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Axiomatization of the AGM Theory of Belief Revision in a Temporal Logic

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Abstract

It is natural to think of belief revision as the interaction of belief and information over time. Thus branching-time temporal logic seems a natural setting for a theory of belief revision. We propose two extensions of a modal logic that, besides the “next-time” temporal operator, contains a belief operator and an information operator. The first logic is shown to provide an axiomatization of the first six postulates of the AGM theory of belief revision, while the second, stronger, logic provides an axiomatization of the full set of AGM postulates.

1 Introduction

There is an unsatisfactory lack of uniformity in the literature between how static beliefs and changes in beliefs are modeled. Starting with Hintikka’s [7] seminal contribution, the notion of static belief has been studied mainly within the context of modal logic. On the syntactic side a belief operator B is introduced, with the intended interpretation of $B\phi$ as “the individual believes that ϕ ”. Various properties of beliefs are then expressed by means of axioms.¹ On the semantic side Kripke structures (Kripke [9]) are used, consisting of a set of states (or possible worlds) Ω together with a binary relation \mathcal{B} on Ω , with the interpretation of $\alpha\mathcal{B}\beta$ as “at state α the individual considers state β possible”. The connection between syntax and semantics is then obtained by means of a valuation that associates with every atomic proposition p the set of states at which p is true. Rules are given for determining the truth of an arbitrary formula at every state of a model; in particular, the formula $B\phi$ is true at state α if and only if ϕ is true at every β such that $\alpha\mathcal{B}\beta$, that is, if ϕ is true at every state

¹For example, the positive introspection axiom $B\phi \rightarrow BB\phi$, which says that if the individual believes ϕ then she believes that she believes ϕ .

that the individual considers possible at α . Often one can show that there is a correspondence between a syntactic axiom and a property of the accessibility relation in the sense that every instance of the axiom is true at every state of every model whose accessibility relation satisfies the property and *vice versa*.² The advantage of the syntactic approach is that it allows one to state properties of beliefs in a clear and transparent way.

On the other hand, with a few exceptions, the tools of modal logic have not been employed in the analysis of belief revision. In their seminal contribution Alchourron, Gärdenfors and Makinson [1] model beliefs as sets of formulas in a given syntactic language and belief revision is construed as an operation that associates with every belief set K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new belief set K_ϕ^* representing the revised beliefs. Several requirements are imposed on this operator in order to capture the notion of “rational” belief change. Their approach has become known as the AGM theory of belief revision and has stimulated a large literature.

The purpose of this paper is to bridge the gap between these two strands of the literature, by representing the AGM postulates as axioms in a modal language. It is natural to think of belief revision as the interaction of belief and information over time. Thus temporal logic is a natural starting point. Besides the “next-time” operator \bigcirc (and its inverse) our language contains a belief operator B and an information operator I . The information operator is not a normal operator and is formally similar to the “only knowing” operator introduced by Levesque in [10]. On the semantic side we consider branching-time frames to represent different possible evolutions of beliefs. For every date t , beliefs and information are represented by binary relations \mathcal{B}_t and \mathcal{I}_t on a set of states Ω . As in the static setting, the link between syntax and semantics is provided by the notion of valuation and model. The truth of a formula in a model is defined at a state-instant pair (ω, t) .

The first logic that we propose axiomatizes the first six AGM postulates (the so-called “basic set”), in the following sense (Proposition 11):

(1) if K is the initial belief set, ϕ is a Boolean (i.e. non-modal) formula and K_ϕ^* is the revised belief set that satisfies the first six AGM postulates then there is a model of the logic, a state $\alpha \in \Omega$ and instants t_1 and t_2 such that: (i) t_2 is an immediate successor of t_1 , (ii) the set of propositions that the individual believes at (α, t_1) coincides with K , (iii) the individual at time t_2 and state α is informed that ϕ , that is, $(\alpha, t_2) \models I\phi$ and (iv) the set of Boolean propositions that the individual believes at (α, t_2) coincides with K_ϕ^* , and

(2) for every model that validates the logic, every state α and every instants t_1 and t_2 such that t_2 is an immediate successor of t_1 , if at time t_2 and state α the individual is informed that ϕ (that is, $(\alpha, t_2) \models I\phi$) then K and K_ϕ^* defined as the sets of Boolean formulas that the individual believes at (α, t_1) and (α, t_2) , respectively, satisfy the first six AGM postulates.

²For example, the positive introspection axiom $B\phi \rightarrow BB\phi$ corresponds to transitivity of the relation \mathcal{B} .

The remaining two AGM postulates deal with comparing how the individual revises his beliefs after learning first that ϕ and then that ψ with his revision when he learns that $\phi \wedge \psi$. This is where the branching-time structure that we use becomes important, since two different evolutions of beliefs need to be compared. The second logic that we propose extends the first by adding two axioms, which are proved to correspond exactly to the last two AGM postulates. We show (Proposition 12) that the stronger logic provides an axiomatization of the full set of AGM axioms, in a sense analogous to the previous result.

The paper is organized as follows. In Section 2 we start with the semantics of temporal belief revision frames. In Section 3 we introduce the basic logic and two extensions of it, which - in Section 4 - are proved to provide an axiomatization of the first six and full set of AGM postulates, respectively. Section 5 concludes.

2 The semantics

On the semantic side we consider branching-time structures with the addition of a belief relation and an information relation for every instant t .

Definition 1 A next-time branching frame is a pair $\langle T, \succ \rangle$ where T is a (possibly infinite) set of instants or dates and \succ is a binary “precedence” relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,

- (1) uniqueness if $t_1 \succ t_3$ and $t_2 \succ t_3$ then $t_1 = t_2$
- (2) acyclicity if $\langle t_1, \dots, t_n \rangle$ is a sequence with $t_i \succ t_{i+1}$
for every $i = 1, \dots, n - 1$, then $t_n \neq t_1$.

The interpretation of $t_1 \succ t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2 : every instant has at most a unique immediate predecessor but can have several immediate successors.

Definition 2 A temporal belief revision frame is a quintuple $\langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ where $\langle T, \succ \rangle$ is a next-time branching frame, Ω is a set of states (or possible worlds) and, for every $t \in T$, \mathcal{B}_t and \mathcal{I}_t are binary relations on Ω .

The interpretation of $\omega \mathcal{B}_t \omega'$ is that at state ω and time t the individual considers state ω' possible (an alternative expression is “ ω' is a doxastic alternative to ω at time t ”). On the other hand, the interpretation of $\omega \mathcal{I}_t \omega'$ is that at state ω and time t , according to the information received, it is possible that the true state is ω' . We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{B}_t \omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{I}_t \omega'\}.$$

Temporal belief frames can be used to describe either a situation where the objective facts describing the world do not change – so that only the beliefs of the agent change over time – or a situation where both the facts and the

doxastic state of the agent change. In the computer science literature the first situation is called belief revision, while the latter is called belief update (see [8]). We shall focus on belief revision.³

We consider a propositional language with five modal operators: the next-time operator \bigcirc and its inverse \bigcirc^{-1} , the belief operator B , the information operator I and the “all state” operator A . The intended interpretation is as follows:

- $\bigcirc\phi$: “at every next instant it will be the case that ϕ ”
- $\bigcirc^{-1}\phi$: “at every previous instant it was the case that ϕ ”
- $B\phi$: “the agent believes that ϕ ”
- $I\phi$: “the agent is informed that ϕ ”
- $A\phi$: “it is true at every state that ϕ ”.

Given a temporal belief revision frame $\langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ one obtains a *model based on it* by adding a function $V : S \rightarrow 2^\Omega$ (where S is the set of atomic propositions and 2^Ω denotes the set of subsets of Ω) that associates with every atomic proposition q the set of states at which q is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition q depends only on the state and not on the time.⁴ Given a model, a state ω , an instant t and a formula ϕ , we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t . Let $\|\phi\|$ denote the truth set of ϕ , that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T : (\omega, t) \models \phi\}$ and let $[\phi]_t \subseteq \Omega$ denote the set of states at which ϕ is true at time t , that is, $[\phi]_t = \{\omega \in \Omega : (\omega, t) \models \phi\}$. Truth at a pair (ω, t) is defined recursively as follows.

- if $q \in S$, $(\omega, t) \models q$ if and only if $\omega \in V(q)$.
- $(\omega, t) \models \neg\phi$ if and only if $(\omega, t) \not\models \phi$.
- $(\omega, t) \models \phi \vee \psi$ if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$ (or both).
- $(\omega, t) \models \bigcirc\phi$ if and only if $(\omega, t') \models \phi$ for every t' such that $t \succ t'$.
- $(\omega, t) \models \bigcirc^{-1}\phi$ if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \succ t$.
- $(\omega, t) \models B\phi$ if and only if $\mathcal{B}_t(\omega) \subseteq [\phi]_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{B}_t(\omega)$.
- $(\omega, t) \models I\phi$ if and only if $\mathcal{I}_t(\omega) = [\phi]_t$, that is, if (1) $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \phi$ then $\omega' \in \mathcal{I}_t(\omega)$.
- $(\omega, t) \models A\phi$ if and only if $\Omega = [\phi]_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq [\phi]_t$ we require equality: $\mathcal{I}_t(\omega) = [\phi]_t$. Thus our information operator is formally similar to the “only knowing” operator discussed in the computer science literature (see [10]), although the interpretation is different.

