the marriage institutions more effective due to the disappearance of the rich proposals among the poor.



The figure shows the expected discounted lifetime-utilities for high- and low-type agents, parameter values $y_H = 1, y_L = 0.25$ and $r = .05$. One can see the interim decrease in V_L in the interval $(\Theta^*, \tilde{\Theta})$. A further analysis shows that $(V_H + V_L)(\Theta)$ is increasing in the vibrancy of the market. I.e. an increase in the vibrancy of the marriage institution has always positive effects on the expected welfare in the economy, nevertheless this may harm low-income consumers in particular situations. The comparative statics according to d show that an increase in d decreases the value of $\tilde{\Theta}$. The other results remain qualitatively unchanged.

3 Welfare Effects of a Redistributive Income Tax

Example: A labor-market with shortsighted workers

The positive welfare effect of a redistributive income tax as mentioned in the introduction can be seen in the following example. Consider a framework of hidden information on worker's types in a labor market setting. There is one firm and workers of mass 1 of different types. A worker can be of two different types which is not observable by the firm. With a probability of 50% a worker is a high-productive type (facing low costs for given effort levels) and with a probability of 50% he is a low-productive type (facing high costs of effort). For simplicity we assume that independent of the productivity type half of the workforce are women while the other half are men. For simplicity we assume that the firm can hire the worker for a full-time (effort is e_H) or a half-time job (effort is e_L) with corresponding per period costs $C^H(e_i)$ for the high type and $C^L(e_i)$ for the low type with $i \in \{H, L\}$. A firm earns Π_H or Π_L if a worker works full-time respectively half-time ($\Pi_H - \Pi_L > 0$). The firm tries to maximize profits by offering an incentive-compatible menu of contracts to the workers. The utility for a worker of rejecting the contracts is assumed to be zero.

In each period after the working decision a marriage market for all workers takes place where the income level y from the working decision serves as the ordinal property of participants. We assume that workers are shortsighted, i.e. they make their decisions in the labor market without taking into account the consequences in the subsequent marriage market.¹⁰ The marriage market is designed in the same way as the market

¹⁰This assumption mainly simplifies the participation constraints. One can show that in a model where firms and workers perfectly anticipate the consequences of their income level on marriage de-

described in section 2.¹¹

In the following we will look at possible equilibria from the firm's as well as from a regulator's point of view by comparing possible profits and welfare. We are able to show that a regulator may increase welfare by a redistributive tax. This is due to the fact that by doing this the regulator can eliminate the negative external effect on the marriage market by different income levels of the workers induced by a separating equilibrium in the labor market. The tax imposed is redistributive in the sense, that everybody is taxed with a proportional income tax. Tax revenues are given back to workers uniformly.

To cancel out welfare or profit effects which would be induced by our cloning assumption from the setup defined in Section 2 we just look at per period utilities instead of lifetime-utilities.¹² Another interpretation would be to look at average utilities of all workers which could serve as a measure for the relative contentedness.

3.1 Pooling Equilibrium

In a pooling equilibrium the firm pays the same wage to all workers. To incentivize them to accept the offer the firm has to meet the participation constraints of both types. As we assume the costs for given effort levels to be higher for the low type in equilibrium the wage is $y = C^L(e_L)$. With such a wage the low-type is indifferent between accepting and rejecting while the high-types make a profit of $C^L(e_L) - C^H(e_L)$. The firm makes a gross gain per worker of Π_L and so the profit of such a pooling equilibrium for the firm is $\Pi_{pooling} = \Pi_L - C^L(e_L)$.

With the notation of Section 2 the per-period-utility of a worker is $rV = y + 1 - \bar{x}$ the welfare function can be formulated in the following way:

$$W_{pooling} = y + 1 - \bar{x} - \frac{C^H(e_L) + C^L(e_L)}{2} + \Pi_{pooling}$$

Parameter \bar{x} is determined endogenously in the marriage market by using the model

cisions the possible negative external effect of a separating equilibrium is internalized by the firms. Therefore, in such a model a welfare increasing income tax does not exist.

¹¹Agents meet each other with a certain probability, observe the two characteristics of their potential partner and make a proposal or decide to stay single. In case of proposals from both sides the match is formed and the emotional rent is realized.

