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Engel's law and growth with directed technical change*

Timo Boppart[†]

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Abstract

This paper presents a tractable endogenous two-sector growth model with non-Gorman intra-temporal preferences and directed technical change. One of the two consumption goods is a necessity, whereas the other is a luxury. If the economy starts with a low initial knowledge stock, households are relatively poor and a high expenditure share is devoted to necessities. Therefore, in early phases of development, technical innovations are mainly directed toward the necessity sector. According to Engel's law, growth in income increases the expenditure share of the luxury sector. Biased technical change constitutes another force that leads to shifts in expenditure shares. The resulting structural change is accompanied by increasing R&D investments in the luxury sector, whereas investments in the necessity sector become less attractive. The asymptotic equilibrium consists of a nonbalanced constant growth path along which the Kaldor facts hold, and growth is mainly driven by the luxury sector.

Keywords: Engel's law, directed technical change, structural change, non-balanced growth, two-sector model, non-Gorman preferences.

JEL classification: O11, O30, O41, D90.

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1 Introduction

It is a well documented, long-standing empirical fact that on a sectoral level economic growth is nonbalanced (see Kuznets (1957)). There are significant differences in total factor productivity growth rates across sectors and we can observe major shifts in sectoral expenditure and employment shares in the long run. These sectoral shifts are summarized under the term “structural change”.¹ Most striking and robust is the fall of the agricultural labor share over time.² Kongsamut, Rebelo and Xi (2001) refer to these nonbalanced features of economic growth as the “Kuznets facts”.

In contrast to these very robust nonbalanced findings, standard growth theory focused on the balanced picture of growth presented by more aggregated variables. It is well-known that the “Kaldor facts” are reflected in aggregate data. These facts state that the growth rate of output, the capital-output ratio, the real interest rate and the labor income share are constant over time (see Kaldor (1961) and Kongsamut, Rebelo and Xi (2001)). In growth theory these stylized facts have become a benchmark and a successful model should (at least) be able to replicate these empirical regularities. The shortcoming that standard growth theory is not able to explain the nonbalanced nature of sectoral economic development has stimulated a small but emerging literature that strives to generate features of nonbalanced growth without significantly deviating from balanced growth on the aggregate level. The theoretical

¹See for instance Kongsamut, Rebelo and Xi (2001), Gollin, Parente and Rogerson (2002) and Greenwood and Uysal (2005).

²In the US the agricultural employment share decreased over the last two centuries from 74% to 2.5% (see Greenwood and Uysal (2005)).

literature on structural change can be divided in two strands. A first strand explains nonbalanced features of development by a demand side mechanism which relies on non-homotheticity of intra-temporal preferences.³ With non-homothetic preferences, expenditure elasticities of demand differ across sectors. As a consequence, expanding consumption expenditure levels (caused by endogenous or exogenous growth) induce a shift in expenditure shares and therefore structural change. This story is consistent with Engel's law, one of the most robust empirical regularity in economics (see Engel (1857), Houthakker (1957) and Houthakker and Taylor (1970)).

A second strand of the literature emphasizes supply side explanations of structural change which rely on differences in productivity growth rates across sectors (see Baumol (1967), Nagai and Pissarides (2006) and Acemoglu and Guerrieri (2008)).⁴ Changes in relative productivities will generate movements in relative prices and therefore changes in expenditure and employment shares. The direction of struc-

³See Matsuyama (1992), Echevarria (1997), Laitner (2000), Kongsamut, Rebelo and Xi (2001) Caselli and Coleman (2001) and Gollin, Parente and Rogerson (2002) which use quasi-homothetic intra-temporal preferences. Another tractable way to generate non-homotheticities is to use hierarchical preferences (see Matsuyama (2002), Foellmi and Zweimueller (2008) and Buera and Kaboski (2009b)). For a nice literature overview see Matsuyama (2005). The model of this paper uses a new approach, with preferences that fall into the "price independent general linearity" (PIGL) class of preferences. Hence, it will be neither based on quasi-homothetic intra-temporal preferences nor on hierarchical preferences.

⁴The paper by Acemoglu and Guerrieri constitutes the first work with a supply side explanation of structural change that determines biased technical change endogenously. Differences in sectoral total factor productivity growth rates are generated by a combination of capital deepening and factor intensity differences. In the working paper version of their model also directed technical change is considered (see Acemoglu and Guerrieri (2006)).

tural change depends crucially on the inter-sectoral elasticity of substitution. If the elasticity of substitution is less than one, the expenditure share of the faster growing sector decreases. If the elasticity of substitution is larger than one, the opposite holds (see Nagai and Pissarides (2006)).⁵

This paper combines both, supply and demand side explanations of structural change. It presents a two-sector dynamic general equilibrium model with non-Gorman intra-temporal preferences and directed technical change. The main contribution is theoretical. I specify intra-temporal preferences that imply an Engel's law and an intra-sectoral elasticity of substitution of less than one. One of the two consumption goods is a necessity with an expenditure elasticity of demand smaller than one, whereas the other good is a luxury with an expenditure elasticity larger than one. Hence, the model comprises the standard demand side explanation of structural change. With biased technical change, the elasticity of substitution is a key variable since it controls the direction of structural change. In the literature there seems to be a consensus that an elasticity of substitution of less than one is the empirically relevant case.⁶

⁵With an elasticity of substitution equal to one, changes in relative prices induce no structural change in terms of expenditure and employment shares.

⁶Baumol, Blackman and Wolff (1985) document a structural change toward the slower growing sector, which is in line with an elasticity of substitution of less than one. Buera and Kaboski (2009a) calibrate their model with an elasticity of substitution equal to 0.5 (their model's best fit is even attained with an elasticity of substitution equal to zero). See also Nagai and Pissarides (2006) and Acemoglu and Guerrieri (2008) which both emphasize the case where the inter-sectoral elasticity of substitution is less than one (in their calibration Acemoglu and Guerrieri (2008) use an elasticity of substitution equal to 0.76).

Although the model assumes non-Gorman preferences aggregation is simple. This is due to the fact that I use preferences that fall into the “price independent general linearity” (PIGL) class of preferences originated by Muellbauer (1975) and Muellbauer (1976). PIGL preferences allow for a “representative household” in Muellbauer’s sense which exhibits the same expenditure shares as the aggregate economy. Income inequality matters, but its effect is fully captured by one summary statistic (in my case by the squared coefficient of variation), which makes the analysis tractable. The analytical solvability of the model does not require an exclusion of income heterogeneity.

The endogenous technical change part of the model draws from the literature on directed technical change (see Acemoglu (1998), Acemoglu and Zilibotti (2001) and Acemoglu (2002)). The bias of technical change is driven by the market size of the two sectors.⁷ Combined with Engel’s law, this generates the following pattern of nonbalanced growth: In early phases of development households are relatively poor. Due to Engel’s law, this implies that a large expenditure share is devoted to necessities. As a consequence, a large fraction of the workforce is employed in the necessity sector. This market size generates an incentive for technical change to be biased toward the necessity sector. Hence, the necessity sector is the main driver of growth at an early stage of development. According to Engel’s law, growth (in expenditure levels) tends to decrease expenditure and employment shares of the necessity sector. Furthermore, because growth is initially biased toward the necessity sector

⁷Furthermore, the literature on directed technical change stresses that technical change is biased toward sector that use relatively expensive production factors more intensively. Such an effect is not present in my model since I only consider one homogeneous type of labor.

and the elasticity of substitution is less than one, also directed technical change tends to increase the expenditure share devoted to luxuries. Overall, this generates a structural change in early phased of development away from necessities toward luxuries. This structural change results in higher incentive for R&D investments in the luxury sector, whereas the attractiveness of the necessity sector is lowered. As a consequence, eventually the luxury sector takes lead as the driver of aggregate growth.

The model is consistent with a globally saddle path stable constant growth path (CGP), where all variables grow at constant rates. Along the CGP employment shares are constant and the “Kaldor facts” hold. Nevertheless, productivity growth rates and growth rates of real output are higher in the luxury than in the necessity sector. This points to the nonbalanced nature of growth even along the CGP. Moreover, even along the CGP Engel’s law applies and biased technical change affects the economic structure. To the best of my knowledge this is the first model with an asymptotic equilibrium in which both explanations of structural change are still at work. A simple simulation illustrates that although transitional dynamic are typically associated with a considerable shift in expenditure and employment shares, the “Kaldor facts” are not significantly violated. This property of the model is similar to Acemoglu and Guerrieri (2008). But in contrast to their model my model includes a demand side explanation of structural change.⁸

⁸Acemoglu and Guerrieri (2008) explicitly point to the importance of a theoretical model that incorporates both, a supply side explanation of structural change and Engel’s law. Their paper concludes with the sentence: “It would be particularly useful to combine the mechanism proposed in this paper with nonhomothetic preferences...” (p. 493).

The remainder of the paper is organized as follows: Section 2 presents the two-sector growth model with non-homothetic intra-temporal preferences and directed technical change. Section 3 provides a simulations of the transitional dynamic for a simple parameter specification. Finally, Section 5 concludes. All proofs are provided in the Appendix A.

2 The Model

2.1 Initial Endowments

There is a mass of n infinitely living households. Each household i is endowed with $l_i > 0$ units of labor and $W_i(0) \geq 0$ units of initial wealth. Households supply their labor inelastically at every instant of time. The aggregate labor supply is denoted by L . Both initial wealth and labor endowments can differ among households.

2.2 Preferences

Each household has the following additively separable representation of inter-temporal preferences

$$\mathcal{U}_i(0) = \int_0^{\infty} \exp(-\rho t) V(P_1(t), P_2(t), e_i(t)) dt, \quad (1)$$

where $\rho > 0$ is the rate of time preference and $V(P_1(t), P_2(t), e_i(t))$ is an indirect instantaneous utility function specified over the prices of the two consumption goods, $P_1(t)$ and $P_2(t)$, and the nominal expenditure level, $e_i(t)$.⁹ The indirect instant-

⁹For a more general preference specification with an inter-temporal CRRA utility function see Appendix C.

neous utility function takes the following form

$$V(P_1(t), P_2(t), e_i(t)) = -\frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)} - \frac{\beta}{1-\alpha+\gamma} \left(\frac{P_2(t)}{P_1(t)}\right)^{1-\alpha+\gamma} + \frac{\beta}{1-\alpha+\gamma}, \quad (2)$$

where $\alpha \in (0, 1]$, $\beta \in [0, \infty)$ and $\gamma \in [0, 1)$.¹⁰ The desirable features of this utility function are discussed below.¹¹ We have the following Lemma:

Lemma 1. *The function (2), (i) is a valid indirect utility specification that represents a locally non-satiated preference relation defined over the two consumption goods if and only if*

$$\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \leq \frac{1}{4} \left[3\alpha - 2 - \gamma + \sqrt{\alpha^2 + (2 + \gamma)^2 - 2\alpha(2 + 3\gamma)} \right], \quad (3)$$

and (ii) is increasing and strictly concave in $e_i(t)$.

Proof: See Appendix A.

In the following I assume that condition (3) is fulfilled for all households at all

¹⁰For $\gamma = 0$ and $\alpha = 1$ we get the limit case

$$V(P_1(t), P_2(t), e_i(t)) = -\frac{P_1(t)}{e_i(t)} - \beta \log \left[\frac{P_2(t)}{P_1(t)} \right].$$

Then, the instantaneous utility function coincides with example 3 introduced by Diewert (1974) page 129.

¹¹The instantaneous utility function is a special instant of the “price independent general linearity” (PIGL) class of preferences and the class of “quadratic expenditure systems” (QES). The PIGL class is specified by Muellbauer (1975) and Muellbauer (1976). PIGL preferences imply that there exists a “representative household” (in Muellbauer’s sense) that exhibits the same expenditure shares as the aggregate economy. The representative expenditure level is independent of prices and just depends on the income distribution. Gorman preferences are also part of the PIGL class. The QES class is specified by Howe, Pollak and Wales (1979) and Van Daal and Merckies (1989).

dates.¹² Households maximize (98) with respect to $\{e_i(t), W_i(t)\}_{t=0}^{\infty}$, subject to the budget constraint

$$\dot{W}_i(t) = r(t)W_i(t) + w(t)l_i - e_i(t),$$

and a standard transversality condition which can be expressed as

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)^2} W_i(t) = 0. \quad (4)$$

$r(t)$ and $w(t)$ are the (nominal) interest and wage rate, respectively, and $W_i(t)$ denotes the wealth of household i at date t . $W_i(0)$ is exogenously given. The result of household's optimization is summarized in the following Lemma.

Lemma 2. (i) *At each point in time we have the following expenditure system*

$$x_1^i(t) = \left[\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{e_i(t)}{P_1(t)} \quad (5)$$

and

$$x_2^i(t) = \left[1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{e_i(t)}{P_2(t)}, \quad (6)$$

where $x_j^i(t)$ is the quantity of good $j = 1, 2$ that household i purchases at date t .

(ii) *Inter-temporal optimization yields the following Euler equation*

$$2g_{e_i}(t) - \alpha g_{P_1}(t) - (1 - \alpha)g_{P_2}(t) = r(t) - \rho, \quad (7)$$

¹²Later, a restriction in terms of exogenous parameters is stated that ensures (3) for all households at all dates. The necessary restriction implies an upper bound on the joint distribution of labor endowments and initial wealth stocks. For $\gamma = 0$, condition (3) can be written as

$$\alpha^2 \geq 4\beta x_1^i(t),$$

where $x_1^i(t)$ denotes the units of necessities consumed by household i . Hence, non-satiation is ensured as long as the number of consumed necessities is not too high.

where $g_{e_i}(t)$ is the growth rate of consumption expenditures of household i and $g_{P_j}(t)$ is the growth rate of prices of good j at date t .

