

A Service of



Leibniz-Informationszentrum Wirtschaft Leibniz Information Centre for Economics

Klüppelberg, Claudia; Kuhn, Gabriel; Peng, Liang

Working Paper Estimating tail dependence of elliptical distributions

Discussion Paper, No. 470

Provided in Cooperation with:

Collaborative Research Center (SFB) 386: Statistical Analysis of discrete structures -Applications in Biometrics and Econometrics, University of Munich (LMU)

Suggested Citation: Klüppelberg, Claudia; Kuhn, Gabriel; Peng, Liang (2006) : Estimating tail dependence of elliptical distributions, Discussion Paper, No. 470, Ludwig-Maximilians-Universität München, Sonderforschungsbereich 386 - Statistische Analyse diskreter Strukturen, München,

https://doi.org/10.5282/ubm/epub.1838

This Version is available at: https://hdl.handle.net/10419/31120

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



WWW.ECONSTOR.EU

Estimating Tail Dependence of Elliptical Distributions

BY CLAUDIA KLÜPPELBERG, GABRIEL KUHN

Center for Mathematical Sciences, Munich University of Technology, D-85747 Garching, Germany. cklu@ma.tum.de_gabriel@ma.tum.de

AND LIANG PENG

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA. peng@math.gatech.edu

SUMMARY

Recently there has been an increasing interest in applying elliptical distributions to risk management. Under weak conditions, Hult and Lindskog (2002) showed that a random vector with an elliptical distribution is in the domain of attraction of a multivariate extreme value distribution. In this chapter we study two estimators for the tail dependence function, which are based on extreme value theory and the structure of an elliptical distribution, respectively. After deriving second order regular variation estimates and proving asymptotic normality for both estimators, we show that the estimator based on the structure of an elliptical distribution is better than that based on extreme value theory in terms of both asymptotic variance and optimal asymptotic mean squared error. Our theoretical results are confirmed by a simulation study.

Keywords: asymptotic normality, elliptical distribution, regular variation, tail copula, tail dependence function.

1 Introduction

Let $(X, Y), (X_1, Y_1), (X_2, Y_2), \cdots$ be independent random vectors with common distribution function F and continuous marginals F_X and F_Y . Define the *tail copula*

$$\lambda(x,y) := \lim_{t \to 0} \frac{1}{t} P\left(1 - F_X(X) \le tx, \ 1 - F_Y(Y) \le ty\right)$$

for $x, y \ge 0$. Then $\lambda(1, 1)$ is called the upper tail dependence coefficient, see e.g. Joe (1997) and, by Huang (1992), $l(x, y) := x + y - \lambda(x, y)$ is called the *tail* dependence function. Assuming that (X, Y) is in the domain of attraction of a bivariate extreme value distribution, there exist several estimators for estimating the tail dependence function l(x, y), see Huang (1992), Einmahl, de Haan and Huang (1993) and de Haan and Resnick (1993). The optimal rate of convergence for estimating l(x, y) is given by Drees and Huang (1998). An alternative method for estimating l(x, y) is via estimating the spectral measure, see Einmahl, de Haan and Sinha (1997) and Einmahl, de Haan and Piterbarg (2001). For modeling dependence of extremes parametrically, we refer to Tawn (1988) and Ledford and Tawn (1997).

Triggered by financial risk management problems we observe an increasing interest in elliptical distributions as natural extensions of the normal family allowing for the modeling of heavy tails and extreme dependence. The vector (X, Y) is *elliptically distributed*, if

$$(X,Y)^T = \boldsymbol{\mu} + G \boldsymbol{A} \boldsymbol{U}^{(2)}, \qquad (1.1)$$

where $\boldsymbol{\mu} = (\mu_X, \mu_Y)^{\mathrm{T}}, G > 0$ is a random variable, called *generating variable*, $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ is a deterministic matrix with

$$\mathbf{A}\mathbf{A}^T =: \mathbf{\Sigma} := \left(\begin{array}{cc} \sigma^2 & \rho \sigma \upsilon \\ \rho \sigma \upsilon & \upsilon^2 \end{array}\right)$$

and rank(Σ) = 2, $U^{(2)}$ is a 2-dimensional random vector uniformly distributed on the unit hyper-sphere $S_2 := \{ \boldsymbol{z} \in \mathbb{R}^2 : ||\boldsymbol{z}|| = 1 \}$, and $U^{(2)}$ is independent of G.

Note that ρ is termed as the linear correlation coefficient of Σ . Under certain conditions, Hult and Lindskog (2002) showed that regular variation of 1 - G with index $\alpha > 0$, i.e., $\lim_{t\to\infty} (1 - G(tx))/(1 - G(t)) = x^{-\alpha}$ for all x > 0, (notation: $1 - G \in \mathcal{R}_{-\alpha}$) is equivalent to regular variation of (X, Y) with the same index $\alpha > 0$ (see Resnick (1987) for the definition of multivariate regular variation). Moreover, if $1 - G \in \mathcal{R}_{-\alpha}$, then

$$\lambda(1,1) = \left(\int_{(\pi/2 - \arcsin\rho)/2}^{\pi/2} (\cos\phi)^{\alpha} d\phi \right) / \left(\int_{0}^{\pi/2} (\cos\phi)^{\alpha} d\phi \right).$$
(1.2)

Here we are interested in estimating the dependence function $\lambda(x, y)$ by assuming that $1 - G \in RV_{-\alpha}$ for some $\alpha > 0$. Since $1 - G \in RV_{-\alpha}$ implies that (X, Y) is in the domain of attraction of an extreme value distribution, a naive estimator is to apply Huang's estimator by ignoring the structure of the elliptical distribution, i.e.,

$$\widehat{\lambda}_{k_{\mathrm{Hu}},n}^{\mathrm{Hu}}(x,y) := \frac{1}{k_{\mathrm{Hu}}} \sum_{i=1}^{n} \boldsymbol{I} \left(X_{i} \ge X_{(n-\lfloor xk_{\mathrm{Hu}}\rfloor,n)}, \ Y_{i} \ge Y_{(n-\lfloor yk_{\mathrm{Hu}}\rfloor,n)} \right), (1.3)$$

where $X_{(1,n)} \leq \cdots \leq X_{(n,n)}$ and $Y_{(1,n)} \leq \cdots \leq Y_{(n,n)}$ denote the order statistics of X_1, \ldots, X_n and Y_1, \ldots, Y_n , respectively, $k_{\text{Hu}} = k_{\text{Hu}}(n) \xrightarrow{n \to \infty} \infty$ and $k_{\text{Hu}}/n \xrightarrow{n \to \infty} 0$. The same estimator has been analysed by Schmidt and Stadtmüller (2005); see their equation (4.14). The aim of this chapter is two-fold. Firstly, we suggest a new estimator, which exploits the structure of an elliptical distribution similar to (1.2). Secondly, we aim at determining the optimal number of order statistics to be used in both estimators. The choice will be based on the asymptotic mean squared error of the estimators.

Our chapter is organized as follows. We first derive an expression for $\lambda(x, y)$, which generalizes equation (1.2), and then construct a new estimator for $\lambda(x, y)$ via this expression; see section 2 for details. After deriving the second order behavior for elliptical distributions and the limiting distributions of both estimators in section 2, we show that the new estimator is better than the naive empirical estimator from Huang in terms of both asymptotic variance and optimal asymptotic mean squared error in section 3. More importantly, the optimal choice of the sample fraction for the new estimator is the same as that for Hill's estimator (Hill (1975)). That is, all data-driven methods for choosing the optimal sample fraction for Hill's estimator can be applied to our new estimator directly. A simulation study is provided in section 3 as well. All proofs are summarized in section 4.

2 Methodology and Main Results

The following theorem gives an expression for $\lambda(x, y)$, which will be employed to construct an estimator.

Theorem 2.1. Suppose (X, Y) defined in (1.1) holds with $\sigma > 0$, $\upsilon > 0$, $|\rho| < 1$ and $1 - G \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Further, define

$$g(t) := \arctan\left((t-\rho)/\sqrt{1-\rho^2}\right) \in [-\arcsin\rho, \pi/2], \quad t \in \mathbb{R}.$$

Then

$$\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} \, \mathrm{d}\phi \right)^{-1} \left(\int_{g((x/y)^{1/\alpha})}^{\pi/2} x(\cos \phi)^{\alpha} \, \mathrm{d}\phi + \int_{-\operatorname{arcsin}\rho}^{g((x/y)^{1/\alpha})} y(\sin(\phi + \arcsin \rho))^{\alpha} \, \mathrm{d}\phi \right).$$

In order to derive the asymptotic normality of $\widehat{\lambda}_{k_{\text{Hu},n}}^{\text{Hu}}(x, y)$, it is known that a second order condition is needed. Here we seek the relation of the second order behavior among the tail copula $\lambda(x, y)$, $\sqrt{X^2 + Y^2}$ and G; see the next two theorems for details.