¹²They just differ by factor $\frac{1}{r}$ and therefore the objectives do not change.

with one circle from Subsection 2.1 as there is a unique wage in a pooling equilibrium. Using $y = C^L(e_L)$ and $\Pi_{pooling} = \Pi_L - C^L(e_L)$ this yields a welfare for a pooling equilibrium of

$$W_{pooling} = \Pi_L + 1 - \bar{x} - \frac{C^H(e_L) + C^L(e_L)}{2}.$$

As \bar{x} depends on the vibrancy of the marriage institution Θ the whole welfare function in this otherwise deterministic model depends on Θ .

3.2 Separating Equilibrium

Depending on the cost-structure of the high-type workers and the corresponding profit of the firm for having full-time employees it may be optimal for the firm to separate the workers according to their productivity.¹³ By the standard approach one has to assure that the participation constraint of the low-type as well as the incentive constraint of the high type are binding, i.e.

$$\begin{aligned} (I) \quad y_H - C^H(e_H) &\stackrel{!}{=} y_L - C^H(e_L) \\ (II) \quad y_L - C^L(e_L) &\stackrel{!}{=} 0 \end{aligned}$$

Hence, the maximization problem of the firm turns out to be easily solved. As long as $C^L(e_H)$ is sufficiently high the following pair of contracts will lead to a separating equilibrium:

$$(y_1, e_1) = (C^L(e_L) + C^H(e_H) - C^H(e_L), e_H) \text{ and } (y_2, e_2) = (C^L(e_L), e_L)$$

The menu of contracts is designed in such a way that optimally high-type workers pick contract (y_1, e_1) and the low-types sign contract (y_2, e_2) . The corresponding profit of the firm is

$$\Pi_{separating} = \frac{1}{2}(\Pi_H - y_1) + \frac{1}{2}(\Pi_L - y_2) = \frac{\Pi_H + \Pi_L}{2} - C^L(e_L) - \frac{C^H(e_H) - C^H(e_L)}{2}.$$

With the welfare function summing up the average per-period-utilities of the workers

¹³For the story of the example to work it is necessary to have sufficiently high costs for the low-type to perform a high effort level $C^L(e_H)$.

and the net-gains from production the welfare determines in this case as

$$W_{separating} = \frac{1}{2}(y_H + 1 - x_H^R(\Theta)) + \frac{1}{2}(y_L + 1 - x_L^P(\Theta)) - \frac{C^H(e_H) + C^L(e_L)}{2} + \Pi_{separating}.$$

Plugging in the expressions for y_H, y_L and $\Pi_{separating}$ of the optimal contract yields

$$W_{separating} = 1 - \frac{1}{2}x_H^R(\Theta) - \frac{1}{2}x_L^P(\Theta) + \frac{\Pi_H + \Pi_L}{2} - \frac{C^H(e_H) + C^L(e_L)}{2}.$$

Again welfare is a function of the vibrancy of the marriage institution Θ as the willingness to compromise x_H^R and x_L^P do.

3.3 The positive Effect of an Income Tax on Welfare

As mentioned above under a redistributive income tax we understand a tax which is a proportional tax-rate imposed on workers after they earned their wages. The tax-rate is known from the beginning. All the tax revenues are afterwards redistributed to workers uniformly. E.g., if half of the workers earned wage y_H and the other half earned wage y_L in the labor market a tax rate of t will lead to a net-income after tax of $(1 - t)y_i$ for a worker of type $i \in \{H, L\}$ and to tax revenues of $t\frac{y_H + y_L}{2}$. Redistributing this to workers gives a net-income after tax and redistribution to workers of $(1 - \frac{t}{2})y_h + \frac{t}{2}y_l$ for high-income workers and $(1 - \frac{t}{2})y_l + \frac{t}{2}y_h$. These two incomes coincide for $t = 1$. This shows that an income tax of 100% would make incentive constraints for the separation of high- and low-productive types impossible. Hence, as long as welfare for a certain pooling equilibrium is higher than for a separating equilibrium without tax, which a firm would like to implement, a redistributive income tax has a positive welfare effect.

