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# A SIMPLE MODEL OF INVESTMENT WITH AN INFINITE NUMBER OF TECHNOLOGIES

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ABSTRACT. This paper investigates technology adoption of a single firm in a continuous time model with an infinite planning horizon and an infinite number of investment opportunities. Technological progress is exogenous and modeled either by a poisson process or a geometric poisson process. For both processes, we characterize the optimal investment strategies. In the case of a poisson process we show that a cyclical investment pattern, that is adopting every  $m$ -th technology is optimal. If technological progress is modeled as a geometric poisson process, we argue that the number of technologies not adopted between two adoptions decreases with time, until finally each new technology will be adopted.

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*Keywords:* Technology adoption; Technological uncertainty; Investment irreversibility; Optimal timing

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## 1. INTRODUCTION

In the course of time firms are confronted with opportunities to invest in advanced technologies which for example lower the cost function by improving efficiency or simply result in higher profits. Postponing investment and waiting for improved future technologies may be beneficial compared to immediate adoption, even if the technology available for adoption outperforms the current technology. The following citation from Sayers [Say50], page 230, illustrates this for the case of ship's engines in Britain around 1923:

“Put in economic terms, the shipowners’ position was that, though total costs of new engines might already be less than running costs of old engines, the profit on engines of 1923 build might be wiped out by the appearance in 1924 of even lower-cost engines, the purchase of which would allow a competitor (who had postponed the decision) to cut freights further. Also there was uncertainty as to which of two types of 1923 engine would prove to work at lower cost.”

More generally, the decision whether to adopt a new technology is subject to different factors as strategic interaction, uncertainty about the arrival time and the value of future technologies or the firm's individual innovation history.

The majority of the literature focusses on the adoption of a limited number of technologies (for a detailed survey of the literature on technology adoption see Hoppe [Hop02]). Models typically have the following structure: At time 0, a firm is equipped with one technology, and may adopt an advanced technology (sometimes called current innovation) or wait for the arrival of a third, even more advanced technology (called future innovation), which the firm may once again adopt. Dependent on the parameter values, four adoption patterns may arise: Adoption of no technology (called *bystander* strategy), adoption of both technologies (called *compulsive* strategy), adoption of the second technology (called

*leapfrog* strategy), and adoption of the current innovation, but no adoption of the future innovation (called *buy-and-hold* strategy).

Grenadier and Weiss [GW97] - from whom we borrowed the naming of the strategy types - allow for a fifth strategy (called *laggard* strategy) by letting the firm adopt the so-called current innovation at a discounted price even after the future innovation arrived. They furthermore incorporate learning: Adoption of a future technology is less costly, if previous technologies have been adopted. Huisman and Kort [HK03] extend the (stochastic) model by Grenadier and Weiss (dropping the uncertainty) to study strategic interaction in a deterministic duopoly model. Strategic interactions in technology adoption are studied by Scherer [Sch67], Reinganum [Rei81a, Rei81b] and Fudenberg and Tirole [FT85] in deterministic oligopoly models. Stenbacka and Tombak [ST94] and Huisman and Kort [HK98] extend these studies by incorporating a stochastic lag between adoption and implementation of the future technology.

In the above mentioned models firms have to choose which of three technologies, i.e. a current technology, a current innovation and a future innovation to use. Farzin et al. [FHK98] and Doraszelski [Dor01] consider an arbitrary but finite number of future innovations that evolve over time. They furthermore allow for uncertainty about both the arrival time and the quality of future innovations. By interpreting the technologies as abatement technologies, such models are applied in environmental economics, for an example see von Döllen and Requate [vDR08]. Doraszelski [Dor04] extends the models in [FHK98] and [Dor01] by distinguishing between innovations and improvements.

In models with an arbitrary finite number of innovation opportunities, some of the above mentioned strategy types could be transferred: the *bystander* adopts no technology, the *compulsive* type adopts every technology, the *buy-and-hold* type adopts the first and skips the following technologies. For the *leapfrog* strategy, we would have to specify the number of technologies that are leapfrogged. We could even think of strategies more complex like

leapfrogging three technologies, adopting two technologies and afterwards leapfrogging four technologies.

The aim of this paper is to analyze the optimal number of technologies to leapfrog (in the following also called *jump width*) in a continuous-time model with an infinite number of advanced technologies. Optimal adoption of an infinite number of technologies has been analyzed by Balcer and Lippman [BL84], whose paper in this respect is closest to ours.<sup>1</sup> Both papers abstract from strategic interaction and in both the arrival time of future technologies is uncertain, while the value of new technologies is known with certainty. Balcer and Lippman incorporate expectations about the likelihood of innovations and revision of these expectations as time passes. Therefore situations may arise, in which a firm after some time may adopt a technology, which it initially chose not to adopt. In our model however, the firm has full knowledge about the distribution of future technologies, implying that - under an optimal investment strategy - a technology is adopted at the time of its arrival, or never adopted at all. While the model of Balcer and Lippman makes very general predictions about firms behaviour, we focus on a single aspect: Identification of patterns in optimal investment strategies (which correspond to the strategy types mentioned above) for specific types of technological progress. We therefore consider the technological progress to be governed either by a poisson process or by a geometric poisson process. For both cases, we will show neither a *bystander* strategy nor a *buy-and-hold* strategy can be optimal. A *compulsive* strategy is optimal for specific parameter values. For the case of a poisson process, we will show that a *constant jump* strategy is optimal, which could be interpreted as a repeated *leapfrog* strategy. In case of a geometric poisson process, we will show that the optimal jump width is decreasing. This could be interpreted as a repeated *leapfrog* strategy with a decreasing number of technologies that are leapfrogged.