In the setting of (1.1) assume that there exists $A(t) \to 0$ such that for all x > 0 and some $\beta \le 0$

$$\lim_{t \to \infty} \frac{P(G \ge tx)/P(G \ge t) - x^{-\alpha}}{A(t)} = x^{-\alpha} \frac{x^{\beta} - 1}{\beta}, \qquad (2.1)$$

where $\beta \leq 0$ is called a *second order regular variation parameter*, see de Haan and Stadtmüller (1996). Additionally, we assume

$$\lim_{t \to \infty} t^2 A(t) =: a \in [-\infty, \infty].$$
(2.2)

Since $A \in \mathcal{R}_{\beta}$, a = 0 for $\beta < -2$ and $|a| = \infty$ for $\beta \in (-2, 0]$.

The following two theorems derive the corresponding second order condition for $\sqrt{X^2 + Y^2}$ and the tail copula $\lambda(x, y)$.

Theorem 2.2. Assume that the conditions of Theorem 2.1, (2.1) and (2.2) hold. Further, define

$$d_1(\phi) = \sigma^2 \cos^2 \phi + v^2 \sin^2(\phi + \arcsin \rho),$$

$$d_2(\phi) = \mu_X \sigma \cos \phi + \mu_Y v \sin(\phi + \arcsin \rho).$$

Then, for all x > 0,

$$\lim_{t \to \infty} \frac{P\{\sqrt{X^2 + Y^2} \ge tx\} / P\{\sqrt{X^2 + Y^2} \ge t\} - x^{-\alpha}}{t^{-2} + |A(t)|}$$

$$= x^{-\alpha} \left(\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi \right)^{-1} \left\{ \frac{a}{1 + |a|} \frac{x^{\beta} - 1}{\beta} \left(\int_{-\pi}^{\pi} (d_1(\phi))^{(\alpha - \beta)/2} d\phi \right) + \frac{1}{1 + |a|} \frac{\alpha}{2} \left(x^{-2} - 1 \right) \int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2 - 1} \times \left[\alpha (d_2(\phi))^2 + d_1(\phi) (\mu_X^2 + \mu_Y^2) \right] d\phi \right\}. (2.3)$$

Also, for all x > 0 and $V(x) := \inf\{y : P(\sqrt{X^2 + Y^2} > y) \le x^{-1}\},\$

$$\lim_{t \to \infty} \frac{V(tx)/V(t) - x^{1/\alpha}}{(F_Y^{\leftarrow}(1 - t^{-1}))^{-2} + |A(F_Y^{\leftarrow}(1 - t^{-1}))|} = x^{1/\alpha} \left(\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi \right)^{-1} \left(\frac{\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi}{v^{\alpha} \int_{0}^{\pi} (\sin \phi)^{\alpha} d\phi} \right)^{-(2 \wedge |\beta|)/\alpha} \times \left\{ \frac{a}{1 + |a|} \frac{x^{\beta/\alpha} - 1}{\alpha\beta} \left(\int_{-\pi}^{\pi} (d_1(\phi))^{(\alpha - \beta)/2} d\phi \right) + \frac{1}{1 + |a|} \frac{1}{2} \left(x^{-2/\alpha} - 1 \right) \int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2 - 1} \times \left[\alpha (d_2(\phi))^2 + d_1(\phi) (\mu_X^2 + \mu_Y^2) \right] d\phi \right\} =: x^{1/\alpha} \mathcal{B}_{(2.4)}(x). \quad (2.4)$$

Especially, when $\mu_X = \mu_Y = 0$, we have for all x > 0

$$\lim_{t \to \infty} \frac{V(tx)/V(t) - x^{1/\alpha}}{A(F_Y^{\leftarrow}(1 - t^{-1}))} = x^{1/\alpha} \frac{x^{\beta/\alpha} - 1}{v^{\beta} \alpha \beta} \left(\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi \right)^{-1} \times \left(\int_{-\pi}^{\pi} (d_1(\phi))^{(\alpha-\beta)/2} d\phi \right) \left(\frac{\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi}{\int_{0}^{\pi} (\sin \phi)^{\alpha} d\phi} \right)^{\beta/\alpha} =: x^{1/\alpha} \mathcal{B}_{(2.5)}(x).$$
(2.5)

Theorem 2.3. Assume that the conditions of Theorem 2.1 and (2.1) hold. Further, define

$$\mathcal{S}_{2}^{+} := \left\{ \boldsymbol{z} \in \mathbb{R}^{2} : \boldsymbol{z} \ge \boldsymbol{0} \text{ and } \|\boldsymbol{z}\| = 1 \right\} \text{ and}$$
$$\mathcal{B}_{(2.6)}(\boldsymbol{x}) := -\boldsymbol{x} \frac{\boldsymbol{x}^{-\beta/\alpha} - 1}{\beta} \left(\int_{0}^{\pi} (\sin \phi)^{\alpha} \mathrm{d} \phi \right)^{-1} \left(\int_{0}^{\pi} (\sin \phi)^{\alpha-\beta} \mathrm{d} \phi \right) (2.6)$$

Then,

$$\lim_{t \to 0} \frac{t^{-1}P(F_X(X) \ge 1 - tx, F_Y(Y) \ge 1 - ty) - \lambda(x, y)}{A(F_Y^{-}(1 - t))} = v^{-\beta} \left\{ \frac{x}{\beta} \int_{g((x/y)^{1/\alpha})}^{\pi/2} \left[x^{-\beta/\alpha} (\cos \phi)^{\alpha-\beta} - (\cos \phi)^{\alpha} \right] d\phi + \frac{y}{\beta} \int_{-\arccos n}^{g((x/y)^{1/\alpha})} \left[y^{-\beta/\alpha} (\sin(\phi + \arcsin \rho))^{\alpha-\beta} - (\sin(\phi + \arcsin \rho))^{\alpha} \right] d\phi + \mathcal{B}_{(2.6)}(x) \int_{g((x/y)^{1/\alpha})}^{\pi/2} (\cos \phi)^{\alpha} d\phi + \mathcal{B}_{(2.6)}(y) \int_{-\arcsin \rho}^{g((x/y)^{1/\alpha})} (\sin(\phi + \arcsin \rho))^{\alpha} d\phi - \lambda(x, y) \frac{1}{\beta} \int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} \left((\cos \phi)^{-\beta} - 1 \right) d\phi \right\} \left(\int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} d\phi \right)^{-1} =: \mathcal{B}_{(2.7)}(x, y) \tag{2.7}$$

holds for all $x, y \ge 0$ and uniformly on \mathcal{S}_2^+ .

Now we are ready to define our new estimator. Put
$$Z_i = \sqrt{X_i^2 + Y_i^2}$$
 for $i = 1, ..., n$ and let $Z_{(1,n)} \leq \cdots \leq Z_{(n,n)}$ denote their order statistics. First we estimate the index α by Hill's estimator, which is defined as

$$\widehat{\alpha}_{k_{\rm El},n}^{\rm H} := \left(\frac{1}{k_{\rm El}} \sum_{i=1}^{k_{\rm El}} \log Z_{(n-i+1,n)} - \log Z_{(n-k_{\rm El},n)}\right)^{-1},$$

where $k_{\rm El} = k_{\rm El}(n) \to \infty$ and $k_{\rm El}/n \to 0$ as $n \to \infty$. Now let (X, Y) and (\tilde{X}, \tilde{Y}) be iid with elliptical distribution. Then, it follows from Hult and Lindskog (2002) that $\tau = (2/\pi) \arcsin \rho$, where τ is called *Kendall's tau* and defined by

$$\tau := P\left(\left(X - \widetilde{X}\right)\left(Y - \widetilde{Y}\right) > 0\right) - P\left(\left(X - \widetilde{X}\right)\left(Y - \widetilde{Y}\right) < 0\right).$$

As usual, we estimate Kendall's tau by

$$\widehat{\tau}_n := \frac{2}{n(n-1)} \sum_{1 \le i,j \le n} \operatorname{sign} \left((X_i - X_j)(Y_i - Y_j) \right),$$

which results in estimating ρ by

$$\widehat{\rho}_n = \sin\left(\frac{\pi}{2}\widehat{\tau}_n\right).$$

Hence, we can estimate $\lambda(x, y)$ by replacing ρ and α in Theorem 2.1 by $\hat{\rho}_n$ and $\hat{\alpha}_{k_{\text{EI}},n}^{\text{H}}$, respectively. Let us denote this estimator by

$$\widehat{\lambda}_{k_{\text{El}},n}^{\text{El}}(x,y). \tag{2.8}$$

We remark that $\widehat{\lambda}_{k_{\text{El}},n}^{\text{El}}(1,1)$ was mentioned by Schmidt (2003), but without further study. The following theorem shows the asymptotic normalities of $\widehat{\lambda}_{k_{\text{Hu}},n}^{\text{Hu}}(x,y)$ and $\widehat{\lambda}_{k_{\text{El}},n}^{\text{El}}(x,y)$, which allows us to compare these two estimators theoretically.