³For example, our analysis would be appropriate to model the evolving belief states of an archaeologist who is trying to learn what truly happened several thousand years ago. New archaeological discoveries provide clues and information about the past, which the scientist uses to update his beliefs. However, the facts he is trying to learn do not change: their truth value was fixed in the distant past.

⁴Belief update would require a valuation to be defined as a function $V : S \rightarrow 2^{\Omega \times T}$.

The non-normality of the I operator is what makes it necessary to add the “all state” operator A (see [6]).

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t) . A formula ϕ is *valid in a frame* if it is valid in every model based on it.

3 The basic logic and two extensions

The formal language is built in the usual way (see [2]) from a countable set of atomic propositions, the connectives \neg and \vee (from which the connectives \wedge , \rightarrow and \leftrightarrow are defined as usual) and the modal operators \bigcirc , \bigcirc^{-1} , B , I and A . Let $\diamond\phi \stackrel{def}{=} \neg \bigcirc \neg\phi$, and $\diamond^{-1}\phi \stackrel{def}{=} \neg \bigcirc^{-1} \neg\phi$. Thus the interpretation of $\diamond\phi$ is “at *some* next instant it will be the case that ϕ ” while the interpretation of $\diamond^{-1}\phi$ is “at some previous instant it was the case that ϕ ”.

We denote by \mathbb{L}_0 the basic logic of belief revision defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.
2. Axiom K for \bigcirc , \bigcirc^{-1} , B and A :

$$(\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi \quad \text{for } \Box \in \{\bigcirc, \bigcirc^{-1}, B, A\} \quad (\text{K})$$

3. Temporal axioms relating \bigcirc and \bigcirc^{-1} :

$$\begin{aligned} \phi &\rightarrow \bigcirc \diamond^{-1} \phi & (\text{O}_1) \\ \phi &\rightarrow \bigcirc^{-1} \diamond \phi & (\text{O}_2) \end{aligned}$$

4. Backward Uniqueness axiom:

$$\diamond^{-1}\phi \rightarrow \bigcirc^{-1}\phi \quad (\text{BU})$$

5. S5 axioms for A :

$$\begin{aligned} A\phi &\rightarrow \phi & (\text{T}_A) \\ \neg A\phi &\rightarrow A\neg A\phi & (\text{5}_A) \end{aligned}$$

6. Inclusion axiom for B (note the absence of an analogous axiom for I):

$$A\phi \rightarrow B\phi \quad (\text{Incl}_B)$$

7. Axioms to capture the non-standard semantics for I :

$$\begin{aligned} (I\phi \wedge I\psi) &\rightarrow A(\phi \leftrightarrow \psi) & (\text{I}_1) \\ A(\phi \leftrightarrow \psi) &\rightarrow (I\phi \leftrightarrow I\psi) & (\text{I}_2) \end{aligned}$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\phi, \phi \rightarrow \psi}{\psi}$ (*MP*)

2. Necessitation for A , \bigcirc and \bigcirc^{-1} :

$$\frac{\phi}{\Box \phi} \quad \text{for } \Box \in \{\bigcirc, \bigcirc^{-1}, A\}. \quad (\text{Nec})$$

Note that from (*MP*), (*Incl_B*) and Necessitation for A one can derive necessitation for B ($\frac{\phi}{B\phi}$). On the other hand, necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

Remark 3 *By (MP), axiom K and necessitation, the following is a derived rule of inference for the operators \bigcirc , \bigcirc^{-1} , B and A : $\frac{\phi \rightarrow \psi}{\Box \phi \rightarrow \Box \psi}$ for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$. We call this rule *RK*. On the other hand, rule *RK* is not a valid rule of inference for the operator I (despite the fact that axiom *K* for I can be shown to be a theorem of \mathbb{L}_0).*

The proof of the following proposition is standard and is relegated to the Appendix.⁵

Proposition 4 *Logic \mathbb{L}_0 is sound with respect to the class of temporal belief revision frames (cf. Definition 2), that is, every theorem of \mathbb{L}_0 is valid in every model based on a temporal belief frame*

Our purpose is to model how the *factual* beliefs of an individual change over time in response to *factual* information. Thus the axioms we introduce are restricted to *Boolean formulas*, which are formulas that do not contain any modal operators. That is, Boolean formulas are defined recursively as follows: (1) every atomic proposition is a Boolean formula, and (2) if ϕ and ψ are Boolean formulas then so are $\neg\phi$ and $(\phi \vee \psi)$. As the following proposition shows, the truth value of a Boolean formula does not change over time: it is only a function of the state. We denote by Φ^B the set of Boolean formulas.

Proposition 5 *Let $\phi \in \Phi^B$. Fix an arbitrary model and suppose that $(\omega, t) \models \phi$. Then, for every $t' \in T$, $(\omega, t') \models \phi$.*

Proof. The proof is by induction on the complexity of ϕ . If $\phi = q$ where q is an atomic proposition, then $(\omega, t) \models q$ if and only if $\omega \in V(q)$ and therefore $(\omega, t') \models q$ for every $t' \in T$. Suppose now that the statement is true for ϕ and ψ , that is, if $(\omega, t) \models \phi$ then $(\omega, t') \models \phi$ for every $t' \in T$ and similarly for ψ . By definition, $(\omega, t) \models \neg\phi$ if and only if $(\omega, t) \not\models \phi$; by the induction hypothesis $(\omega, t) \not\models \phi$ if and only if $(\omega, t') \not\models \phi$ for every $t' \in T$. Hence $(\omega, t') \models \neg\phi$ for every

⁵Completeness issues are not relevant for the results of this paper and are dealt with in a separate paper that studies several extensions of \mathbb{L}_0 besides the two considered here.

$t' \in T$. By definition, $(\omega, t) \models \phi \vee \psi$ if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$; by the induction hypothesis, if $(\omega, t) \models \phi$ then $(\omega, t') \models \phi$ for every $t' \in T$ and if $(\omega, t) \models \psi$ then $(\omega, t') \models \psi$ for every $t' \in T$. Thus $(\omega, t') \models \phi \vee \psi$ for every $t' \in T$. ■

We now introduce two sets of axioms that provide two extensions of logic \mathbb{L}_0 , one of which will be shown to axiomatize the basic sets of AGM postulates and the other the full set. *Note that all of the following axioms apply only to Boolean formulas.*

The first axiom says that factual information is believed. This is known in the literature as Success or Acceptance (‘A’ stands for ‘Acceptance’): if ϕ is a Boolean formula,

$$I\phi \rightarrow B\phi. \quad (A)$$

The second axiom requires the individual not to drop any of his current factual beliefs at any next instant where he is informed of some fact that he currently considers possible (‘ND’ stands for ‘Not Drop’): if ϕ and ψ are Boolean formulas

$$(\neg B\neg\phi \wedge B\psi) \rightarrow \bigcirc(I\phi \rightarrow B\psi). \quad (ND)$$

The third axiom requires that if the individual considers it possible that $(\phi \wedge \neg\psi)$ then at any next instant where he is informed that ϕ he does not believe that ψ , that is, he cannot add new factual beliefs, unless they are implied by the old beliefs and the information received (‘NA’ stands for ‘Not Add’): if ϕ and ψ are Boolean formulas,

$$\neg B\neg(\phi \wedge \neg\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi). \quad (NA)$$

The fourth axiom says that if the individual receives consistent information then his beliefs are consistent, in the sense that he does not simultaneously believe a formula and its negation (‘WC’ stands for ‘Weak Consistency’): if ϕ is a Boolean formula,

$$(I\phi \wedge \neg A\neg\phi) \rightarrow (B\psi \rightarrow \neg B\neg\psi). \quad (WC)$$

We call the following property of temporal belief frames “Qualitative Bayes Rule” (*QBR*): $\forall t_1, t_2 \in T, \forall \omega \in \Omega$,

$$\text{if } t_1 \rightarrow t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \emptyset \text{ then } \mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega). \quad (QBR)$$

The expression “Qualitative Bayes Rule” is motivated by the following observation (see [3]). In a probabilistic setting, let P_{ω, t_1} be the probability measure over a set of states Ω representing the individual’s beliefs at state ω and time t_1 , let

$F \subseteq \Omega$ be an event representing the information received by the individual at a later date t_2 and let P_{ω, t_2} be the posterior probability measure representing the revised beliefs at state ω and date t_2 . Bayes' rule requires that, if $P_{\omega, t_1}(F) > 0$, then, for every event $E \subseteq \Omega$, $P_{\omega, t_2}(E) = \frac{P_{\omega, t_1}(E \cap F)}{P_{\omega, t_1}(F)}$. Bayes' rule thus implies the following (where $\text{supp}(P)$ denotes the support of the probability measure P):

$$\text{if } \text{supp}(P_{\omega, t_1}) \cap F \neq \emptyset, \text{ then } \text{supp}(P_{\omega, t_2}) = \text{supp}(P_{\omega, t_1}) \cap F.$$

If we set $\mathcal{B}_{t_1}(\omega) = \text{supp}(P_{\omega, t_1})$, $F = \mathcal{I}_{t_2}(\omega)$, with $t_1 \rightsquigarrow t_2$, and $\mathcal{B}_{t_2}(\omega) = \text{supp}(P_{\omega, t_2})$ then we get the Qualitative Bayes Rule as stated above. Thus in a probabilistic setting the proposition ‘‘at date t the individual believes ϕ ’’ would be interpreted as ‘‘the individual assigns probability 1 to the event $[\phi]_t \subseteq \Omega$ ’’.

