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## Rational expectations models with anticipated shocks and optimal policy: a general solution method and a New Keynesian example

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## **Working Papers**

Kiel Institute  
for the World Economy



**Rational Expectations Models with  
Anticipated Shocks and Optimal  
Policy: A General Solution Method  
and a New Keynesian Example**

**by Hans-Werner Wohltmann and  
Roland Winkler**

**No. 1507 | April 2009**

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## **Rational Expectations Models with Anticipated Shocks and Optimal Policy: A General Solution Method and a New Keynesian Example\***

Hans-Werner Wohltmann and Roland Winkler

**Abstract:** The purpose of this paper is to show how to solve linear dynamic rational expectations models with anticipated shocks by using the generalized Schur decomposition method. Furthermore, we determine the optimal unrestricted and restricted policy responses to anticipated shocks. We demonstrate our solution method by means of a micro-founded hybrid New Keynesian model and show that anticipated cost-push shocks entail higher welfare losses than unanticipated shocks of equal size.

**Keywords:** Anticipated Shocks, Optimal Monetary Policy, Rational Expectations, Generalized Schur Decomposition, Welfare Effects

**JEL classification:** C61, C63, E52

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# 1 Introduction

Recently, a number of macroeconometric studies emphasized the role of anticipated shocks as sources of macroeconomic fluctuations. Beaudry and Portier (2006) find that more than one-half of business cycle fluctuations are caused by news concerning future technological opportunities. Davis (2007) and Fujiwara, Hirose, and Shintani (2008) analyze the importance of anticipated shocks in medium-scale New Keynesian DSGE models and find that these disturbances are important components of aggregate fluctuations. Schmitt-Grohé and Uribe (2008) conduct a Bayesian estimation of a real-business cycle model and find that anticipated shocks are the most important source of aggregate fluctuations. In particular, they show that anticipated shocks explain two thirds of the volatility in consumption, output, investment, and employment.

In light of these findings, we attempt to investigate, whether the anticipation of future cost-push shocks has a stabilizing and therefore welfare-enhancing effect when compared to unanticipated shocks. In order to conduct an analysis of the (welfare) effects of anticipated shocks, this paper presents a general solution method for linear dynamic rational expectations models with anticipated shocks and optimal policy. Our method extends the work of Söderlind (1999), who uses the generalized Schur decomposition method, advocated by Klein (2000), to solve linear rational expectations models with optimal policy. However, Söderlind (1999) only considers stochastic models with white noise shocks which are, by definition, unpredictable. In the case of anticipated shocks, the occurrence of all future shocks is known exactly at the time when the solution of the model is computed. Our method also contains unanticipated shocks as a limiting case.

As an economic example, we lay out a calibrated New Keynesian model for a closed and cashless economy with internal habit formation in consumption preferences, a variant of Calvo price staggering with partial indexation to past inflation and a time-varying wage mark-up which represents a typical cost-push shock. We compare the effects of mark-up shocks under optimal monetary policy for different lengths of the anticipation period. Our main finding is that anticipated cost shocks entail higher welfare losses than unanticipated cost disturbances of equal magnitude.

The paper is organized as follows. Section 2 discusses optimal policies in RE models with *anticipated* temporary shocks. We first determine the optimal unrestricted policy under precommitment and calculate the minimum value of the intertemporal loss function. We then consider (optimal) simple rules and demonstrate how the Schur decomposition can be used to solve the model under these conditions. Section 3 derives the hybrid New Keynesian model, presents the welfare-theoretic loss function and discusses the effects of anticipated and unanticipated cost-push shocks. Finally, Section 4 provides concluding remarks.

## 2 The Model

In this paper we discuss the following linear expectational difference equations

$$A \begin{pmatrix} w_{t+1} \\ E_t v_{t+1} \end{pmatrix} = B \begin{pmatrix} w_t \\ v_t \end{pmatrix} + C u_t + D \nu_{t+1} , \quad (1)$$

where  $w_t$  is an  $n_1 \times 1$  vector of predetermined variables, assuming  $w_0$  given,  $v_t$  an  $n_2 \times 1$  vector of non-predetermined variables,  $u_t$  an  $m \times 1$  vector of policy instruments, and  $\nu_{t+1}$  an  $r \times 1$  vector of exogenous shocks. The matrices  $A$  and  $B$  are  $n \times n$  (where  $n = n_1 + n_2$ ), while the matrices  $C$  and  $D$  are  $n \times m$  and  $n \times r$  respectively. We allow matrix  $A$  to be singular which is the case if static (intratemporal) equations are included within the dynamic relationships. The vector  $w$ , composed of backward-looking variables, can include exogenous variables, following autoregressive processes.  $E_t v_{t+1}$  denotes model consistent (rational) expectations of  $v_{t+1}$  formed at time  $t$ . We assume that the shocks are anticipated by the public in advance and take the following form

$$\nu_t = \begin{cases} \bar{\nu} & \text{for } t = \tau > 0 \\ 0 & \text{for } t \neq \tau , \end{cases} \quad (2)$$

where  $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_r)'$  is a constant non-zero  $r \times 1$  vector. It is assumed that at time  $t = 0$  the public anticipates a shock of the form outlined in (2) to take place at some

future date  $\tau > 0$ . Note that  $\tau$  also defines the lengths of the anticipation period. Since shocks are anticipated by the public we have  $E_t \nu_{t+1} = \nu_{t+1}$ . For notational convenience, we define the  $n \times 1$  vector  $k_t = (w'_t, v'_t)'$ . Assume that the policy maker's welfare loss at time  $t$  is given by

$$J_t = \frac{1}{2} E_t \sum_{i=0}^{\infty} \lambda^i \{k'_{t+i} \tilde{W} k_{t+i} + 2k'_{t+i} P u_{t+i} + u'_{t+i} R u_{t+i}\}, \quad (3)$$

where  $\tilde{W}$  and  $R$  are symmetric and non-negative definite and  $P$  is  $n \times m$ .