Theorem 2.4. Assume that the conditions of Theorem 2.1 and (2.1) hold. Suppose $k_{\text{Hu}} = k_{\text{Hu}}(n) \xrightarrow{n \to \infty} \infty$, $k_{\text{Hu}}/n \xrightarrow{n \to \infty} 0$ and

$$\sqrt{k_{\mathrm{Hu}}}A(F_Y^{\leftarrow}(1-k_{\mathrm{Hu}}/n)) \xrightarrow{n \to \infty} \mathcal{K}_{\mathrm{Hu}},$$

for $|\mathcal{K}_{Hu}| < \infty$. Then, as $n \to \infty$,

$$\sup_{0 \le x, y \le T} \left| \sqrt{k_{\mathrm{Hu}}} \left(\widehat{\lambda}_{k_{\mathrm{Hu}}, n}^{\mathrm{Hu}}(x, y) - \lambda(x, y) \right) - \mathcal{K}_{\mathrm{Hu}} \mathcal{B}_{(2.7)}(x, y) - B(x, y) \right|^{2} = o_{p}(1),$$

for any T > 0, where $\mathcal{B}_{(2.7)}(x, y)$ is defined in Theorem 2.3,

$$B(x,y) = W(x,y) - \left(1 - \frac{\partial \lambda(x,y)}{\partial x}\right) W(x,0) - \left(1 - \frac{\partial \lambda(x,y)}{\partial y}\right) W(0,y),$$

and W(x, y) is a Wiener process with zero mean and covariance structure

$$E(W(x_1, y_1)W(x_2, y_2))$$

= $x_1 \wedge x_2 + y_1 \wedge y_2 - \lambda(x_1 \wedge x_2, y_1) - \lambda(x_1 \wedge x_2, y_2) - \lambda(x_1, y_1 \wedge y_2)$
 $-\lambda(x_2, y_1 \wedge y_2) + \lambda(x_1, y_2) + \lambda(x_2, y_1) + \lambda(x_1 \wedge x_2, y_1 \wedge y_2).$

Therefore, for any fixed x, y > 0,

$$\sqrt{k_{\mathrm{Hu}}} \left(\widehat{\lambda}_{k_{\mathrm{Hu}},n}^{\mathrm{Hu}}(x,y) - \lambda(x,y) \right) \stackrel{d}{\longrightarrow} \mathcal{N} \left(\mathcal{K}_{\mathrm{Hu}} \mathcal{B}_{(2.7)}(x,y), \sigma_{\mathrm{Hu}}^2 \right)$$

as $n \to \infty$, where

$$\sigma_{\rm Hu}^2 = x \left(\frac{\partial}{\partial x}\lambda(x,y)\right)^2 + y \left(\frac{\partial}{\partial y}\lambda(x,y)\right)^2 + 2\lambda(x,y) \times \qquad (2.10)$$
$$\times \left(\frac{1}{2} - \frac{\partial}{\partial x}\lambda(x,y) - \frac{\partial}{\partial y}\lambda(x,y) + \left(\frac{\partial\lambda(x,y)}{\partial x}\right)\left(\frac{\partial\lambda(x,y)}{\partial y}\right)\right),$$

$$\frac{\partial}{\partial x}\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^{\alpha} d\phi\right)^{-1} \int_{g((x/y)^{1/\alpha})}^{\pi/2} (\cos\phi)^{\alpha} d\phi \quad and (2.11)$$

$$\frac{\partial}{\partial y}\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^{\alpha} d\phi\right)^{-1} \times (2.12)$$

$$\times \int_{-\operatorname{arcsin}\rho}^{g((x/y)^{1/\alpha})} (\sin(\phi + \arcsin\rho))^{\alpha} d\phi.$$

Theorem 2.5. Assume that the conditions of Theorem 2.1 and (2.1) hold. Further assume (2.2) holds when $\mu \neq 0$. Suppose $k_{\text{El}} = k_{\text{El}}(n, \mu) \xrightarrow{n \to \infty} \infty$, $k_{\text{El}}/n \xrightarrow{n \to \infty} 0$ and

$$\sqrt{k_{\rm El}} \left(\left(F_Y^{\leftarrow} (1 - k_{\rm El}/n) \right)^{-2} + \left| A(F_Y^{\leftarrow} (1 - k_{\rm El}/n)) \right| \right) \stackrel{n \to \infty}{\longrightarrow} : \mathcal{K}_{\rm El}, \quad \boldsymbol{\mu} \neq 0,$$
$$\sqrt{k_{\rm El}} A(F_Y^{\leftarrow} (1 - k_{\rm El}/n)) \stackrel{n \to \infty}{\longrightarrow} : \mathcal{K}_{\rm El}, \quad \boldsymbol{\mu} = \mathbf{0},$$

for $|\mathcal{K}_{\rm El}| < \infty$ Then, as $n \to \infty$,

$$\sup_{0 \le x, y \le T} \left| \sqrt{k_{\text{El}}} \left(\widehat{\lambda}_{k_{\text{El}}, n}^{\text{El}}(x, y) - \lambda(x, y) \right) - \mathcal{B}_{(2.15)}(x, y) Z_0 \right| = o_p(1)(2.13)$$

where $Z_0 \sim \mathcal{N} \left(-\alpha^2 \mathcal{K}_{El} \mathcal{B}_{(2.14)}, \, \alpha^2 \right)$ with

$$\mathcal{B}_{(2.14)} := \begin{cases} \int_0^1 \mathcal{B}_{(2.4)}(1/s) \, \mathrm{d}s, & \mu \neq \mathbf{0}, \\ \int_0^1 \mathcal{B}_{(2.5)}(1/s) \, \mathrm{d}s, & \mu = \mathbf{0}, \end{cases}$$
(2.14)

 $\mathcal{B}_{(2.4)}(s)$ and $\mathcal{B}_{(2.5)}(s)$ are defined in Theorem 2.2 and

$$\mathcal{B}_{(2.15)}(x,y) := \left\{ \int_{g((x/y)^{1/\alpha})}^{\pi/2} x(\cos\phi)^{\alpha} \ln(\cos\phi) \,\mathrm{d}\phi \right. \\ \left. + \int_{-\operatorname{arcsin}\rho}^{g((x/y)^{1/\alpha})} y(\sin(\phi + \arcsin\rho))^{\alpha} \ln(\sin(\phi + \arcsin\rho)) \,\mathrm{d}\phi \right. \\ \left. - \lambda(x,y) \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^{\alpha} \ln(\cos\phi) \,\mathrm{d}\phi \right) \right\} \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^{\alpha} \,\mathrm{d}\phi \right)^{-1} (2.15)$$

Therefore, for any fixed x, y > 0,

$$\sqrt{k_{\rm El}} \left(\widehat{\lambda}_{k_{\rm El},n}^{\rm El}(x,y) - \lambda(x,y) \right)
\xrightarrow{d} \mathcal{N} \left(-\alpha^2 \mathcal{K}_{\rm El} \mathcal{B}_{(2.14)} \mathcal{B}_{(2.15)}(x,y), \, \alpha^2 \left(\mathcal{B}_{(2.15)}(x,y) \right)^2 \right). \qquad \Box$$

The next corollary gives the optimal choice of sample fraction for both estimators. As criterion we use the *asymptotic mean squared error* of $\widehat{\lambda}_{k_{\text{Hu}},n}^{\text{Hu}}$ and $\widehat{\lambda}_{k_{\text{El}},n}^{\text{El}}$, denoted by $\text{amse}_{\text{Hu}}(k_{\text{Hu}})$ and $\text{amse}_{\text{El}}(k_{\text{El}})$, respectively.

Corollary 2.6. Assume that the conditions of Theorems 2.4 and 2.5 hold. Further, suppose that

$$A(F_Y^{\leftarrow}(1-t)) \sim b_0 t^{-\beta/\alpha},$$

$$(F_Y^{\leftarrow}(1-t))^{-2} + |A(F_Y^{\leftarrow}(1-t))| \sim b_1 t^{(2\wedge(-\beta))/\alpha}$$

for some $b_0, b_1 > 0$ as $t \to 0$ and define

$$b_2 t^{-\beta_2/\alpha} := \begin{cases} b_1 t^{(2\wedge(-\beta))/\alpha} & \boldsymbol{\mu} \neq \boldsymbol{0}, \\ b_0 t^{-\beta/\alpha}, & \boldsymbol{\mu} = 0. \end{cases}$$

Then

amse_{Hu}(k_{Hu}) =
$$\sigma_{Hu}^2 k_{Hu}^{-1} + (b(k_{Hu}/n)^{-\beta/\alpha} \mathcal{B}_{(2.7)}(x,y))^2$$

and

amse_{El}(k_{El}) =
$$(\mathcal{B}_{(2.15)}(x,y))^2 \left(\alpha^2 k_{\rm El}^{-1} + \left(\alpha^2 b_2 (k_{\rm El}/n)^{-\beta_2/\alpha} \mathcal{B}_{(2.14)} \right)^2 \right).$$