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<sup>1</sup>From a technical point of view, both models differ in the way time is modeled: In Balcer and Lippmans model time is discrete, while in our model time is continuous.

The remainder of this paper is organized as follows. In Section 2, we present the model and state the firms maximization problem in a general fashion. In Section 3 we assume that technological progress follows a poisson process and give closed expressions both for the optimal jump width and the value function. Furthermore, for both the effects of changes in parameter values are analyzed. The case where the technology follows a geometric poisson process is studied in Section 4. The finding that the optimal jump width decreases is illustrated by an example with specific parameter values. Section 5 concludes.

## 2. MODEL

Consider a firm owning a technology that yields a profit  $\Pi > 0$  per unit of time. The discounted future profit stream of this technology (which we will denote  $\Pi$  in the following) is

$$\int_0^\infty e^{-rt} \Pi dt = \frac{\Pi}{r}, \tag{1}$$

where  $r > 0$  is the constant discount rate.

New technologies  $\Pi_i^*, i \in \mathbb{N}_0$  become available in the course of time, i.e.  $\Pi_i^*$  emerges at time  $t_i > 0$  with  $t_i < t_{i+1}$  for all  $i \in \mathbb{N}_0$ .<sup>2</sup> To adopt technology  $\Pi_i^*$ , the firm has to pay the investment cost  $I^*(t_i)$ . At each point in time only the latest technology is available for purchase, but once adopted, each technology has an infinite lifespan. The firm can employ one technology at a time, therefore by adopting a new technology, the firm forfeits its current technology.<sup>3</sup>

For each technology  $\Pi_i^*$ , the binary decision to skip or adopt the respective technology can be represented by  $x_i = 0$ , if the technology is skipped, and by  $x_i = 1$ , if the technology is adopted. Each sequence of decisions corresponds to a vector  $x \in \{0, 1\}^{\mathbb{N}_0}$  and recursively defines the technology actually employed by the firm in the period  $[t_i, t_{i+1}]$  (denoted by

<sup>2</sup>We assume  $t_0 = 0$  and  $t_i \rightarrow \infty$  for  $i \rightarrow \infty$ .

<sup>3</sup>By omitting the adoption of all future technologies, the firm could choose to remain with the same technology until eternity.

$\Pi_i, i \in \mathbb{N}_0$ ) through

$$\Pi_i(x) := \Pi_{i-1}(x) + x_i(\Pi_i^* - \Pi_{i-1}(x)) = \begin{cases} \Pi_{i-1}(x), & \text{if } x_i = 0 \\ \Pi_i^*, & \text{if } x_i = 1 \end{cases} \quad (2)$$

for all  $i \in \mathbb{N}$  and  $\Pi_0(x) = \Pi + x_0(\Pi_0^* - \Pi)$ . Similarly, the investment cost at time  $t_i$  can be written as

$$I_i(x) = I(t_i) := x_i I^*(t_i). \quad (3)$$

With the employed technology  $\Pi_i, i \in \mathbb{N}_0$  defined by equation 2 and the investment cost  $I_i, i \in \mathbb{N}_0$  defined by equation 3, the payoff associated with the decision vector  $x$  is

$$\sum_{i=0}^{\infty} \left( \int_{t_i}^{t_{i+1}} e^{-rt} \Pi_i(x) dt - e^{-rt_i} I_i(x) \right). \quad (4)$$

The firms optimization problem is

$$\max_{x \in \{0,1\}^{\mathbb{N}_0}} \sum_{i=0}^{\infty} \left( \int_{t_i}^{t_{i+1}} e^{-rt} \Pi_i(x) dt - e^{-rt_i} I_i(x) \right) \quad (5)$$

subject to

$$\Pi_i(x) := \Pi_{i-1}(x) + x_i(\Pi_i^* - \Pi_{i-1}(x)) \quad (6)$$

$$\Pi_0(x) := \Pi + x_0(\Pi_0^* - \Pi) \quad (7)$$

$$I_i(x) := x_i I^*(t_i). \quad (8)$$

for all  $i \in \mathbb{N}$ .