Using this interpretation of a redistributive tax an example of a welfare increasing income tax is equivalent to looking for a situation, i.e. parameters $\Pi_H, \Pi_L, C^H(e_H), C^H(e_L), C^L(e_H), C^L(e_L)$ and Θ are chosen, such that the following two conditions hold:

- (1) $\Pi_{separating} > \Pi_{pooling}$
- (2) $W_{separating} < W_{pooling}$

The following Proposition shows a condition which yields an environment in which a redistributive tax increases welfare:

Proposition 4 *Positive Welfare Effects of a Redistributive Income Tax*

In a labor-market setting with a subsequent marriage market and shortsighted workers a redistributive income tax which maximizes welfare exists if

$$0 < \Pi_H - \Pi_L - (C^H(e_H) - C^H(e_L)) < x_H^R(\Theta) + x_L^P(\Theta) - 2\bar{x}(\Theta).$$

We now have to check that the omitted participation constraint for the high-type and the incentive constraint for the low-type are fulfilled. This is obvious as for the incentive constraint of the low-type we are not restricted in any kind in the choice of $C^L(e_H)$. The participation constraint of the high-type is also of no matter as the inequality of Proposition 4 shows that we are not restricted in the base level of wages but only in the wage-differential.

Hence, this example shows that in such a sequence of markets a redistributing income tax may have a positive welfare effect. With a smaller wage-differential people are relatively more concerned on the quality of their emotional match in the marriage market as the financial compromise for the decisive part of the marriage becomes smaller or even vanishes. The example shows that the gains in welfare induced by having a similar wage structure may outweigh the losses in efficient production in the labor market.¹⁴

4 Conclusion

In this paper we study an important aspect of many bilateral matching markets with search frictions like marriage, employment, housing etc. which is widely overlooked in the literature on marriage markets. The existing literature is exclusively concerned with one attribute of each potential partner.¹⁵ All agents' preferences on this characteristic are identical and all agents agree who is the most attractive partner on the other side with respect to education, money, beauty or whatsoever. In most if not all circumstances concerning the match of two agents in real life some personal component matters, e.g. sharing hobbies with your partner, having regional preferences as

¹⁴Note that we do not claim, that the pooling equilibrium is the best possible outcome in these specific cases of Proposition 4. Our argument rather shows that a tax *can* have a positive welfare effect as the extreme case of the maximum tax-rate has a positive welfare effect.

¹⁵The only paper with a similar subject in a static context to the authors' knowledge so far is Konrad and Lommerud (2010).

an employer or employee or even some silent cultural agreements with your business partner.

Our framework offers aside from a vertical characteristic where all agents preferences agree on a second subjective characteristic. A match with another agent then generates an additional rent if agents have a similar occurrence in the second characteristic.¹⁶ This second characteristic differs fundamentally from the first one in the sense that all characteristics belong to either objective or subjective valuations of a potential partner's attributes. The first attribute captures all more or less measurable properties like income, beauty, education while the second one is somehow the personal position of an agent in the market, e.g. location, taste etc..

In the paper the analysis of such a model with search frictions is done for only one or two potential values of the objective characteristic while having a continuum in the taste dimension. The key inside is that the willingness to compromise, i.e. the potential partners an individual accepts, heavily depends on the matching institutions which are characterized by search frictions. Generally the willingness to compromise decreases the better the matching institutions are. However, it may be the case that agents with the lower objective attribute suffer from better marriage institutions. This effect is due to high-type agents which lower their willingness to compromise in response to a higher vibrancy of the marriage market.

From the analysis one aspect becomes very clear. The closer the gap between high- and low-types in the objective dimension the more important the match quality with regard to the subjective attribute becomes. As per assumption a match generates some extra rent it may become attractive from a welfare perspective to redistribute some of the income. This reduces the heterogeneity of agents with respect to the horizontal measure in formed matches and thereby a regulator may achieve a better average matching quality. Our example incorporates a labor market on the first stage and a marriage market on the second stage where each agent is endowed with his income from the labor market. We are able to show that it may be the case that the losses from introducing taxes on income and thereby inducing a distortion in the productive sector are outweighed by efficiency gains in the matching market. In general, the regulator has no interest in equalling wages in the economy, but as a side-effect of having low income differentials people are able to adjust their behavior to their personal preferences.

¹⁶The same model can be applied for cases of "Opposites attract."

An outline for future research is easily set. On the empirical side one may be interested whether possible predictions of the model hold true. One such possible prediction would be that in countries or communities with a high income-gap the tendency of observing marriages within the same caste is more likely than in others. Another prediction could be that regional preferences matter in all kinds of employment relationships. If one assumes decreasing marginal utility of increasing wages this effect should sustain onto the best-compensated jobs. Another line of future research is more theoretical. As our framework only offers a solution for a discrete number of income levels (to be precise there are only two) a natural extension is to look at a continuum of possible values of the objective characteristic.¹⁷ The results in such a framework should be similar to ours but the approach will differ as it is not possible to distinguish between a certain number of different agents and to solve the problem step by step.