## 2.1 Optimal Policy with Precommitment

We are now going to develop the policy maker's optimal policy rule at time  $t = 0$ . It is assumed that the policy maker is able to commit to such a rule. From the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} E_0 \sum_{t=0}^{\infty} \lambda^t \{k'_t \tilde{W} k_t + 2k'_t P u_t + u'_t R u_t + 2\rho'_{t+1} [B k_t + C u_t + D \nu_{t+1} - A k_{t+1}]\} \quad (4)$$

with the  $n \times 1$  multiplier  $\rho_{t+1}$ , we get the first-order conditions with respect to  $\rho_{t+1}$ ,  $k_t$ , and  $u_t$ :

$$\begin{pmatrix} A & 0_{n \times m} & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times m} & \lambda B' \\ 0_{m \times n} & 0_{m \times m} & -C' \end{pmatrix} \begin{pmatrix} k_{t+1} \\ u_{t+1} \\ \rho_{t+1} \end{pmatrix} = \begin{pmatrix} B & C & 0_{n \times n} \\ -\lambda \tilde{W} & -\lambda P & A' \\ P' & R & 0_{m \times n} \end{pmatrix} \begin{pmatrix} k_t \\ u_t \\ \rho_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{n \times r} \\ 0_{m \times r} \end{pmatrix} \nu_{t+1}. \quad (5)$$

To solve the system of equations in (5), expand the state and costate vector  $k_t$  and  $\rho_t$  as  $(w'_t, v'_t)'$  and  $(\rho'_{wt}, \rho'_{vt})'$  respectively and rearrange the rows of the  $(2n + m) \times 1$  vector  $(k'_t, u'_t, \rho'_t)'$  by placing the predetermined vector  $\rho_{vt}$  after  $w_t$ . Since  $v_t$  is forward-looking with an arbitrarily chosen initial value  $v_0$ , the corresponding Lagrange multiplier  $\rho_{vt}$  is predetermined with an initial value  $\rho_{v0} = 0$ . Rearrange the columns of the  $(2n + m) \times$

$(2n + m)$  matrices in (5) according to the re-ordering of  $(k'_t, u'_t, \rho_t)'$  and write the result as

$$F \begin{pmatrix} \tilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{pmatrix} = G \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{n \times r} \\ 0_{m \times r} \end{pmatrix} \nu_{t+1} , \quad (6)$$

where  $\tilde{w}_t = (w'_t, \rho'_{vt})'$  and  $\tilde{v}_t = (v'_t, u'_t, \rho'_{wt})'$ . The  $n \times 1$  vector  $\tilde{w}_t$  contains the 'backward-looking' variables of (5) while the  $(n + m) \times 1$  vector  $\tilde{v}_t$  contains the 'forward-looking' variables.

Equation (5) implies that the  $(2n+m) \times (2n+m)$  matrix  $F$  is singular. To solve equation (6) we apply the generalized Schur decomposition method (Söderlind, 1999; Klein, 2000). The decomposition of the square matrices  $F$  and  $G$  is given by  $F = \overline{Q}' S \overline{Z}'$ ,  $G = \overline{Q}' T \overline{Z}'$  or equivalently  $Q F Z = S$ ,  $Q G Z = T$ , where  $Q, Z, S$ , and  $T$  are square matrices of complex numbers,  $S$  and  $T$  are upper triangular and  $Q$  and  $Z$  are unitary, i.e.  $Q \cdot \overline{Q}' = \overline{Q}' \cdot Q = I_{(2n+m) \times (2n+m)} = Z \cdot \overline{Z}' = \overline{Z}' \cdot Z$ , where the non-singular matrix  $\overline{Q}'$  is the transpose of  $\overline{Q}$ , which denotes the complex conjugate of  $Q$ .  $\overline{Z}'$  is the transpose of the complex conjugate of  $Z$ . The matrices  $S$  and  $T$  can be arranged in such a way that the block with the stable generalized eigenvalues (the  $i$ th diagonal element of  $T$  divided by the  $i$ th diagonal element of  $S$ ) comes first. Premultiply both sides of equation (6) with  $Q$  and define auxiliary variables  $\tilde{z}_t$  and  $\tilde{x}_t$  so that

$$\begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} = \overline{Z}' \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} . \quad (7)$$

Partitioning the triangular matrices  $S$  and  $T$  in order to conform with  $\tilde{z}$  and  $\tilde{x}$ . Then set

$$Q \begin{pmatrix} D \\ 0_{n \times r} \\ 0_{m \times r} \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} , \quad (8)$$

where  $Q_1$  is  $n \times r$  and  $Q_2$  is  $(n+m) \times r$ . We then obtain the equivalent system

$$\begin{pmatrix} S_{11} & S_{12} \\ 0_{(n+m) \times n} & S_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0_{(n+m) \times n} & T_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \nu_{t+1}, \quad (9)$$

where the  $n \times n$  matrix  $S_{11}$  and the  $(n+m) \times (n+m)$  matrix  $T_{22}$  are invertible while  $S_{22}$  is singular. The square matrix  $T_{11}$  may also be singular. The lower block of equation (9) contains the unstable generalized eigenvalues and must be solved forward. Since

$$\tilde{x}_{t+s} = M_2 \tilde{x}_{t+s+1} - T_{22}^{-1} Q_2 \nu_{t+s+1} \quad (s = 0, 1, 2, \dots), \quad (10)$$

where  $M_2 = T_{22}^{-1} S_{22}$ , the unique stable solution for  $\tilde{x}_t$  is given by

$$\tilde{x}_t = - \sum_{s=0}^{\infty} M_2^s T_{22}^{-1} Q_2 E_t \nu_{t+s+1} = \begin{cases} -M_2^{\tau-1-t} T_{22}^{-1} Q_2 \bar{\nu} & \text{for } 0 \leq t < \tau \\ 0 & \text{for } t \geq \tau. \end{cases} \quad (11)$$