Proof: See Appendix A.

The squared brackets in (5) and (6) constitute the expenditure shares of the two goods. With $\beta = 0$ expenditure shares reduce to α and $1 - \alpha$, respectively, which corresponds to the Cobb-Douglas case. This illustrates that (2) can be viewed as a generalization of Cobb-Douglas preferences. Clearly, in the Cobb-Douglas case ($\beta = 0$) the expenditure elasticities of demand and the elasticity of substitution are all equal to one, which excludes an Engel's law or an effect of biased technical change on the demand structure. Therefore, I focus in the following on the more interesting cases which allow for changing expenditure shares. This is done by assuming¹³

$$\beta > 0. \tag{8}$$

For $\beta > 0$, we see that the Engel curves are quadratic functions of the expenditure level, $e_i(t)$. The Engel curves are depicted in Figure 1. This highlights that we have non-Gorman preferences, where the marginal propensities to consume depend on the expenditure level.¹⁴ Moreover, the terms in squared brackets of (5) and (6) reveal that the expenditure shares are linear functions of the expenditure level, where

¹³For the sake of completeness, the equilibrium dynamic of the Cobb-Douglas case ($\beta = 0$) is provided in Appendix B.

¹⁴Demand systems that are quadratic in expenditure levels have attracted some attention in the empirical literature (See Howe, Pollak and Wales (1979) and Ryan and Wales (1999)). Quadratic expenditure systems fit the data much better than linear ones. For instance Ryan and Wales (1999), p. 285, conclude: "...the coefficients on the quadratic terms are found to be highly significant, and

β controls their slope (Figure 2 draws the expenditure shares as a function of the expenditure level). Thus β can be interpreted as a measure of the degree of non-homotheticity of preferences. α , in contrast, is the expenditure share of necessities of a household with a marginally positive expenditure level (i.e. $e_i(t) \rightarrow 0$).

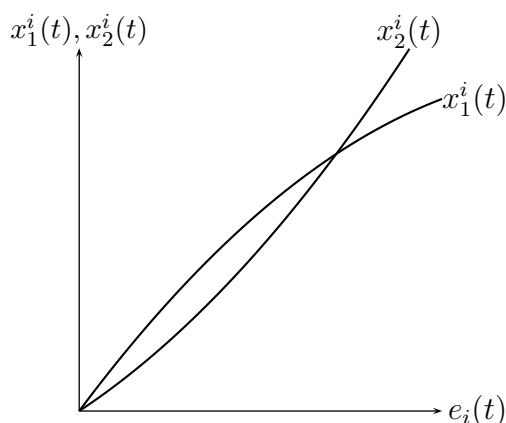


Figure 1: Engel curves

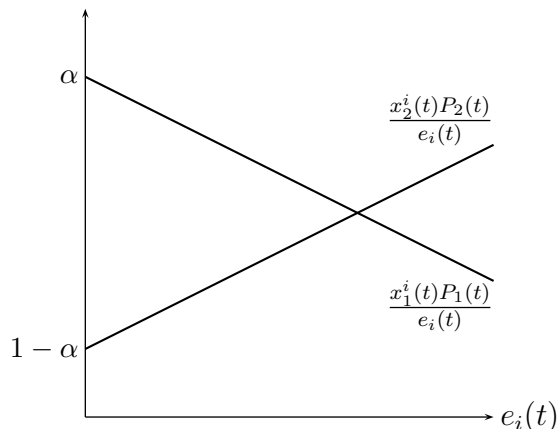


Figure 2: Expenditure shares

The demand system exhibits another special feature: The expenditure shares just depend on the ratio $\frac{e_i(t)}{P_1(t)^{1+\gamma}P_2(t)^{-\gamma}}$. As a consequence, when $P_1(t)^{1+\gamma}P_2(t)^{-\gamma}$ and $e_i(t)$ grow pari passu expenditure shares are constant. This allows for a constant (non-balanced) growth path in an endogenous growth model despite non-homotheticity of preferences. Furthermore, the condition that ensures non-satiation, (3), can be consistent with a constant $\frac{e_i(t)}{P_1(t)^{1+\gamma}P_2(t)^{-\gamma}}$. Thus, the functional form of preferences allows for unbounded growth without causing satiation.

the inclusion of these terms yields marginal budget shares and own-price elasticities that behave quite differently from those obtained when Engel curves are restricted to be linear.” Given this empirical evidence and the tractability of quadratic expenditure systems, it is surprising that they are so far ignored by macroeconomic theory.

Although we have non-Gorman preferences, part (ii) of Lemma 2 shows that household's inter-temporal optimization results in a relatively simple Euler equation. It directly follows from (7) that growth rates of expenditure levels must be the same for all households at a given date, t (i.e. $g_{e_i}(t) = g_e \forall t$).

When we derive the expenditure elasticities of demand and the elasticity of substitution we get the following Lemma.

Lemma 3. *With $\beta > 0$, (i) the elasticity of substitution between the two goods is strictly less than one. (ii) The expenditure elasticities of demand are*

$$\frac{\partial x_1^i(t)}{\partial e_i(t)} \frac{e_i(t)}{x_1^i(t)} = 1 - \frac{\beta e_i(t)}{\alpha P_1(t)^{1+\gamma} P_2(t)^{-\gamma} - \beta e_i(t)} \in [0, 1),$$

and

$$\frac{\partial x_2^i(t)}{\partial e_i(t)} \frac{e_i(t)}{x_2^i(t)} = 1 + \frac{\beta e_i(t)}{(1 - \alpha) P_1(t)^{1+\gamma} P_2(t)^{-\gamma} + \beta e_i(t)} > 1.$$

Proof: See Appendix A.

This Lemma reflects two important features.

First, the elasticity of substitution is less than one. The elasticity of substitution is the key parameter that controls which sector will benefit (in terms of expenditure shares) from biased technical change (see Nagai and Pissarides (2006)). An elasticity of substitution of less than one implies that the expenditure share of the slower growing sector tends to increase. In the literature there is a consensus that this is the empirically relevant case.

Second, because the expenditure elasticity of demand for good one is strictly less than one, it is called henceforth “necessity”. Although being less than one, note that the expenditure elasticity of good one is always greater than zero, which expresses

the exact formulation of Engel's law.¹⁵ The expenditure elasticity of demand for good two is greater than one. Therefore, I refer to good two as "luxury".

The next Lemma describes aggregate market demands.

Lemma 4. (i) *Aggregate demands can be written as*

$$X_1(t) = \left[\alpha - \beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{E(t)}{P_1(t)}, \quad (9)$$

$$X_2(t) = \left[(1 - \alpha) + \beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{E(t)}{P_2(t)}, \quad (10)$$

where $\phi_e(t)$ is the squared coefficient of variation of consumption expenditures across households and $E(t)$ is the aggregate consumption expenditure at date t .

(ii) A household with $e_i(t) = \frac{E(t)[1+\phi_e(t)]}{n} \equiv e^{RA}(t)$ is the representative agent in Muellbauer's sense.

Proof: See Appendix A.

This Lemma highlights the tractability of preferences which fall into the "price independent general linearity" (PIGL) class. Aggregate demands can be written as functions of the aggregate expenditure level, the per capita expenditure level and a (scale invariant) measure of inequality of consumption expenditure levels across households (the coefficient of variation). This constitutes a tractable way to deal

¹⁵See for instance Browning (2008): "Engel's law states that food is not a luxury. This is one of the earliest empirical regularities in economics and also one of the most robust. The widespread finding is that regressions of food expenditures, quantities or budget shares on income or total expenditure and other variables such as prices, demographics and regional dummies uniformly imply that the income elasticity of food is less than 1 (and greater than zero)."

with inequality in a dynamic general equilibrium model.¹⁶ The representative agent is defined after Muellbauer (1975) and possesses the same expenditure shares as the aggregate economy. As long as there is household heterogeneity, the expenditure level of the representative agent is larger than per capita expenditures, $\frac{E(t)}{n}$. Because the marginal propensity to consume luxuries increases in the expenditure level, aggregate demand for luxuries increases in the per capita expenditure level and in inequality, $\phi_e(t)$.

Because the Euler equation, (7), implies that growth rates of expenditure levels across households are equalized at a given date, and because the relevant inequality measurement is scale invariant we can write

$$\phi_e(t) = \phi_e, \forall t.$$

This simplifies the equilibrium analysis dramatically.

To reduce the number of model parameters I assume henceforth $\alpha = 1$. Empirically this is a plausible case, since it implies that a household just consumes necessities as his level of consumption expenditures goes to zero. Moreover, it gives rise to a simple case which is well-suited to illustrate the model's key features. Qualitatively, as Appendix C shows, these central features are unchanged when we have $\alpha < 1$.

¹⁶With the exception of Foellmi and Zweimüller's (2006) model with 0/1 preferences, I am not aware of any other dynamic general equilibrium model that features a demand sided effect of inequality (by assuming non-Gorman preferences) and does not have to exclude households heterogeneity. Moreover, in contrast to Foellmi and Zweimüller (2006), where two income groups are considered, my model allows for any - possibly continuous - distribution of expenditure levels with an upper bound.

With $\alpha = 1$ condition (3) simplifies to

$$2\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \leq (1 - \gamma). \quad (11)$$

For the Euler equation we get

$$2g_{e_i}(t) - g_{P_1}(t) = r(t) - \rho, \quad (12)$$

and the aggregate demand functions are

$$X_1(t) = \left[1 - \beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{E(t)}{P_1(t)}, \quad (13)$$

and

$$X_2(t) = \left[\beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{E(t)}{P_2(t)}. \quad (14)$$

2.3 Production

The two final goods are produced competitively by a continuum of sector specific intermediate inputs. We have the following standard CES production functions

$$Y_1(t) = \left[\int_0^{M_1(t)} y_{1,\omega}(t)^{\frac{1}{2}} d\omega \right]^2 \quad \text{and} \quad Y_2(t) = \left[\int_0^{M_2(t)} y_{2,\omega}(t)^{\frac{1}{2}} d\omega \right]^2,$$

$y_{1,\omega}(t)$'s and $y_{2,\omega}(t)$'s denote the amount of intermediate inputs used at date t in sector 1 and 2, respectively.¹⁷ $M_1(t)$ and $M_2(t)$ are the numbers of different sector specific input varieties that are available at date t (I also refer to $M_j(t)$ as the

¹⁷The elasticity of substitution between intermediate inputs is assumed to be 2. If the elasticity of substitution is unequal to two (but constant) the growth rates of output, $Y_j(t)$, differ from the growth rates of the sector specific knowledge stock, $M_j(t)$, by a constant factor. This complicates the notation without changing the main results or delivering new insights.

knowledge stock of sector j). $M_1(0)$ and $M_2(0)$ are exogenously given. The sector specificity of intermediate inputs allow for separated growth rates.

Each machine type ω suitable for sector $j = 1, 2$ is produced by monopolistically competitive firms according to the following production functions

$$y_{1,\omega}(t) = l_{1,\omega}(t) \text{ and } y_{2,\omega}(t) = l_{2,\omega}(t), \quad (15)$$

where $l_{j,\omega}(t)$, $j = 1, 2$ is employed labor by firm j, ω . To ease notation, I define

$$L_1(t) \equiv \int_0^{M_1(t)} l_{1,\omega}(t) d\omega \text{ and } L_2(t) \equiv \int_0^{M_2(t)} l_{2,\omega}(t) d\omega. \quad (16)$$

Furthermore, I define $\eta_{j,\omega}$ as the price of intermediate input ω of sector j . Under these assumptions, the static market equilibrium outcome is characterized by the following Lemma.

Lemma 5. *Equilibrium output of sector j can be written as*

$$Y_j(t) = L_j(t)M_j(t), \quad j = 1, 2. \quad (17)$$

Firms instantaneous profits, $\pi_{j,\omega}(t)$, are given by

$$\pi_{j,\omega}(t) = \frac{1}{2}P_j(t)L_j(t), \quad \forall \omega, \quad j = 1, 2, \quad (18)$$

and the wage rate can be expressed as

$$w(t) = \frac{1}{2}P_j(t)M_j(t). \quad (19)$$

Wage equalization across sectors implies

$$P_1(t)M_1(t) = P_2(t)M_2(t). \quad (20)$$

Proof: See Appendix A.

This Lemma shows that output of sector j can be written as the product of the stock of knowledge, $M_j(t)$, to a constant power and the number of labor employed in sector j , $L_j(t)$. The output of final goods, $Y_j(t)$, $j = 1, 2$, is consumed. Labor is either used to produce intermediate inputs or is employed in R&D. The R&D process is described next.

2.4 Innovation Possibilities Frontiers

In order to create a new type of intermediate input suitable for sector j , $\frac{f}{M_j(t)}$ units of labor are needed. f is a positive constant. The fixed investment costs decrease in $M_j(t)$ because of intra-sectoral spillovers.¹⁸ Inter-sectoral spillovers are excluded.¹⁹ An innovator gets a monopoly position for the new variety of intermediate input until there is a “bad shock” which destroys his innovation. Such a “bad shock” occurs for a intermediate input variety suitable for sector j with probability δ_j . We have $\delta_1 > 0$.²⁰ Under these assumptions, the innovation possibilities frontiers of the

¹⁸I prefer a spillover to a lab-equipment framework because in this case R&D activities in the two sector do not affect the relative market size of the sectors, $\frac{L_1(t)}{L_2(t)}$. Hence, in this spillover model, relative market size is fully determined by preferences.