Let \mathbb{L}_b be the logic obtained by adding the above four axioms to the basic logic \mathbb{L}_0 . We denote this by writing $\mathbb{L}_b = \mathbb{L}_0 + A + ND + NA + WC$ (the subscript ‘b’ was chosen because, as shown later, logic \mathbb{L}_b provides an axiomatization of the *basic* set of AGM postulates).

Definition 6 *An \mathbb{L}_b -frame is a temporal belief revision frame that satisfies the following properties:*

- (1) *the Qualitative Bayes Rule,*
- (2) $\forall \omega \in \Omega, \forall t \in T, \mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega),$
- (3) $\forall \omega \in \Omega, \forall t \in T, \text{if } \mathcal{I}_t(\omega) \neq \emptyset \text{ then } \mathcal{B}_t(\omega) \neq \emptyset.$

An \mathbb{L}_b -model is a model based on an \mathbb{L}_b -frame.

Proposition 7 *Logic \mathbb{L}_b is sound with respect to the class of \mathbb{L}_b -frames. That is, every theorem of \mathbb{L}_b is valid in every \mathbb{L}_b -model.*

Proof. By Proposition 4 it is enough to show that the four axioms A , ND , NA and WC are valid in an arbitrary model based on a frame that satisfies the above three properties. Fix an arbitrary such model.

Validity of A . Let $\omega \in \Omega$, $t \in T$ and $\phi \in \Phi^B$ be such that $(\alpha, t) \models I\phi$. Then $\mathcal{I}_t(\alpha) = [\phi]_t$. Hence, by property (2), $\mathcal{B}_t(\alpha) \subseteq [\phi]_t$, that is, $(\alpha, t) \models B\phi$.

Validity of ND . Let $\omega \in \Omega$, $t_1 \in T$ and $\phi, \psi \in \Phi^B$ be such that $(\omega, t_1) \models \neg B\neg\phi \wedge B\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightsquigarrow t_2$ and $(\omega, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\omega) = [\phi]_{t_2}$. Since $(\omega, t_1) \models \neg B\neg\phi$, there exists an $\omega' \in \mathcal{B}_{t_1}(\omega)$ such that $(\omega', t_1) \models \phi$. Since ϕ is Boolean, by Proposition 5, $(\omega', t_2) \models \phi$ and, therefore, $\omega' \in \mathcal{I}_{t_2}(\omega)$. Thus $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \emptyset$ and, by (QBR) , $\mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_1}(\omega)$. Fix an arbitrary $\omega'' \in \mathcal{B}_{t_2}(\omega)$. Then $\omega'' \in \mathcal{B}_{t_1}(\omega)$ and, since $(\omega, t_1) \models B\psi$, $(\omega'', t_1) \models \psi$. Since ψ is Boolean, by Proposition 5, $(\omega'', t_2) \models \psi$. Hence $(\omega, t_2) \models B\psi$.

Validity of NA . Let $\omega \in \Omega$, $t_1 \in T$ and $\phi, \psi \in \Phi^B$ be such that $(\omega, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightsquigarrow t_2$ and suppose that $(\omega, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\omega) = [\phi]_{t_2}$. Since $(\omega, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$, there exists an $\omega' \in \mathcal{B}_{t_1}(\omega)$ such that $(\omega', t_1) \models \phi \wedge \neg\psi$. Since ϕ and ψ are Boolean, by Proposition 5, $(\omega', t_2) \models \phi \wedge \neg\psi$. Thus $\omega' \in \mathcal{I}_{t_2}(\omega)$, so that $\omega' \in \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega)$. By (QBR) , $\omega' \in \mathcal{B}_{t_2}(\omega)$. Thus, since $(\omega', t_2) \models \neg\psi$, $(\omega, t_2) \models \neg B\psi$.

Validity of *WC*. Let $\omega \in \Omega$, $t \in T$ and $\phi \in \Phi^B$ be such that $(\alpha, t) \models I\phi \wedge \neg A\neg\phi$. Then $\mathcal{I}_t(\alpha) = \lceil \phi \rceil_t$ and there exists a β such that $(\beta, t) \models \phi$. Thus $\mathcal{I}_t(\alpha) \neq \emptyset$ and by property (3) $\mathcal{B}_t(\alpha) \neq \emptyset$. Fix an arbitrary formula ψ and suppose that $(\alpha, t) \models B\psi$. Then, $\forall \omega \in \mathcal{B}_t(\alpha)$, $(\omega, t) \models \psi$. Since $\mathcal{B}_t(\alpha) \neq \emptyset$, there exists a $\gamma \in \mathcal{B}_t(\alpha)$. Thus $(\gamma, t) \models \psi$ and hence $(\alpha, t) \models \neg B\neg\psi$. ■

We now strengthen logic \mathbb{L}_b by adding two more axioms.

The first axiom says that if there is a next instant where the individual is informed that $\phi \wedge \psi$ and believes that χ , then at every next instant it must be the case that if the individual is informed that ϕ then he must believe that $(\phi \wedge \psi) \rightarrow \chi$ (we call this axiom *K7* because, as we will show later, it corresponds to AGM postulate (*K*7*): if ϕ , ψ and χ are Boolean formulas,

$$\diamond(I(\phi \wedge \psi) \wedge B\chi) \rightarrow \bigcirc(I\phi \rightarrow B((\phi \wedge \psi) \rightarrow \chi)). \quad (K7)$$

The second axiom says that if there is a next instant where the individual is informed that ϕ , considers $\phi \wedge \psi$ possible and believes that $\psi \rightarrow \chi$, then at every next instant it must be the case that if the individual is informed that $\phi \wedge \psi$ then he believes that χ (we call this axiom *K8* because it corresponds to AGM postulate (*K*8*): if ϕ , ψ and χ are Boolean formulas,

$$\diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)) \rightarrow \bigcirc(I(\phi \wedge \psi) \rightarrow B\chi). \quad (K8)$$

Let \mathbb{L}_{AGM} be the logic obtained by adding the above two axioms to \mathbb{L}_b . Thus $\mathbb{L}_{AGM} = \mathbb{L}_0 + A + ND + NA + WC + K7 + K8$ (the subscript ‘AGM’ was chosen because, as shown later, logic \mathbb{L}_{AGM} provides an axiomatization of the full set of AGM postulates).

Definition 8 An \mathbb{L}_{AGM} -frame is an \mathbb{L}_b -frame (cf. Definition 6) that satisfies the following additional property: $\forall \omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$,

$$\begin{aligned} & \text{if } t_1 \succ t_2, t_1 \succ t_3, \mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha) \text{ and } \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \emptyset \\ & \text{then } \mathcal{B}_{t_3}(\alpha) = \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha). \end{aligned} \quad (CAB)$$

An \mathbb{L}_{AGM} -model is a model based on an \mathbb{L}_{AGM} -frame.⁶

Proposition 9 Logic \mathbb{L}_{AGM} is sound with respect to the class of \mathbb{L}_{AGM} -frames. That is, every theorem of \mathbb{L}_{AGM} is valid in every \mathbb{L}_{AGM} -model.

Proof. By Proposition 7 it is enough to show that axioms *K7* and *K8* are valid in an arbitrary model based on a frame that satisfies (CAB). Fix an arbitrary such model.

Validity of *K7*. Let α and t_1 be such that $(\alpha, t_1) \models \diamond(I(\phi \wedge \psi) \wedge B\chi)$, where ϕ , ψ and χ are Boolean formulas. Then there exists a t_3 such that $t_1 \succ t_3$

⁶In the above property ‘CAB’ stands for ‘Comparison Across Branches’

and $(\alpha, t_3) \models I(\phi \wedge \psi) \wedge B\chi$. Then $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$. Fix an arbitrary t_2 such that $t_1 \succ t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$. Since ϕ and ψ are Boolean, by Proposition 5 $\lceil \phi \wedge \psi \rceil_{t_3} = \lceil \phi \wedge \psi \rceil_{t_2}$. Thus, since $\lceil \phi \wedge \psi \rceil_{t_2} \subseteq \lceil \phi \rceil_{t_2}$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. If $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) = \emptyset$, then, for every $\omega \in \mathcal{B}_{t_2}(\alpha)$, $(\omega, t_2) \models \neg(\phi \wedge \psi)$ and thus $(\omega, t_2) \models (\phi \wedge \psi) \rightarrow \chi$, so that $(\alpha, t_2) \models B((\phi \wedge \psi) \rightarrow \chi)$. If, on the other hand, $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \emptyset$, then, by (CAB), $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) = \mathcal{B}_{t_3}(\alpha)$. Fix an arbitrary $\beta \in \mathcal{B}_{t_2}(\alpha)$. If $(\beta, t_2) \models \neg(\phi \wedge \psi)$ then $(\beta, t_2) \models (\phi \wedge \psi) \rightarrow \chi$. If $(\beta, t_2) \models \phi \wedge \psi$, then, by Proposition 5, $(\beta, t_3) \models \phi \wedge \psi$ and, therefore, $\beta \in \mathcal{I}_{t_3}(\alpha)$. Hence $\beta \in \mathcal{B}_{t_3}(\alpha)$. Since $(\alpha, t_3) \models B\chi$, $(\beta, t_3) \models \chi$ and, therefore, $(\beta, t_3) \models (\phi \wedge \psi) \rightarrow \chi$. Since $(\phi \wedge \psi \rightarrow \chi)$ is Boolean (because ϕ , ψ and χ are), by Proposition 5, $(\beta, t_2) \models (\phi \wedge \psi) \rightarrow \chi$. Thus, since $\beta \in \mathcal{B}_{t_2}(\alpha)$ was chosen arbitrarily, $(\alpha, t_2) \models B(\phi \wedge \psi \rightarrow \chi)$.