The upper block of (9) contains the stable generalized eigenvalues and can be solved backward. Since

$$\tilde{z}_{t+1} = M_1 \tilde{z}_t + S_{11}^{-1} (T_{12} \tilde{x}_t - S_{12} \tilde{x}_{t+1}) + S_{11}^{-1} Q_1 \nu_{t+1}, \quad (12)$$

where  $M_1 = S_{11}^{-1} T_{11}$  (which in general is not invertible), the general solution is given by

$$\begin{aligned} \tilde{z}_t &= M_1^t K + \sum_{s=0}^{t-1} M_1^{t-s-1} S_{11}^{-1} (T_{12} \tilde{x}_s - S_{12} \tilde{x}_{s+1} + Q_1 \nu_{s+1}) \\ &= \begin{cases} M_1^t K + \sum_{s=0}^{t-1} M_1^{t-s-1} S_{11}^{-1} (T_{12} \tilde{x}_s - S_{12} \tilde{x}_{s+1}) & \text{for } 0 \leq t < \tau \\ M_1^t K + \sum_{s=0}^{\tau-1} M_1^{t-s-1} S_{11}^{-1} (T_{12} \tilde{x}_s - S_{12} \tilde{x}_{s+1}) \\ \quad + M_1^{t-\tau} S_{11}^{-1} Q_1 \bar{\nu} & \text{for } t \geq \tau, \end{cases} \end{aligned} \quad (13)$$

where  $\tilde{x}_s$  is defined in (11).



We can rewrite (13) as follows:

$$\tilde{z}_t = M_1^{t-\tau} \tilde{K} \quad \text{for } t \geq \tau, \quad (14)$$

where

$$\tilde{K} = M_1^\tau K + S_{11}^{-1} Q_1 \bar{\nu} + \sum_{s=0}^{\tau-1} M_1^{\tau-s-1} S_{11}^{-1} (T_{12} \tilde{x}_s - S_{12} \tilde{x}_{s+1}). \quad (15)$$

Since

$$\tilde{x}_s = \begin{cases} -M_2^{\tau-1-s} T_{22}^{-1} Q_2 \bar{\nu} & \text{for } 0 \leq s < \tau \\ 0 & \text{for } s \geq \tau, \end{cases} \quad (16)$$

we can write  $\tilde{K}$  as

$$\tilde{K} = M_1^\tau K + S_{11}^{-1} Q_1 \bar{\nu} + [-\tilde{W}_1 + M_1 \tilde{W}_2] T_{22}^{-1} Q_2 \bar{\nu}, \quad (17)$$

where

$$\tilde{W}_1 = \sum_{s=0}^{\tau-1} M_1^{\tau-s-1} S_{11}^{-1} T_{12} M_2^{\tau-s-1} = \sum_{k=0}^{\tau-1} M_1^k S_{11}^{-1} T_{12} M_2^k \quad (18)$$

and

$$\tilde{W}_2 = \sum_{s=0}^{\tau-2} M_1^{\tau-s-2} S_{11}^{-1} S_{12} M_2^{\tau-s-2} = \sum_{k=0}^{\tau-2} M_1^k S_{11}^{-1} S_{12} M_2^k. \quad (19)$$

$\tilde{W}_1$  as well as  $\tilde{W}_2$  is a finite geometric sum of matrices and can be written as

$$\tilde{W}_1 = S_{11}^{-1} T_{12} - M_1^\tau S_{11}^{-1} T_{12} M_2^\tau + M_1 \tilde{W}_1 M_2 \quad (20)$$

and

$$\tilde{W}_2 = S_{11}^{-1}S_{12} - M_1^{\tau-1}S_{11}^{-1}S_{12}M_2^{\tau-1} + M_1\tilde{W}_2M_2. \quad (21)$$

To solve for both,  $\tilde{W}_1$  and  $\tilde{W}_2$ , we use the matrix identities (Rudebusch and Svensson, 1999; Klein, 2000)  $\text{vec}(A + B) = \text{vec}(A) + \text{vec}(B)$  and  $\text{vec}(ABC) = [C' \otimes A] \text{vec}(B)$  where  $\text{vec}(A)$  denotes the vector of stacked column vectors of the matrix  $A$  and  $\otimes$  denotes the Kronecker product of matrices.

We then obtain from (20) and (21)

$$\text{vec } \tilde{W}_1 - [M_2' \otimes M_1] \text{vec } \tilde{W}_1 = \text{vec} [S_{11}^{-1}T_{12} - M_1^{\tau}S_{11}^{-1}T_{12}M_2^{\tau}] \quad (22)$$

and

$$\text{vec } \tilde{W}_2 - [M_2' \otimes M_1] \text{vec } \tilde{W}_2 = \text{vec} [S_{11}^{-1}S_{12} - M_1^{\tau-1}S_{11}^{-1}S_{12}M_2^{\tau-1}] \quad (23)$$

with the solution

$$\text{vec } \tilde{W}_1 = [I - M_2' \otimes M_1]^{-1} \cdot \text{vec} [S_{11}^{-1}T_{12} - M_1^{\tau}S_{11}^{-1}T_{12}M_2^{\tau}] \quad (24)$$

$$\text{vec } \tilde{W}_2 = [I - M_2' \otimes M_1]^{-1} \cdot \text{vec} [S_{11}^{-1}S_{12} - M_1^{\tau-1}S_{11}^{-1}S_{12}M_2^{\tau-1}]. \quad (25)$$