¹⁹This is important, because it allows me to generate a constant growth path along which growth rates of the two sectors differ. See also the model of Acemoglu and Guerrieri (2006), which relies on the same assumption.

²⁰Examples for such bad shocks are natural disasters, new regulations or major changes in production circumstances. A positive exit probability in sector one is needed to allow for a decreasing stock of knowledge in the necessity sector. A similar constant and exogenous exit probability is assumed in Melitz (2003). As long as we have $\gamma > 0$ or we approach the CGP from below, the equilibrium dynamic is unchanged with $\delta_1 = 0$. In any case, δ_2 is not restricted to be (strictly)

two sectors can be written as follows

$$\dot{M}_j(t) = L_j^R(t) \frac{M_j(t)}{f} - \delta_j M_j(t), \quad j = 1, 2, \quad (21)$$

where $L_j^R(t)$ is the number of “researchers” that are employed in R&D for sector j at date t .

It is worth noting, that with respect to technology, the two sectors are fully symmetric (with the exception of possible differences in the exit probability). Technically, both sectors can constitute an “engine of growth”. Thus the production side allows for sustainable growth in both sectors independent of the other sector.

There is free market entry, which implies that in equilibrium, R&D activity in sector j is either zero or is determined by a zero profit condition.

To ease notation, $P_2(t)$ is chosen henceforth as numéraire, i.e.

$$P_2(t) = 1, \quad \forall t. \quad (22)$$

Then, R&D activity is characterized by the following Lemma.

Lemma 6. *Under free market entry we must have*

$$f \geq \frac{L_1(t)}{r(t) + \delta_1 - g_2(t) + g_1(t)}, \quad (23)$$

$$f \geq \frac{L_2(t)}{r(t) + \delta_2}, \quad (24)$$

$$L_j^R(t) \geq 0, \quad j = 1, 2, \quad \forall t, \quad (25)$$

$$L_1^R(t) \left[f - \frac{L_1(t)}{r(t) + \delta_1 - g_2(t) + g_1(t)} \right] = 0, \quad (26)$$

positive.

and

$$L_2^R(t) \left[f - \frac{L_2(t)}{r(t) + \delta_2} \right] = 0, \quad (27)$$

where $g_j(t) \equiv \frac{\dot{M}_j(t)}{M_j(t)}$.

Proof: See Appendix A.

$g_j(t)$ is the growth rate of the knowledge stock in sector j . The incentive to innovate in sector j is determined by market size, $L_j(t)$, the nominal interest rate, $r(t)$, the exit probability, δ_j and the price dynamic of good j .²¹

2.5 Equilibrium

2.5.1 Definitions

In this economy an equilibrium is defined as follows:

Definition 1. A *dynamic market allocation* is a time path of individual consumption expenditures $\{e_i(t)\}_{t=0}^{\infty}$, $\forall i$; a time path of aggregate R&D investments and stock of knowledge in both sectors, $\{L_j^R(t), M_j(t)\}_{t=0}^{\infty}$, $j = 1, 2$; a time path of prices of final goods $\{P_1(t), P_2(t)\}_{t=0}^{\infty}$; a time path of prices and quantities of all intermediate inputs $\{\eta_{j,\omega}(t), y_{j,\omega}(t)\}_{t=0}^{\infty}$, $\forall \omega$, and $j = 1, 2$; and a time path of nominal interest rate and wages $\{r(t), w(t)\}_{t=0}^{\infty}$.

Definition 2. A *dynamic market equilibrium* is an allocation in which monopolistically competitive intermediate input producers chose prices and quantities, such that the discounted value of profits is maximized, the time paths of $P_1(t)$, $P_2(t)$,

²¹Note that $g_2(t) - g_1(t)$ is the growth rate of the price of the necessity, since $P_2(t)$, $\forall t$ is chosen as numéraire.

$r(t)$ and $w(t)$ are consistent with market clearing and perfect mobility in the labor market, and the evolution of $e_i(t)$, $\forall i$, $L_j^R(t)$ and $M_j(t)$ $j = 1, 2$ are consistent with optimization, free market entry, market clearing and perfect competition in final good sectors.

2.5.2 Equilibrium Conditions

Labor is used in R&D and in production of intermediate inputs. Then, labor market clearing implies

$$L = L_1(t) + L_2(t) + L_1^R(t) + L_2^R(t). \quad (28)$$

Market clearing of good markets is given by

$$Y_j(t) = X_j(t), \quad j = 1, 2. \quad (29)$$

The next Lemma states a collapsed system of equations, that characterizes the equilibrium dynamic.

Lemma 7. *The equilibrium can be characterized by the following equations:*

$$f \geq \frac{L - L_2(t) - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]}{r(t) + \delta_1 - g_2(t) + g_1(t)}, \quad (30)$$

$$f \geq \frac{L_2(t)}{r(t) + \delta_2}, \quad (31)$$

$$f[g_j(t) + \delta_j] \geq 0, \quad j = 1, 2, \quad (32)$$

$$f[g_1(t) + \delta_1] \left[f - \frac{L - L_2(t) - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]}{r(t) + \delta_1 - g_2(t) + g_1(t)} \right] = 0, \quad (33)$$

$$f[g_2(t) + \delta_2] \left[f - \frac{L_2(t)}{r(t) + \delta_2} \right] = 0, \quad (34)$$

$$g_2(t) + g_1(t) - 2 \frac{f\dot{g}_1(t) + f\dot{g}_2(t)}{L - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]} = r(t) - \rho, \quad (35)$$

$$L_2(t) = \beta \frac{[1 + \phi_e]}{n} m(t) [L - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]]^2, \quad (36)$$

and

$$\dot{m}(t) = [(1 + \gamma)g_1(t) - \gamma g_2(t)] m(t), \quad (37)$$

where $m(t) \equiv \frac{M_1(t)^{1+\gamma}}{M_2(t)^\gamma}$.

Proof: See Appendix A.

We have the following endogenous variables: $r(t)$, $g_1(t)$, $g_2(t)$, $\dot{g}_1(t)$, $\dot{g}_2(t)$, $m(t)$ and $L_2(t)$. $m(0) = \frac{M_1(0)^{1+\gamma}}{M_2(0)^\gamma}$ is inherited from the past. (30) - (34) are the free entry slackness conditions of the two sectors. (35) denotes the Euler equation, whereas (36) guarantees market clearing in the second sector.²² Finally the transversality condition, (4), has to hold. With $\alpha = 1$ this can be rewritten as

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \frac{P_1(t)}{e_i(t)^2} W_i(t) = 0. \quad (38)$$

2.6 Constant Growth Path

Definition 3. A *constant growth path* (CGP) is an equilibrium path, along which growth rates of all variables are constant.

This definition is according to Acemoglu and Guerrieri (2008). Kongsamut, Rebelo and Xi (2001) uses the terminology “generalized balanced growth path” for the same concept. It is worthwhile to note, that growth rates must be constant but can (and actually will) differ across sectors. Let us assume the following condition holds

$$L > 2f(\delta_1 + \delta_2 + \rho). \quad (39)$$

²²Walras’ law ensures that the market for necessities clears.

This condition ensures positive growth along the CGP.

Then, the next Proposition states that there exists a unique CGP. CGP values are denoted by asterisks.

Proposition 1. *If assumption (8), (11) and (39) hold, there exists a unique CGP with*

$$g_1^* = \frac{\gamma [L - 2f(\delta_1 + \delta_2 + \rho)]}{2f(1 + 3\gamma)}, \quad (40)$$

$$g_2^* = \frac{(1 + \gamma) [L - 2f(\delta_1 + \delta_2 + \rho)]}{2f(1 + 3\gamma)}, \quad (41)$$

$$m^* = \frac{2n(1 + 3\gamma) [L(1 + 2\gamma) + 2f(\gamma\delta_2 + \gamma\rho - (1 + 2\gamma)\delta_1)]}{\beta(1 + \phi_e) [L(1 + 4\gamma) - 2f(\gamma(\delta_1 + \delta_2) - \rho(1 + 2\gamma))]^2}, \quad (42)$$

$$\left[\frac{E(t)}{M_2(t)} \right]^* = \frac{(1 + 4\gamma)L - 2f\gamma(\delta_1 + \delta_2) + 2f(2\gamma + 1)\rho}{2(1 + 3\gamma)}, \quad (43)$$

$$L_1^* = \frac{\gamma L + f[(\delta_1 + \rho)(1 + \gamma) - 2\gamma\delta_2]}{2(1 + 3\gamma)}, \quad (44)$$

$$L_2^* = \frac{(1 + 2\gamma)L + 2f\gamma(\delta_2 + \rho) - 2f(1 + 2\gamma)\delta_1}{2(1 + 3\gamma)}, \quad (45)$$

$$L_1^{R*} = \frac{\gamma [L - 2f(\delta_2 + \rho)] + 2f(1 + 2\gamma)\delta_1}{2(1 + 3\gamma)},$$

and

$$L_2^{R*} = \frac{(1 + \gamma) [L - 2f(\delta_1 + \rho)] + 4f\gamma\delta_2}{2(1 + 3\gamma)}.$$

Proof: See Appendix A.

We see that condition (39) ensures positive growth. Moreover, (39) guarantees that m^* is positive.

Several features of the CGP are worth noting. First, the algebraic structure is relatively simple. Despite the fact of non-Gorman preferences, per capita consumption

expenditures and inequality do not affect growth rates along the CGP. The distribution of expenditure levels only matters for m^* . The growth rate of both sector increase in L due to a standard scale effect present in the model. Moreover, g_j^* $j = 1, 2$ decrease in the exit probabilities, δ_j , the rate of time preference, ρ , and the R&D investments costs, f .

Second, along the CGP the “Kaldor facts” are fulfilled. The growth rates of output, the capital-output ratio, the share of capital income in GDP and the real interest rate are constant over time.

Third, the employment shares of the two sectors are constant along the CGP. Thus, there is no structural change in terms of employment. The intuition for this is that there are two counterbalancing effects at work. On the one hand, since there is positive growth in income and expenditure levels, Engel’s law applies and the expenditure share of the luxury sector tends to be increased whereas the expenditure share of the necessity sector tends to be lowered.²³ On the other hand there is an effect of biased technical change that goes in the opposite direction. The growth rate of the luxury sector is higher. Since the inter-sectoral elasticity of substitution

²³It is important to emphasize that there is an Engel’s law in the CGP. This contrasts to models that use quasi-homothetic preferences to generate structural change for which Engel’s law vanishes asymptotically. This undesirable implication of quasi-homothetic preferences explains why the standard model does a poor job in fitting the data. Buera and Kaboski (2009) acknowledge: “A major problem is that the relative price of services to industry was rising during this period but at a slower rate than the relative shares. Thus, in this later period, both the relative quantity and relative price of services were rising relative to industry. Explaining this would require a large, delayed income effect toward services. This is not possible with the Stone–Geary preferences, where the endowments and subsistence requirements are most important at low levels of income.”

is less than one, biased technical change tends to shift expenditure shares from the luxury to the necessity sector. Along the CGP these two effects exactly cancel each other.²⁴

Finally, there are features of nonbalanced growth even along the CGP. Productivity growth rates differ (growth is higher in the luxury sector). Moreover, real output of necessities expands at a lower rate than the real output of luxuries. This sectoral difference is solely generated by the non-homotheticity of preferences (because otherwise the two sectors are fully symmetric).

The CGP bears some resemblance to the asymptotic equilibrium in Acemoglu and Guerrieri (2008). In both models, although the growth rates differ across sectors and there is a structural change in terms of real output, the dynamic is consistent with the “Kaldor facts”. But note also the important differences. In contrast to Acemoglu and Guerrieri (2008), the CGP does not only exist asymptotically and the relative employment share of one sector did not become vanishingly small. And most important, there is also an Engel’s law present in this model.

Finally, we are now ready to specify assumption (11) in terms of exogenous parameters. Suppose the economy starts directly along its CGP (i.e. $m(0) = m^*$). Furthermore, suppose household i' is the one with the highest consumption expen-

²⁴This feature is graphically illustrated and discussed below for a simple case where $\gamma = 0$.

diture level along the CGP, i.e.²⁵

$$e_{i'}^*(t) \geq e_i^*(t), \quad \forall i. \quad (46)$$

Let us write

$$e_{i'}^*(t) = \kappa E(t)^*, \quad (47)$$

where $\kappa \geq \frac{1}{n}$ is the share of aggregate consumption expenditure that is spent by the richest household. When we use (20), (2.6.1), (2.6.1), (2.6.1) we can rewrite (11) as

$$\frac{2 [L(1 + 2\gamma) + 2f(\gamma\delta_2 + \gamma\rho - (1 + 2\gamma)\delta_1)]}{[L(1 + 4\gamma) - 2f(\gamma(\delta_1 + \delta_2) - \rho(1 + 2\gamma))]} \frac{\kappa n}{(1 + \phi_e)} \leq 1 - \gamma. \quad (48)$$

Without inequality this can be expressed as²⁶

$$L(1 + \gamma + 4\gamma^2) \leq 2f [(2 + \gamma)(1 + \gamma)\delta_1 - \gamma(3 - \gamma)\delta_2 + (1 - \gamma - 2\gamma^2)\rho]. \quad (49)$$

This condition implies that there is an upper bound on the distribution of per capita expenditure levels at $t = 0$. Note that (39) and (48) are not mutually exclusive.²⁷

So far we just analyzed the equilibrium dynamic along the CGP. Clearly, it would be a sheer coincidence if the knowledge stocks inherited from the past imply $m(0) = m^*$.