To get a first impression, consider the following basic example. The firm employs its initial technology  $\Pi$  (skipping  $\Pi_0^*$ ) until  $t_1$ , then adopts technology  $\Pi_1^*$  and thereafter omitts the adoption of any new technology. The corresponding decision vector is  $x = (0, 1, 0, 0, \dots)$ , the employed technology  $\Pi_i, i \in \mathbb{N}_0$  is given by  $\Pi_0 = \Pi$  and  $\Pi_i = \Pi_1^*$  for all

$i \in \mathbb{N}$ . This investment strategy yields the discounted stream of profits

$$\begin{aligned} \int_0^{t_1} e^{-rt} \Pi dt + \int_{t_1}^{\infty} e^{-rt} \Pi_1 dt - e^{-rt_1} I(t_1) &= (1 - e^{-rt_1}) \frac{\Pi}{r} + e^{-rt_1} \left( \frac{\Pi_1^*}{r} - I(t_1) \right) \\ &= \frac{\Pi}{r} + e^{-rt_1} \left( \frac{\Pi_1^* - \Pi}{r} - I(t_1) \right). \end{aligned} \quad (9)$$

As we see in the above equation, the payoff can also be phrased in terms of the technologies increments, so let

$$Z_i^* := \Pi_i^* - \Pi_{i-1}^* \quad (10)$$

for all  $i \in \mathbb{N}$ , with  $Z_0^* = \Pi_0^* - \Pi$ . Then, of course, each technology  $\Pi_i^*$  can be written as

$$\Pi_i^* = \Pi + \sum_{k=0}^i Z_k^* \quad (11)$$

for all  $i \in \mathbb{N}_0$ . Similar to equation 2 a sequence of decisions  $x$  defines the increments of the employed technologies as

$$Z_i(x) := \Pi_i(x) - \Pi_{i-1}(x) \quad (12)$$

for all  $i \in \mathbb{N}$ , with  $Z_0(x) = \Pi_0(x) - \Pi$  and, as above,

$$\Pi_i(x) = \Pi + \sum_{k=0}^i Z_k(x). \quad (13)$$

Using this representation, we can write the discounted stream of profits associated with a strategy  $x$ , depending only depends on increments:

$$\sum_{i=0}^{\infty} \left( \int_{t_i}^{t_{i+1}} e^{-rt} \Pi_i(x) dt - e^{-rt_i} I_i(x) \right) = \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i(x)}{r} - I_i(x) \right). \quad (14)$$



The derivation is given in the appendix. Hence, the optimization problem given by equations 5 to 8 can be reformulated as

$$\max_{x \in \{0,1\}^{\mathbb{N}_0}} \left[ \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i(x)}{r} - I_i(x) \right) \right] \tag{15}$$

subject to

$$Z_i(x) = x_i \left( \sum_{k=0}^i Z_k^* - \sum_{k=0}^{i-1} Z_k(x) \right), \tag{16}$$

$$Z_0(x) = x_0 Z_0^*, \tag{17}$$

$$I_i(x) = x_i I^*(t_i), \tag{18}$$

for all  $i \in \mathbb{N}_0$ .

Until now, we did not specify the dynamics of  $t$ ,  $\Pi^*$  and  $I^*$  explicitly. We will use the above framework in the following to study optimal investment strategies when  $\Pi^*$  either follows a poisson process or a geometric poisson process. Without loss of generality, we assume the investment cost  $I^*$  to be constant and incorporate the relative changes in the dynamics of the technology.<sup>4</sup> To simplify notation, we suppress the dependency on  $x$  in the remainder of this article by writing  $\Pi_i, i \in \mathbb{N}_0$  and  $I_i, i \in \mathbb{N}_0$ .

### 3. POISSON PROCESS

Consider the inter-arrival times  $t_{i+1} - t_i, i \in \mathbb{N}_0$  to be independent identically distributed exponential random variables having mean  $1/\lambda$  (with  $\lambda > 0$ ). Let the technologies increase by a constant amount  $\alpha > 0$ , that is  $Z_i^* = \alpha$  or

$$\Pi_{i+1}^* = \Pi_i^* + \alpha \tag{19}$$

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<sup>4</sup>This is due to the fact that the model is overspecified. In terms of the parameters, we are only interested in some (non-trivial) relation between possible profits from adoption ( $\Pi^*$ ) and investment cost  $I^*$ . Letting  $I^*$  be constant can be interpreted as a normalization with respect to the investment cost.

for all  $i \in \mathbb{N}_0$  and  $\Pi_0^* = \Pi + \alpha$ , hence

$$\Pi_i^* = \Pi + \alpha(i + 1) \tag{20}$$

for all  $i \in \mathbb{N}_0$ . Note that, with a minor change in (or abuse of) notation, the (discrete) technology sequence  $\Pi_i^*, i \in \mathbb{N}_0$  can be represented by a continuous-time process as follows: Redefining  $\Pi_t^* := \Pi_i^*$  for all  $t \in [t_i, t_{i+1})$  and for all  $i \in \mathbb{N}_0$ , the (now defined) continuous-time process  $\Pi_t^*, t \geq 0$  is a poisson process with jump amplitude  $\alpha > 0$ , arrival rate  $\lambda$  and starting point  $\Pi + \alpha$ , i.e.

$$\Pi_t^* = \alpha P_t + \Pi + \alpha, \tag{21}$$

where  $P_t$  is a poisson process with arrival rate  $\lambda$ , i.e.

$$dP_t = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases} \tag{22}$$

for all  $t \geq 0$ . For this reason, we may occasionally call  $\Pi_i^*, i \in \mathbb{N}_0$  a poisson process despite its discrete nature/index set.