According to (13) and (16), the solution of  $\tilde{z}_t$  over the anticipation interval  $0 < t < \tau$  can be rewritten as

$$\tilde{z}_t = M_1^t K + [-W_{1t}^* + W_{2t}^*] T_{22}^{-1} Q_2 \bar{\nu} \quad \text{for } 0 \leq t < \tau \quad (26)$$

with

$$W_{1t}^* = \sum_{s=0}^{t-1} M_1^{t-s-1} S_{11}^{-1} T_{12} M_2^{\tau-s-1} = \sum_{k=0}^{t-1} M_1^k S_{11}^{-1} T_{12} M_2^{\tau-t+k} \quad (27)$$

and

$$W_{2t}^* = \sum_{s=1}^t M_1^{t-s} S_{11}^{-1} S_{12} M_2^{\tau-s-1} = \sum_{k=0}^{t-1} M_1^k S_{11}^{-1} S_{12} M_2^{\tau-1-t+k}. \quad (28)$$

$W_{1t}^*$  satisfies the matrix equation<sup>1</sup>

$$W_{1t}^* = S_{11}^{-1} T_{12} M_2^{\tau-t} - M_1^t S_{11}^{-1} T_{12} M_2^\tau + M_1 W_{1t}^* M_2 \quad (0 \leq t < \tau) \quad (29)$$

with the solution

$$\text{vec } W_{1t}^* = [I - M_2' \otimes M_1]^{-1} \cdot \text{vec } (S_{11}^{-1} T_{12} M_2^{\tau-t} - M_1^t S_{11}^{-1} T_{12} M_2^\tau). \quad (30)$$

The matrix  $W_{2t}^*$  satisfies the equation<sup>2</sup>

$$W_{2t}^* = S_{11}^{-1} S_{12} M_2^{\tau-1-t} - M_1^t S_{11}^{-1} S_{12} M_2^{\tau-1} + M_1 W_{2t}^* M_2 \quad (0 \leq t < \tau) \quad (31)$$

with the solution

$$\text{vec } W_{2t}^* = [I - M_2' \otimes M_1]^{-1} \cdot \text{vec } (S_{11}^{-1} S_{12} M_2^{\tau-1-t} - M_1^t S_{11}^{-1} S_{12} M_2^{\tau-1}). \quad (32)$$

The constant  $K$  can be determined using the initial value of the predetermined vector  $\tilde{w}$ . By premultiplying equation (7) with  $Z$  and by partitioning the matrix  $Z$  to conform with the dimension of  $\tilde{z}$  and  $\tilde{x}$ , we obtain

$$\begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} \quad (33)$$

and therefore  $\tilde{w}_0 = Z_{11}\tilde{z}_0 + Z_{12}\tilde{x}_0$  with  $\tilde{w}_0 = (w'_0, 0'_{n_2 \times 1})'$ ,  $\tilde{z}_0 = K$ , and  $\tilde{x}_0 = -M_2^{\tau-1} T_{22}^{-1} Q_2 \bar{v}$ ,

<sup>1</sup>Note that equation (29) is also well-defined for  $t = \tau$ . In this case it is equivalent to (20) implying  $W_{1\tau}^* = \tilde{W}_1$ .

<sup>2</sup>For  $t = \tau - 1$  equation (31) is equivalent to (21) so that  $W_{2\tau-1}^* = \tilde{W}_2$ . Then, according to (17),  $\tilde{K} = \tilde{z}_\tau = M_1^\tau K + S_{11}^{-1} Q_1 \bar{v} + [-W_{1\tau}^* + M_1 W_{2\tau-1}^*] T_{22}^{-1} Q_2 \bar{v}$ .

where it is assumed that  $\tau > 0$ .<sup>3</sup> The constant  $K$  is then given by

$$K = Z_{11}^{-1}\tilde{w}_0 - Z_{11}^{-1}Z_{12}\tilde{x}_0, \quad (34)$$

provided the inverse  $Z_{11}^{-1}$  exists. A necessary condition is that the dynamic system (6) has the saddle path property, i.e., that the number of backward-looking variables ( $n_1 + n_2 = n$ ) coincides with the number of stable generalized eigenvalues (Söderlind, 1999; Klein, 2000).

In the case  $\tau > 0$  we can assume  $w_0 = 0$  so that the constant  $K$  can be written as

$$K = Z_{11}^{-1}Z_{12}M_2^{\tau-1}T_{22}^{-1}Q_2\bar{\nu}. \quad (35)$$

The solution to the state vector  $(\tilde{z}_t, \tilde{x}_t)'$  for  $0 \leq t < \tau$  now reads as follows

$$\begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} = \Xi_t T_{22}^{-1} Q_2 \bar{\nu} \quad \text{for } 0 \leq t < \tau, \quad (36)$$

where

$$\Xi_t = \begin{pmatrix} M_1^t Z_{11}^{-1} Z_{12} M_2^{\tau-1} - W_{1t}^* + W_{2t}^* \\ -M_2^{\tau-1-t} \end{pmatrix} \quad (0 \leq t < \tau). \quad (37)$$

If  $Z_{11}$  is invertible, equation (33) implies

$$\tilde{v}_t = Z_{21}\tilde{z}_t + Z_{22}\tilde{x}_t = Z_{21}(Z_{11}^{-1}\tilde{w}_t - Z_{11}^{-1}Z_{12}\tilde{x}_t) + Z_{22}\tilde{x}_t = N\tilde{w}_t + \hat{Z}\tilde{x}_t, \quad (38)$$

where  $N = Z_{21}Z_{11}^{-1}$  and  $\hat{Z} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ . Write equation (38) as

$$\begin{pmatrix} v_t \\ u_t \\ \rho_{wt} \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \\ N_{31} & N_{32} \end{pmatrix} \begin{pmatrix} w_t \\ \rho_{vt} \end{pmatrix} + \begin{pmatrix} \hat{Z}_1 \\ \hat{Z}_2 \\ \hat{Z}_3 \end{pmatrix} \tilde{x}_t \quad (39)$$