Therefore, in the next Section transitional dynamic is analyzed and global saddle path stability of the CGP is proven. But before doing, this the CGP of a very

²⁵Along the CGP, the propensity to consume out of labor income and out of wealth income are constant and given by 1 and $\frac{\rho}{r^*}$, respectively. This implies that

$$e_i^*(t) = l_i w(t) + W_i(t)(\rho),$$

where $w(t) = \frac{1}{2}M_2(t)$ and $W_i(t)$ grows at constant rate g_2^* . Then, for a given joint distribution of initial wealth and labor endowments, i' can be identified.

²⁶Note that no heterogeneity ($\phi_e = 0$) would imply $\kappa = \frac{1}{n}$.

²⁷For instance with $\gamma = 0$, (49) can be written as $L \leq 4f\delta_1 + 2f\rho$.

simple case ($\gamma = 0$) is illustrated. It is shown that the dynamic of this simple case is identical to a prominent example in the literature.

2.6.1 The CGP for a very simple case ($\gamma = 0$)

The CGP simplifies to a very simple case if we have $\gamma = 0$. Then the CGP values are given by

$$\begin{aligned} g_1^* &= 0, \\ g_2^* &= \frac{L}{2f} - \delta_1 - \delta_2 - \rho, \\ M_1^* &= \frac{n}{\beta(1 + \phi_e)} \frac{2(L - 2f\delta_1)}{(L + 2f\rho)^2}, \\ \left[\frac{E(t)}{M_2(t)} \right]^* &= \frac{L}{2} + f\rho, \\ L_1^* &= (\rho + \delta_1)f, \\ L_2^* &= \frac{L}{2} - f\delta_1 \\ L_1^{R*} &= \delta_1 f, \end{aligned}$$

and

$$L_2^{R*} = \frac{L}{2} - f[\delta_1 + \rho].$$

This illustrates that the model is then algebraically as simple as a standard textbook growth model. For this simple case productivity growth along the CGP is entirely driven by the luxury sector and we have $m(t) = M_1(t)$. It is worth noting to note that then the dynamic along the CGP is exactly the same as in a well-known example introduced by Baumol (1967). We have one “progressive” sector with positive productivity growth and one “nonprogressive” sector with constant labor

productivity. But in contrast to Baumol (1967), here sectoral differences in growth rates are not exogenously assumed but determined endogenously. With $\gamma = 0$ the intra-temporal utility maximization along the CGP is illustrated in Figure 3. Point A depicts the optimal consumption point for a given initial expenditure level, $e_i(t)$, and a given initial relative price of the necessity sector, $P_1(t)$ (note that the price of the luxury has been normalized to one). Since we only have growth in the luxury sector along the CGP, $e_i(t)$ and $P_1(t)$ grow *pari passu* over time. Then, at a later date t' the budgeted line is steeper but has the same intercept on the horizontal axis. For $\gamma = 0$, the individual demand system (5) and (6) reveals that as long as $\frac{e_i(t)}{P_1(t)}$ is constant the expenditure share do not change. Hence, the new optimal consumption bundle C must contain the same amount of necessities as the initial consumption bundle A . So the adjustment of the consumption structure from point A to C looks as if the household has Cobb-Douglas preferences. This is the nice feature of the assumed preferences, what makes the model consistent with a GCP. But when we decompose the total effect in an income and substitution effect, the important deviations from Cobb-Douglas preferences become apparent. First, we see that the income effect from B to C includes an effect on the demand structure, which manifests the Engel's law. Moreover, the substitution effect from A to B is smaller than it would be in the Cobb-Douglas case (in the Cobb-Douglas case the substitution effect would go from A to \tilde{B}). This visualizes that the substitution effect is less than one.

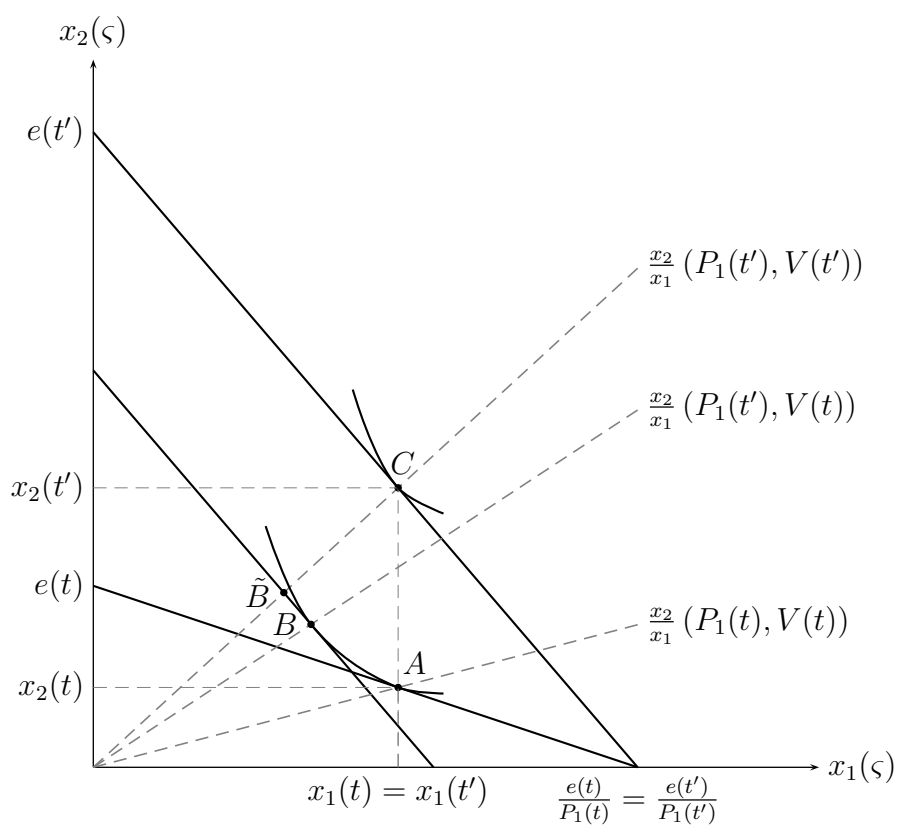


Figure 3: Income and substitution effects along the CGP

2.7 Transitional dynamic

The next Proposition analyses the transitional dynamic.

Proposition 2. *If (11) and (39) hold for all household in every instant of time, then (i) the CGP is globally saddle path stable. And (ii) within finite time there must be strictly positive investments in both sectors. Then, (iii) the dynamic is characterized by*

$$\dot{g}_1(t) = \left[\rho + (1 + 2\gamma)\frac{L}{2f} - (1 + \gamma)\sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}} - 2\gamma g_1(t) - \gamma(\delta_1 + \delta_2) \right] \left[\frac{L}{2f} - g_1(t) - \delta_1 \right], \quad (50)$$

and

$$\dot{m}(t) = \left[(1 + 2\gamma)g_1(t) - \gamma \left[\frac{L}{f} - \delta_1 - \delta_2 - \sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}} \right] \right] m(t), \quad (51)$$

where $\tilde{\beta} \equiv \frac{(1+\phi_e)\beta}{n}$.

Proof: See Appendix A.

Part (i) of this Proposition ensures that the economy converges to the CGP no matter what the initial $m(0)$ is. With equation (50) and (51) the transitional dynamic can be illustrated in a phase diagram. This is done in Figure 4.²⁸ Suppose

²⁸We cannot exclude an upward sloping $\dot{g}_1 = 0$ locus or a downward sloping $\dot{m} = 0$ locus. But there is just one intersection of the two isoclines and in this point the $\dot{m} = 0$ locus is steeper than the $\dot{g}_1 = 0$ locus. Together with the dynamic above and below the respective isocline, this proves saddle path stability (see Appendix A which provides a formal proof of Proposition 2). $g_1(t) > \frac{L}{2f} - \delta_1$ would imply $E(t) < 0$. Therefore $g_1(t) > \frac{L}{2f} - \delta_1$ is not feasible. With $\gamma = 0$ a

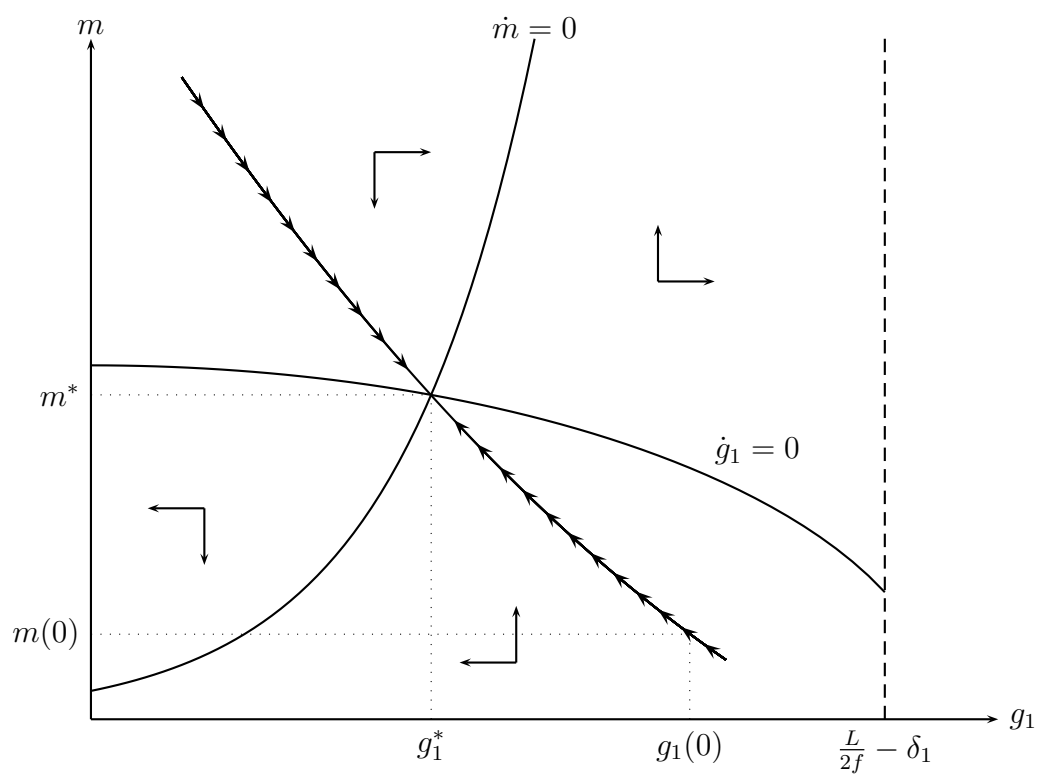


Figure 4: Phase diagram

the economy starts with an initial $m(0) < m^*$. This means that the initial stock of knowledge of the necessity sector is relatively low. As a consequence households devote a high expenditure share to necessities. This high expenditure share results in a high employment share in the necessity sector. This high employment and expenditure share constitutes a relatively large market for intermediate goods suitable for the necessity sector and therefore a higher R&D incentive in this sector. As a consequence we observe an initial productivity growth rate in the necessity sector that is above its CGP level, whereas growth in the luxury sector is relatively low (compared to CGP). According to Engel's law, growth in income and expenditure levels is associated with a decrease in the expenditure share devoted to the necessity sector. Moreover, since technical change is directed toward the necessity sector and the elasticity of substitution is less than one, also biased technical change shifts expenditure shares from the necessity toward the luxury sector. Overall we then observe a fast initial structural change in employment and expenditure shares. But this structural change lets the R&D incentives not unaffected. Due to the changing market sizes technical change will be more and more biased toward the luxury sector. Since the elasticity of substitution is less than one biased technical change will then eventually slow down the structural change. The transitional dynamic with $m(0) < m^*$ is summarized in the following Proposition.

Proposition 3. *When the economy starts with $m(0) < m^*$, $L_1(t)$ and $g_1(t)$ decrease and $L_2(t)$, $g_2(t)$ and $m(t)$ increase over time. When we have $m(0) > m^*$, the opposite*

simple case emerges, where the $\dot{m} = 0$ locus is a vertical line at $g_1 = g_1^* = 0$ and the $\dot{g}_1 = 0$ is a negative sloped straight line intercepting the horizontal axis at $g_1 = \frac{L}{2f} - \delta_1$.

holds.

Proof: see Appendix A.

This illustrates that the model can generate structural change in terms of employment shares and relative growth rates along the transition. With $m(0) < m^*$, the model's prediction is that we observe a shift in the employment share away from the necessity toward the luxury sector and that growth is more and more biased toward the luxury sector (i.e. $g_2(t) - g_1(t)$ increases over time).

It remains to show that assumption (11) holds along the transition. This is done in the next Proposition.

Proposition 4. *If we have $m(0) \leq m^*$ condition (48) is sufficient to ensure (11).*

Proof: See Appendix A.

As long as we approach the CGP from below, condition (48) is sufficient to ensure non-satiation along the transition. When we have $m(0) \geq m^*$, (11) holds as long as $m(0)$ is not too high.

Clearly, the “Kaldor facts” only hold asymptotically. Hence it remains to show that the model can generate a long-lasting and considerable structural change without significantly violating the stylized facts of balanced growth. This is done in next section.