¹⁷The framework is extendable to a higher discrete number of income levels without any conceptual changes.

A Appendix

Proof of Proposition 1

(a) Plugging $rV = y + 1 - x$ into the expectation operator one gets

$$rV = y + \Theta E(\max\{0, \bar{x} - x\})$$

and as $E(\max\{0, \bar{x} - x\}) = \int_0^{\bar{x}} \bar{x} - x dx = \frac{1}{2}\bar{x}^2$ this yields the result.

(b) Equating (I) and (II) gives $1 - \bar{x} = \frac{1}{2}\Theta\bar{x}^2$. Solving this for \bar{x} shows

$$\bar{x}_{1,2} = -\frac{1}{\Theta} \pm \sqrt{\frac{1}{\Theta^2} + \frac{2}{\Theta}}.$$

The relevant root is the positive. Hence, one gets $\bar{x} = \frac{1}{\Theta}(\sqrt{1 + 2\Theta} - 1)$.

Obviously \bar{x} is always positive.

$\bar{x} \leq 1$ is equivalent to $1 + 2\Theta \leq (1 + \theta)^2$ which is fulfilled for every non-negative Θ .

(c) Applying l'Hopital's rule shows

$$\lim_{\Theta \rightarrow 0} \frac{\sqrt{1 + 2\Theta} - 1}{\Theta} = \lim_{\Theta \rightarrow 0} (1 + 2\Theta)^{-\frac{1}{2}} = 1.$$

Clearly it holds $\lim_{\Theta \rightarrow \infty} \bar{x} = 0$ as the denominator dominates. ■

Proof of Corollary 1

The proof directly follows from the body text. ■

Proof of Proposition 2

The proof directly follows from the body text. ■

Proof of Proposition 3

First we will show the formulae of (a_1) , (b_1) , (c_1) and (d_1) . Then we will go on to derive the other statements.

The formulae for the left parts of x_H^R , x_H^P and V_H can be determined by solving the equation system given in Corollary 1. From (II) and (III) one gets $x_H^R - x_H^P = \frac{d}{2} \Leftrightarrow x_H^R = x_L^P + \frac{d}{2}$, the indifference condition in the willingness to compromise. Equating (I) and (II) gives

$$\begin{aligned}
& y_H + \frac{1}{4}\Theta[(x_H^P + \frac{d}{2})^2 + x_H^{P2}] && \stackrel{!}{=} y_H + 1 - x_H^P - \frac{d}{2} \\
\Leftrightarrow & \frac{1}{4}\Theta \left[2x_H^{P2} + dx_H^P + \frac{d^2}{4} \right] && = 1 - x_H^P - \frac{d}{2} \\
\Leftrightarrow & \frac{1}{2}\Theta x_H^{P2} + \frac{1}{4}d\Theta x_H^P + \frac{d^2}{16}\Theta && = 1 - x_H^P - \frac{d}{2} \\
\Leftrightarrow & \frac{1}{2}\Theta x_H^{P2} + (\frac{1}{4}d\Theta + 1)x_H^P + (\frac{d^2\Theta}{16} + \frac{d}{2} - 1) && = 0 \\
\Leftrightarrow & x_H^{P2} + (\frac{1}{2}d + \frac{2}{\Theta})x_H^P + (\frac{d^2}{8} + \frac{d}{\Theta} - \frac{2}{\Theta}) && = 0
\end{aligned}$$

Solving the quadratic expression gives a single positive root which is given by

$$x_H^P = -\left(\frac{1}{4}d + \frac{1}{\Theta}\right) + \sqrt{\left(\frac{1}{4}d + \frac{1}{\Theta}\right)^2 - \frac{d^2}{8} - \frac{d}{\Theta} + \frac{2}{\Theta}}.$$

The expression under the square root can be reformulated as

$$\left(\frac{1}{4}d + \frac{1}{\Theta}\right)^2 - \frac{d^2}{8} - \frac{d}{\Theta} + \frac{2}{\Theta} = \frac{-d^2\Theta^2 - 8d\Theta + 32\Theta + 16}{16\Theta^2}$$

and hence the solution is given by

$$\begin{aligned}
x_H^P(\Theta, d) &= \frac{-4 - d\Theta + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta} \\
&= -\frac{d}{4} + \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta}.
\end{aligned}$$

It follows

$$x_H^R(\Theta, d) = x_H^P(\Theta, d) + \frac{d}{2} = \frac{d}{4} + \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta}.$$

It can be verified easily that X_H^P is positive iff $\Theta \leq \tilde{\Theta} = 8\frac{2-d}{d^2}$ which has to be the case per construction of the model.