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<sup>3</sup>In the special case  $\tau = 0$  (unanticipated shocks) we have  $\tilde{x}_0 = 0$  and  $\tilde{z}_t = (S_{11}^{-1}T_{11})^t K + (S_{11}^{-1}T_{11})^t S_{11}^{-1}Q_1\bar{\nu}$  implying  $\tilde{z}_0 = K + S_{11}^{-1}Q_1\bar{\nu}$  and  $K = Z_{11}^{-1}\tilde{w}_0 - S_{11}^{-1}Q_1\bar{\nu}$  with  $w_0 \neq 0$ . By contrast, the initial value  $w_0$  can be normalized to zero if  $\tau > 0$ .

and assume the  $n_2 \times n_2$  matrix  $N_{12}$  is invertible. The optimal policy rule under commitment can then be written as

$$\begin{aligned} u_t &= N_{21}w_t + N_{22}\rho_{vt} + \hat{Z}_2\tilde{x}_t = N_{21}w_t + N_{22}N_{12}^{-1}(v_t - N_{11}w_t - \hat{Z}_1\tilde{x}_t) + \hat{Z}_2\tilde{x}_t \\ &= N_{22}N_{12}^{-1}v_t + (N_{21} - N_{22}N_{12}^{-1}N_{11})w_t + (\hat{Z}_2 - N_{22}N_{12}^{-1}\hat{Z}_1)\tilde{x}_t, \end{aligned} \quad (40)$$

where  $\tilde{x}_t$  is given by (11). For  $t < \tau$ ,  $u_t$  depends on the auxiliary variable  $\tilde{x}_t$ , while for  $t \geq \tau$ ,  $u_t$  is only a linear function of the predetermined state variables  $w_t$  and  $\rho_{vt}$ , where  $\rho_{vt}$  can be substituted by the original state variables  $v_t$  and  $w_t$ .

### Minimum Value of the Loss Function

To determine the minimum value of the loss function  $J_t$  at time  $t = 0$ , we express  $J_t$  as function of  $\tilde{w}$  and  $\tilde{v}$ . The loss function (3) can be written as

$$J_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i (k'_{t+i}, u'_{t+i}) H \begin{pmatrix} k_{t+i} \\ u_{t+i} \end{pmatrix} = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i (w'_{t+i}, v'_{t+i}, u'_{t+i}) H \begin{pmatrix} w_{t+i} \\ v_{t+i} \\ u_{t+i} \end{pmatrix}, \quad (41)$$

where the  $(n+m) \times (n+m)$  matrix  $H$  is given by

$$H = \begin{pmatrix} \tilde{W} & P \\ P' & R \end{pmatrix} \quad (42)$$

with  $H = H'$ . Define the  $n_1 \times n$  matrix  $\tilde{D}_1$  and the  $(n_2+m) \times (n+m)$  matrix  $\tilde{D}_2$  by  $\tilde{D}_1 = (I_{n_1 \times n_1}, 0_{n_1 \times n_2})$  and  $\tilde{D}_2 = (I_{(n_2+m) \times (n_2+m)}, 0_{(n_2+m) \times n_1})$ , respectively. Then  $w = \tilde{D}_1(w', \rho'_v)' = \tilde{D}_1\tilde{w}'$ ,  $(v', u')' = \tilde{D}_2(v', u', \rho'_w)' = \tilde{D}_2\tilde{v}'$ ,  $(w', v', u')' = \tilde{D}(\tilde{w}', \tilde{v}')'$  with

$$\tilde{D} = \begin{pmatrix} \tilde{D}_1 & 0_{n_1 \times (n+m)} \\ 0_{(n_2+m) \times n} & \tilde{D}_2 \end{pmatrix} = \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} & 0_{n_1 \times (n_2+m)} & 0_{n_1 \times n_1} \\ 0_{(n_2+m) \times n_1} & 0_{(n_2+m) \times n_2} & I_{(n_2+m) \times (n_2+m)} & 0_{(n_2+m) \times n_1} \end{pmatrix}, \quad (43)$$

which is a  $(n+m) \times (2n+m)$  matrix. The loss function  $J_t$  can now be rewritten as

$$J_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i (\tilde{w}'_{t+i}, \tilde{v}'_{t+i}) \tilde{D}' H \tilde{D} \begin{pmatrix} \tilde{w}_{t+i} \\ \tilde{v}_{t+i} \end{pmatrix} = J_t^{(1)} + J_t^{(2)}, \quad (44)$$

where

$$J_t^{(1)} = \frac{1}{2} \sum_{i=0}^{\tau-1} \lambda^i (\tilde{w}'_{t+i}, \tilde{v}'_{t+i}) \tilde{D}' H \tilde{D} \begin{pmatrix} \tilde{w}_{t+i} \\ \tilde{v}_{t+i} \end{pmatrix} \quad (45)$$

and

$$J_t^{(2)} = \frac{1}{2} \sum_{i=\tau}^{\infty} \lambda^i (\tilde{w}'_{t+i}, \tilde{v}'_{t+i}) \tilde{D}' H \tilde{D} \begin{pmatrix} \tilde{w}_{t+i} \\ \tilde{v}_{t+i} \end{pmatrix}. \quad (46)$$