Validity of K8. Let ϕ , ψ and χ be Boolean formulas and let α and t_1 be such that $(\alpha, t_1) \models \diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi))$. Then there exists a t_2 such that $t_1 \succ t_2$ and $(\alpha, t_2) \models I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)$. Thus $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$ and there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $(\beta, t_2) \models \phi \wedge \psi$. Fix an arbitrary t_3 such that $t_1 \succ t_3$ and suppose that $(\alpha, t_3) \models I(\phi \wedge \psi)$. Then $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$. Since $\phi \wedge \psi$ is a Boolean formula and $(\beta, t_2) \models \phi \wedge \psi$, by Proposition 5 $(\beta, t_3) \models \phi \wedge \psi$ and therefore $\beta \in \mathcal{I}_{t_3}(\alpha)$. Hence $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \emptyset$. Furthermore, since ϕ is Boolean, by Proposition 5 $\lceil \phi \rceil_{t_3} = \lceil \phi \rceil_{t_2}$. Thus, since $\lceil \phi \wedge \psi \rceil_{t_3} \subseteq \lceil \phi \rceil_{t_3}$ it follows that $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by (CAB), $\mathcal{B}_{t_3}(\alpha) = \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Fix an arbitrary $\gamma \in \mathcal{B}_{t_3}(\alpha)$. Then $\gamma \in \mathcal{B}_{t_2}(\alpha)$ and, since $(\alpha, t_2) \models B(\psi \rightarrow \chi)$, $(\gamma, t_2) \models \psi \rightarrow \chi$. Since $\psi \rightarrow \chi$ is a Boolean formula, by Proposition 5 $(\gamma, t_3) \models \psi \rightarrow \chi$. Since $\mathcal{B}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_3}(\alpha)$ (by definition of \mathbb{L}_{AGM} -frame) and $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$, $(\gamma, t_3) \models \psi$. Thus $(\gamma, t_3) \models \chi$. Hence $(\alpha, t_3) \models B\chi$. ■

We end this section with a Lemma that will be used later.

Lemma 10 *In any logic where B is a normal operator (that is, it satisfies axiom K and the rule of necessitation) the following is a theorem:*

$$(B\phi \wedge \neg B\neg\psi) \rightarrow \neg B\neg(\phi \wedge \psi).$$

Proof. ('PL' stands for 'Propositional Logic')

- | | |
|--|-----------------------|
| 1. $B\phi \wedge B(\phi \rightarrow \neg\psi) \rightarrow B\neg\psi$ | Axiom K for B |
| 2. $B\phi \rightarrow (B(\phi \rightarrow \neg\psi) \rightarrow B\neg\psi)$ | 1, PL |
| 3. $(B(\phi \rightarrow \neg\psi) \rightarrow B\neg\psi) \rightarrow (\neg B\neg\psi \rightarrow \neg B(\phi \rightarrow \neg\psi))$ | tautology |
| 4. $B\phi \rightarrow (\neg B\neg\psi \rightarrow \neg B(\phi \rightarrow \neg\psi))$ | 2, 3, PL |
| 5. $(B\phi \wedge \neg B\neg\psi) \rightarrow \neg B(\phi \rightarrow \neg\psi)$ | 4, PL |
| 6. $\neg(\phi \wedge \psi) \rightarrow (\phi \rightarrow \neg\psi)$ | tautology |
| 7. $B\neg(\phi \wedge \psi) \rightarrow B(\phi \rightarrow \neg\psi)$ | 6, RK (cf. Remark. 3) |
| 8. $\neg B(\phi \rightarrow \neg\psi) \rightarrow \neg B\neg(\phi \wedge \psi)$ | 7, PL |
| 9. $(B\phi \wedge \neg B\neg\psi) \rightarrow \neg B\neg(\phi \wedge \psi)$ | 5, 8, PL. ■ |

4 Axiomatization of the AGM theory

The AGM theory of belief revision was developed within the framework of belief sets. Let Γ be the set of formulas in a propositional language. Given a subset $F \subseteq \Gamma$, its PL-deductive closure $[F]^{PL}$ is defined as follows: $\psi \in [F]^{PL}$ if and only if there exist $\phi_1, \dots, \phi_n \in F$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $F \subseteq \Gamma$ is *consistent* if $[F]^{PL} \neq \Gamma$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[F]^{PL}$). A set $F \subseteq \Gamma$ is *deductively closed* if $F = [F]^{PL}$. Given a consistent and deductively closed set K (thought of as the initial beliefs of the individual) and a formula ϕ (thought of as a new piece of information), the *revision of K by ϕ* , denoted by K_ϕ^* , is a subset of Γ that satisfies the following conditions, known as the AGM postulates:

- (K*1) K_ϕ^* is deductively closed
- (K*2) $\phi \in K_\phi^*$
- (K*3) $K_\phi^* \subseteq [K \cup \{\phi\}]^{PL}$
- (K*4) if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K_\phi^*$
- (K*5) $K_\phi^* = \Gamma$ if and only if ϕ is a contradiction
- (K*6) if $\phi \leftrightarrow \psi$ is a tautology then $K_\phi^* = K_\psi^*$
- (K*7) $K_{\phi \wedge \psi}^* \subseteq [K_\phi^* \cup \{\psi\}]^{PL}$
- (K*8) if $\neg\psi \notin K_\phi^*$, then $[K_\phi^* \cup \{\psi\}]^{PL} \subseteq K_{\phi \wedge \psi}^*$.

(K*1) requires the revised belief set to be deductively closed. In our framework this corresponds to requiring the B operator to be a normal operator, that is, to satisfy axiom K and the inference rule Necessitation.

(K*2) requires that the information be believed. In our framework, this corresponds to imposing the Acceptance axiom (for Boolean ϕ): $I\phi \rightarrow B\phi$.

(K*3) says that beliefs should be revised minimally, in the sense that no new belief should be added unless it can be deduced from the information received and the initial beliefs. As shown below, this requirement corresponds to our axiom NA (for Boolean ϕ and ψ): $\neg B\neg(\phi \wedge \neg\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi)$.

(K*4) says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs. As shown below, this requirement corresponds to our axiom ND (for Boolean ϕ and ψ): $(\neg B\neg\phi \wedge B\psi) \rightarrow \bigcirc(I\phi \rightarrow B\psi)$.

(K*5) requires the revised beliefs to be consistent, unless the information ϕ is contradictory (that is, $\neg\phi$ is a tautology). This corresponds to our axiom WC (for Boolean ϕ): $(I\phi \wedge \neg A\neg\phi) \rightarrow (B\psi \rightarrow \neg B\neg\psi)$.

(K*6) is automatically satisfied in our framework, since if $\phi \leftrightarrow \psi$ is a tautology then $\|\phi\| = \|\psi\|$ in every model and therefore the formula $I\phi \leftrightarrow I\psi$ is valid. Hence revision based on ϕ must coincide with revision based on ψ .

(K*7) and (K*8) are a generalization of (K*3) and (K*4) that

“applies to *iterated* changes of belief. The idea is that if K_ϕ^* is a revision of K and K_ϕ^* is to be changed by adding further sentences, such a change should be made by using expansions of K_ϕ^* whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $K_{\phi \wedge \psi}^*$) ought to be the same as the expansion of K_ϕ^* by ψ , so long as ψ does not contradict the beliefs in K_ϕ^* ” (Gärdenfors [5], p. 55).⁷

We will show below that (K*7) corresponds to our axiom $K7$ and (K*8) to axiom $K8$.

The set of postulates (K*1) through (K*6) is called the *basic set* of postulates for belief revision (Gärdenfors, [5] p. 55).

The following proposition shows that logic \mathbb{L}_b provides an axiomatization of the set of basic postulates.

Proposition 11 *Logic \mathbb{L}_b axiomatizes the set of basic AGM postulates (K*1)-(K*6), in the sense that both (A) and (B) below hold:*

(A) *Fix an arbitrary \mathbb{L}_b -model and let $t_1, t_2 \in T$ and $\alpha \in \Omega$ be such that $t_1 \succ t_2$ and $(\alpha, t_2) \models I\phi$, with $\phi \in \Phi^B$ (where $\Phi^B \subseteq \Phi$ denotes the subset of Boolean formulas). Define $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$ and $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$. Then K_ϕ^* satisfies AGM postulates (K*1)-(K*6).*

(B) *Let $K \subseteq \Phi^B$ be a consistent and deductively closed set and $\phi \in \Phi^B$. If $K_\phi^* \subseteq \Phi^B$ satisfies AGM postulates (K*1)-(K*6) then there is an \mathbb{L}_b -model, $t_1, t_2 \in T$ and $\alpha \in \Omega$ such that*

(B.1) $t_1 \succ t_2$

(B.2) $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$

(B.3) $(\alpha, t_2) \models I\phi$

(B.4) $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$

(B.5) *if ϕ is consistent then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$.*

Proof. (A) Fix an \mathbb{L}_b -model and let $t_1, t_2 \in T$ and $\alpha \in \Omega$ be such that $t_1 \succ t_2$ and $(\alpha, t_2) \models I\phi$, with $\phi \in \Phi^B$. Let $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$ and $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$. We need to prove that AGM postulates (K*1)-(K*6) are satisfied.

(K*1): we need to show that K_ϕ^* is deductively closed, that is, $K_\phi^* = [K_\phi^*]^{PL}$.