Since the arrival times are stochastic, the deterministic objective function 15 changes to

$$E \left[ \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i}{r} - I_i \right) \right]. \tag{23}$$

Let  $V(\cdot, \cdot)$  denote the value function. The first argument is the current technology, the second argument keeps track of the technology currently available for adoption, so

$$V(\Pi, \Pi + \alpha) = \max_{x \in \{0,1\}^{\mathbb{N}_0}} E \left[ \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i}{r} - I_i \right) \right]. \tag{24}$$

*Remark.* It is sufficient to study the problem for an initial technology  $\Pi = 0$ , since

$$V(\Pi, \Pi + \alpha) = \max_{x \in \{0,1\}^{\mathbb{N}_0}} E \left[ \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i}{r} - I_i \right) \right] \quad (25)$$

$$= \frac{\Pi}{r} + \max_{x \in \{0,1\}^{\mathbb{N}_0}} E \left[ \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i}{r} - I_i \right) \right] \quad (26)$$

$$= \frac{\Pi}{r} + V(0, \alpha). \quad (27)$$

To analyze the optimal adoption strategy, we proceed in small steps. First we show that adoption of some technology will take place, whatever the parameter values are. In models with a finite number of technologies (such as Farzin et al. [FHK98] or Grenadier and Weiss [GW97]) a typical result is that for specific parameter values no adoption takes place, that is the *bystander* strategy is optimal. In our model the number of technologies that become available for adoption is infinite and they are unbounded while investment cost are constant, therefore no adoption can not be an optimal strategy.

**Lemma 3.1.** *Staying with the initial technology is not optimal.*

Lemma 3.1 shows that, no matter how high investment costs are, at some point in time a new technology will be adopted (and as we will see later, even an infinite number of adoptions will take place). Since at least one technology is adopted, there also is a *first* technology that is adopted. Therefore we may ask which is the first technology that will be adopted, or equivalently how many technologies should be skipped before the first adoption. In the following, we will call the number of skipped technologies between two adoptions *jump width* (not to be confused with the jump amplitude of the poisson process) and the optimal number of skipped technologies before the first adoption *optimal jump width for the first jump*.

**Lemma 3.2.** *Let  $m \in \mathbb{N}$  be the optimal jump width for the first jump and let  $\beta = \lambda/(r+\lambda)$ . For all  $k \in \{1, \dots, m\}$  we have*

$$V(0, \alpha) = \beta^{k-1}V(0, \alpha k). \tag{28}$$

Using Lemma 3.2, we give a closed form expression for the value function and characterize the optimal investment strategy in the following theorem.

**Theorem 3.3.** *Let  $m \in \mathbb{N}$  be the optimal jump width for the first jump. Then*

$$V(0, \alpha) = \frac{\beta^{m-1}}{1 - \beta^m} \left( \frac{\alpha m}{r} - I \right). \tag{29}$$

*Furthermore, the optimal jump width is constant.*

From Theorem 3.3 we can derive the optimal jump width.

**Theorem 3.4.** *The optimal jump width is*

$$m = \frac{rI}{\alpha} + \frac{1}{-\ln \beta} \left[ 1 + W \left( -\frac{\beta^{rI/\alpha}}{e} \right) \right], \tag{30}$$

*where  $W$  is the Lambert  $W$  function.*

The Lambert  $W$  function is the inverse of the function  $w \mapsto we^w$ . For more a details on the Lambert  $W$  function see [CGH<sup>+</sup>96].

Of course, the derived jump width will hardly ever be an integer. To get the optimal integer jump width, it is necessary to identify which of the two nearest integers gives the higher profit stream. This can easily be done by plugging  $\lfloor m \rfloor$  and  $\lceil m \rceil$  into the value function and comparing the respective results. Either of them is the optimal integer jump width, since the value function is unimodal in the jump width. Without substantial change in the conclusion, theorems 3.5 and 3.6 below could be modified to account for the fact that  $m$  typically is not an integer, but for the sake of simplicity, we ignore this detail. The next theorem describes how changes in the parameters affect the optimal jump width.

**Theorem 3.5.** *The optimal jump width is*

- (i) *decreasing in  $\alpha$ ,*
- (ii) *increasing in  $I$ ,*
- (iii) *decreasing in  $1/\lambda$ ,*
- (iv) *increasing in  $r$ .*

The interpretation of Theorem 3.5 is straightforward: The higher the additional benefit from advanced technologies (that is, the higher  $\alpha$ ), the more often the firm will adopt. In contrast, the higher the investment cost  $I$ , the longer it takes until the firm is compensated for the investment cost payed, hence new technologies are adopted with a lower frequency (in terms of jump width). A higher discount rate  $r$  reduces the incentive to invest because the present value of the future profits of the new technology decreases with increasing  $r$ . Finally, consider an increase in  $1/\lambda$ : The expected arrival times of future technologies increase, therefore the current technology generates a higher stream of profit per period. Hence it is profitable to buy new technologies “more often” in the sense that a smaller number of technologies is skipped.<sup>5</sup>

**Theorem 3.6.** *The value function is*

- (i) *increasing in  $\alpha$ ,*
- (ii) *decreasing in  $I$ ,*
- (iii) *decreasing in  $1/\lambda$ ,*
- (iv) *decreasing in  $r$ .*

The results are once again in line with intuition. The better the new technologies (the higher  $\alpha$ ) the higher is the discounted profit stream from optimal adoption. The more costly innovations are, the lower the discounted profit stream. The result that the firms profits are

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<sup>5</sup>Note that “more often” does not relate to the actual time between two adoptions, but rather to the number of technologies skipped between two adoptions. A detailed discussion about the relation between jump width and actual time will be part of a subsequent version of this paper.

decreasing with  $1/\lambda$  may look puzzling. A higher value  $1/\lambda$  results in a faster technology improvement. At the same time - according to Theorem 3.5 - faster improvement decreases the optimal jump width. Therefore each technology is employed for shorter time period and - compared to a situation with a slow technological improvement - gives a lower total payoff before it is replaced. Finally a higher discount rate  $r$  gives a lower discounted stream of profits since future profits become less valuable.