First, we calculate  $J_t^{(2)}$ . For  $t \geq \tau$ , we have  $\tilde{v}_t = N \tilde{w}_t$  and  $\tilde{w}_t = Z_{11} \tilde{z}_t$ , where  $N = Z_{21} Z_{11}^{-1}$ . We then obtain  $(\tilde{w}'_t, \tilde{v}'_t)' = \tilde{N} \tilde{w}_t = \tilde{N} Z_{11} \tilde{z}_t$ , where  $\tilde{N} = (I_{n \times n}, N)'$  is a  $(2n+m) \times n$  matrix.  $J_t^{(2)}$  can then be rewritten as

$$J_t^{(2)} = \frac{1}{2} \sum_{i=\tau}^{\infty} \lambda^i Z'_{11} \tilde{z}'_{t+i} \tilde{N}' \tilde{D}' H \tilde{D} \tilde{N} Z_{11} \tilde{z}_{t+i} = \frac{1}{2} \sum_{i=\tau}^{\infty} \lambda^i Z'_{11} \tilde{z}'_{t+i} H^* Z_{11} \tilde{z}_{t+i} \quad (47)$$

with  $H^* = \tilde{N}' \tilde{D}' H \tilde{D} \tilde{N}$  is a symmetric  $n \times n$  matrix. Inserting the solution formula (14) for  $\tilde{z}_t$  in (47) we obtain

$$\begin{aligned} J_t^{(2)} &= \frac{1}{2} (M_1^t \tilde{K})' \lambda^\tau \left( \sum_{i=\tau}^{\infty} \lambda^{i-\tau} (Z_{11} M_1^{i-\tau})' H^* (Z_{11} M_1^{i-\tau}) \right) M_1^t \tilde{K} \\ &= \frac{1}{2} \lambda^\tau \varphi_t' V^* \varphi_t = \frac{1}{2} \lambda^\tau \text{trace}(V^* \varphi_t \varphi_t') , \end{aligned} \quad (48)$$

where  $\varphi_t = M_1^t \tilde{K}$  and  $V^*$  is the convergent geometric sum of matrices

$$V^* = \sum_{i=\tau}^{\infty} \lambda^{i-\tau} (Z_{11} M_1^{i-\tau})' H^* (Z_{11} M_1^{i-\tau}), \quad (49)$$

which is of dimension  $n \times n$  and satisfies the matrix equation

$$V^* = Z'_{11} H^* Z_{11} + \lambda M'_1 V^* M_1 \quad (50)$$

with the solution

$$\text{vec}(V^*) = [I - \lambda M'_1 \otimes M_1]^{-1} \text{vec}(Z'_{11} H^* Z_{11}) . \quad (51)$$

For  $t = 0$  we obtain from (48)

$$J_0^{(2)} = \frac{1}{2} \lambda^\tau \text{trace}(V^* \varphi_0 \varphi'_0) = \frac{1}{2} \lambda^\tau \text{trace}(V^* \tilde{K} \tilde{K}') \quad (52)$$

with  $\tilde{K}$  given by (17).

The next step is the calculation of the finite sum  $J_t^{(1)}$  as defined in (45). Because  $(\tilde{w}'_t, \tilde{v}'_t)' = Z(\tilde{z}'_t, \tilde{x}'_t)'$ , we can write  $J_0^{(1)}$  as

$$J_0^{(1)} = \frac{1}{2} \sum_{i=0}^{\tau-1} \lambda^i (\tilde{z}'_i, \tilde{x}'_i) Z' \tilde{D}' H \tilde{D} Z \begin{pmatrix} \tilde{z}_i \\ \tilde{x}_i \end{pmatrix} = \frac{1}{2} \sum_{t=0}^{\tau-1} \lambda^t (\tilde{z}'_t, \tilde{x}'_t) \tilde{H} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} , \quad (53)$$

where  $\tilde{H} = Z' \tilde{D}' H \tilde{D} Z$ .

Inserting the solution formula (36) for  $(\tilde{z}'_t, \tilde{x}'_t)'$  in (53), we obtain the expression

$$J_0^{(1)} = \frac{1}{2} (T_{22}^{-1} Q_2 \bar{\nu})' \left[ \sum_{t=0}^{\tau-1} \lambda^t \Xi'_t \tilde{H} \Xi_t \right] (T_{22}^{-1} Q_2 \bar{\nu}) = \frac{1}{2} \mu' W^* \mu = \frac{1}{2} \text{trace}(W^* \mu \mu') , \quad (54)$$

where  $\mu = T_{22}^{-1} Q_2 \bar{\nu}$  and  $W^* = \sum_{t=0}^{\tau-1} \lambda^t \Xi'_t \tilde{H} \Xi_t$ .

The total loss under the optimal unrestricted policy under commitment is now given by

$$J_0 = J_0^{(1)} + J_0^{(2)} = \frac{1}{2} \text{trace}(W^* \mu \mu') + \frac{1}{2} \lambda^\tau \text{trace}(V^* \tilde{K} \tilde{K}') . \quad (55)$$

Obviously, the value of  $J_0$  depends on the size of the lead time  $\tau$ . In New Keynesian models we often have a hump-shaped pattern for the function  $J_0 = J_0(\tau)$  where  $J_0$  is

increasing in  $\tau$  for small values of  $\tau$  (see Section 3).