If $\psi \in K_\phi^*$ then $\psi \in [K_\phi^*]^{PL}$, because $\psi \rightarrow \psi$ is a tautology. Now let $\psi \in [K_\phi^*]^{PL}$. Then there exist $\phi_1, \dots, \phi_n \in K_\phi^*$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology, hence a theorem of \mathbb{L}_b . Then, by necessitation for B and Proposition 7, $(\alpha, t_2) \models B((\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi)$. By definition of K_ϕ^* , since $\phi_1, \dots, \phi_n \in K_\phi^*$,

⁷The expansion of K_ϕ^* by ψ is $[K_\phi^* \cup \{\psi\}]^{PL}$.

$(\alpha, t_2) \models B(\phi_1 \wedge \dots \wedge \phi_n)$. By axiom K for B and Proposition 7, $(\alpha, t_2) \models B((\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi) \wedge B(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow B\psi$. Thus $(\alpha, t_2) \models B\psi$, that is, $\psi \in K_\phi^*$.

(K*2): we need to show that $\phi \in K_\phi^*$, that is, $(\alpha, t_2) \models B\phi$. By Axiom A and Proposition 7, $(\alpha, t_2) \models I\phi \rightarrow B\phi$ and by hypothesis $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models B\phi$.

(K*3): we need to show that $K_\phi^* \subseteq [K \cup \{\phi\}]^{PL}$. Let $\psi \in K_\phi^*$, i.e. $(\alpha, t_2) \models B\psi$. First of all, note that axiom NA is tautologically equivalent to $\diamond(I\phi \wedge B\psi) \rightarrow B(\phi \rightarrow \psi)$. Thus, by Proposition 7, $(\alpha, t_1) \models \diamond(I\phi \wedge B\psi) \rightarrow B(\phi \rightarrow \psi)$. By hypothesis, $(\alpha, t_2) \models I\phi \wedge B\psi$ and $t_1 \succ t_2$. Thus $(\alpha, t_1) \models \diamond(I\phi \wedge B\psi)$ and, therefore, $(\alpha, t_1) \models B(\phi \rightarrow \psi)$, that is $(\phi \rightarrow \psi) \in K$. Hence $\{\phi, (\phi \rightarrow \psi)\} \in K \cup \{\phi\}$ so that, since $(\phi \wedge (\phi \rightarrow \psi)) \rightarrow \psi$ is a tautology, $\psi \in [K \cup \{\phi\}]^{PL}$.

(K*4): we need to show that if $\neg\phi \notin K$ then $[K \cup \{\phi\}]^{PL} \subseteq K_\phi^*$. Suppose that $\neg\phi \notin K$, that is, $(\alpha, t_1) \models \neg B\neg\phi$. First of all, note that axiom ND is tautologically equivalent to $\neg B\neg\phi \rightarrow (B\psi \rightarrow \bigcirc(I\phi \rightarrow B\psi))$. Thus, by Proposition 7, $(\alpha, t_1) \models \neg B\neg\phi \rightarrow (B\psi \rightarrow \bigcirc(I\phi \rightarrow B\psi))$. Hence

$$(\alpha, t_1) \models B\psi \rightarrow \bigcirc(I\phi \rightarrow B\psi), \text{ for every Boolean formula } \psi. \quad (*)$$

Let $\chi \in [K \cup \{\phi\}]^{PL}$, that is, there exist $\phi_1, \dots, \phi_n \in K \cup \{\phi\}$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi$ is a tautology. We want to show that $\chi \in K_\phi^*$, i.e. $(\alpha, t_2) \models B\chi$. Since $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi$ is a tautology, by necessitation for B and Proposition 7, $(\alpha, t_1) \models B((\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi)$. If $\phi_i \in K$ for every $i = 1, \dots, n$, then $(\alpha, t_1) \models B(\phi_1 \wedge \dots \wedge \phi_n)$ and therefore (using axiom K for B and Proposition 7) $(\alpha, t_1) \models B\chi$. Thus, by (*), $(\alpha, t_1) \models \bigcirc(I\phi \rightarrow B\chi)$ so that, since $t_1 \succ t_2$, $(\alpha, t_2) \models I\phi \rightarrow B\chi$. Since, by hypothesis, $(\alpha, t_2) \models I\phi$, it follows that $(\alpha, t_2) \models B\chi$, i.e. $\chi \in K_\phi^*$. If $\phi_i \notin K$, for some $i = 1, \dots, n$ then we can assume (renumbering the formulas, if necessary) that $\phi_n \notin K$, which implies that $\phi_n = \phi$, and $\phi_1, \dots, \phi_{n-1} \in K$, so that $(\alpha, t_1) \models B(\phi_1 \wedge \dots \wedge \phi_{n-1})$. Since, by hypothesis, $(\phi_1 \wedge \dots \wedge \phi_{n-1} \wedge \phi) \rightarrow \chi$ is a tautology and is tautologically equivalent to $(\phi_1 \wedge \dots \wedge \phi_{n-1}) \rightarrow (\phi \rightarrow \chi)$, by necessitation for B and Proposition 7 $(\alpha, t_1) \models B((\phi_1 \wedge \dots \wedge \phi_{n-1}) \rightarrow (\phi \rightarrow \chi))$. Thus $(\alpha, t_1) \models B(\phi \rightarrow \chi)$ (appealing, once again, to axiom K for B and Proposition 7). Hence, by (*) (with $\psi = (\phi \rightarrow \chi)$), $(\alpha, t_1) \models \bigcirc(I\phi \rightarrow B(\phi \rightarrow \chi))$. Since $t_1 \succ t_2$, $(\alpha, t_2) \models I\phi \rightarrow B(\phi \rightarrow \chi)$. By hypothesis, $\alpha \models I\phi$ and by (K*2) (proved above), $(\alpha, t_2) \models B\phi$. Thus $(\alpha, t_2) \models B(\phi \rightarrow \chi) \wedge B\phi$. By axiom K and Proposition 7, $(\alpha, t_2) \models (B(\phi \rightarrow \chi) \wedge B\phi) \rightarrow B\chi$. Hence $(\alpha, t_2) \models B\chi$, i.e. $\chi \in K_\phi^*$.

(K*5): we have to show that $K_\phi^* \neq \Phi^B$, unless ϕ is a contradiction (that is, $\neg\phi$ is a tautology). If ϕ is a contradiction, then $\|\phi\| = \emptyset$ and since, by hypothesis, $(\alpha, t_2) \models I\phi$, $\mathcal{I}_{t_2}(\alpha) = \emptyset$. By definition of \mathbb{L}_b -model, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_2}(\alpha) = \emptyset$ and therefore $(\alpha, t_2) \models B\psi$ for every formula ψ . Hence $K_\phi^* = \Phi^B$. If ϕ is not a contradiction, then, by hypothesis (B.5), $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$. Since ϕ is Boolean, by Proposition 5, $(\beta, t_2) \models \phi$. Thus $(\alpha, t_2) \models \neg A\neg\phi$. By hypothesis, $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models I\phi \wedge \neg A\neg\phi$. By axiom WC and Proposition 7, $(\alpha, t_2) \models (I\phi \wedge \neg A\neg\phi) \rightarrow (B\psi \rightarrow \neg B\neg\psi)$. Thus

$(\alpha, t_2) \models B\psi \rightarrow \neg B\neg\psi$ for every formula ψ . Hence if $\psi \in K_\phi^*$ then $\neg\psi \notin K_\phi^*$. Since, by (K*2) (proved above), $\phi \in K_\phi^*$, it follows that $\neg\phi \notin K_\phi^*$ and therefore $K_\phi^* \neq \Phi$.

(K*6): we have to show that if $\phi \leftrightarrow \psi$ is a tautology then $K_\phi^* = K_\psi^*$. If $\phi \leftrightarrow \psi$ is a tautology, then $\|\phi \leftrightarrow \psi\| = \Omega \times T$, so that $[\phi]_{t_2} = [\psi]_{t_2}$. Thus $(\alpha, t_2) \models I\phi$ if and only if $(\alpha, t_2) \models I\psi$. Hence, by definition, $K_\phi^* = K_\psi^*$.

(B) Next we prove the converse, namely that if $K \subseteq \Phi^B$ is a consistent and deductively closed set, $\phi \in \Phi^B$ and $K_\phi^* \subseteq \Phi^B$ satisfies AGM postulates (K*1)-(K*6) then there is an \mathbb{L}_b -model, $t_1, t_2 \in T$ and $\alpha \in \Omega$ such that (B.1)-(B.5) above are satisfied. Let \mathbb{M}_B^{PL} be the set of maximally consistent sets of formulas for a propositional logic whose set of formulas is Φ^B . For any $F \subseteq \Phi^B$ let $\mathbb{M}_F = \{\omega \in \mathbb{M}_B^{PL} : F \subseteq \omega\}$. By Lindenbaum's lemma, $\mathbb{M}_F \neq \emptyset$ if and only if F is a consistent set, that is, $[F]^{PL} \neq \Phi^B$.

Define the following belief revision frame: $T = \{t_1, t_2\}$, $\succrightarrow = \{(t_1, t_2)\}$, $\Omega = \mathbb{M}_B^{PL}$ and, for every $\omega \in \Omega$,

$$\begin{aligned} \mathcal{B}_{t_1}(\omega) &= \mathcal{I}_{t_1}(\omega) = \mathbb{M}_K \\ \mathcal{I}_{t_2}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi & \text{otherwise} \end{cases} \\ \mathcal{B}_{t_2}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K_\phi^*} & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases} \end{aligned}$$

First we show that this frame is an \mathbb{L}_b -frame (cf. Definition 6).