Let k_{Hu}^{opt} and k_{El}^{opt} denote the optimal sample fraction in the sense of minimizing $amse_{Hu}$ and $amse_{El}$, respectively. Then

$$k_{\rm Hu}^{\rm opt} = \left(\frac{-\alpha \sigma_{\rm Hu}^2}{2\beta b_0^2 \left(\mathcal{B}_{(2.7)}(x,y)\right)^2}\right)^{\alpha/(\alpha-2\beta)} n^{-2\beta/(\alpha-2\beta)},$$

$$k_{\rm El}^{\rm opt} = \left(-2\beta_2 \alpha b_2^2 \left(\mathcal{B}_{(2.14)}\right)^2\right)^{-\alpha/(\alpha-2\beta_2)} n^{-2\beta_2/(\alpha-2\beta_2)},$$

$$amse_{\rm Hu}^{\rm opt} := amse_{\rm Hu} \left(k_{\rm Hu}^{\rm opt}\right) = n^{2\beta/(\alpha-2\beta)} b_0^{2\alpha/(\alpha-2\beta)} \left(1 - \frac{\alpha}{2\beta}\right) \times \left(\left(\sigma_{\rm Hu}^2\right)^{-\beta/\alpha} \mathcal{B}_{(2.7)}(x,y) \sqrt{-2\beta/\alpha}\right)^{2\alpha/(\alpha-2\beta)} and$$

$$amse_{\rm El}^{\rm opt} := amse_{\rm El} \left(k_{\rm El}^{\rm opt}\right) = n^{2\beta_2/(\alpha-2\beta_2)} b_2^{2\alpha/(\alpha-2\beta_2)} \left(1 - \frac{\alpha}{2\beta_2}\right) \times \alpha^2 (\mathcal{B}_{(2.15)}(x,y))^2 \left(\sqrt{-2\alpha\beta_2} \mathcal{B}_{(2.14)}\right)^{2\alpha/(\alpha-2\beta_2)}.$$

	_		
_	_	_	

Remark 2.7. Note that $k_{\text{El}}^{\text{opt}}$ is independent of x and y, but $k_{\text{Hu}}^{\text{opt}}$ depends on xand y. In case of $\boldsymbol{\mu} = \boldsymbol{0}$, both $\operatorname{amse}_{\text{Hu}}^{\text{opt}}$ and $\operatorname{amse}_{\text{El}}^{\text{opt}}$ depend on n, α , β , ρ , v, x, yand b_0 , $\operatorname{amse}_{\text{El}}^{\text{opt}}$ additionally depends on σ , but the ratio $\operatorname{amse}_{\text{Hu}}^{\text{opt}}/\operatorname{amse}_{\text{El}}^{\text{opt}}$ is independent of n and b_0 . Since the optimal $k_{\text{El}}^{\text{opt}}$ is the same as that for Hill's estimator, when $\boldsymbol{\mu} = 0$, all data-driven methods for choosing the optimal sample fraction for Hill's estimator can be applied to $\widehat{\lambda}_{k_{\text{El},n}}^{\text{El}}(x, y)$ directly. Note that $\boldsymbol{\mu}$ is the median of (X, Y) and the mean of (X, Y) when $\alpha > 1$. Hence, we could estimate $\boldsymbol{\mu}$ by the sample median, say $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_X, \widehat{\mu}_Y)$. Therefore, consider the new estimator $\widehat{\lambda}_{k_{\text{El},n}}^{\text{El}}(x, y)$ with $Z_i = \sqrt{X_i^2 + Y_i^2}$ replaced by $\sqrt{(X_i - \widehat{\mu}_X)^2 + (Y_i - \widehat{\mu}_Y)^2}$. Similar to the proofs in Ling and Peng (2004), we can show that Theorem 2.5 and Corollary 2.6 hold with $\boldsymbol{\mu} = 0$ for this new estimator.

3 Comparisons and Simulation Study

First we compare $\sigma_{\text{Hu}}^2, \sigma_{\text{El}}^2$ given in Theorem 2.4 and 2.5. Note that both only depend on α , ρ , x and y. In Figure 1, we plot the ratio $\sigma_{\text{El}}^2(\alpha)/\sigma_{\text{Hu}}^2(\alpha)$ for x = y = 1 as a function of α , and each curve therein corresponds to a different correlation $\rho \in \{0.1, \ldots, 0.9\}$. From Figure 1, we conclude that $\widehat{\lambda}_{k,n}^{\text{El}}$ is always better in terms of asymptotic variance.

Second, we compare the two estimators in terms of optimal asymptotic mean squared errors. Since the ratio of the optimal asymptotic mean squared error depends on α , β , Σ , μ , x, y, we consider elliptical distributions with $\sigma = v = 1$, $\mu_X = \mu_Y = 0$. In Figure 2, we consider $G \sim \text{Fréchet}(\alpha)$, i.e. $P(G > x) = \exp(-x^{-\alpha})$, x > 0, hence (2.1) is satisfied with $\beta = -\alpha$. In Figure 3, we consider $G \sim \text{Pareto}(\alpha)$, i.e. $P(G > x) = (1 + x)^{-\alpha}$ for x > 0, therefore, (2.1) is satisfied with $\beta = -1$. Under the above setup, the ratio of optimal asymptotic mean squared errors only depends on α , ρ , x, y. Similar to Figure 1, we plot the ratio amse^{opt}_{El}(α)/amse^{opt}_{Hu}(α) for x = y = 1 as a function of α for different ρ 's in Figures 2 and 3. We conclude from both Figures that $\widehat{\lambda}_{k,n}^{\text{El}}$ always performs better than $\widehat{\lambda}_{k,n}^{\text{Hu}}$ in terms of optimal asymptotic mean squared errors as well.

Third, we examine the influence of x and y to the ratio of asymptotic mean squared error. We plot the ratio $\operatorname{amse}_{\operatorname{El}}^{\operatorname{opt}}(\alpha)/\operatorname{amse}_{\operatorname{Hu}}^{\operatorname{opt}}(\alpha)$ for $||(x,y)|| = \sqrt{2}$ and $G \sim \operatorname{Pareto}(\alpha)$ in Figure 4, where each curve corresponds to a different pair of $(\alpha, \rho) \in \{(20, 0.9), (10, 0.6), (5, 0.3), (1, 0.1)\}$. This figure further confirms that $\widehat{\lambda}_{k,n}^{\operatorname{El}}$ always has a smaller optimal asymptotic mean squared error than $\widehat{\lambda}_{k,n}^{\operatorname{Hu}}$. Finally, we study the finite sample behavior of the two estimators $\widehat{\lambda}_{k,n}^{\text{El}}(x, y)$ and $\widehat{\lambda}_{k,n}^{\text{Hu}}(x, y)$. As above, we consider two elliptical distributions with $\sigma = \upsilon = 1$, $\mu_X = \mu_Y = 0, \ G \sim \text{Fréchet}(\alpha)$ in Figure 5 and $G \sim \text{Pareto}(\alpha)$ in Figure 6. We generate 1000 random samples of size n = 1000 from these elliptical distributions with $(\alpha, \rho) \in \{(20, 0.9), (10, 0.6), (5, 0.3), (1, 0.1)\}$, and plot $\widehat{\lambda}_{k,n}^{\text{El}}(1, 1)$ and $\widehat{\lambda}_{k,n}^{\text{Hu}}(1, 1)$ against $k = 1, \ldots, 300$ for different pairs (α, ρ) in Figures 5 and 6, where the solid line corresponds to $\widehat{\lambda}_{k,n}^{\text{El}}(1, 1)$ and the dashed line to $\widehat{\lambda}_{k,n}^{\text{Hu}}(1, 1)$. This simulation study also confirms the better performance of $\widehat{\lambda}_{k,n}^{\text{El}}$.



Figure 1: Ratio $\sigma_{\rm El}^2(\alpha)/\sigma_{\rm Hu}^2(\alpha)$ for different correlations ρ .



Figure 2: Ratio $\operatorname{amse}_{\operatorname{Hu}}^{\operatorname{opt}}(\alpha)/\operatorname{amse}_{\operatorname{El}}^{\operatorname{opt}}(\alpha)$ for different correlations ρ and $\beta = -\alpha$.



Figure 3: Ratio $\operatorname{amse}_{\operatorname{Hu}}^{\operatorname{opt}}(\alpha)/\operatorname{amse}_{\operatorname{El}}^{\operatorname{opt}}(\alpha)$ for different correlations ρ and $\beta = -1$.



Figure 4: Ratio $\operatorname{amse}_{\operatorname{El}}^{\operatorname{opt}}(\alpha)/\operatorname{amse}_{\operatorname{Hu}}^{\operatorname{opt}}(\alpha), x^2 + y^2 = 2$, for different (α, ρ) and $\beta = -1$.



Figure 5: Mean of estimators $\widehat{\lambda}_{k,n}^{\text{Hu}}(1,1)$ and $\widehat{\lambda}_{k,n}^{\text{El}}(1,1)$ for 1 000 samples of length $n = 1\,000$ and different k with $\sigma = \upsilon = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $G \sim \text{Fréchet}(\alpha)$, and different (α, ρ) .