#### 4. GEOMETRIC POISSON PROCESS

Let  $\Pi_t^*, t \geq 0$  follow a geometric poisson process

$$d\Pi_t^* = \alpha\Pi_t^*dP_t \tag{31}$$

where  $P_t$  is a poisson process with arrival rate  $\lambda$ , or equivalently

$$\frac{d\Pi_t^*}{\Pi_t^*} = \alpha dP_t, \tag{32}$$

and let  $\Pi_0^* = \alpha\Pi$ . If we assume that the poisson process  $P_t, t \geq 0$  jumps in  $s \geq 0$ , we get

$$\Pi_s^* - \Pi_{s-}^* = \alpha\Pi_{s-}^*, \tag{33}$$

and therefore

$$\Pi_s^* = (1 + \alpha)\Pi_{s-}^*. \tag{34}$$

Switching from the continuous-time representation  $\Pi_t^*, t \geq 0$  to the discrete representation  $\Pi_i^*, i \in \mathbb{N}_0$ , we get

$$\Pi_i^* = (1 + \alpha)^{i+1}\Pi. \tag{35}$$

While in the previous section the technology increments were constant, in this case the increments

$$Z_i^* = \Pi_i^* - \Pi_{i-1}^* = (1 + \alpha)^{i+1}\Pi - (1 + \alpha)^i\Pi = (1 + \alpha)^i\alpha\Pi \quad (36)$$

are increasing at an exponential rate. This implies that with each new technology the profitability increases exponentially and furthermore, after some point in time every technology will be adopted. More formally, let  $\Pi_n^*$  be a technology, such that every technology  $\Pi_k^*$  with  $k > n$  will be adopted. Since technology  $\Pi_{n+1}^*$  will be adopted, the additional expected payoff from adopting technology  $n$  is bounded from above by

$$E [(e^{-rt_n} - e^{-rt_{n+1}}) Z_n - I] = \frac{1}{r + \lambda} Z_n - I \quad (37)$$

This is negative, if

$$Z_n < (r + \lambda)I. \quad (38)$$

Since  $Z_n \geq Z_n^* = (1 + \alpha)^{n-1}\alpha\Pi$ , we get

$$(1 + \alpha)^{n-1} < \frac{(r + \lambda)I}{\alpha\Pi} \quad (39)$$

which is equivalent to

$$n < 1 + \frac{\ln((r + \lambda)I) - \ln(\alpha\Pi)}{\ln(1 + \alpha)}. \quad (40)$$

This sufficient condition simplifies the task of finding the optimal investment strategy  $x \in \{0, 1\}^{\mathbb{N}_0}$ , which corresponds to an infinite number of decisions, to finding the optimal (finite)  $x \in \{0, 1\}^n$ . Still this is non-trivial, since the investment incentives are path-dependent. However, this path-dependence is limited to the last technology adoption. Therefore, the

optimal investment strategy can be found by a dynamic programming algorithm.<sup>6</sup> The following example illustrates the above for specific parameter values.

**Example.** For parameter values  $\alpha = 0.1, i = 6, 1/\lambda = 4, \Pi = 1, r = 0.05$ , we get the condition

$$n < 31.326, \tag{41}$$

therefore we know that every technology after the 31st will be adopted. The dynamic programming algorithm gives the optimal investment strategy

$$x = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, \dots) \tag{42}$$

and the value

$$V = 32.9304. \tag{43}$$

An interesting observation can be made by rewriting the sequence of optimal jump widths of the above example in a vector

$$(5, 4, 3, 2, 2, 1, 1, 1, 1, 1, 0, \dots). \tag{44}$$

Without giving a proof (which will be part of a subsequent version of this paper) we state the following:

**Claim.** *The optimal jump width is non-increasing.*

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<sup>6</sup>The algorithm is a standard dynamic programming algorithm which uses the fact, that the path-dependency is limited to the last technology adoption. Details are available from the author upon request.



## 5. CONCLUSION

This paper studies optimal investment patterns when an infinite number of advanced technologies become available over time. If technological progress is governed by a poisson process, we show that the optimal strategy is to adopt every  $m$ -th technology, where we derive  $m$  as the solution to the firms maximization problem. Closed expressions for  $m$  and the value of the problem are calculated and the effects of changes in parameter values are described. If technological progress is governed by a geometric poisson process, we show that after the arrival of some technology, each technology will be adopted and therefore the problem of finding the optimal infinite sequence of investment  $x \in \{0, 1\}^{\mathbb{N}_0}$  simplifies to the problem of finding the optimal finite sequence  $x \in \{0, 1\}^n$ . An example shows a typical pattern for this case. The claim that the optimal jump width is non-increasing remains to be proven.