3 A Simple Simulation

(50) and (51) shows that $m(t)$ enters only multiplicatively with $\tilde{\beta}$. Moreover, m^* is proportional to $\frac{1}{\tilde{\beta}}$. This implies that the model is fully specified if we know ρ , f , L , δ_1 , δ_2 , γ and $\frac{m^*-m(0)}{m^*}$. Figure 5-10 illustrates the transitional dynamic of the key variables under the following parameter specification: $\rho = 0.02$, $f = 10$, $L = 2$, $\delta_1 = 0.04$, $\delta_2 = 0$, $\gamma = 0.1$ and $\frac{m^*-m(0)}{m^*} = 0.4$. The equilibrium dynamic is simulated by a shooting algorithm (see Judd (1998, Chapter 10)), where the differential equations are solved by a fourth-order Runge-Kutta method.

First, Figure 5 reveals that the model generates a considerable structural change in terms of employment shares (the fraction of the labor force employed in production and R&D of the luxury sector increases from 0.15 to 0.36). Furthermore the transitional dynamic is slow, taking several decades. Figure 6 shows that development goes hand in hand with a significant shift in the technical bias of growth. The growth rate of the luxury sector grows from 0.7 to 3.3 percent, whereas the growth rate of the necessity sector decreases from 2 to 0.3 percent. Next, Figure 7-9 illustrate that the “Kaldor facts” are not significantly violated although the model generates structural change along several dimensions. The evolution of the saving rate, the real interest rate and the growth rate of real consumption expenditure is non-monotonic and very flat.

Finally, since expenditure shares differ across individuals, the relevant true cost of living price index is individual specific.²⁹ As a consequence, although all nominal

²⁹The growth rate in the true cost of living index of household i at date t is given by

$$\frac{\dot{P}_i(t)}{P_i(t)} = \frac{\partial e(P_1(t), P_2(t), v_i(t))}{\partial P_1(t)} \frac{\dot{P}_1(t)}{e_i(t)} + \frac{\partial e(P_1(t), P_2(t), v_i(t))}{\partial P_2(t)} \frac{\dot{P}_2(t)}{e_i(t)} = \left[1 - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma}} \right] (g_2(t) - g_1(t)),$$

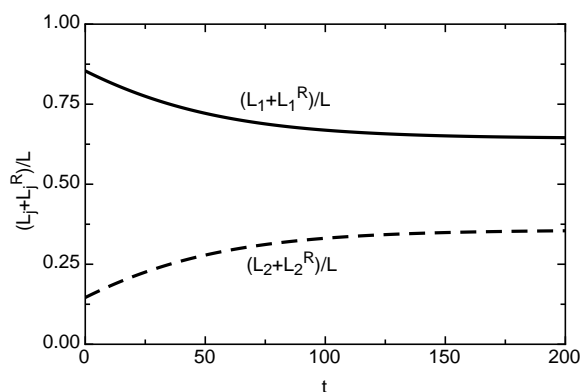


Figure 5: Employment shares

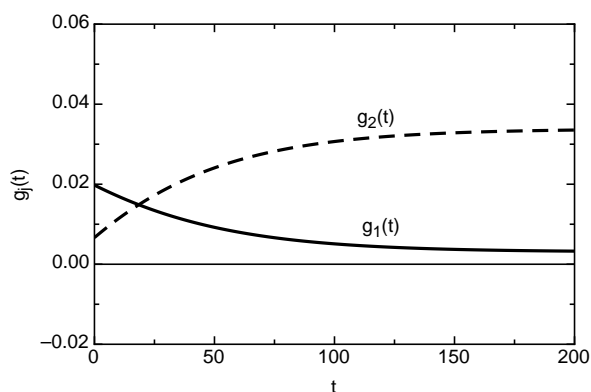


Figure 6: Sectoral growth rates

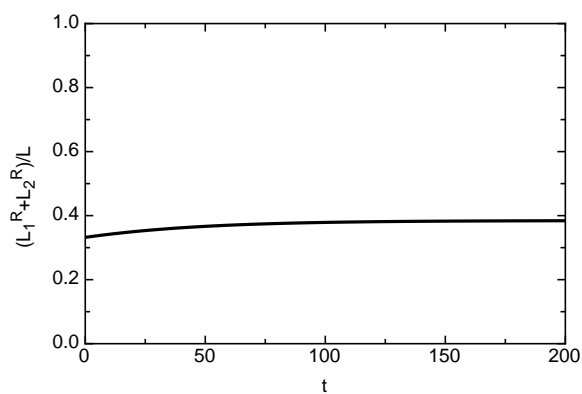


Figure 7: Saving rate

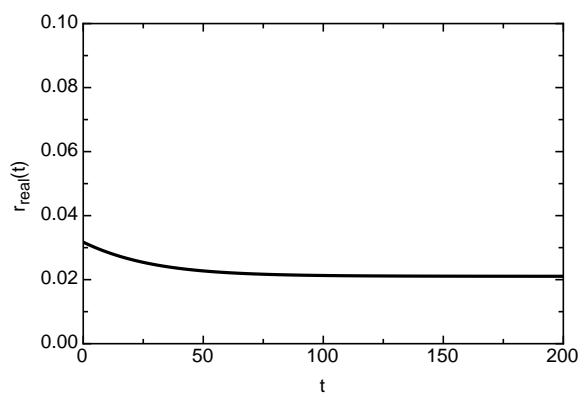


Figure 8: Real interest rate

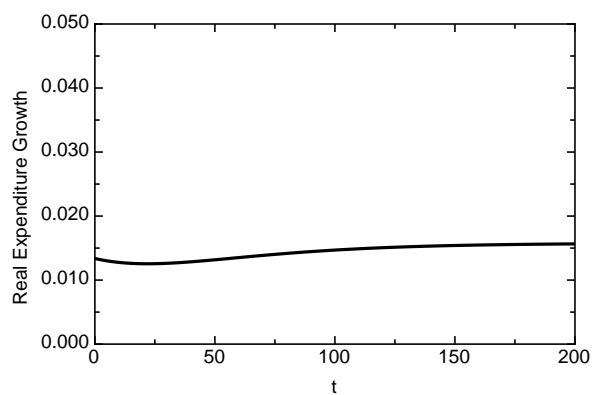


Figure 9: Real expenditure level (of the representative household)

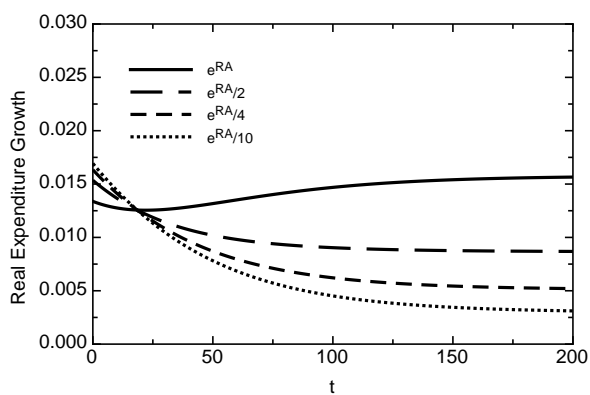


Figure 10: Real expenditure growth for different expenditure levels.

expenditure levels grow at the same rate in a given point in time. Real expenditure growth differs across expenditure levels. This is visualized in Figure 10. Because growth is first biased toward the necessity sector real expenditure of relatively poor individuals grows faster. But since the sectoral bias of growth shifts along transition toward the luxury sector, real expenditure growth of richer households will eventually be higher. Since Engel’s law applies also in the CGP, different real expenditure growth can also be observed in the asymptotic equilibrium. I am not aware of any other dynamic general equilibrium model that has an analytically solvable asymptotic equilibrium with this feature.

4 Conclusion

Differences in sectoral growth rates and systematic shifts in sectoral expenditure and employment share are very robust findings of an old economic literature. However, it is also well known that aggregated variables present a balanced picture of economic growth (captured by the “Kaldor facts”). Due to technical difficulties there was a long lack of theoretical models that reconcile the balanced and nonbalanced nature of growth in a general equilibrium model. This paper presented a tractable two sector endogenous growth model which can generate a considerable structural change without significantly deviating from the “Kaldor facts”. The main building block of the model are Engel’s law and biased technical change, both empirically well-founded explanations of structural change. The model predicts systematic dif-

 where $v_i(t)$ is the attained instantaneous utility level of household i at date t . This is the partial true cost of living index defined by Pollak (1975).

ferences in the evolution of expenditure shares and sectoral productivity growth rates between luxuries and necessities. It is left to further research to test these hypothesis.

The paper relies on a closed economy, dynamic general equilibrium model and therefore abstracts from open economy phenomena as trade and international technology diffusion. Introducing such open economy effects would be an interesting area for further research. For instance, for technologically lagging countries technology adoption can constitute an attractive alternative to R&D investments. Moreover, it is worth noting that international trade in final goods would allow for sectoral specialization irrespective of the degree of development. But when we look at consumption data, the model predicts that an Engel's law should still be observed.

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Appendix A: Proof of Lemmas and Propositions

Proof of Lemma 1

(i) According to Proposition 3.E.2 of Mas-Colell, Whinston and Green (1995), for an expenditure function to represent a locally non-satiated preference relation it must be (a) homogeneous of degree one in prices, (b) strictly increasing in $V_i(t)$ and nondecreasing in all prices, (c) concave in prices and (d) continuous in prices and $V_i(t)$.

(2) corresponds to the following expenditure function³⁰

$$e(P_1(t), P_2(t), V_i(t)) = \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{\left[-V_i(t) + \frac{\beta}{1-\alpha+\gamma} - \frac{\beta}{1-\alpha+\gamma} \left(\frac{P_2(t)}{P_1(t)}\right)^{1-\alpha+\gamma}\right]}. \quad (52)$$

It is readily to see, that (a) and (d) are fulfilled. So I start proving (b). For the first derivative of $e(P_1(t), P_2(t), V_i(t))$ with respect to $V_i(t)$ we get

$$\frac{\partial e(P_1(t), P_2(t), V_i(t))}{\partial V_i(t)} = \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{\left[-V_i(t) + \frac{\beta}{1-\alpha+\gamma} - \frac{\beta}{1-\alpha+\gamma} \left(\frac{P_2(t)}{P_1(t)}\right)^{1-\alpha+\gamma}\right]^2} > 0.$$

The first derivative with respect to $P_1(t)$ yields (use again (2) to write the derivative in terms of $e_i(t)$)

$$\frac{e_i(t)}{P_1(t)} \left[\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right]$$

This is non-negative as long as

$$\alpha \geq \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}. \quad (53)$$

³⁰For $\alpha = 1 + \gamma$ we get again the limit case

$$e(P_1(t), P_2(t), V_i(t)) = P_1(t) \left[-V_i(t) - \beta \log \left[\frac{P_2(t)}{P_1(t)} \right] \right]^{-1}.$$

To ease notation this case differentiation is neglected in the proof.

Differentiating with respect to $P_2(t)$ gives (use again (2) to write the derivative in terms of $e_i(t)$)

$$\frac{e_i(t)}{P_2(t)} \left[1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \geq 0.$$

This is always non-negative. Thus, (b) is fulfilled if and only if (53) holds.

To show (c) we must prove that the Hessian is negative semidefinite. We get the following Hessian

$$H = \Xi \begin{pmatrix} -P_1(t)^{-2} & P_1(t)^{-1} P_2(t)^{-1} \\ P_1(t)^{-1} P_2(t)^{-1} & -P_2(t)^{-2} \end{pmatrix},$$

where

$$\Xi = \left[(1 - \alpha)\alpha - 2 \left[\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right]^2 - (2 - 3\alpha + \gamma) \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] e_i(t).$$

The eigenvalues are 0 and $-\Xi [P_1(t)^{-2} + P_2(t)^{-2}]$. For negative semi-definiteness these eigenvalues must be less or equal to zero (which requires $\Xi \geq 0$). Since $\alpha \in (0, 1]$ this is fulfilled if and only if

$$\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \leq \frac{1}{4} \left[3\alpha - 2 - \gamma + \sqrt{\alpha^2 + (2 + \gamma)^2 - 2\alpha(2 + 3\gamma)} \right]. \quad (54)$$

As long as $\gamma \geq 0$ we have

$$\alpha > \frac{\alpha}{2} \geq \frac{1}{4} \left[3\alpha - 2 - \gamma + \sqrt{\alpha^2 + (2 + \gamma)^2 - 2\alpha(2 + 3\gamma)} \right].$$

As a consequence (54) implies

$$\alpha > \frac{\alpha}{2} \geq \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}, \quad (55)$$

and is therefore sufficient for (53). Thus properties (b) and (c) are fulfilled if (54) holds. For $\beta = 0$ this condition is always fulfilled. ■

(ii) For the first derivative of the indirect utility function with respect to $e_i(t)$, we obtain

$$\frac{\partial V(P_1(t), P_2(t), e_i(t))}{\partial e_i(t)} = \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)^2} > 0.$$

Moreover, we have

$$\frac{\partial^2 V(P_1(t), P_2(t), e_i(t))}{\partial e_i(t)^2} = -2 \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)^3} < 0.$$

■

Proof of Lemma 2

(i) The derivation of the demand system is just an application of Roy's identity. ■

(ii) The corresponding current value Hamiltonian is

$$\mathcal{H} = V(P_1(t), P_2(t), e_i(t)) + \lambda(t) [r(t)W_i(t) + w(t)l_i - e_i(t)].$$

We can then derive the following first-order conditions

$$\dot{\lambda}(t) = \lambda(t) [\rho - r(t)] \quad \text{and} \quad \frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)^2} = \lambda(t). \quad (56)$$