Figure 6: Mean of estimators $\widehat{\lambda}_{k,n}^{\text{Hu}}(1,1)$ and $\widehat{\lambda}_{k,n}^{\text{El}}(1,1)$ for 1 000 samples of length $n = 1\,000$ and different k with $\sigma = \upsilon = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $G \sim \text{Pareto}(\alpha)$ and different (α, ρ) .

4 Proofs

Proof of Theorem 2.1: Without loss of generality, we assume $\boldsymbol{\mu} = \boldsymbol{0}$. Let $\Phi \sim \operatorname{unif}(-\pi,\pi)$ be independent of G and $F_i^{\leftarrow}(x)$ denote the inverse of $F_i(x)$, i = 1, 2. Then, by Hult and Lindskog (2002),

$$F_X^{\leftarrow}(u) = \frac{\sigma}{v} F_Y^{\leftarrow}(u), \qquad \text{for } 0 < u < 1,$$

$$\lim_{t \to \infty} (1 - F_i(tx)) / (1 - F_i(t)) = x^{-\alpha}, \qquad \text{for } x > 0 \text{ and } i = 1, 2,$$

$$(X, Y) \stackrel{d}{=} (\sigma G \cos \Phi, v G \sin (\arcsin \rho + \Phi)).$$
(4.16)

Therefore,

$$t^{-1}P\left(F_X(X) \ge 1 - tx, \ F_Y(Y) \ge 1 - ty\right)$$

= $t^{-1}P\left(G\cos\Phi \ge F_Y^{\leftarrow}(1 - tx)/v, G\sin(\arcsin\rho + \Phi) \ge F_Y^{\leftarrow}(1 - ty)/v\right)$
= $\frac{1}{2\pi t} \int_{-\arcsin\rho}^{\pi/2} P\left(G \ge \frac{F_Y^{\leftarrow}(1 - tx)}{v\cos\phi} \lor \frac{F_Y^{\leftarrow}(1 - ty)}{v\sin(\arcsin\rho + \phi)}\right) d\phi.$
(4.17)

Note that

$$t = P\left(X > F_X^{\leftarrow}(1-t)\right) = P\left(G\cos\Phi > F_Y^{\leftarrow}(1-t)/\upsilon\right)$$
$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} P\left(G > \frac{F_Y^{\leftarrow}(1-t)}{\upsilon\cos\phi}\right) \,\mathrm{d}\phi.$$

Further, $1 \ge P(G > x/\cos \phi) / P(G \ge x) \xrightarrow{x \to \infty} (\cos \phi)^{\alpha}$. Hence, in the following formula we can apply the dominated convergence theorem and obtain

$$\frac{1}{\mathcal{B}_{(4.18)}(t)} := \frac{1}{2\pi t} P\left(G \ge F_Y^{\leftarrow}(1-t)/\upsilon\right)$$
$$\xrightarrow{t \to 0} \left(\int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} \,\mathrm{d}\phi\right)^{-1} =: \frac{1}{\mathcal{B}_{(4.18)}}.$$
(4.18)

Next, we obtain for $\phi \in (-\arcsin \rho, \pi/2)$

$$\frac{F_Y^{\leftarrow}(1-tx)}{\upsilon\cos\phi} \ge \frac{F_Y^{\leftarrow}(1-ty)}{\upsilon\sin(\arcsin\rho+\phi)} \iff \frac{F_Y^{\leftarrow}(1-ty)}{F_Y^{\leftarrow}(1-tx)} \le \frac{\sin(\arcsin\rho+\phi)}{\cos\phi}.$$

Note that $\sin(\arcsin \rho + \phi) / \cos \phi$ is strictly increasing, hence its inverse exists and equals $\arctan\left((\cdot - \rho) / \sqrt{1 - \rho^2}\right)$. Therefore,

$$\frac{F_Y^{\leftarrow}(1-tx)}{v\cos\phi} \ge \frac{F_Y^{\leftarrow}(1-ty)}{v\sin(\arcsin\rho+\phi)}$$

$$\Leftrightarrow \quad \phi \ge \arctan\left(\frac{F_Y^{\leftarrow}(1-ty)/F_Y^{\leftarrow}(1-tx)-\rho}{\sqrt{1-\rho^2}}\right)$$

$$=: g\left(\frac{F_Y^{\leftarrow}(1-ty)}{F_Y^{\leftarrow}(1-tx)}\right) =: h(x,y,t).$$
(4.19)

Since $1 - F_Y \in \mathcal{R}_{-\alpha}$, by Proposition 1.7(9) of Geluk and de Haan (1987) $F_Y^{\leftarrow}(1 - tx)/F_Y^{\leftarrow}(1-t) \xrightarrow{t \to 0} x^{-1/\alpha}$, i.e.,

$$h(x, y, t) \xrightarrow{t \to 0} g\left((x/y)^{1/\alpha}\right). \tag{4.20}$$

It follows from (4.17) and (4.19) that

$$t^{-1}P(F_{X}(X) \ge 1 - tx, \ F_{Y}(Y) \ge 1 - ty)$$

$$= \frac{1}{\mathcal{B}_{(4.18)}(t)} \int_{h(x,y,t)}^{\pi/2} \frac{P\left(G \ge \frac{F_{Y}^{\leftarrow}(1-t)}{v\cos\phi} \frac{F_{Y}^{\leftarrow}(1-tx)}{F_{Y}^{\leftarrow}(1-t)}\right)}{P(G \ge F_{Y}^{\leftarrow}(1-t)/v)} d\phi$$

$$+ \frac{1}{\mathcal{B}_{(4.18)}(t)} \int_{-\arcsin\rho}^{h(x,y,t)} \frac{P\left(G \ge \frac{F_{Y}^{\leftarrow}(1-t)}{v\sin(\arcsin\rho+\phi)} \frac{F_{Y}^{\leftarrow}(1-ty)}{F_{Y}^{\leftarrow}(1-t)}\right)}{P(G \ge F_{Y}^{\leftarrow}(1-t)/v)} d\phi.$$
(4.21)

Hence, the theorem follows from (4.18), (4.20) and Potter's inequality, e.g. see (1.20) in Geluk and de Haan (1987).

Proof of Theorem 2.2: Since

$$(X,Y) \stackrel{d}{=} (\mu_X + \sigma G \cos \Phi, \ \mu_Y + \nu G \sin(\Phi + \arcsin \rho)),$$

we have $X^2 + Y^2 \stackrel{d}{=} G^2 d_1(\Phi) + 2G d_2(\Phi) + \mu_X^2 + \mu_Y^2$. Define

$$d_3(x,\phi) := \frac{1}{d_1(\phi)} \left(-d_2(\phi) + \sqrt{d_2^2(\phi) - d_1(\phi) \left(\mu_X^2 + \mu_Y^2 - x^2\right)} \right).$$

Since
$$P(X^2 + Y^2 \ge t) = P(G \ge d_3(t, \Phi))$$
 holds for large t , we obtain

$$\frac{P(X^2 + Y^2 \ge t^2 x^2)}{P(X^2 + Y^2 \ge t^2)} - x^{-\alpha}$$

$$= \left(\int_{-\pi}^{\pi} \frac{P(G \ge d_3(tx, \phi))}{P(G \ge t)} d\phi \right) \left(\int_{-\pi}^{\pi} \frac{P(G \ge d_3(t, \phi))}{P(G \ge t)} d\phi \right)^{-1} - x^{-\alpha}$$

$$= \left\{ \int_{-\pi}^{\pi} \left[\frac{P(G \ge d_3(tx, \phi))}{P(G \ge t)} - \left(\frac{1}{t} d_3(tx, \phi) \right)^{-\alpha} \right] d\phi + \int_{-\pi}^{\pi} \left[-x^{-\alpha} \frac{P(G \ge d_3(t, \phi))}{P(G \ge t)} + x^{-\alpha} \left(\frac{1}{t} d_3(t, \phi) \right)^{-\alpha} \right] d\phi + \int_{-\pi}^{\pi} \left[\left(\frac{1}{t} d_3(tx, \phi) \right)^{-\alpha} - x^{-\alpha} \left(\frac{1}{t} d_3(t, \phi) \right)^{-\alpha} \right] d\phi \right\} \times \\ \times \left(\int_{-\pi}^{\pi} \frac{P(G \ge d_3(t, \phi))}{P(G \ge t)} d\phi \right)^{-1}$$

$$=: \left(\mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t) \right) \left(\int_{-\pi}^{\pi} \frac{P(G \ge d_3(t, \phi))}{P(G \ge t)} d\phi \right)^{-1}.$$

Since $|\rho|<1,$ it is straightforward to check that

$$\lim_{t \to \infty} t^{-1} d_3(t, \phi) = (d_1(\phi))^{-1/2},$$

$$0 < \sup_{-\pi \le \phi \le \pi} d_1(\phi) < \infty, \text{ and}$$

$$\sup_{-\pi \le \phi \le \pi} |d_2(\phi)| < \infty.$$
(4.22)