Apart from that, several extensions should be carried out. First of all, the speed of adoption until now is only described in terms of the number of adoptions, not in terms of the actual time. Since we can calculate the optimal jump width in the case of a poisson process, it would be interesting to analyze the impact of a change in the speed of innovation (that is  $\lambda$ ) on the actual speed of implementation. Even without the constraint on integer-valued jump widths, the effect on the number of adoptions in a given time interval is unclear. Furthermore, in the case of a geometric poisson process further numerical simulations will give additional insight an the observed patterns. One could for instance conjecture, that not only the jump width is non-increasing, but also that the number of times the same jump widths is optimal is decreasing. The analysis should further contain a third process, which is increasing, but has decreasing increments. Further possible generalizations would include uncertainty about the value of new technologies and non-constant inter-arrival times between the arrival of technologies.

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6. APPENDIX

Derivation for the second representation of the optimization problem:

$$G = \sum_{i=0}^{\infty} \left( \int_{t_i}^{t_{i+1}} e^{-rt} \Pi_i dt - e^{-rt_i} I_i \right) \quad (45)$$

$$= \sum_{i=0}^{\infty} (e^{-rt_i} - e^{-rt_{i+1}}) \frac{\Pi_i}{r} - e^{-rt_i} I_i \quad (46)$$

$$= \sum_{i=0}^{\infty} e^{-rt_i} \frac{\Pi_i}{r} - \sum_{i=0}^{\infty} e^{-rt_{i+1}} \frac{\Pi_i}{r} - \sum_{i=0}^{\infty} e^{-rt_i} I_i \quad (47)$$

$$= e^{-rt_0} \frac{\Pi_0}{r} + \sum_{i=1}^{\infty} e^{-rt_i} \frac{\Pi_{i-1} + Z_i}{r} - \sum_{i=1}^{\infty} e^{-rt_i} \frac{\Pi_{i-1}}{r} - \sum_{i=0}^{\infty} e^{-rt_i} I_i \quad (48)$$

$$= \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \frac{Z_i}{r} - \sum_{i=0}^{\infty} e^{-rt_i} I_i \quad (49)$$

$$= \frac{\Pi}{r} + \sum_{i=0}^{\infty} e^{-rt_i} \left( \frac{Z_i}{r} - I_i \right). \quad (50)$$

*Proof of Lemma 3.1.* Skipping the first  $n-1$  technologies and adopting the  $n$ -th technology (and not adopting any further technologies) yields an additional discounted payoff

$$e^{-rt_n} \left( \frac{\alpha n}{r} - I \right), \quad (51)$$

hence compared to staying with the initial technology, it is profitable to buy the  $n$ -th technology if

$$n > \frac{rI}{\alpha}. \quad (52)$$

Since  $r, \alpha$  and  $I$  are constant, the existence of such an  $n \in \mathbb{N}$  is obvious. □

*Proof of Lemma 3.2.* For  $m = 1$  there is nothing to prove. Let  $m \geq 2$ . The bellman equation in  $t_0$  is

$$V(0, \alpha) = \max \{V(\alpha, \alpha) - I, E [e^{-rt}V(0, \alpha_2)]\}. \quad (53)$$

Since adoption of the first technology is not optimal,

$$V(0, \alpha) = E [e^{-rt}V(0, \alpha_2)] = E [e^{-rt}] V(0, \alpha_2) = \beta V(0, \alpha_2). \quad (54)$$

Iteration concludes the proof.  $\square$

*Proof of Theorem 3.3.* Let  $m$  be the optimal jump width of the first jump. By Lemma 3.2,

$$V(0, \alpha) = \beta^{m-1}V(0, \alpha m). \quad (55)$$

Since adoption of technology  $m + 1$  is optimal, the bellman equation in  $t_m$

$$V(0, \alpha m) = \max \{V(\alpha m, \alpha m) - I, E [e^{-rt}V(0, \alpha(m + 1))]\}. \quad (56)$$

simplifies to

$$V(0, \alpha m) = V(\alpha m, \alpha m) - I = \frac{\alpha m}{r} - I + V(0, 0). \quad (57)$$

Since the arrival of the next technology is again exponentially distributed, we know

$$V(0, 0) = E[e^{-rt}V(0, \alpha)] = E[e^{-rt}]V(0, \alpha) = \beta V(0, \alpha), \quad (58)$$

and therefore

$$V(0, \alpha) = \beta^{m-1}V(0, \alpha m) \quad (59)$$

$$= \beta^{m-1} \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \quad (60)$$

$$= \beta^{m-1} \left( \frac{\alpha m}{r} - I + \beta V(0, \alpha) \right). \quad (61)$$

At this point we may conclude that the optimal jump width is constant, since at time  $t_{m+1}$  the firm is facing the initial optimization problem. Solving for the value function gives

$$V(0, \alpha) = \frac{\beta^{m-1}}{1 - \beta^m} \left( \frac{\alpha m}{r} - I \right). \quad (62)$$

□

*A second proof of Theorem 3.3.* The maximization problem can be reformulated as follows:

In terms of the optimal jump width of the first jump, the bellman equation is

$$V(0, \alpha) = \max_{m \in \mathbb{N}_0} E \left[ 0 + e^{-rt_m} \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \right] \quad (63)$$

$$= \max_{m \in \mathbb{N}_0} E \left[ e^{-rt_m} \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \right]. \quad (64)$$