The formula for the left part of x_L^P can be determined by solving the equation system given in Corollary 1. From (IV) and (V) by using the expression for x_H^P . Equating the

expressions gives

$$1 - x_L^P = \frac{1}{2}\Theta x_H^P \left[\frac{d}{2} + x_L^P - \frac{x_H^P}{2} \right] + \frac{1}{4}\Theta(x_L^P)^2.$$

To get this in quadratic form one can determine

$$(x_L^P)^2 + (2x_H^P + \frac{4}{\Theta})x_L^P - (x_H^P)^2 + dx_H^P - \frac{4}{\Theta} = 0.$$

Solving this for x_L^P taking the correct (positive) root one gets

$$x_L^P = \frac{-x_H^P\Theta - 2 + \sqrt{2\Theta^2(x_H^P)^2 + \Theta^2(\frac{4}{\Theta} - d)x_H^P + 4 + 4\Theta}}{\Theta}.$$

Now one can plug in the expression for x_H^P and after some simple rearranging one gets

$$x_L^P = \frac{1}{4\Theta} \left(-4 + d\Theta - \sqrt{16 - \Theta(-32 + d(8 + d\Theta))} + 2\sqrt{16 + \Theta(32 + d^2\Theta - 2d\sqrt{16 - \Theta(-32 + d(8 + d\Theta))})} \right).$$

The right parts of the functions are clear by definition and by the two circles adaption of Proposition 1. The formulae for the discounted life-time-utilities directly follow from (II) and (V) of the equation system given in Proposition 2. Hence, (a₁), (b₁), (c₁) and (d₁) are shown.

(a₂) For the continuity one has to show the equality of $x_H^R(\tilde{\Theta})$ for both parts of the function which directly yields the result. The value at $\tilde{\Theta}$ equals both times $\frac{d}{2}$.

(a₃) For $\Theta \geq \tilde{\Theta}$ the willingness to compromise is $x_H^R(\Theta) = \frac{2}{\Theta}(\sqrt{1 + \Theta} - 1)$. The derivative is

$$\frac{\partial x_H^R}{\partial \Theta} = -\frac{2}{\Theta^2}(\sqrt{1 + \Theta} - 1) + \frac{1}{\Theta\sqrt{1 + \Theta}}.$$

Using $\sqrt{1 + \tilde{\Theta}} = \frac{4-d}{d}$ one gets

$$\frac{\partial x_H^R}{\partial \Theta}(\tilde{\Theta}) = -\frac{d^3}{16(4-d)}.$$

For $\Theta \leq \tilde{\Theta}$ the willingness to compromise is $x_H^R = \frac{d}{4} + \frac{\sqrt{16+32\Theta-8d\Theta-d^2\Theta^2}}{4\Theta} - \frac{1}{\Theta}$. The derivative

is

$$\frac{\partial x_H^R}{\partial \Theta} = \frac{1}{\Theta^2} + \frac{32 - 8d - 2d^2\Theta}{8\Theta\sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}} - \frac{\sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta^2}.$$

Using $\sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2} = \frac{4(4-d)}{d}$ one gets again

$$\frac{\partial x_H^R}{\partial \Theta}(\tilde{\Theta}) = -\frac{d^3}{16(4-d)}.$$

Hence, x_H^R is differentiable for all positive values of Θ . It can be shown generally that the derivative is always negative.

(a₄) By applying the rule of l'Hopital one gets

$$\lim_{\Theta \rightarrow 0} \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta} = 1 - \frac{1}{4}d.$$

Hence, $\lim_{\Theta \rightarrow 0} x_H^R(\Theta) = 1$. The other claim directly follows from part c) of Proposition 1.

(b₂) For the continuity one has to show the equality of $x_L^P(\tilde{\Theta})$ for both parts of the function which directly yields the result. The value at $\tilde{\Theta}$ equals both times $\frac{d}{2}$.