Hence, similarly to the proof of Theorem 2.1,

$$\lim_{t \to \infty} \int_{-\pi}^{\pi} \frac{P(G \ge d_3(t, \phi))}{P(G \ge t)} \, \mathrm{d}\phi = \int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} \, \mathrm{d}\phi.$$
(4.23)

Similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \ge t_0, d_3(tx, \phi) \ge t_0$

$$\frac{\left|\frac{P\left(G \ge d_{3}(tx,\phi)\right)}{P(G \ge t)} - \left(\frac{1}{t}d_{3}(tx,\phi)\right)^{-\alpha}}{A(t)} - \left(\frac{1}{t}d_{3}(tx,\phi)\right)^{-\alpha} \frac{\left(\frac{1}{t}d_{3}(tx,\phi)\right)^{\beta} - 1}{\beta}\right|$$

$$\leq \varepsilon \left(1 + \left(\frac{1}{t}d_{3}(tx,\phi)\right)^{-\alpha} + \left(\frac{1}{t}d_{3}(tx,\phi)\right)^{-\alpha+\beta} \exp\left\{\varepsilon \left|\ln\left(\frac{1}{t}d_{3}(tx,\phi)\right)\right|\right\}\right).$$

$$(4.24)$$

Using (4.22), for any fixed x > 0, we can choose t_0 large enough such that $d_3(tx, \phi) \ge t_0$ uniformly for $\phi \in [-\pi, \pi]$. That is, for any fixed x > 0, (4.24) holds uniformly for $\phi \in [-\pi, \pi]$. Therefore, by dominated convergence theorem and (4.23), for x > 0,

$$\lim_{t \to \infty} \frac{\mathcal{J}_{1}(t)}{A(t)} = \frac{1}{x^{\alpha}\beta} \int_{-\pi}^{\pi} \left(x^{\beta} \left(d_{1}(\phi) \right)^{(\alpha-\beta)/2} - \left(d_{1}(\phi) \right)^{\alpha/2} \right) d\phi \quad \text{and} 4.25)$$
$$\lim_{t \to \infty} \frac{\mathcal{J}_{2}(t)}{A(t)} = -\frac{1}{x^{\alpha}\beta} \int_{-\pi}^{\pi} \left((d_{1}(\phi))^{(\alpha-\beta)/2} - (d_{1}(\phi))^{\alpha/2} \right) d\phi. \quad (4.26)$$

It follows from (4.22) and a Taylor expansion, for x > 0, that

$$\mathcal{J}_{3}(t) = \frac{\alpha}{tx^{\alpha}} \left(x^{-1} - 1\right) \int_{-\pi}^{\pi} \left(d_{1}(\phi)\right)^{\alpha/2 - 1/2} d_{2}(\phi) \,\mathrm{d}\phi + o\left(t^{-2}\right) + \frac{\alpha}{2t^{2}x^{\alpha}} \left(x^{-2} - 1\right) \int_{-\pi}^{\pi} \left(d_{1}(\phi)\right)^{\alpha/2 - 1} \left[\alpha(d_{2}(\phi))^{2} + d_{1}(\phi)(\mu_{X}^{2} + \mu_{Y}^{2})\right] \,\mathrm{d}\phi.$$

$$(4.27)$$

Note that $\sin(\phi + \arcsin \rho) = \sqrt{1 - \rho^2} \sin \phi + \rho \cos \phi$. Then, splitting the integral into $[-\pi, -\pi/2)$, $[-\pi/2, 0)$, $[0, \pi/2)$, $[\pi/2, \pi]$ and using the symmetry of sin and cos, we obtain

$$\int_{-\pi}^{\pi} \left(d_1(\phi) \right)^{(\alpha-1)/2} d_2(\phi) \, \mathrm{d}\phi = 0.$$
(4.28)

Hence (2.3) follows from (4.25), (4.26), (4.27) and (4.28). Note that

$$\lim_{t \to \infty} P\left(\sqrt{X^2 + Y^2} > t\right) / P(G > t) = \int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi$$

and, since $Y \stackrel{d}{=} \mu_Y + vG\sin\Phi$ with $\Phi \sim unif(-\pi, \pi)$ holds,

$$\lim_{t \to \infty} P(Y > t) / P(G > t) = v^{\alpha} \int_0^{\pi} (\sin \phi)^{\alpha} d\phi$$

Therefore, we have

$$V(t) \sim \inf \left\{ y : P(G > y) \le t^{-1} / \int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} \, d\phi \right\} \text{ and } F_Y^{\leftarrow}(1 - t^{-1}) \sim \inf \left\{ y : P(G > y) \le t^{-1} / \left(v^{\alpha} \int_0^{\pi} (\sin \phi)^{\alpha} \, d\phi \right) \right\}.$$

Hence,

$$\lim_{t \to \infty} \frac{V(t)}{F_Y^{\leftarrow}(1-t^{-1})} = \left(\frac{\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi}{v^{\alpha} \int_0^{\pi} (\sin \phi)^{\alpha} d\phi}\right)^{1/\alpha},$$

i.e., since $t^2 |A(t)| \xrightarrow{t \to \infty} \infty$ for $-2 < \beta \le 0$,

$$\lim_{t \to \infty} \frac{(V(t))^{-2} + |A(V(t))|}{(F_Y^{\leftarrow}(1 - t^{-1}))^{-2} + |A(F_Y^{\leftarrow}(1 - t^{-1}))|} = \left(\frac{\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2} d\phi}{v^{\alpha} \int_0^{\pi} (\sin \phi)^{\alpha} d\phi}\right)^{-(2 \wedge |\beta|)/\alpha}.$$
(4.29)

Note that, by Taylor expansion,

$$\left(\frac{V(tx)}{V(t)}\right)^{-\alpha} = \frac{1}{x} - \frac{\alpha}{x^{1+1/\alpha}} \left(\frac{V(tx)}{V(t)} - x^{1/\alpha}\right) + o\left(1/V(t) + |A(V(t))|\right).$$
(4.30)

Therefore, replacing t and x in (2.3) by V(t) and V(tx)/V(t), respectively, and using (4.29) and (4.30), we obtain (2.4). Let $\mu_X = \mu_Y = 0$, then $\mathcal{J}_3(t) = 0$ and we obtain (2.5).

Proof of Theorem 2.3: In order to prove Theorem 2.3, we can assume $\mu_X = \mu_Y = 0$ since $\lambda(x, y)$ is independent of margins. We also set v = 1 and give the correction at the end of the proof. Using an upper-triangle decomposition of Σ yields $Y \stackrel{d}{=} G \sin \Phi$, where $\Phi \sim unif(-\pi, \pi)$ and is independent of G. Then, write

$$\frac{P(Y \ge tx)}{P(Y \ge t)} - x^{-\alpha} = \frac{\int_0^{\pi} P(G \ge tx/\sin\phi) \,\mathrm{d}\phi}{\int_0^{\pi} P(G \ge t/\sin\phi) \,\mathrm{d}\phi} - x^{-\alpha}$$
$$= \left(\int_0^{\pi} \frac{P(G \ge t/\sin\phi)}{P(G \ge t)} \,\mathrm{d}\phi\right)^{-1} \left\{\int_0^{\pi} \left[\frac{P(G \ge tx/\sin\phi)}{P(G \ge t)} - \left(\frac{x}{\sin\phi}\right)^{-\alpha}\right] - x^{-\alpha} \left[\frac{P(G \ge t/\sin\phi)}{P(G \ge t)} - \left(\frac{1}{\sin\phi}\right)^{-\alpha}\right] \,\mathrm{d}\phi\right\}.$$

Then, by (2.1), we have for x > 0

$$\lim_{t \to \infty} \left(\frac{P(Y \ge tx)}{P(Y \ge t)} - x^{-\alpha} \right) / A(t)$$

= $x^{-\alpha} \frac{x^{\beta} - 1}{\beta} \left(\int_0^{\pi} (\sin \phi)^{\alpha} d\phi \right)^{-1} \left(\int_0^{\pi} (\sin \phi)^{\alpha - \beta} d\phi \right).$

Replacing t and x in the latter equation by $F_Y^{\leftarrow}(1-s)$ and $F_Y^{\leftarrow}(1-sy)/F_Y^{\leftarrow}(1-s)$, respectively, we obtain, for y > 0,

$$\lim_{s \to 0} \left(\left(\frac{F_Y^{\leftarrow}(1-sy)}{F_Y^{\leftarrow}(1-s)} \right)^{-\alpha} - y \right) / A \left(F_Y^{\leftarrow}(1-s) \right) = \mathcal{B}_{(2.6)}(y).$$
(4.31)