The Qualitative Bayes Rule is clearly satisfied, since $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, in which case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K = \mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_1}(\omega)$.

The property that $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$ (for every ω and t) is also satisfied: the only case where, possibly, $\mathcal{B}_t(\omega) \neq \mathcal{I}_t(\omega)$ is when $t = t_2$ and ϕ is a consistent formula. In this case, there are two possibilities: (1) $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$ and (2) $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$. In case (1) $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K \subseteq \mathbb{M}_\phi = \mathcal{I}_{t_2}(\omega)$. In case (2) $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K_\phi^*}$ and $\mathcal{I}_{t_2}(\omega) = \mathbb{M}_\phi$. Now, if $\omega \in \mathbb{M}_{K_\phi^*}$ then $K_\phi^* \subseteq \omega$ and, since by AGM postulate (K*2), $\phi \in K_\phi^*$, it follows that $\phi \in \omega$, that is, $\omega \in \mathbb{M}_\phi$. Hence $\mathbb{M}_{K_\phi^*} \subseteq \mathbb{M}_\phi$.

Finally, the property that, for every ω and t , $\mathcal{B}_t(\omega) \neq \emptyset$ whenever $\mathcal{I}_t(\omega) \neq \emptyset$ is also satisfied. If $t = t_1$, trivially because $\mathcal{B}_{t_1}(\omega) = \mathcal{I}_{t_1}(\omega)$. If $t = t_2$, $\mathcal{I}_{t_2}(\omega) \neq \emptyset$ if and only if ϕ is a consistent formula; in this case either $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$, if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, or $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K_\phi^*}$, in which case by AGM postulate (K*5) K_ϕ^* is a consistent set and therefore, by Lindenbaum's lemma, $\mathbb{M}_{K_\phi^*} \neq \emptyset$.

Now define the following model based on this frame: for every atomic proposition q , for every $\omega \in \Omega$ and for every $t \in T$, $(\omega, t) \models q$ if and only if $q \in \omega$. By a straightforward induction argument (cf. the proof of Proposition 5) it can be shown that, for every Boolean formula $\psi \in \Phi^B$, for every $\omega \in \Omega$ and for every $t \in T$,

$$(\omega, t) \models \psi \text{ if and only if } \psi \in \omega. \quad (**)$$

We also note the following (see Theorem 2.20 (1) in Chellas, [4], p. 57): $\forall F \subseteq \Phi^B, \forall \psi \in \Phi^B,$

$$\psi \in [F]^{PL} \text{ if and only if } \psi \in \omega, \forall \omega \in \mathbb{M}_F. \quad (***)$$

Fix an arbitrary $\alpha \in \Omega$. We want to show that properties (B.1)-(B.5) are satisfied.

(1): $t_1 \rightsquigarrow t_2$ by construction.

(2): we need to show that $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$. First we show that $K \subseteq \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$. Let $\psi \in K$. Then $\psi \in \omega$ for every $\omega \in \mathbb{M}_K$. Thus, since $\mathcal{B}_{t_1}(\alpha) = \mathbb{M}_K$, it follows from (***) that $(\alpha, t_1) \models B\psi$. Next we show that if $\psi \in \Phi^B$ and $(\alpha, t_1) \models B\psi$ then $\psi \in K$. Since $\mathcal{B}_{t_1}(\alpha) = \mathbb{M}_K$, by (***) $\psi \in \omega$ for every $\omega \in \mathbb{M}_K$. Thus, by (***), $\psi \in [K]^{PL}$. By hypothesis, K is deductively closed, that is, $K = [K]^{PL}$. Hence $\psi \in K$.

(3): we need to show that $(\alpha, t_2) \models I\phi$. By (**), for every $\omega \in \Omega$, $(\omega, t_2) \models \phi$ if and only if $\phi \in \omega$, that is, $\omega \in \mathbb{M}_\phi$. Since, by construction, $\mathcal{I}_{t_2}(\alpha) = \mathbb{M}_\phi$, it follows that $(\alpha, t_2) \models I\phi$.

(4): we need to show that $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$. There are several cases to be considered.

(4.i) ϕ is a contradiction. Then, by AGM postulate (K*5), $K_\phi^* = \Phi^B$ and, by construction, $\mathcal{B}_{t_2}(\alpha) = \emptyset$, so that $(\alpha, t_2) \models B\psi$ for every formula ψ . Hence $\{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\} = \Phi^B = K_\phi^*$.

(4.ii) ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$. In this case, by construction, $\mathcal{B}_{t_2}(\alpha) = \mathbb{M}_{K_\phi^*}$. If $\psi \in K_\phi^*$ then $\psi \in \omega$ for all $\omega \in \mathbb{M}_{K_\phi^*}$ and, therefore, $(\alpha, t_2) \models B\psi$. Conversely, if $(\alpha, t_2) \models B\psi$ then by (***) $\psi \in \omega$ for all $\omega \in \mathbb{M}_{K_\phi^*}$ and, therefore,

by (***) $\psi \in [K_\phi^*]^{PL}$. By AGM postulate (K*1), $K_\phi^* = [K_\phi^*]^{PL}$. Thus $\psi \in K_\phi^*$.

(4.iii) ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, in which case $\mathcal{B}_{t_2}(\alpha) = \mathbb{M}_\phi \cap \mathbb{M}_K$. First of all, note that $\mathbb{M}_\phi \cap \mathbb{M}_K = \mathbb{M}_{K \cup \{\phi\}}$. Secondly, it must be that $\neg\phi \notin K$ (if $\neg\phi \in K$ then $\neg\phi \in \omega$ for every $\omega \in \mathbb{M}_K$ and therefore $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$). Hence, by AGM postulates (K*3) and (K*4), $K_\phi^* = [K \cup \{\phi\}]^{PL}$. By (***), for every Boolean formula ψ , $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $\psi \in \omega$, for all $\omega \in \mathbb{M}_{K \cup \{\phi\}}$. Hence $\{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\} = [K \cup \{\phi\}]^{PL} = K_\phi^*$.

(5): we need to show that, if ϕ is consistent, then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$. If ϕ is consistent, then by Lindenbaum's lemma, there exists a $\beta \in \mathbb{M}_\phi^{PL}$ such that $\phi \in \beta$. By (**), $(\beta, t) \models \phi$ for all $t \in T$. ■

The following proposition shows that logic \mathbb{L}_{AGM} provides an axiomatization of the AGM theory of belief revision.

Proposition 12 *Logic \mathbb{L}_{AGM} provides an axiomatization of the full set of AGM postulates (K*1)-(K*8), in the sense that both (A) and (B) below hold:*

(A) *Fix an arbitrary \mathbb{L}_{AGM} -model and let $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ be such that $t_1 \rightsquigarrow t_2$, $t_1 \rightsquigarrow t_3$, $(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$ with $\phi, \psi \in \Phi^B$.*

Define $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$, $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$ and $K_{\phi \wedge \psi}^* = \{\chi \in \Phi^B : (\alpha, t_3) \models B\chi\}$. Then K_ϕ^* and $K_{\phi \wedge \psi}^*$ satisfy AGM postulates (K*1)-(K*8).

(B). Let $K \subseteq \Phi^B$ be a consistent and deductively closed set and $\phi, \psi \in \Phi^B$. If $K_\phi^*, K_{\phi \wedge \psi}^* \subseteq \Phi^B$ satisfy AGM postulates (K*1)-(K*8) then there is an \mathbb{L}_{AGM} -model, $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ such that

(B.1) $t_1 \succ t_2$

(B.2) $K = \{\chi \in \Phi^B : (\alpha, t_1) \models B\chi\}$

(B.3) $(\alpha, t_2) \models I\phi$

(B.4) $K_\phi^* = \{\chi \in \Phi^B : (\alpha, t_2) \models B\chi\}$

(B.5) if ϕ is consistent then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$

(B.6) $t_1 \succ t_3$

(B.7) $(\alpha, t_3) \models I(\phi \wedge \psi)$

(B.8) $K_{\phi \wedge \psi}^* = \{\chi \in \Phi^B : (\alpha, t_3) \models B\chi\}$.

Proof. (A) Fix an arbitrary \mathbb{L}_{AGM} -model and let $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ be such that $t_1 \succ t_2, t_1 \succ t_3, (\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$ with $\phi, \psi \in \Phi^B$. Define $K = \{\psi \in \Phi^B : (\alpha, t_1) \models B\psi\}$, $K_\phi^* = \{\psi \in \Phi^B : (\alpha, t_2) \models B\psi\}$ and $K_{\phi \wedge \psi}^* = \{\chi \in \Phi^B : (\alpha, t_3) \models B\chi\}$. We need to show that K_ϕ^* and $K_{\phi \wedge \psi}^*$ satisfy AGM postulates (K*1)-(K*8). The proof that AGM postulates (K*1)-(K*6) are satisfied is the same as in Proposition 11 (every \mathbb{L}_{AGM} -model is an \mathbb{L}_b -model). Thus we shall only prove that AGM postulates (K*7) and (K*8) are satisfied.