The transversality condition can be written as $\lim_{t \rightarrow \infty} \lambda(t)W_i(t) \exp(-\rho t) = 0$. The first-order conditions, (56), simplify to the following optimality condition

$$2g_{e_i}(t) - \alpha g_{P_1}(t) - (1 - \alpha)g_{P_2}(t) = r(t) - \rho.$$

■

Proof of Lemma 3

(i) In view of (5), (6) and (52), the Hicksian demand functions can be written as

$$x_1^{i,H}(t) = [\alpha - \beta Z] Z \left[\frac{P_1(t)}{P_2(t)} \right]^\gamma \quad \text{and} \quad x_2^{i,H}(t) = [1 - \alpha + \beta Z] Z \left[\frac{P_1(t)}{P_2(t)} \right]^{1+\gamma},$$

where $Z \equiv \left[\frac{P_2(t)}{P_1(t)} \right]^{1-\alpha+\gamma} \left[-V_i(t) + \frac{\beta}{1-\alpha+\gamma} - \frac{\beta}{1-\alpha+\gamma} \left(\frac{P_2(t)}{P_1(t)} \right)^{1-\alpha+\gamma} \right]^{-1}$. With the Allen-Uzawa formula, the elasticity of substitution can be written as³¹

$$\frac{\partial x_1^{i,H}(t)}{\partial P_2(t)} \frac{P_1(t)x_1^{i,H}(t) + P_2(t)x_2^{i,H}(t)}{x_1^{i,H}(t)x_2^{i,H}(t)}.$$

When we apply this formula, we obtain for the elasticity of substitution

$$\frac{(1-\alpha)\alpha - 2 \left[\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right]^2 - (2-3\alpha+\gamma) \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}}{\left[\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \left[1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right]} \geq 0. \quad (57)$$

The denominator of this expression is larger than zero because of (54). The numerator is non-negative because of (55). (57) can be rewritten as

$$1 - \frac{\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \left[1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} + \gamma \right]}{\left[\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \left[1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right]} < 1, \quad (58)$$

which shows that the elasticity of substitution is less than one (since $\alpha \leq 1$ and $\gamma \geq 0$). ■

(ii) We get $\frac{\partial x_1^i(t)}{\partial e_i(t)} \frac{e_i(t)}{x_1^i(t)} = \frac{\alpha - 2\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}}{\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}}$. This is between zero and one since

$$\beta e_i(t) \leq \frac{\alpha}{2} P_1(t)^{1+\gamma} P_2(t)^{-\gamma} \text{ (see (4)). Moreover, we get } \frac{\partial x_2^i(t)}{\partial e_i(t)} \frac{e_i(t)}{x_2^i(t)} = \frac{1 - \alpha + 2\beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}}{1 - \alpha + \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}},$$

which is greater than one for $\beta > 0$. ■

Proof of Lemma 4

Aggregation of individual demands yields

$$X_1(t) = \alpha \int_0^n \frac{e_i(t)}{P_1(t)} di - \beta \int_0^n \frac{e_i(t)^2}{P_1(t)^{2+\gamma} P_2(t)^{-\gamma}} di = \alpha \frac{E(t)}{P_1(t)} - \beta \frac{1}{P_1(t)^{2+\gamma} P_2(t)^{-\gamma}} \int_0^n e_i(t)^2 di,$$

³¹Note that in the two good case, the Allen-Uzawa elasticity of substitution coincides with the Hicksian elasticity of substitution.

where $E(t) \equiv \int_0^n e_i(t) di$. We can then use $\int_0^n e_i(t)^2 di = n \left[\text{var}(e_i(t)) + \left[\frac{E(t)}{n} \right]^2 \right]$, which gives

$$X_1(t) = \alpha \frac{E(t)}{P_1(t)} - \beta \frac{n}{P_1(t)^{2+\gamma} P_2(t)^{-\gamma}} \left[\text{var}(e_i(t)) + \left[\frac{E(t)}{n} \right]^2 \right].$$

For the last step we define $\phi_e(t) \equiv \frac{n^2 \text{var}(e_i(t))}{E(t)^2}$ so that $\text{var}(e_i(t)) = \phi_e(t) \left[\frac{E(t)}{n} \right]^2$.

Then, we get for aggregate demand

$$X_1(t) = \left[\alpha - \beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}} \right] \frac{E(t)}{P_1(t)}.$$

Aggregate demand for good two can be derived in a fully analogous way. The aggregate expenditure share for good one is then given by $\alpha - \beta \frac{E(t) [1 + \phi_e(t)]}{n} \frac{1}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}$, whereas the individual expenditure share is $\alpha - \beta \frac{e_i(t)}{P_1(t)^{1+\gamma} P_2(t)^{-\gamma}}$. The two are equalized if $e_i(t) = \frac{E(t) [1 + \phi_e(t)]}{n}$. Thus, the representative agent is the one with $e^{RA}(t) = \frac{E(t) [1 + \phi_e(t)]}{n}$. ■

Proof of Lemma 5

Because production in the two sectors is fully symmetric, it is enough to derive optimality conditions for one sector $j = 1, 2$. Let $\eta_{j,\omega}(t)$ denote the price of intermediate input ω of sector j at date t . Because of perfect competition, final good producers take $\eta_{j,\omega}(t)$ as given and solve the following cost minimization problem,

$$\min_{\{y_{j,\omega}\}_{\omega=0}^{M_j(t)}} \int_0^{M_j(t)} \eta_{j,\omega}(t) y_{j,\omega}(t) d\omega,$$

subject to the constraint

$$Y_j(t) - \left[\int_0^{M_j(t)} y_{j,\omega}(t)^{\frac{1}{2}} d\omega \right]^2 = 0, \quad (59)$$

where $Y_j(t)$ is an exogenously given output level. The first-order conditions of this minimization are

$$\eta_{j,\omega}(t) = \mu(t)y_{j,\omega}(t)^{-\frac{1}{2}} \left[\int_0^{M_j(t)} y_{j,\omega'}(t)^{\frac{1}{2}} d\omega' \right], \quad \forall \omega, \quad (60)$$

where $\mu(t)$ is the multiplier attached to the constraint (59). Zero profits imply $\mu(t) = P_j(t)$. Then, the first-order condition can be rewritten as

$$y_{j,\omega}(t)^{\frac{1}{2}} = \left[\frac{P_j(t)}{\eta_{j,\omega}(t)} \right] \left[\int_0^{M_j(t)} y_{j,\omega'}(t)^{\frac{1}{2}} d\omega' \right], \quad \forall \omega, \quad (61)$$

or

$$y_{j,\omega}(t) = \left[\frac{P_j(t)}{\eta_{j,\omega}(t)} \right]^2 Y_j(t), \quad \forall \omega. \quad (62)$$

When we sum up over all intermediate inputs ω and solve for $P_j(t)$ we obtain

$$P_j(t) = \left[\int_0^{M_j(t)} \eta_{j,\omega}(t)^{-1} d\omega \right]^{-\frac{1}{2}}. \quad (63)$$

Each machine producer maximizes its instantaneous profits,

$$\max_{\eta_{j,\omega}(t), y_{j,\omega}(t)} \pi_{j,\omega}(t) = [\eta_{j,\omega}(t) - w(t)] y_{j,\omega}(t),$$

subject to demand (62). This gives the following first-order condition:

$$\eta_{j,\omega}(t) = 2w(t). \quad (64)$$

This holds for both sectors, every machine type ω and every date t . If we plug (64) into (63) we get

$$P_j(t) = 2w(t)M_j(t)^{-1}. \quad (65)$$

When we use (64) and (65) in (62) we get

$$l_{j,\omega}(t) = y_{j,\omega}(t) = M_j(t)^{-2}Y_j(t), \quad \forall \omega. \quad (66)$$

Because of symmetry we have $M_j(t)l_{j,\omega}(t) = L_j(t)$. Instantaneous profits are given by

$$\pi_{j,\omega}(t) = y_{j,\omega}(t)w(t) = \frac{1}{2}P_j(t)L_j(t), \quad \forall \omega.$$

■

Proof of Lemma 6

The instantaneous profits are given by (see (18))

$$\pi_{j,\omega}(t) = \frac{1}{2}L_j(t)P_j(t).$$

Cost of an innovation in sector j are (see (19))

$$\frac{f}{M_j(t)}w(t) = \frac{f}{2}P_j(t). \quad (67)$$

Suppose there is an interval $t' \in [t_0, t_0 + s]$ with positive R&D investments in sector j . Then, we must have

$$f = \frac{1}{P_j(t')} \int_{t'}^{\infty} L_j(\varsigma)P_j(\varsigma)e^{-\int_{t'}^{\varsigma}[r(\nu)+\delta_j]d\nu} d\varsigma, \quad \forall t' \in [t_0, t_0 + s]. \quad (68)$$

This is equivalent to

$$f = \frac{1}{P_j(t')} \int_{t'}^t L_j(\varsigma)P_j(\varsigma)e^{-\int_{t'}^{\varsigma}[r(\nu)+\delta_j]d\nu} d\varsigma + e^{-\int_{t'}^t[r(\nu)+\delta_j]d\nu} \frac{1}{P_j(t')} \int_t^{\infty} L_j(\varsigma)P_j(\varsigma)e^{-\int_t^{\varsigma}[r(\nu)+\delta_j]d\nu} d\varsigma. \quad (69)$$

with $t \in (t_0, t_0 + s]$. In view of (68) equation (69) can be rewritten as

$$f = \frac{1}{P_j(t')} \int_{t'}^t L_j(\varsigma)P_j(\varsigma)e^{-\int_{t'}^{\varsigma}[r(\nu)+\delta_j]d\nu} d\varsigma + e^{-\int_{t'}^t[r(\nu)+\delta_j]d\nu} \frac{P_j(t)}{P_j(t')} f. \quad (70)$$

Then, we differentiate both sides of (70) with respect to t . This gives

$$0 = L_j(t) \frac{P_j(t)}{P_j(t')} e^{-\int_{t'}^t[r(\nu)+\delta_j]d\nu} + e^{-\int_{t'}^t[r(\nu)+\delta_j]d\nu} \frac{f}{P_j(t')} \left[\dot{P}_j(t) - (r(t) + \delta_j)P_j(t) \right].$$

This can be simplified to

$$L_j(t) = f \left[r(t) + \delta_j - \frac{\dot{P}_j(t)}{P_j(t)} \right]. \quad (71)$$

Note that (20) and the choice of numéraire implies that $P_1(t) = \frac{M_2(t)}{M_1(t)}$. Then, for $j = 1$ we can rewrite (71) as

$$f = \frac{L_1(t)}{r(t) + \delta_1 - g_2(t) + g_1(t)},$$

and for $j = 2$ we get

$$f = \frac{L_2(t)}{r(t) + \delta_2}.$$

In general we must have

$$f \geq \frac{L_1(t)}{r(t) + \delta_1 - g_2(t) + g_1(t)} \text{ and } f \geq \frac{L_2(t)}{r(t) + \delta_2},$$

where the equations hold with equality, if there are positive investments. ■

Proof of Lemma 7

(21) implies

$$L_j^R(t) = f [g_j(t) + \delta_j], \quad j = 1, 2. \quad (72)$$

If we use that and the labor market clearing condition, (28), in equations (23) - (27), we obtain (30) - (34). By definition we must have $E(t) = X_1(t)P_1(t) + X_2(t)P_2(t)$.

With (17), (20), (22), (28) and (29) this can be rewritten as

$$E(t) = M_2(t) [L - L_1^R(t) - L_2^R(t)],$$

or as

$$\frac{E(t)}{M_2(t)} = L - f [g_1(t) + \delta_1] - f [g_2(t) + \delta_2]. \quad (73)$$

When we use (72), differentiate with respect to time and combine it with (12), (20) and (22), and set $\nu = 2$, we get (35). If we plug (73) into (14) and set it equal to (17), we finally obtain (36). ■

Proof of Proposition 1

Proof of existence:

It is easy to verify that all the equations of Lemma 7 hold if we plug in the values of the constant growth path. In addition, the transversality condition (38) has to be fulfilled too. Next note that along the CGP, wealth and consumption expenditures of all households grow at the same constant rate g_2^* . This implies that it is enough to show that the transversality condition holds in the aggregate. There are strictly positive investments in both sectors at every date along the CGP (note that (39) implies $g_2^* > 0$ and $g_1^* \geq 0$). Therefore, the value of a firm must be equalized to the cost of innovating at all dates. For the aggregate wealth stock we then get (see (20), (22) and (67))

$$W(t) = \int_0^n W_i(t) di = M_2(t)f.$$

In view of (73) we get in the aggregate

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \frac{P_1(t)}{E(t)^2} W(t) = \frac{f}{M_1(0) [L - f(\delta_1 + g_1^* + \delta_2 + g_2^*)]^2} \lim_{t \rightarrow \infty} \exp(-g_1^* t - \rho t).$$

Because we have $g_1^* + \rho > 0$ the limit is equal to zero and the transversality condition is fulfilled.

Proof of uniqueness:

First, I state and prove a Claim that turns out to be useful.

Claim 1. *If (39) and (11) holds for all households we must have $L_1^R(t) > 0, \forall t$.*

Proof of Claim 1: Suppose we have $L_1^R(t) = 0$ which is identical to $g_1(t) = -\delta_1$.