In the limiting case of unanticipated shocks ( $\tau = 0$ ), the total loss is given by

$$J_0 = J_0^{(2)} = \frac{1}{2} \tilde{K}' V^* \tilde{K} , \quad (56)$$

where

$$\tilde{K} = K \Big|_{\tau=0} + S_{11}^{-1} Q_1 \bar{\nu} = Z_{11}^{-1} \tilde{w}_0 - S_{11}^{-1} Q_1 \bar{\nu} + S_{11}^{-1} Q_1 \bar{\nu} = Z_{11}^{-1} \tilde{w}_0 . \quad (57)$$

Then

$$J_0 = \frac{1}{2} \tilde{w}_0' Z_{11}^{-1'} V^* Z_{11}^{-1} \tilde{w}_0 = \frac{1}{2} \tilde{w}_0' V \tilde{w}_0 = \frac{1}{2} \text{trace}(V \tilde{w}_0 \tilde{w}_0') , \quad (58)$$

where

$$\tilde{w}_0 \tilde{w}_0' = \begin{pmatrix} w_0 \\ \rho_v 0 \end{pmatrix} (w_0', \rho_v' 0) = \begin{pmatrix} w_0 w_0' & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{pmatrix} \quad (59)$$

and  $V = Z_{11}^{-1'} V^* Z_{11}^{-1}$  satisfies the matrix equation

$$\begin{aligned} V &= Z_{11}^{-1'} V^* Z_{11}^{-1} = H^* + \lambda Z_{11}^{-1'} M_1' V^* M_1 Z_{11}^{-1} \\ &= H^* + \lambda Z_{11}^{-1'} M_1' Z_{11}^{-1} Z_{11}^{-1'} V^* Z_{11}^{-1} Z_{11} M_1 Z_{11}^{-1} = H^* + \lambda \Gamma' V \Gamma \end{aligned} \quad (60)$$

with  $\Gamma = Z_{11} M_1 Z_{11}^{-1}$ .

## 2.2 (Optimal) Simple Rules

The policy maker could alternatively commit to a suboptimal simple rule of the form

$$u_t = \Lambda k_t + \Psi E_t k_{t+1} , \quad (61)$$

where the constant matrices  $\Lambda$  and  $\Psi$  are  $m \times n$ . Assuming rational expectations and exogenous shocks of the form (2) which are anticipated in  $t = 0$ , we obtain the dynamic



system

$$\begin{pmatrix} A & 0_{n \times m} \\ \Psi & 0_{m \times m} \end{pmatrix} \begin{pmatrix} k_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} B & C \\ -\Lambda & I_{m \times m} \end{pmatrix} \begin{pmatrix} k_t \\ u_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{m \times r} \end{pmatrix} \nu_{t+1} . \quad (62)$$

The generalized Schur decomposition yields the system of equations

$$F \begin{pmatrix} \tilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{pmatrix} = G \begin{pmatrix} \tilde{w}_t \\ \tilde{v}_t \end{pmatrix} + \begin{pmatrix} D \\ 0_{m \times r} \end{pmatrix} \nu_{t+1} , \quad (63)$$

where  $\tilde{w} = w$  is an  $n_1 \times 1$  vector,  $\tilde{v} = (v', u')'$  is an  $(n_2 + m) \times 1$  vector and where the square matrices  $F$  and  $G$  are  $(n + m) \times (n + m)$  with the decomposition  $QFZ = S$  and  $QGZ = T$ , where  $Q$ ,  $Z$ ,  $S$ , and  $T$  are  $(n + m) \times (n + m)$  matrices. Since

$$\begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{x} \end{pmatrix} , \quad (64)$$

the matrices  $Z_{11}$ ,  $Z_{12}$ ,  $Z_{21}$ , and  $Z_{22}$  are now  $n_1 \times n_1$ ,  $n_1 \times (n_2 + m)$ ,  $(n_2 + m) \times n_1$ , and  $(n_2 + m) \times (n_2 + m)$  respectively. The auxiliary variables  $\tilde{z}$  and  $\tilde{x}$  satisfy the following system of equations

$$\begin{pmatrix} S_{11} & S_{12} \\ 0_{(n_2+m) \times n_1} & S_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_{t+1} \\ \tilde{x}_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0_{(n_2+m) \times n_1} & T_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_t \\ \tilde{x}_t \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \nu_{t+1} , \quad (65)$$

where  $S_{11}$  and  $T_{11}$  are  $n_1 \times n_1$  matrices,  $S_{22}$  and  $T_{22}$  are  $(n_2 + m) \times (n_2 + m)$  and  $S_{12}$  and  $T_{12}$  are  $n_1 \times (n_2 + m)$ . The matrices  $Q_1$  and  $Q_2$  are  $n_1 \times r$  and  $(n_2 + m) \times r$  respectively with

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = Q \begin{pmatrix} D \\ 0_{m \times r} \end{pmatrix} . \quad (66)$$

The solution of (65) is given by (11) and (13). For  $t \geq \tau$ , we obtain  $\tilde{v}_t = N\tilde{w}_t = Nw_t$ , where  $N = Z_{21}Z_{11}^{-1}$  is now an  $(n_2 + m) \times n_1$  matrix.

The loss function (44) simplifies to

$$J_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda^i (w'_{t+i}, \tilde{v}'_{t+i}) H \begin{pmatrix} w_{t+i} \\ \tilde{v}_{t+i} \end{pmatrix} \quad (67)$$

since  $\tilde{D}_1 = I_{n_1 \times n_1}$ ,  $\tilde{D}_2 = I_{(n_2+m) \times (n_2+m)}$  and therefore  $\tilde{D} = I_{(n+m) \times (n+m)}$  (cf. (43)).  $J_t$  can be partitioned using (44).  $J_t^{(2)}$  can be written as (47) with  $H^* = \tilde{N}' H \tilde{N}$  and  $\tilde{N} = (I_{n_1 \times n_1}, N')'$ . The value of the loss function  $J_0$  for given matrices  $\Lambda$  and  $\Psi$  is given by  $J_0 = J_0^{(1)} + J_0^{(2)}$ , where  $J_0^{(1)}$  and  $J_0^{(2)}$  are defined in (52) and (54) respectively.