First we show that (K*7) is satisfied, that is, that $K_{\phi \wedge \psi}^* \subseteq [K_\phi^* \cup \{\psi\}]^{PL}$. Fix an arbitrary $\chi \in K_{\phi \wedge \psi}^*$, that is, $(\alpha, t_3) \models B\chi$. By hypothesis, $(\alpha, t_3) \models I(\phi \wedge \psi)$. Thus $(\alpha, t_3) \models I(\phi \wedge \psi) \wedge B\chi$ and, since $t_1 \succ t_3, (\alpha, t_1) \models \diamond(I(\phi \wedge \psi) \wedge B\chi)$. By axiom K7 and Proposition 9, $(\alpha, t_1) \models \diamond(I(\phi \wedge \psi) \wedge B\chi) \rightarrow \bigcirc(I\phi \rightarrow B((\phi \wedge \psi) \rightarrow \chi))$. Hence $(\alpha, t_1) \models \bigcirc(I\phi \rightarrow B((\phi \wedge \psi) \rightarrow \chi))$, from which it follows, since $t_1 \succ t_2$, that $(\alpha, t_2) \models I\phi \rightarrow B((\phi \wedge \psi) \rightarrow \chi)$. By hypothesis, $(\alpha, t_2) \models I\phi$. Hence $(\alpha, t_2) \models B(\phi \rightarrow (\psi \rightarrow \chi))$ [since $(\phi \wedge \psi) \rightarrow \chi$ is tautologically equivalent to $\phi \rightarrow (\psi \rightarrow \chi)$]. By axiom A and Proposition 9, $(\alpha, t_2) \models I\phi \rightarrow B\phi$ and by hypothesis $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models B\phi$. By axiom K and Proposition 9, $(\alpha, t_2) \models (B(\phi \rightarrow (\psi \rightarrow \chi)) \wedge B\phi) \rightarrow B(\psi \rightarrow \chi)$. Thus $(\alpha, t_2) \models B(\psi \rightarrow \chi)$, that is, $(\psi \rightarrow \chi) \in K_\phi^*$. Hence $\chi \in [K_\phi^* \cup \{\psi\}]^{PL}$.

Next we prove that (K*8) is satisfied, that is, that if $\neg\psi \notin K_\phi^*$ then $[K_\phi^* \cup \{\psi\}]^{PL} \subseteq K_{\phi \wedge \psi}^*$. Fix an arbitrary $\chi \in [K_\phi^* \cup \{\psi\}]^{PL}$. Then there exist $\phi_1, \dots, \phi_n \in K_\phi^* \cup \{\psi\}$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi$ is a tautology. If $\phi_i \in K_\phi^*$ for every $i = 1, \dots, n$ then, since by AGM postulate (K*1) K_ϕ^* is deductively closed (that is, $K_\phi^* = [K_\phi^*]^{PL}$), $\chi \in K_\phi^*$ and thus $(\psi \rightarrow \chi) \in K_\phi^*$ (since $\chi \rightarrow (\psi \rightarrow \chi)$ is a tautology). If $\phi_i \notin K_\phi^*$ for some i then we can assume (renumbering the formulas, if necessary) that $\phi_n \notin K_\phi^*$, from which it follows that $\phi_n = \psi$. Since, by hypothesis, $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi$ is a tautology and it is tautologically equivalent to $(\phi_1 \wedge \dots \wedge \phi_{n-1}) \rightarrow (\phi_n \rightarrow \chi)$ and $\phi_n = \psi$, it follows that $(\psi \rightarrow \chi) \in [K_\phi^*]^{PL} = K_\phi^*$. Thus

$$(\psi \rightarrow \chi) \in K_\phi^*, \text{ that is, } (\alpha, t_2) \models B(\psi \rightarrow \chi). \quad (\dagger)$$

By hypothesis, $\neg\psi \notin K_\phi^*$, that is, $(\alpha, t_2) \models \neg B\neg\psi$. By axiom *A* and Proposition 9, $(\alpha, t_2) \models I\phi \rightarrow B\phi$ and by hypothesis $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models B\phi$ and, therefore, $(\alpha, t_2) \models B\phi \wedge \neg B\neg\psi$. By Lemma 10 and Proposition 9, $(\alpha, t_2) \models (B\phi \wedge \neg B\neg\psi) \rightarrow \neg B\neg(\phi \wedge \psi)$. Thus

$$(\alpha, t_2) \models \neg B\neg(\phi \wedge \psi). \quad (\dagger\dagger)$$

By hypothesis, $(\alpha, t_2) \models I\phi$. This, together with (\dagger) and $(\dagger\dagger)$ yields $(\alpha, t_2) \models I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)$. Hence, since $t_1 \succ t_2$, $(\alpha, t_1) \models \diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi))$. By axiom *K8* and Proposition 9, $(\alpha, t_1) \models \diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)) \rightarrow \bigcirc(I(\phi \wedge \psi) \rightarrow B\chi)$. Thus $(\alpha, t_1) \models \bigcirc(I(\phi \wedge \psi) \rightarrow B\chi)$ from which it follows, since $t_1 \succ t_3$, that $(\alpha, t_3) \models I(\phi \wedge \psi) \rightarrow B\chi$. By hypothesis, $(\alpha, t_3) \models I(\phi \wedge \psi)$. Hence $(\alpha, t_3) \models B\chi$, that is, $\chi \in K_{\phi \wedge \psi}^*$.

(B) Next we prove the converse, namely that if $K \subseteq \Phi^B$ is consistent and deductively closed, $\phi, \psi \in \Phi^B$ and $K_\phi^*, K_{\phi \wedge \psi}^* \subseteq \Phi^B$ satisfy AGM postulates (K*1)-(K*8) then there is an \mathbb{L}_{AGM} -model, $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ such that (B.1)-(B.8) are satisfied. We proceed as in the proof of Proposition 11. Thus \mathbb{M}_B^{PL} denotes the set of maximally consistent sets of formulas for a propositional logic whose set of formulas is Φ^B and, for $F \subseteq \Phi^B$, $\mathbb{M}_F = \{\omega \in \mathbb{M}_B^{PL} : F \subseteq \omega\}$. Define the following belief revision frame: $T = \{t_1, t_2, t_3\}$, $\succ = \{(t_1, t_2), (t_1, t_3)\}$, $\Omega = \mathbb{M}_B^{PL}$ and, for every $\omega \in \Omega$,

$$\begin{aligned} \mathcal{B}_{t_1}(\omega) &= \mathcal{I}_{t_1}(\omega) = \mathbb{M}_K \\ \mathcal{I}_{t_2}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi & \text{otherwise} \end{cases} \\ \mathcal{B}_{t_2}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K_\phi^*} & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases} \\ \mathcal{I}_{t_3}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \wedge \psi \text{ is a contradiction} \\ \mathbb{M}_{\phi \wedge \psi} & \text{otherwise} \end{cases} \\ \mathcal{B}_{t_3}(\omega) =_{t_2}(\omega) &= \begin{cases} \emptyset & \text{if } \phi \wedge \psi \text{ is a contradiction} \\ \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K & \text{if } \phi \wedge \psi \text{ is consistent and } \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K_{\phi \wedge \psi}^*} & \text{if } \phi \wedge \psi \text{ is consistent and } \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \emptyset \end{cases} \end{aligned}$$

First we show that this frame is an \mathbb{L}_{AGM} -frame (cf. Definition 8). Note that \mathcal{B}_{t_1} , \mathcal{I}_{t_1} , \mathcal{B}_{t_2} and \mathcal{I}_{t_2} are the same as in the \mathbb{L}_b -frame defined in the proof of Proposition 11. Thus we only need to focus on the additional elements.

The Qualitative Bayes Rule is satisfied, since $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_K \cap \mathbb{M}_{\phi \wedge \psi} \neq \emptyset$, in which case $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_K \cap \mathbb{M}_{\phi \wedge \psi} = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_3}(\omega)$.

The property that, for every ω and t , $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$ is also satisfied. The only case left to examine is the case where $t = t_3$ and $\phi \wedge \psi$ is a consistent formula. If $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K \neq \emptyset$, then $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K \subseteq \mathbb{M}_{\phi \wedge \psi} = \mathcal{I}_{t_3}(\omega)$. If $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \emptyset$ then $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K_{\phi \wedge \psi}^*}$ and $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi}$. Now, if $\omega \in \mathbb{M}_{K_{\phi \wedge \psi}^*}$ then $K_{\phi \wedge \psi}^* \subseteq \omega$ and, since by AGM postulate (K*2), $\phi \wedge \psi \in K_{\phi \wedge \psi}^*$, it follows that $\phi \wedge \psi \in \omega$, that is, $\omega \in \mathbb{M}_{\phi \wedge \psi}$. Hence $\mathbb{M}_{K_{\phi \wedge \psi}^*} \subseteq \mathbb{M}_{\phi \wedge \psi}$.

The property that, for every ω and t , $\mathcal{B}_t(\omega) \neq \emptyset$ whenever $\mathcal{I}_t(\omega) \neq \emptyset$ is also satisfied. The only case left to examine is the case where $t = t_3$. Now, $\mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\phi \wedge \psi$ is a consistent formula; in this case either $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K$, if $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K \neq \emptyset$, or $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K_{\phi \wedge \psi}^*}$, in which case by AGM postulate (K*5) $K_{\phi \wedge \psi}^*$ is a consistent set and therefore, by Lindenbaum's lemma, $\mathbb{M}_{K_{\phi \wedge \psi}^*} \neq \emptyset$.

Next we have to show that the \mathbb{L}_{AGM} -specific property is satisfied, namely that if t_1, t, t' and ω are such that $t_1 \succ t$, $t_1 \succ t'$, $\mathcal{I}_{t'}(\omega) \subseteq \mathcal{I}_t(\omega)$ and $\mathcal{I}_{t'}(\omega) \cap \mathcal{B}_t(\omega) \neq \emptyset$ then $\mathcal{B}_{t'}(\omega) = \mathcal{I}_{t'}(\omega) \cap \mathcal{B}_t(\omega)$.