First suppose we have $g_1(t) = -\delta_1$ and $g_2(t) = -\delta_2$. Then we must have

$$f \geq \frac{L - L_2(t)}{\rho - \delta_1} \text{ and } f \geq \frac{L_2(t)}{\rho - \delta_1}. \quad (74)$$

This implies $L - 2f\rho + 2f\delta_1 \leq 0$, which constitutes a contradiction to (39).

Next suppose we have $g_1(t) = -\delta_1$ and $g_2(t) > -\delta_2$. Then, we must have (see (30) - (34))

$$f \geq \frac{L - L_2(t) - f[g_2(t) + \delta_2]}{r(t) - g_2(t)} \text{ and } f = \frac{L_2(t)}{r(t) + \delta_2},$$

which implies

$$L_2(t) \geq \frac{1}{2}L. \quad (75)$$

Note that we have

$$L_2(t) = \beta \frac{[1 + \phi_e]}{n} m(t) [L - f[g_2(t) + \delta_2]]^2. \quad (76)$$

Next note that there exists a household i for all t with $e_i(t) \geq \frac{[1 + \phi_e]}{n} E(t) = e^{RA}(t)$.

If we use $\alpha = 1$ and this expenditure level of the representative household and $E(t) = M_2(t) [L - f[g_2(t) + \delta_2]]$ in (11), we get

$$2\beta \frac{[1 + \phi_e]}{n} m(t) [L - f[g_2(t) + \delta_2]] \leq 1 - \gamma. \quad (77)$$

If we substitute this into (76), we get

$$L_2(t) \leq \frac{1 - \gamma}{2} [L - f[g_2(t) + \delta_2]].$$

Because we have $\gamma \geq 0$ and $g_2(t) > -\delta_2$ by assumption, this constitutes a contradiction to (75) (since $\delta_2 < 0$). Hence, $L_1^R(t) = 0$ cannot occur. ■

When we differentiate the labor market clearing condition with respect to time we get

$$L_1(t)g_{L_1}(t) + L_2(t)g_{L_2}(t) + L_1^R(t)g_{L_1^R}(t) + L_2^R(t)g_{L_2^R}(t) = 0, \quad (78)$$

where $g_{L_j}(t)$ and $g_{L_j^R}(t)$ are the growth rates of the respective variables L_j and L_j^R , $j = 1, 2$. In a CGP these growth rates must be constant. But because we must have $L_j(t) \leq L$ and $L_j^R(t) \leq L$ for $j = 1, 2$ and all t we see that these growth rates cannot be constantly positive. Then, (78) implies that these growth rates must all be zero along a CGP. Since along a CGP $g_1(t)$ and $g_2(t)$ are constant too, $g_{L_2}(t) = 0$ implies $g_m(t) = 0$ (see (36)). Then, we must have along a CGP $g_1 = \frac{\gamma}{1+\gamma}g_2$ (see (37)). Hence, in order to prove uniqueness we have to prove that there is only one constellation of constant growth rates that is consistent with the equations in Lemma 7 and $g_1 = \frac{\gamma}{1+\gamma}g_2$. From Claim 1 we know that we must have $g_1 > -\delta_1$. Thus, we can distinguish two different regimes: regime \mathcal{A} with $g_2 > -\delta_2$ and regime \mathcal{B} with $g_2 = -\delta_2$.

Suppose we have regime \mathcal{B} with $g_2 = -\delta_2$. This implies

$$f = \frac{L - L_2 - f(g_1 + \delta_1)}{2g_1 + \delta_1 + \rho}, \text{ and } f \geq \frac{L_2}{g_1 + \rho}. \quad (79)$$

These two equations imply $4fg_1 + 2\rho f + 2f\delta_1 \geq L$. With $g_1 = \frac{\gamma}{1+\gamma}g_2 = -\frac{\gamma}{1+\gamma}\delta_2$ this can be rewritten as $2\rho f + 2f\delta_1 \geq L + 4f\frac{\gamma}{1+\gamma}\delta_2$, which constitutes a contradiction to (39). Hence we must have regime \mathcal{A} with $L_1^R > 0$ and $L_2^R > 0$ along a CGP. Hence, equations (30) and (31) hold with equality. When we then set $\dot{g}_j(t) = 0$, $j = 1, 2$ and $g_1 = \frac{\gamma}{1+\gamma}g_2$ and solve the system of equations of Lemma 7 for g_2 we get the unique solution

$$g_2^* = \frac{(1 + \gamma) [L - 2f(\delta_1 + \delta_2 + \rho)]}{2f(1 + 3\gamma)}. \quad (80)$$

This proves the uniqueness of the CGP. ■

Proof of Proposition 2

Again, we can discriminate between the following two regimes: regime \mathcal{A} with $g_2(t) > -\delta_2$ and regime \mathcal{B} with $g_2(t) = -\delta_2$. We know that we must have $g_1(t) > -\delta_1$ due to Claim (1) in the proof of Proposition 1. First the dynamics of the two regimes is analyzed.

Equilibrium dynamic of regime \mathcal{A} (proof of part (iii))

If we have strictly positive investments in both sectors, we know that (30) and (31) have to hold with equality. When we solve (30) and (31) for $L_2(t)$ and set it equal to (36), we get

$$L_2(t) = \tilde{\beta}m(t) [L - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]]^2 = \frac{L}{2} - f[g_1(t) + \delta_1], \quad (81)$$

where $\tilde{\beta} \equiv \frac{(1+\phi_e)\beta}{n}$. Moreover, (35) gives

$$2f \frac{\dot{g}_1(t) + \dot{g}_2(t)}{L - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]} = g_2(t) + 2g_1(t) - \frac{L}{2f} + \rho + \delta_1 + \delta_2. \quad (82)$$

Solving (81) for $g_2(t)$ we obtain

$$g_2(t) = -g_1(t) + \frac{L}{f} - \delta_1 - \delta_2 - \sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}}, \quad (83)$$

where I have chosen the relevant solution.³² Then, differentiating (81) with respect to time and using (81) again yields

$$2f \frac{\dot{g}_2(t) + \dot{g}_1(t)}{L - f[g_1(t) + \delta_1] - f[g_2(t) + \delta_2]} = \frac{f\dot{g}_1(t)}{\frac{L}{2} - f[g_1(t) + \delta_1]} + (1 + \gamma)g_1(t) - \gamma g_2(t). \quad (84)$$

When we solve (82), (83) and (84) for $\dot{g}_1(t)$, we get

$$\dot{g}_1(t) = \left[\rho + (1 + 2\gamma)\frac{L}{2f} - (1 + \gamma)\sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}} - 2\gamma g_1(t) - \gamma(\delta_1 + \delta_2) \right] \left[\frac{L}{2f} - g_1(t) - \delta_1 \right], \quad (85)$$

Finally, when we use (83) in (37) we get (51).

Necessary conditions for regime \mathcal{B}

When we use $g_2(t) = -\delta_2$ and $g_1(t) > -\delta_1$ in (30) - (34) we get

$$f = \frac{L - L_2(t) - f[g_1(t) + \delta_1]}{r(t) + \delta_1 + \delta_2 + g_1(t)} \text{ and } f \geq \frac{L_2(t)}{r(t) + \delta_2}.$$

These two equations imply

$$L_2(t) \leq \frac{1}{2} [L - 2f(g_1(t) + \delta_1)], \quad (86)$$

which is identical to

$$L_1(t) \geq \frac{1}{2}L. \quad (87)$$

We are now ready to proof global saddle path stability. This is done in two steps.

The first step proves saddle path stability of the CGP within regime \mathcal{A} . The second

³²The other solution would imply a negative aggregate consumption level

$$E(t) = M_2(t) [L - f[\delta_1 + \delta_2 + g_1(t) + g_2(t)]].$$

step proves that regime \mathcal{A} has to prevail within finite time (part (ii) of Proposition 2).

Step 1: Saddle path stability of the CGP under regime \mathcal{A}

Under regime \mathcal{A} the dynamic is characterized by (85) and

$$\dot{m}(t) = \left[(1 + 2\gamma)g_1(t) - \gamma \left[\frac{L}{f} - \delta_1 - \delta_2 - \sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}} \right] \right] m(t). \quad (88)$$

Clearly we must have $L - 2f(g_1(t) + \delta_1) > 0$ (which ensures $E(t) > 0$). When we differentiate (85) with respect to $m(t)$ we get

$$\frac{\partial \dot{g}_1(t)}{\partial m(t)} = f\tilde{\beta}(1 + \gamma) \left[\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f} \right]^{\frac{3}{2}} > 0. \quad (89)$$

Hence, we have $\dot{g}_1(t) > 0$ if $m(t)$ is “large”. When we differentiate (88) with respect to $m(t)$ and evaluate it along the $\dot{m}(t) = 0$ isocline, we get

$$\left. \frac{\partial \dot{m}(t)}{\partial m(t)} \right|_{\dot{m}(t)=0} = -\frac{\gamma}{2} \sqrt{\frac{\frac{L}{2f} - g_1(t) - \delta_1}{\tilde{\beta}m(t)f}} \leq 0. \quad (90)$$

Thus, with $\gamma > 0$, we have $\dot{m}(t) > 0$ if $m(t)$ is “small”.³³

For saddle path stability we need that the slope of the $\dot{m}(t) = 0$ locus is larger than the slope of the $\dot{g}_1(t) = 0$ locus in the CGP. For the difference between the slope of the $\dot{m}(t) = 0$ locus and the slope of the $\dot{g}_1(t) = 0$ locus we get

$$\frac{8f(1 + 3\gamma)^3 [(1 + 2\gamma) [L - 2f\delta_1] + 2f\gamma(\delta_2 + \rho)]}{\tilde{\beta}\gamma(1 + \gamma) [L(1 + 4\gamma) - 2f\gamma(\delta_1 + \delta_2) + (1 - 2\gamma)\rho]^3} > 0, \quad (91)$$

which is positive because of (39). This proves that the CGP is globally saddle path stable if regime \mathcal{A} prevails forever. Next I show that eventually we must be in regime

³³With $\gamma = 0$ the $\dot{m} = 0$ isocline is given by $g_1(t) = 0$ and $m(t)$ is increasing if $g_1(t) > 0 = g_1^*$.

\mathcal{A} (and stay there).

Step 2: Proof of predominance of regime \mathcal{B} within finite time

In general we can discriminate between three cases: A first case where regime \mathcal{A} holds forever, a second case where regime \mathcal{B} holds forever and a the third case where we have a recurring interchange of regime \mathcal{A} and \mathcal{B} . I proof that cases two and three cannot occur.

Exclusion of case two (predominance of regime \mathcal{B} forever)

First, note that the CGP is unique and only consistent with regime \mathcal{A} . As a consequence, $g_1(t)$ cannot be constant or converge to a constant level under regime \mathcal{B} . Moreover, a steadily increasing or decreasing (and non-converging) $g_1(t)$ is impossible since $g_1(t)$ has an upper and lower bound. Hence, we must have cycle in $g_1(t)$ (phases with $\dot{g}_1(t) > 0$ as well as phases with $\dot{g}_1(t) < 0$). First note that we cannot have $g_1(t) > -\delta_2 \frac{\gamma}{1+\gamma}, \forall t$ because this implies a steadily growing $m(t)$ (and therefore eventually a violation of condition (87)). Hence, because we must have cycles in $g_1(t)$ we must have phases with $g_1(t) \leq -\delta_2 \frac{\gamma}{1+\gamma}$ and $\dot{g}_1(t) \geq 0$, which clearly implies that we have $r(t) < r^*$ and $r(t) - g_2(t) + g_1(t) \leq 2g_1(t) + \rho < r^* - g_2^* + g_1^* = 2g_1^* + \rho$. But $g_1(t) \leq -\delta_2 \frac{\gamma}{1+\gamma}$ and $g_2(t) = -\delta_2$ implies $L_1(t) + L_2(t) > L_1^* + L_2^*$. Hence there must be either $L_1(t) > L_1^*$ or $L_2(t) > L_2^*$. But since $r(t) - g_2(t) + g_1(t) < r^* - g_2^* + g_1^*$ and $r(t) < r^*$, at least one of the free-entry conditions have to be violated. This excludes the possibility of a infinite predominance of regime \mathcal{B} .

Exclusion of case three (repeated interchanges between regime \mathcal{A} and \mathcal{B})

First I show that if we have a regime switch, we must have $\dot{g}_1(t) < 0$ and $\dot{m}(t) > 0$.

At a regime switch we cannot have discontinuities in $g_j(t)$, $j = 1, 2$.³⁴ Hence right before/after the regime switch we must have $g_2(t) = -\delta_2$, $\dot{g}_2(t) = 0$ and regime \mathcal{A} . This implies that we must have (see (81))

$$\tilde{\beta}m(t) [L - f [g_1(t) + \delta_1]]^2 = \frac{L}{2} - f [g_1(t) + \delta_1], \quad (92)$$

Because of (11), this equation defines a negative relation between $m(t)$ and $g_1(t)$.³⁵ Then we can discriminate three cases: First, suppose we have $m(t) > m^*$. This implies $g_1(t) < g_1^*$ (see (92)). Moreover, this leads to $L_2(t) > L_2^*$ (note that $L_2(t)$ is increasing in both $E(t)$ and $m(t)$). For this to be compatible with regime \mathcal{A} we need $r(t) > r^*$. This is only possible with $\dot{g}_1(t) < 0$ (since $\dot{g}_2(t) = 0$, $g_1(t) < g_1^*$ and $g_2(t) = -\delta_2 < g_2^*$). Now we can rewrite (84) as

$$-f^2 \frac{\dot{g}_1(t) [g_1(t) + \delta_1]}{[L - f [g_1(t) + \delta_1]] [\frac{L}{2} - f [g_1(t) + \delta_1]]} = \frac{\dot{m}(t)}{m(t)}. \quad (93)$$

Then we see, that $\dot{g}_1(t) < 0$ and $\dot{g}_2(t) = 0$ implies $\dot{m}(t) > 0$.