Denote $f(t) := F_Y^{\leftarrow}(1-t)$. Then, by (4.21), we can write

$$\begin{split} t^{-1}P\left(F_{X}(X) \geq 1 - tx, \ F_{Y}(Y) \geq 1 - ty\right) \\ &= \frac{1}{\mathcal{B}_{(4.18)}(t)} \left\{ \int_{h(x,y,t)}^{\pi/2} \left[\frac{P\left(G \geq \frac{f(t)}{\cos\phi} \frac{f(tx)}{f(t)}\right)}{P\left(G \geq f(t)\right)} - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} \right] \mathrm{d}\phi \right. \\ &+ \int_{h(x,y,t)}^{\pi/2} \left[\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} - x(\cos\phi)^{\alpha} \right] \mathrm{d}\phi + \int_{h(x,y,t)}^{h(x,y,0)} x(\cos\phi)^{\alpha} \, \mathrm{d}\phi \\ &+ \int_{-\operatorname{arcsin}\rho}^{h(x,y,t)} \left[\frac{P\left(G \geq \frac{f(t)}{\sin(\operatorname{arcsin}\rho + \phi)} \frac{f(ty)}{f(t)}\right)}{P\left(G \geq f(t)\right)} \\ &- \left(\frac{f(ty)}{f(t)\sin(\operatorname{arcsin}\rho + \phi)}\right)^{-\alpha} \right] \mathrm{d}\phi \\ &+ \int_{-\operatorname{arcsin}\rho}^{h(x,y,t)} \left[\left(\frac{f(ty)}{f(t)\sin(\operatorname{arcsin}\rho + \phi)}\right)^{-\alpha} - y\left(\sin\left(\operatorname{arcsin}\rho + \phi\right)\right)^{\alpha} \right] \mathrm{d}\phi \\ &+ \int_{h(x,y,0)}^{h(x,y,0)} y\left(\sin\left(\operatorname{arcsin}\rho + \phi\right)\right)^{\alpha} \, \mathrm{d}\phi + \int_{h(x,y,0)}^{\pi/2} x(\cos\phi)^{\alpha} \, \mathrm{d}\phi \\ &+ \int_{-\operatorname{arcsin}\rho}^{h(x,y,0)} y\left(\sin\left(\operatorname{arcsin}\rho + \phi\right)\right)^{\alpha} \, \mathrm{d}\phi \right\} \\ &=: \frac{1}{\mathcal{B}_{(4.18)}(t)} \left(\sum_{i=1}^{6} \mathcal{J}_{i}(t) + \mathcal{J}_{7} + \mathcal{J}_{8}\right). \end{split}$$
(4.32)

Note that $1/|\cos \phi| \ge 1$ and v is given, using Potter's bound and similar arguments as in the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists some small $t_0 > 0$ such that for all $f(t) \ge f(t_0)$, $f(tx) \ge f(t_0)$ and

$$\phi \in \left[-\pi/2, \pi/2\right] \\ \left| \frac{P\left(G \ge \frac{f(tx)}{\cos\phi}\right) / P\left(G \ge f(t)\right) - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha}}{A(f(t))} - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} \frac{\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{\beta} - 1}{\beta} \right| \\ \le \varepsilon \left(1 + \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} + \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha+\beta} \exp\left\{\varepsilon \left|\ln\frac{f(tx)}{f(t)\cos\phi}\right|\right\}\right),$$

$$(4.33)$$

and for all $t \leq t_0$ and $tx \leq t_0$,

$$(1-\varepsilon)x^{-1/\alpha}\exp(-\varepsilon|\log x|) \leq \frac{f(tx)}{f(t)} \leq (1+\epsilon)x^{-1/\alpha}\exp(\varepsilon|\log x|).$$
(4.34)

Since $f(t) \ge t_0$ and $t \le t_0$ imply that $f(tx) \ge t_0$ and $tx \le t_0$ for all $0 \le x \le 1$, respectively, by (4.33), (4.34), (4.20) and dominated convergence, we have

$$\lim_{t \to 0} \frac{\mathcal{J}_1(t)}{A(f(t))} = \frac{x}{\beta} \int_{h(x,y,0)}^{\pi/2} \left[x^{-\beta/\alpha} (\cos \phi)^{\alpha-\beta} - (\cos \phi)^{\alpha} \right] \mathrm{d}\phi \qquad (4.35)$$

holds for all $x, y \ge 0$ and uniformly on \mathcal{S}_2^+ . Similarly,

$$\lim_{t \to 0} \frac{\mathcal{J}_4(t)}{A(f(t))} = \frac{y}{\beta} \int_{-\arccos \rho}^{h(x,y,0)} \left[y^{-\beta/\alpha} (\sin(\phi + \arcsin \rho))^{\alpha-\beta} - (\sin(\phi + \arcsin \rho))^{\alpha} \right] d\phi$$

$$(4.36)$$

holds for all $x, y \ge 0$ and uniformly on \mathcal{S}_2^+ .

Using (4.31) and a way similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \le t_0$ and $tx \le t_0$

$$\left|\frac{(F_Y^{\leftarrow}(1-tx)/F_Y^{\leftarrow}(1-t))^{-\alpha}-x}{A(F_Y^{\leftarrow}(1-s))} - \mathcal{B}_{(2.6)}(x)\right|$$

$$\leq \varepsilon \left(C_1 + C_2 x + C_3 x^{1-\beta/\alpha} \exp(\varepsilon |\ln x|)\right), \qquad (4.37)$$

where the constants $C_1 > 0, C_2 > 0, C_3 > 0$ are independent of x and t. Hence, it follows from (4.20) and (4.37) that

$$\lim_{t \to 0} \frac{\mathcal{J}_2(t)}{A\left(F_Y^{\leftarrow}(1-t)\right)} = \mathcal{B}_{(2.6)}(x) \int_{h(x,y,0)}^{\pi/2} (\cos \phi)^{\alpha} d\phi \quad \text{and} \tag{4.38}$$

$$\lim_{t \to 0} \frac{\mathcal{J}_5(t)}{A\left(F_Y^{\leftarrow}(1-t)\right)} = \mathcal{B}_{(2.6)}(y) \int_{-\arcsin\rho}^{h(x,y,0)} (\sin(\phi + \arcsin\rho))^{\alpha} d\phi \ (4.39)$$

holds for all $x, y \ge 0$ and uniformly on \mathcal{S}_2^+ .

Note that

$$\frac{1}{A(f(t))} \left(\left(\frac{f(ty)}{f(tx)} \right)^{-\alpha} - \frac{y}{x} \right)$$

$$= \frac{1}{A(f(t))} \left[\frac{1}{x} \left(\left(\frac{f(ty)}{f(t)} \right)^{-\alpha} - y \right) - \frac{1}{x} \left(\frac{f(ty)}{f(tx)} \right)^{-\alpha} \left(\left(\frac{f(tx)}{f(t)} \right)^{-\alpha} - x \right) \right]$$

$$\stackrel{t \to 0}{\longrightarrow} \frac{1}{x} \mathcal{B}_{(2.6)}(y) - \frac{y}{x^2} \mathcal{B}_{(2.6)}(x).$$

Similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \le t_0, tx \le t_0, ty \le t_0$

$$\left| \frac{1}{A(f(t))} \left(\left(\frac{f(ty)}{f(tx)} \right)^{-\alpha} - \frac{y}{x} \right) - \frac{1}{x} \mathcal{B}_{(2.6)}(y) + \frac{y}{x^2} \mathcal{B}_{(2.6)}(x) \right|$$

$$\leq \frac{1}{x} \varepsilon \left(C_1 + C_2 y + C_3 y^{1-\beta/\alpha} e^{\varepsilon |\log y|} \right)$$

$$+ \frac{1}{x} \left(\frac{y}{x} \right) \left(C_1 + C_2 x + C_3 x^{1-\beta/\alpha} \exp(\varepsilon |\log x|) \right)$$

$$+ \frac{1}{x} \left(\frac{y}{x} \right) \exp(\varepsilon |\log(y/x)|) \left(C_1 + C_2 x + C_3 x^{1-\beta/\alpha} \exp(\varepsilon |\log x|) \right), (4.40)$$

where constants $C_1 > 0, C_2 > 0, C_3 > 0$ are independent of t, x, y. Using (4.40),

$$\begin{cases} \limsup_{z \to 0} |g'(z^{-1/\alpha})z^{2/\alpha}| < \infty \\ \limsup_{z \to \infty} |g'(z^{-1/\alpha})| < \infty \\ \limsup_{z \to \infty} [\sin(g(z^{-1/\alpha}) + \arcsin\rho)]^{\alpha}z < \infty \end{cases}$$

and applying a Taylor expansion to $g(z^{-1/\alpha})$, we can show that

$$\lim_{t \to 0} \frac{\mathcal{J}_{3}(t)}{A(f(t))} = \lim_{t \to 0} \frac{1}{A(f(t))} \int_{g(f(ty)/f(tx))}^{g((x/y)^{1/\alpha})} x(\cos \phi)^{\alpha} \, \mathrm{d}\phi$$
$$= \frac{x}{\alpha} \left[\cos \left(g \left((x/y)^{1/\alpha} \right) \right) \right]^{\alpha} g' \left((x/y)^{1/\alpha} \right) \left(\frac{\mathcal{B}_{(2.6)}(y)}{y} - \frac{\mathcal{B}_{(2.6)}(x)}{x} \right) \left(\frac{x}{y} \right)^{1/\alpha}$$
(4.41)