Let  $m$  be the optimal jump width for the first jump. Then

$$V(0, \alpha) = E \left[ e^{-rt_m} \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \right] \quad (65)$$

$$= E \left[ e^{-rt_m} \right] \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \quad (66)$$

$$= \beta^m \left( \frac{\alpha m}{r} - I + V(0, 0) \right) \quad (67)$$

$$= \beta^m \left( \frac{\alpha m}{r} - I + \beta V(0, \alpha) \right) \quad (68)$$

Solving for  $V(0, \alpha)$  completes the proof. □

*Proof of Theorem 3.4.* The value function is increasing for all  $j < m$  and decreasing for all  $j > m$ . The derivative of the value function with respect to  $m$  is

$$\frac{\partial V}{\partial m} = \frac{\beta^{m-1}}{(1 - \beta^m)^2} \left[ (1 - \beta^m) \frac{\alpha}{r} + \left( \frac{\alpha m}{r} - I \right) \ln \beta \right].$$

We prove that this derivative is positive for all  $j < m$ . The derivative  $\partial V / \partial m$  is positive, iff

$$(1 - \beta^j) \frac{\alpha}{r} > - \left( \frac{\alpha j}{r} - I \right) \ln \beta \quad (69)$$

or

$$\beta^j < \left( j - \frac{rI}{\alpha} - I \right) \ln \beta + 1 = j \ln \beta + 1 - \frac{rI}{\alpha} \ln \beta = j \ln \beta + c. \quad (70)$$

with  $c = 1 - \frac{rI}{\alpha} \ln \beta$ . Substituting  $t = -j - \frac{c}{\ln \beta}$  (or  $j = -t - \frac{c}{\ln \beta}$ ) yields

$$\beta^{-t - \frac{c}{\ln \beta}} < -t \ln \beta \quad (71)$$

or

$$\beta^{-\frac{c}{\ln \beta}} < -t \beta^t \ln \beta \quad (72)$$

or (since  $\beta < 1$  and  $\ln \beta < 0$ )

$$t \beta^t > -\frac{1}{\ln \beta} \beta^{-\frac{c}{\ln \beta}} =: \zeta. \quad (73)$$

Then

$$t > \frac{W(\zeta \ln \beta)}{\ln \beta} \quad (74)$$

and hence

$$-j - \frac{c}{\ln \beta} > \frac{W(\zeta \ln \beta)}{\ln \beta} \quad (75)$$

or

$$j < -\frac{W(\zeta \ln \beta)}{\ln \beta} - \frac{c}{\ln \beta}. \quad (76)$$

Since  $\zeta \ln \beta = -\frac{\beta^{\frac{rI}{\alpha}}}{e}$  we get

$$j < -\frac{1}{\ln \beta} \left( 1 - \frac{rI}{\alpha} \ln \beta + W \left( -\frac{\beta^{\frac{rI}{\alpha}}}{e} \right) \right) = m. \quad (77)$$

The same line of reasoning shows that  $V$  is decreasing for  $j > m$  and therefore that  $m$  maximizes  $V$ .  $\square$

*Proof of Theorem 3.5.* (i) Let

$$h(\alpha) = \frac{rI}{\alpha} \quad (78)$$

and

$$g(x) = -\frac{\beta^x}{e}. \quad (79)$$

Then

$$m = h(\alpha) + \frac{1}{-\ln \beta} [1 + W(g(h(\alpha)))], \quad (80)$$

and

$$\frac{\partial T}{\partial \alpha} = h'(\alpha) + \frac{1}{-\ln \beta} \left[ \frac{dW(g(h(\alpha)))}{d\alpha} \right]. \quad (81)$$

Applying the chain rule to the last item gives

$$\frac{dW(g(h(\alpha)))}{d\alpha} = W'(g(h(\alpha)))g'(h(\alpha))h'(\alpha). \quad (82)$$

We analyze the factors from right to left. Obviously  $h'(\alpha) = -\frac{rI}{\alpha^2} < 0$  and

$$g'(h(\alpha)) = -\frac{1}{e}\beta^{h(\alpha)} \ln \beta > 0, \quad (83)$$

since  $\beta < 1$ . To analyze the first term, note that

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}. \quad (84)$$

Since  $h(\alpha) > 0$ , we have  $g(h(\alpha)) \in (-1/e, 0)$ . Due to  $W(z) \in (-1, 0)$  for  $z \in (-1/e, 0)$ , the numerator in the above equation is negative. The denominator (as a product of



a positive and a negative ) is negative, hence  $W'(g(h(\alpha)))$  is positive. Therefore

$$\frac{dW(g(h(\alpha)))}{d\alpha} = W'(g(h(\alpha)))g'(h(\alpha))h'(\alpha) < 0, \quad (85)$$

and due to  $\beta < 0$  we may finally conclude

$$\frac{\partial m}{\partial \alpha} = h'(\alpha) + \frac{1}{-\ln \beta} \left[ \frac{dW(g(h(\alpha)))}{d\alpha} \right] < 0. \quad (86)$$

(ii) Consider once again

$$m = \frac{rI}{\alpha} + \frac{1}{-\ln \beta} \left[ 1 + W \left( g \left( \frac{rI}{\alpha} \right) \right) \right]. \quad (87)$$