We start with $t = t_3$ and $t' = t_2$. In this case the joint condition $\mathcal{I}_{t_2}(\omega) \subseteq \mathcal{I}_{t_3}(\omega)$ and $\mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_3}(\omega) \neq \emptyset$ holds if and only if $(\phi \wedge \psi)$ is consistent (implying that ϕ is consistent) and $\mathbb{M}_\phi \subseteq \mathbb{M}_{\phi \wedge \psi}$, which implies that $\mathbb{M}_{\phi \wedge \psi} = \mathbb{M}_\phi$. This, in turn, implies that $(\phi \wedge \psi) \leftrightarrow \phi$ is a tautology, so that, by AGM postulate (K*6), $K_{\phi \wedge \psi}^* = K_\phi^*$. Then

$$\mathcal{B}_{t_2}(\omega) = \begin{cases} \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K_\phi^*} & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases}$$

and

$$\mathcal{B}_{t_3}(\omega) = \begin{cases} \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K_{\phi \wedge \psi}^*} = \mathbb{M}_{K_\phi^*} & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases}$$

Thus $\mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_3}(\omega)$. Hence, since $\mathcal{B}_{t_2}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ (proved above for all t), it follows that $\mathcal{B}_{t_2}(\omega) = \mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_3}(\omega)$.

Next we consider the case where $t = t_2$ and $t' = t_3$. In this case we do have that $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ (in fact, $\mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\phi \wedge \psi$ is consistent, in which case ϕ must be consistent and then $\mathcal{I}_{t_2}(\omega) = \mathbb{M}_\phi$ and $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi}$ and $\mathbb{M}_{\phi \wedge \psi} \subseteq \mathbb{M}_\phi$). Now, $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$ only if $\phi \wedge \psi$ is consistent in which case $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi}$. Assume, therefore, that $\phi \wedge \psi$ is consistent (which implies that ϕ is consistent). We need to consider several cases.

(i) $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K \neq \emptyset$. Then, since $\mathbb{M}_{\phi \wedge \psi} \subseteq \mathbb{M}_\phi$, $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$; it follows, by construction, that $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$ and $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K$ so that (since $\mathbb{M}_{\phi \wedge \psi} \subseteq \mathbb{M}_\phi$) $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$.

(ii) $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \emptyset$ but $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$. In this case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$ and thus $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) = \mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \emptyset$ and therefore there is nothing to prove, since the requirement that $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$ only holds if $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$.

(iii) $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$, which implies that $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_K = \emptyset$. In this case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K_\phi^*}$ and $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K_{\phi \wedge \psi}^*}$, so that $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_{K_\phi^*} \neq \emptyset$. Assume this. Then it must be that $\neg\psi \notin K_\phi^*$. Thus, by AGM postulates (K*7) and (K*8), $K_{\phi \wedge \psi}^* = [K_\phi^* \cup \{\psi\}]^{PL}$. We need to show that $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$, that is, that $\mathbb{M}_{K_{\phi \wedge \psi}^*} = \mathbb{M}_{K_\phi^*} \cap \mathbb{M}_{\phi \wedge \psi}$. Let $\omega \in \mathbb{M}_{K_{\phi \wedge \psi}^*}$. Then $\omega \supseteq K_{\phi \wedge \psi}^* = [K_\phi^* \cup \{\psi\}]^{PL} \supseteq K_\phi^*$. Thus $\omega \in \mathbb{M}_{K_\phi^*}$. Furthermore, by AGM postulate (K*2), $(\phi \wedge \psi) \in K_{\phi \wedge \psi}^*$, so that $\mathbb{M}_{K_{\phi \wedge \psi}^*} \subseteq \mathbb{M}_{\phi \wedge \psi}$. Thus $\mathbb{M}_{K_{\phi \wedge \psi}^*} \subseteq$

⁸By AGM postulate (K*2), $\phi \in K_\phi^*$. Thus if it were the case that $\neg\psi \in K_\phi^*$, then we would have that $(\phi \wedge \neg\psi) \in \omega$ for every $\omega \in \mathbb{M}_{K_\phi^*}$, contradicting the fact that $\mathbb{M}_{\phi \wedge \psi} \cap \mathbb{M}_{K_\phi^*} \neq \emptyset$.

$\mathbb{M}_{K_\phi^*} \cap \mathbb{M}_{\phi \wedge \psi}$. To complete the proof we need to show that $\mathbb{M}_{K_\phi^*} \cap \mathbb{M}_{\phi \wedge \psi} \subseteq \mathbb{M}_{K_{\phi \wedge \psi}^*}$. Since, by AGM postulate (K*2), $\phi \in K_\phi^*$, $[K_\phi^* \cup \{\psi\}]^{PL} = [K_\phi^* \cup \{\phi \wedge \psi\}]^{PL}$. Thus, if $\omega \in \mathbb{M}_{K_\phi^*} \cap \mathbb{M}_{\phi \wedge \psi}$ then $\omega \supseteq [K_\phi^* \cup \{\phi \wedge \psi\}]^{PL} = [K_\phi^* \cup \{\psi\}]^{PL} = K_{\phi \wedge \psi}^*$, that is, $\omega \in \mathbb{M}_{K_{\phi \wedge \psi}^*}$. ■

5 Conclusion

We proposed a temporal logic where information and beliefs are modeled explicitly by means of two modal operators I and B , respectively. This logic can accommodate not only the AGM theory of belief revision but also iterated revision, a topic that has received considerable attention in recent years (see, for example, [11]). As remarked in the introduction, there are two advantages to modeling belief revision in a modal framework: (1) one achieves a uniform treatment of static and dynamic beliefs, thus providing a unified theory of both, and (2) the approach allows one to state properties of beliefs in a clear and transparent way by means of syntactic axioms.

Some of the ideas contained in this paper (in particular the modeling of information by means of a non-normal modal operator) were first put forward in [3]. The framework in that paper was different, however, since it was not based on branching-time structures and only two dates were considered with two associated belief operators, B_0 (representing initial beliefs) and B_1 (representing revised beliefs). The main contribution of that paper was a soundness and completeness result for the proposed logic with respect to the class of frames that satisfy the Qualitative Bayes Rule.

For a detailed discussion of literature that is somewhat related to the general approach discussed here, the reader is referred to [3].

A Appendix

Proof of Proposition 4. The proof that \mathbb{L}_0 is sound with respect to the class of temporal belief revision frames is along the usual lines (see [2] or [4]). We need to show that (1) the rules of inference are validity preserving and (2) the axioms of \mathbb{L}_0 are valid in an arbitrary temporal belief revision frame. The proof of (1) is entirely standard and is omitted. The proof of validity of axiom K for \bigcirc , \bigcirc^{-1} and A and for the temporal axioms (O_1) and (O_2) is also standard and is omitted.

Validity of the backward uniqueness axiom (BU) is an immediate consequence of the fact that in a belief revision frame every instant t has at most a unique immediate predecessor: if $(\omega, t_2) \models \diamond^{-1}\phi$ then there exist a t_1 such that $t_1 \succ t_2$ and $(\omega, t_1) \models \phi$. Since, for every $t \in T$, $t \succ t_2$ if and only if $t = t_1$, it follows that $(\omega, t_2) \models \bigcirc^{-1}\phi$.

Validity of the S5 axioms for A is also straightforward. Suppose that $(\alpha, t) \models A\phi$. Then $(\omega, t) \models \phi$ for every $\omega \in \Omega$, thus in particular for $\omega = \alpha$. Similarly,

if $(\alpha, t) \models \neg A\phi$ then there exists a $\beta \in \Omega$ such that $(\beta, t) \models \neg\phi$. Hence $(\omega, t) \models \neg A\phi$ for every $\omega \in \Omega$ and therefore $(\alpha, t) \models A\neg A\phi$.

The proof that the inclusion axiom for B (Incl_B) is valid is straightforward and is omitted.

Validity of axiom I_1 : $I\phi \wedge I\psi \rightarrow A(\phi \leftrightarrow \psi)$. Suppose that $(\alpha, t) \models I\phi \wedge I\psi$. Then $\mathcal{I}_t(\alpha) = \lceil \phi \rceil_t$ and $\mathcal{I}(\alpha) = \lceil \psi \rceil_t$. Thus $\lceil \phi \rceil_t = \lceil \psi \rceil_t$ and hence $\lceil \phi \leftrightarrow \psi \rceil_t = \Omega$, yielding $(\alpha, t) \models A(\phi \leftrightarrow \psi)$.

Validity of axiom I_2 : $A(\phi \leftrightarrow \psi) \rightarrow (I\phi \leftrightarrow I\psi)$. Suppose that $(\alpha, t) \models A(\phi \leftrightarrow \psi)$. Then $\lceil \phi \leftrightarrow \psi \rceil_t = \Omega$ and, therefore, $\lceil \phi \rceil_t = \lceil \psi \rceil_t$. Thus, $(\alpha, t) \models I\phi$ if and only if $\mathcal{I}_t(\alpha) = \lceil \phi \rceil_t$ if and only if $\mathcal{I}_t(\alpha) = \lceil \psi \rceil_t$, if and only if $(\alpha, t) \models I\psi$. Hence $(\alpha, t) \models I\phi \leftrightarrow I\psi$. ■

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