(b₃) Clearly the function is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$. The function is not differentiable in $\tilde{\Theta}$ as the left-side and the right-side derivative give different values. For the right-side derivative this value is $-\frac{d^3}{16(4-d)}$ and for the left-side it is $\frac{d^3(4-3d)}{16(4-d)^2}$. As the derivative is negative at 0 and positive at $\tilde{\Theta}$ by the continuity of the derivative and by the intermediate theorem a value Θ^* with the required properties exists.

(b₄) As

$$\frac{\partial}{\partial \Theta} \left(-4 + d\Theta - \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2} + 2\sqrt{16 + \Theta(32 + d^2\Theta - 2d\sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2})} \right) (\tilde{\Theta}) = 4$$

l'Hopital's rule yields $\lim_{\Theta \rightarrow 0} x_L^P(\Theta) = 1$. The other claim directly follows from part c) of Proposition 1.

(c₂) The continuity of the function follows directly from $x_L^R(\tilde{\Theta}) = x_H^P(\tilde{\Theta}) = 0$.

(c₃) Clearly the function is differentiable for all $\Theta \in \mathbb{R}_+ \setminus \{\tilde{\Theta}\}$. $x_H^P(\Theta)$ is not differentiable in $\tilde{\Theta}$ as

$$\frac{\partial x_H^P}{\partial \Theta}(\tilde{\Theta}) = \frac{\partial x_H^R}{\partial \Theta}(\tilde{\Theta}) < 0.$$

(c₄) The only thing to show is that $\lim_{\Theta \rightarrow 0} x_H^P(\Theta) = 1 - \frac{d}{2}$. As

$$\lim_{\Theta \rightarrow 0} \frac{-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4\Theta} = 1 - \frac{1}{4}d$$

one gets

$$\lim_{\Theta \rightarrow 0} -\frac{d-4 + \sqrt{16 + 32\Theta - 8d\Theta - d^2\Theta^2}}{4 \cdot 4\Theta} = 1 - \frac{d}{2}.$$

This limit lies in the interval $[0, 1]$ as d is bounded by 1.

(d₂) This follows directly from equations (II) and (V) of Proposition 2.

(d₃) This follows directly from (a₃) and (b₃).

(d₄) This follows directly from comparing the different values of the derivatives for the left-hand side of $x_H^R(\Theta)$ and $x_L^P(\Theta)$. From part (a₃) and part (b₃) of this proof it is enough to show that

$$\left| \frac{d^3}{16(4-d)} \right| > \left| \frac{d^3(4-3d)}{16(4-d)^2} \right| \Leftrightarrow 4-d > 4-3d.$$

All other claims directly follow from the form of the function and the properties of the underlying parts. ■

Proof of Proposition 4

The firm-profit condition $\Pi_{separating} > \Pi_{pooling}$ can be written as

$$\Pi_L - C^L(e_L) < \frac{\Pi_H + \Pi_L}{2} - C^L(e_L) - \frac{1}{2}C^H(e_H) + \frac{1}{2}C^H(e_H).$$

This can be reformulated to get $\Pi_H - \Pi_L > C^H(e_H) - C^H(e_L)$ which is the condition for a separating equilibrium to be profit-maximizing from firm's perspective.

The welfare condition $W_{separating} < W_{pooling}$ can be written as

$$1 - \bar{x}(\Theta) + \Pi_L - \frac{C^H(e_L) + C^L(e_L)}{2} > 1 - \frac{1}{2}x_H^R(\Theta) - \frac{1}{2}x_L^P(\Theta) + \frac{\Pi_H + \Pi_L}{2} - \frac{C^H(e_H) + C^L(e_L)}{2}.$$

This can be reformulated as $\Pi_H - \Pi_L < x_H^R(\Theta) + x_L^P(\Theta) - 2\bar{x}(\Theta) + C^H(e_H) - C^H(e_L)$. These two inequalities together and $\Pi_H > \Pi_L$ per assumption show the inequality given in the Proposition. The existence of such parameter values can be seen by showing that the function $x_H^R(\Theta) + x_L^P(\Theta) - 2\bar{x}(\Theta)$ always gives some positive values as long as $\Theta < \tilde{\Theta}$. This can be checked easily as for all Θ at $d = 0$ the function has a value of 0 and is increasing in d . ■

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