With  $g'(rI/\alpha) = -\frac{1}{e}\beta^{rI/\alpha} \ln \beta$  we get

$$\frac{\partial m}{\partial I} = \frac{r}{\alpha} + \frac{1}{-\ln \beta} \left[ \frac{dW(g(rI/\alpha))}{dI} \right] \quad (88)$$

$$= \frac{r}{\alpha} + \frac{1}{-\ln \beta} \left[ W' \left( g \left( \frac{rI}{\alpha} \right) \right) g' \left( \frac{rI}{\alpha} \right) \frac{r}{\alpha} \right] \quad (89)$$

$$= \frac{r}{\alpha} \left( 1 + \frac{1}{-\ln \beta} \left[ W' \left( g \left( \frac{rI}{\alpha} \right) \right) g' \left( \frac{rI}{\alpha} \right) \right] \right) \quad (90)$$

$$= \frac{r}{\alpha} \left( 1 + \frac{1}{-\ln \beta} \left[ W' \left( g \left( \frac{rI}{\alpha} \right) \right) \frac{1}{e} \beta^{rI/\alpha} (-\ln \beta) \right] \right) \quad (91)$$

$$= \frac{r}{\alpha} \left( 1 + \frac{1}{e} \beta^{rI/\alpha} W' \left( g \left( \frac{rI}{\alpha} \right) \right) \right). \quad (92)$$

From (i), we know that  $W'(g(rI/\alpha)) > 0$ , hence  $\frac{\partial m}{\partial I} > 0$ .

The calculations for (iii) and (iv) are a little more complicated and will be omitted here.  $\square$

*Proof of Theorem 3.6.* Recall that

$$V(0, \alpha) = \frac{\beta^{m-1}}{1 - \beta^m} \left( \frac{\alpha m}{r} - I \right), \quad (93)$$

and furthermore note that for any parameter the indirect effect vanishes, since  $V$  has a local maximum at  $m$ .

(i) For the parameter  $\alpha$  we therefore get

$$\frac{dV}{d\alpha} = \frac{\partial V}{\partial m} \frac{dm}{d\alpha} + \frac{\partial V}{\partial \alpha} = \frac{\partial V}{\partial \alpha} = \frac{\beta^{m-1}}{1-\beta^m} \frac{m}{r} > 0. \quad (94)$$

(ii) For the parameter  $I$  we therefore get

$$\frac{dV}{dI} = \frac{\partial V}{\partial m} \frac{dm}{dI} + \frac{\partial V}{\partial I} = \frac{\partial V}{\partial I} = -\frac{\beta^{m-1}}{1-\beta^m} < 0. \quad (95)$$

(iii) Omitted.

(iv) The calculations for  $r$  are a little more complicated. First, we have

$$\frac{dV}{dr} = \frac{\partial V}{\partial m} \frac{dm}{dr} + \frac{\partial V}{\partial r} = \frac{\partial V}{\partial r} = \frac{\partial b}{\partial r} \left( \frac{\alpha m}{r} - I \right) - b \frac{\alpha m}{r^2}. \quad (96)$$

with  $b = \frac{\beta^{m-1}}{1-\beta^m}$ . Since

$$\frac{\partial b}{\partial r} = -\frac{1}{\lambda} \frac{\beta^m}{(1-\beta^m)^2} (m-1+\beta^m) = -b \frac{1}{r+\lambda} \frac{1}{1-\beta^m} (m-1+\beta^m) \quad (97)$$

we get

$$\frac{dV}{dr} = -b \frac{1}{r} \frac{\beta}{1-\beta^m} (m-1+\beta^m) \left( \frac{\alpha m}{r} - I \right) - b \frac{\alpha m}{r^2} \quad (98)$$

$$= -\frac{b}{r} \left[ \frac{\beta}{1-\beta^m} (m-1+\beta^m) \left( \frac{\alpha m}{r} - I \right) + \frac{\alpha m}{r} \right] \quad (99)$$

To get the sign of this expression, first note that the term in brackets is positive if  $y = -\frac{\beta}{1-\beta^m} (m-1+\beta^m)$  is negative. Therefore consider the case of a positive  $y$ .

Then

$$-y \left( \frac{\alpha m}{r} - I \right) + \frac{\alpha m}{r} = -y \frac{\alpha m}{r} + yI + \frac{\alpha m}{r} \quad (100)$$

$$= (1-y) \frac{\alpha m}{r} + yI \quad (101)$$

This is positive iff

$$y = -\frac{\beta}{1 - \beta^m} (m - 1 + \beta^m) < 1, \quad (102)$$

which is equivalent to

$$-\beta \left( \frac{m}{1 - \beta^m} - 1 \right) < 1 \quad (103)$$

or

$$\frac{m}{1 - \beta^m} > 1 - \frac{1}{\beta} \quad (104)$$

or

$$m > (1 - \beta^m) \frac{\beta - 1}{\beta}. \quad (105)$$

This is fulfilled, since the right hand side is negative and

$$m = \frac{rI}{\alpha} - \frac{1 + W(z)}{\ln \beta} > 0. \quad (106)$$

with  $z = -e^{-1} \beta^{rI/\alpha}$ .

□