and

$$\lim_{t \to 0} \frac{\mathcal{J}_{6}(t)}{A(f(t))} = \lim_{t \to 0} \frac{1}{A(f(t))} \int_{g((x/y)^{1/\alpha})}^{g(f(ty)/f(tx))} y(\sin(\phi + \arcsin\rho))^{\alpha} d\phi$$
$$= -\frac{y}{\alpha} \left[\sin\left(g\left((x/y)^{1/\alpha}\right) + \arcsin\rho\right) \right]^{\alpha} g'\left((x/y)^{1/\alpha}\right) \times \left(\frac{\mathcal{B}_{(2.6)}(y)}{y} - \frac{\mathcal{B}_{(2.6)}(x)}{x}\right) \left(\frac{x}{y}\right)^{1/\alpha}$$
(4.42)

holds for all $x, y \ge 0$ and uniformly on \mathcal{S}_2^+ . Since

$$x \left[\cos \left(g \left((x/y)^{1/\alpha} \right) \right) \right]^{\alpha} = y \left[\sin \left(g \left((x/y)^{1/\alpha} \right) + \arcsin \rho \right) \right]^{\alpha}, (4.43)$$

we obtain $\lim_{t\to 0} (\mathcal{J}_3(t) + \mathcal{J}_6(t))/A(f(t)) = 0.$

By Theorem 2.1, $\lambda(x, y) = (\mathcal{J}_7 + \mathcal{J}_8)/\mathcal{B}_{(4.18)}$, hence

$$\lim_{t \to 0} \frac{1}{A(f(t))} \left(\frac{1}{\mathcal{B}_{(4.18)}(t)} (\mathcal{J}_{7} + \mathcal{J}_{8}) - \lambda(x, y) \right) \\
= \lim_{t \to 0} \frac{1}{A(f(t))} \left(-\frac{\lambda(x, y)}{\mathcal{B}_{(4.18)}(t)} \left(\mathcal{B}_{(4.18)}(t) - \mathcal{B}_{(4.18)} \right) \right) \\
= -\lambda(x, y) \frac{1}{\beta} \left(\int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} \left((\cos \phi)^{-\beta} - 1 \right) d\phi \right) \left(\int_{-\pi/2}^{\pi/2} (\cos \phi)^{\alpha} d\phi \right)^{-1},$$
(4.44)

which obviously holds uniformly on \mathcal{S}_2^+ since $\sup_{\mathcal{S}_2^+} \lambda(x, y) < \infty$. Note that

$$A\left(F_Y^{\leftarrow}(1-t)\right)/A\left(F_{\upsilon Y}^{\leftarrow}(1-t)\right) \xrightarrow{t \to 0} \upsilon^{-\beta}.$$
(4.45)

Hence the theorem follows from (4.35), (4.36), (4.38), (4.39), (4.41), (4.42), (4.44) and (4.45).

Proof of Theorem 2.4: Similar to Huang (1992) or Einmahl, de Haan and Li (2004), we have, as $n \to \infty$,

$$\sup_{0 \le x, y \le T} \left| \sqrt{k_{\mathrm{Hu}}} \left\{ x + y - \widehat{\lambda}_{k_{\mathrm{Hu}},n}^{Hu}(x,y) - l(x,y) \right\} - \mathcal{K}_{\mathrm{Hu}} \mathcal{B}_{(2.7)}(x,y) - B(x,y) \right|$$

= $o_p(1),$

where

$$B(x,y) = W(x,y) - \left(1 - \frac{\partial \lambda(x,y)}{\partial x}\right) W(x,0) - \left(1 - \frac{\partial \lambda(x,y)}{\partial y}\right) W(0,y),$$

and W(x, y) is a Wiener process with zero mean and covariance structure

$$E(W(x_1, y_1)W(x_2, y_2)) = l(x_1 \wedge x_2, y_1)l(x_1 \wedge x_2, y_2) - l(x_1, y_1 \wedge y_2) + l(x_2, y_1 \wedge y_2) - l(x_1, y_2) - l(x_2, y_1) - l(x_1 \wedge x_2, y_1 \wedge y_2).$$

Hence (2.9) follows from $\lambda(x, y) = x + y - l(x, y)$. It is straightforward to check that (2.10), (2.11) and (2.12) hold. Note that the result can also be obtained from Schmidt and Stadtmüller (2005) by taking the bias into account.

Proof of Theorem 2.5: The result follows directly from

$$\sqrt{k_{\rm El}} \left(\widehat{\alpha}_{k_{\rm El},n}^{\rm H} - \alpha \right) \xrightarrow{d} \mathcal{N} \left(-\alpha^2 \mathcal{K}_{\rm El} \mathcal{B}_{(2.14)}, \alpha^2 \right)$$

(see de Haan and Peng (1998)), $\hat{\tau}_n - \tau = o_p \left(k_{\text{El}}^{-1/2}\right)$ and the delta method for the expression of $\lambda(x, y)$ given in Theorem 2.1.

Acknowledgments

Peng's research was supported by NSF grant DMS-04-03443 and a Humboldt Research Fellowship.

References

- DRAISMA, G., DE HAAN, L., PENG, L. AND PEREIRA, T. (1999) A Bootstrap-based Method to Achieve Optimality in Estimating the Extremevalue Index. *Extremes* 2(4), pp. 367-404.
- [2] DREES, H. AND HUANG, X. (1998) Best Attainable Rates of Convergence for Estimates of the Stable Tail Dependence Functions. J. Mult. Anal. 64(1), pp. 25-47.
- [3] EINMAHL, J., DE HAAN, L. AND HUANG, X. (1993) Estimating a Multidimensional Extreme Value Distribution. J. Mult. Anal. 47(1), pp. 35-47.
- [4] EINMAHL, J., DE HAAN, L. AND LI, D. (2004) Weighted Approximations of Tail Copula Processes with Application to Testing the Multivariate Extreme Condition. Submitted. Available at www.few.eur.nl/few/people/ldehaan/cv.htm
- [5] EINMAHL, J., DE HAAN, L. AND PITERBARG, V. (2001) Nonparametric Estimation of the Spectral Measure of an Extreme Value Distribution. Ann. Statist. 29(5), pp. 1401-1423.
- [6] EINMAHL, J., DE HAAN, L. AND SINHA, A. (1997) Estimating the Spectral Measure of an Extreme Value Distribution. *Stochastic Process. Appl.* 70(2), pp. 143-171.
- [7] GELUK, J. AND DE HAAN, L. (1987) Regular Variation, Extensions and Tauberian Theorems. CWI Tract 40.
- [8] DE HAAN, L. AND PENG, L. (1998) Comparison of Tail Index Estimators. Statist. Neerlandica 52(1), pp. 60-70.
- [9] DE HAAN, L. AND RESNICK, S. (1993) Estimating the Limit Distribution of Multivariate Extremes. *Commun. Statist. Stochastic Models* 9(2), pp. 275-309.
- [10] DE HAAN, L. AND STADTMÜLLER, U. (1996) Generalized Regular Variation of Second Order. J. Australian Math. Soc. Ser. A 61(3), pp. 381-395.
- [11] HILL, B. (1975) A Simple General Approach to Inference About the Tail of a Distribution. Ann. Statist. 3(5), pp. 1163-1174.

- [12] HUANG, X. (1992) Statistics of Bivariate Extremes. Ph.D. Thesis, Erasmus University Rotterdam, Tinbergen Research Institute, no. 22.
- [13] HULT, H. AND LINDSKOG, F. (2002) Multivariate Extremes, Aggregation and Dependence in Elliptical Distributions. Adv. Appl. Prob. 34(3), pp. 587-608.
- [14] JOE, H. (1997) Multivariate Models and Dependence Concepts. Chapman&Hall, London.
- [15] LEDFORD, A. AND TAWN, J. (1997) Modelling Dependence within Joint Tail Regions. Statistics for near Independence in Multivariate Extreme Values. J. Royal Statist. Soc. B, 59(2), pp. 475-499.
- [16] LING, S. AND PENG, L. (2004) Hill's Estimator for the Tail Index of an ARMA Model. Journal of Statistical Planning and Inference, 123(2), pp. 279 293.
- [17] RESNICK, S. (1987) Extreme Values, Regular Variations and Point Processes. Springer, New York.
- [18] SCHMIDT, R. (2003) Credit Risk Modelling and Estimation via Elliptical Copulae. In *Credit Risk: Measurement, Evaluation and Management*, ed. Bohl, Nakhaeizadeh, Rachev, Ridder and Vollmer. Physica Verlag Heidelberg, pp. 267-289.
- [19] Schmidt, R. and Stadtmüller, U. (2005) Nonparametric Estimation of Tail Dependence. Scand. J. of Stat. To appear.
- [20] TAWN, J. (1988) Bivariate Extreme Value Theory: Models and Estimation. Biometrika 75(3), 397-415.