Second, suppose we have a regime switch with $m(t) < m^*$ and $g_1(t) > g_1^*$. $g_1(t) > g_1^*$ and $g_2(t) = -\delta_2$ implies $\dot{m}(t) > 0$ (see (37)). In view of (84) we get again $\dot{g}_1(t) < 0$. Finally, suppose we have $m(t) < m^*$ and $g_1(t) < g_1^*$. This implies $L_1(t) > L_1^*$ (because of (11) $L_2(t)$ is increasing in $E(t)$ and decreasing in $m(t)$). Then, we must

³⁴The Euler equation implies continuity in $E(t)$. Thus the sum $g_1(t) + g_2(t)$ must be continuous. As a consequence $L_j(t)$, $j = 1, 2$ are continuous too. We then see that strictly binding free entry conditions imply continuity in $g_1(t) - g_2(t)$.

³⁵To see this note that

$$\tilde{\beta}m(t) [L - f [g_1(t) + \delta_1]]^2 - \frac{L}{2} + f [g_1(t) + \delta_1],$$

is increasing in $g_1(t)$ because of (11).

have $r(t) + \delta_1 + \delta_2 + g_1(t) > r^* + \delta_1 - g_2^* + g_1^*$ which gives (see (30) and (35))

$$2g_1(t) - f \frac{\dot{g}_1(t)}{L - f[g_1(t) + \delta_1]} > 2g_1^*. \quad (94)$$

Clearly, this implies $\dot{g}_1(t) < 0$ (and again $\dot{m}(t) > 0$).

Thus we have seen that if we have a regime switch we must have $\dot{g}_1(t) < 0$ and $\dot{m}(t) > 0$. Next I proof that a regime switch from \mathcal{B} to \mathcal{A} cannot be followed by a switch back to \mathcal{B} . Suppose we have a regime switch from \mathcal{B} to \mathcal{A} at t' and a switch back to \mathcal{B} at $t'' > t'$. From the analysis above we know that we must have $\dot{g}_1(\varsigma) < 0$ and $\dot{m}(\varsigma) > 0$ at $\varsigma = t', t''$. Then a closer look on the phase diagram shows that this implies $\dot{g}_1(\varsigma) < 0$ and $\dot{m}(\varsigma) > 0$ for all $\varsigma \in [t', t'']$. As a consequence, we must have $m(t'') > m(t')$ and $g_1(t'') < g_1(t')$ which implies $L_2(t'') > L_2(t')$. Then, we must have $r(t'') > r(t')$ which gives

$$g_1(t'') - f \frac{\dot{g}_1(t'')}{L - f[g_1(t'') + \delta_1]} > g_1(t') - f \frac{\dot{g}_1(t')}{L - f[g_1(t') + \delta_1]}. \quad (95)$$

But this cannot be the case since $g_1(t'') < g_1(t')$ and $m(t'') > m(t')$ implies $0 > \dot{g}_1(t'') > \dot{g}_1(t')$ (see (84)). Hence, regime \mathcal{B} cannot prevail forever and we cannot have repeated interchanges between regime \mathcal{B} and \mathcal{A} . Therefore we conclude, that within finite time we must have regime \mathcal{A} (and stay there). This proves part (i). ■

Proof of Proposition 3

The phase diagram already shows that $g_1(t)$ must be decreasing and $m(t)$ must be increasing if we start with $m(0) < m^*$. Then, a decreasing $g_1(t)$ implies an increasing $L_2(t)$ (see (81)). An increasing $L_2(t)$ implies an increasing $r(t)$ and therefore we must have $r(t) \leq r^*$ (see (31) and note that it has to hold with equality). Note that both

$g_j(t)$ have to converge (i.e. $\lim_{t \rightarrow \infty} \dot{g}_j(t) = 0$). Then we cannot have $\dot{g}_1(t) < 0$ and $\dot{g}_2(t) < 0$ at the same time since this cannot be consistent with a decreasing $r(t)$ and converging $g_j(t)$ (see (35)). Hence we must have $\dot{g}_2(t) > 0$. Finally, we have $L_1(t) = f[r(t) + \delta_1 - g_2(t) + g_1(t)]$ (see (30) which holds with equality). Because $r(t)$ is decreasing, $g_2(t)$ is increasing and $g_1(t)$ is decreasing this is clearly decreasing over time. ■

Proof of Proposition 4

Note that we have along the transition (see (83))

$$E(t) = M_2(t) \sqrt{\frac{\frac{L}{2} - f g_1(t) - f \delta_1}{\tilde{\beta} m(t)}}.$$

Then, with (20), (22) and (47) we can rewrite (11) as

$$2\kappa \sqrt{\left[\frac{L}{2} - f g_1(t) - f \delta_1 \right] \beta m(t) \frac{n}{1 + \phi_e}} \leq 1 - \gamma.$$

We see that the left-hand side is strictly increasing in $m(t)$ and strictly decreasing in $g_1(t)$. When the economy approaches the CGP from below, we know that we have $m(t) \leq m^*$ and $g_1(t) \geq g_1^* = 0 \forall t$. Hence, (11) is fulfilled along the transition if it is fulfilled along the CGP (i.e. as long as (48) holds). ■

Appendix B: Dynamic with $\beta = 0$

With Cobb-Douglas preferences we have

$$L_1(t) = \alpha [L - f [g_1(t) + \delta_1] - f [g_2(t) + \delta_2]]$$

and

$$L_2(t) = (1 - \alpha) [L - f [g_1(t) + \delta_1] - f [g_2(t) + \delta_2]].$$

Transitional dynamics and shifts in expenditure and employment shares never occur (comparable with an AK model). We can distinguish four cases: (A); there are no investments in both sectors, (B); there are positive investments in both sectors, (C); there are just positive investments in sector one, and (D); there are just positive investments in sector two.

Case (A):

This case applies if we have

$$f \geq \frac{\alpha L}{\rho - \alpha\delta_1 - (1 - \alpha)\delta_2} \text{ and } f \geq \frac{(1 - \alpha)L}{\rho - \alpha\delta_1 - (1 - \alpha)\delta_2}.$$

Then, there are no investments and we have $r = \rho - \alpha\delta_1 - (2 - \alpha)\delta_2$ and $E(t) = M_2(t)L$ and both sectors shrink at rate δ_j . The transversality condition holds as long as $\rho > (1 - \alpha)\delta_2 + \alpha\delta_1$.

Case (B):

We must have

$$f = \frac{\alpha [L - f [g_1 + \delta_1] - f [g_2 + \delta_2]]}{(1 - \alpha)g_2 + (1 + \alpha)g_1 + \rho + \delta_1} \text{ and } f = \frac{(1 - \alpha) [L - f [g_1 + \delta_1] - f [g_2 + \delta_2]]}{(2 - \alpha)g_2 + \alpha g_1 + \rho + \delta_2}.$$

If we solve these two equations we get

$$g_1 = \frac{L[(2-\alpha)\alpha - 1] - 2f[\delta_1 - (1-\alpha)[- \delta_2 + \alpha(\delta_1 + \delta_2) + \rho]]}{2f[1 + 2(1-\alpha)\alpha]}$$

and

$$g_2 = \frac{L(1 - 2\alpha^2) - 2f[\delta_2 + \alpha[\delta_2 - \alpha(\delta_1 + \delta_2) + \rho]]}{2f[1 + 2(1-\alpha)\alpha]}.$$

This case can occur if we have

$$L[(2-\alpha)\alpha - 1] - 2f(1-\alpha)[\alpha(\delta_1 - \delta_2) + \delta_2 - \rho] > 0,$$

and

$$L(1 - 2\alpha^2) + 2f\alpha[\alpha(\delta_1 - \delta_2) + \delta_2 - \rho] > 0.$$

Utility is finite as long as $\rho + (1-\alpha)g_2 + \alpha g_1 > 0$.

Case (C):

We then have

$$f = \frac{\alpha[L - f[g_1 + \delta_1]]}{-(1-\alpha)\delta_2 + (1+\alpha)g_1 + \rho + \delta_1} \text{ and } f \geq \frac{(1-\alpha)[L - f[g_1 + \delta_1]]}{-(1-\alpha)\delta_2 + \alpha g_1 + \rho}.$$

If we solve for g_1 we get

$$g_1 = \frac{\alpha L - f((1+\alpha)\delta_1 - (1-\alpha)\delta_2 + \rho)}{f(1+2\alpha)}.$$

This case occurs if we have

$$\alpha L + f[\alpha(\delta_1 - \delta_2) + \delta_2 - \rho] > 0,$$

and

$$L(1 - 2\alpha^2) + 2f\alpha[\alpha(\delta_1 - \delta_2) + \delta_2 - \rho] \leq 0.$$

With $\alpha = 1$, we get the standard one sector spillover model.³⁶ Utility is finite as long as $\rho > (1 - \alpha)\delta_2 - \alpha g_1$.

Case (D) is fully symmetric to case (C).

³⁶Preferences are then given by

$$\mathcal{U}_i(0) = \int_0^\infty e^{-\rho t} \left[-\frac{P_1(t)}{e_i(t)} \right] dt.$$

Note that this is identical to the standard one sector model with an inter-temporal elasticity of substitution equal to 0.5. If we normalize P_1 to one, we can rewrite preferences as

$$\mathcal{U}_i(0) = \int_0^\infty e^{-\rho t} \frac{1}{1-\sigma} [e_i(t)]^{1-\sigma} dt,$$

where $\sigma = 2$.

Appendix C: Possible Generalizations of the Model

The aim of this Appendix is to shortly sketch along which dimension the model can be generalized.

4.1 $\alpha \neq 1$ and $f_1 \neq f_2$

The main features of the CGP remain unchanged when we have $\alpha \neq 1$ or we have ex ante sectoral differences in the productivity of R& D investments ($f_1 \neq f_2$). Then, the CGP is associated with

$$g_1^* = \gamma \frac{L - 2f_1\delta_1 - 2f_2\delta_2 - (f_1 + f_2)\rho}{f_1(1 - \alpha + 3\gamma) + f_2(3(1 + \gamma) - \alpha)}. \quad (96)$$

All other CGP values can then be obtained.³⁷ I do not provide an analytical proof of global stability of the CGP for this case. But simulations show saddle path stability for reasonable parameter specifications.

Introducing a parameter that controls the inter-temporal elasticity of substitution

With $\gamma \in (0, 1)$ the instantaneous utility function can be rewritten without the last term as

$$V(P_1(t), P_2(t), e_i(t)) = -\frac{P_1(t)^\alpha P_2(t)^{1-\alpha}}{e_i(t)} - \frac{\beta}{1 - \alpha + \gamma} \left(\frac{P_2(t)}{P_1(t)} \right)^{1-\alpha+\gamma}. \quad (97)$$

Note that this change lets the optimal expenditure system and the derivative of instantaneous utility with respect to time unaffected. Then, the instantaneous util-

³⁷For instance we have $g_2^* = \frac{1+\gamma}{\gamma}g_1^*$ and $r^* = \alpha g_1^* + (2 - \alpha)g_2^* + \rho$.

ity level grows along the CGP at constant rate $- \left[1 - \alpha + \alpha \frac{\gamma}{1+\gamma} \right] g_2^*$.³⁸ Under this modified instantaneous utility function the following CRRA inter-temporal utility function is consistent with a CGP

$$\mathcal{U}_i(0) = \int_0^\infty \exp(-\rho t) \frac{1}{1-\sigma} V(P_1(t), P_2(t), e_i(t))^{1-\sigma} dt, \quad (98)$$

where $\sigma \in (0, \infty)$. Then we have an additional parameter σ in the model that controls the inter-temporal elasticity of substitution.

Adding a third sector that is neither a luxury nor a necessity

Changing the instantaneous utility function to

$$V(\cdot) = - \frac{P_1(t)^{\nu\alpha} P_2(t)^{\nu(1-\alpha)} P_3(t)^{(1-\nu)}}{e_i(t)} - \frac{\beta}{\nu(1-\alpha) + \gamma} \left(\frac{P_2(t)}{P_1(t)} \right)^{\nu(1-\alpha) + \gamma} + \frac{\beta}{\nu(1-\alpha) + \gamma}. \quad (99)$$

Then, a constant fraction $1 - \nu$ of expenditure is devoted to the third neutral sector.

The rest of the model remains almost unaffected and we still get a CGP along with

$$\tilde{m} = \frac{M_1(t)^{\nu+\gamma} M_3(t)^{1-\nu}}{M_2(t)^\gamma} \text{ is constant.}$$

Introducing population growth

Suppose the labor force L grows at constant rate n . This requires an adjustment of the innovation possibilities frontiers to

$$\dot{M}_j(t) = L_j^R(t) \frac{M_j(t)^\epsilon}{f} - \delta_j M_j(t), \quad j = 1, 2, \quad (100)$$

³⁸To see this note that we have $g_1^* = \frac{\gamma}{1+\gamma} g_2^*$. Since the attained utility value is negative, a negative growth rate implies that the attained instantaneous utility level is growing over time.

where ϵ is less than one. When we assume that population growth happens within households, we have to adjust the discount rate to $\rho - n$. Then, there still exists a CGP which exhibits the same key features.