The minimization of  $J_0$  with respect to the coefficients of the matrices  $\Lambda$  and  $\Psi$  yields an optimal simple rule of the form (61).

### 3 Example: A Hybrid New Keynesian Model

The model is a standard New Keynesian model for a closed and cashless economy with the additional features of internal habit formation in consumption preferences and a variant of the Calvo (1983) mechanism with partial indexation of non-optimized prices to past inflation.<sup>4</sup> The economy consists of final goods producers, labor bundlers, households, and intermediate goods producers.

Final goods producers use a continuum of intermediate goods  $Y_t(i)$  to produce the homogenous final good  $Y_t$  in a perfectly competitive market. A final goods producer maximizes his profits  $P_t Y_t - \int_0^1 P_t(i) Y_t(i) di$ , subjected to the following CES production function

$$Y_t = \left( \int_0^1 Y_t(i)^{\frac{1}{1+\lambda_p}} di \right)^{1+\lambda_p}, \quad (68)$$

where  $P_t$  is the price of the final good,  $P_t(i)$  is the price of the intermediate good  $i$ , and  $(1 + \lambda_p)$  is the mark-up in the intermediate goods market.

The first-order condition for profit maximization yields the demand function for an

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<sup>4</sup>Similar models are used by Smets and Wouters (2003), Giannoni and Woodford (2004), or Casares (2006).

intermediate good  $i$  which is given by

$$Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\frac{(1+\lambda_p)}{\lambda_p}} Y_t \quad (69)$$

and the equation for the marginal costs

$$P_t = \left( \int_0^1 P_t(i)^{-\frac{1}{\lambda_p}} di \right)^{-\lambda_p}. \quad (70)$$

Analogously to final goods producers, labor bundlers buy differentiated labor types  $N_t(j)$ , aggregate them to  $N_t$  and sell it to the intermediate goods producers under perfectly competitive conditions. A bundler maximizes his profits  $W_t N_t - \int_0^1 W_t(j) N_t(j) dj$ , subjected to the following CES aggregation function

$$N_t = \left( \int_0^1 N_t(j)^{\frac{1}{1+\lambda_{w,t}}} dj \right)^{1+\lambda_{w,t}}. \quad (71)$$

$W_t$  is the price of the labor bundle  $N_t$ ,  $W_t(j)$  denotes the price of labor type  $j$  and  $(1+\lambda_{w,t})$  is the time-varying wage mark-up.

The first-order condition for profit maximization yields the demand function for the labor type  $j$

$$N_t(j) = \left( \frac{W_t(j)}{W_t} \right)^{-\frac{(1+\lambda_{w,t})}{\lambda_{w,t}}} N_t \quad (72)$$

and the wage index equation

$$W_t = \left( \int_0^1 W_t(j)^{-\frac{1}{\lambda_{w,t}}} dj \right)^{-\lambda_{w,t}}. \quad (73)$$

The economy is made up by a continuum of households, indexed by  $j \in [0, 1]$ . Each household  $j$  is a monopolistic supplier of labor type  $N_t(j)$ . The household determines the amount of the final good  $C_t(j)$  for consumption, its one-period nominal bond holdings  $B_t(j)$ , and chooses the wage for its labor type  $W_t(j)$  in order to maximize its lifetime

utility given by

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left( \frac{1}{1-\sigma} (C_t(j) - hC_{t-1}(j))^{1-\sigma} - \frac{1}{1+\eta} N_t(j)^{1+\eta} \right), \quad (74)$$

where  $\beta$  is the discount factor,  $\sigma > 0$  is the inverse of the intertemporal elasticity of substitution in consumption, and  $\eta > 0$  is the inverse of the labor supply elasticity.  $C_{t-1}(j)$  is the consumption of the  $j$ th household in period  $t-1$  and  $N_t(j)$  are the total hours worked. We assume  $h \geq 0$  to allow for internal habit formation in consumption. Maximization of (74) is subjected to the labor demand function (72) and the households' period-by-period budget constraint which is given by

$$C_t(j) + \frac{B_t(j)}{P_t} = \frac{W_t(j)}{P_t} N_t(j) + \frac{R_{t-1} B_{t-1}(j)}{P_t} + D_t^r(j), \quad (75)$$

where  $R_t$  is the one-period gross nominal interest rate on households'  $j$ th nominal bond holdings  $B_t(j)$  and  $D_t^r(j)$  are dividends, expressed in real terms.

The first-order conditions for this maximization problem are given by

$$\beta R_t \mathbb{E}_t \pi_{t+1}^{-1} = \mathbb{E}_t \left[ \frac{(C_t - hC_{t-1})^{-\sigma} - h\beta(C_{t+1} - hC_t)^{-\sigma}}{(C_{t+1} - hC_t)^{-\sigma} - h\beta(C_{t+2} - hC_{t+1})^{-\sigma}} \right], \quad (76)$$

$$\frac{W_t}{P_t} = (1 + \lambda_{w,t}) \mathbb{E}_t \left[ \frac{N_t^\eta}{(C_t - hC_{t-1})^{-\sigma} - h\beta(C_{t+1} - hC_t)^{-\sigma}} \right], \quad (77)$$

where  $\pi_t = P_t/P_{t-1}$  is the gross rate of price inflation. We make use of the fact that all households are faced with the same optimization problem and hence, choose the same amount of consumption  $C_t(j) = C_t$ , the same nominal wage  $W_t(j) = W_t$ , and supply the same amount of labor  $N_t(j) = N_t$ .

Each intermediate goods producer is a monopolistic supplier of the intermediate good  $i \in [0, 1]$ . Firm  $i$  uses the amount  $N_t(i)$  of homogenous labor and the constant returns to scale technology  $Y_t(i) = N_t(i)$ , to produce his intermediate good  $Y_t(i)$ . Real marginal costs are the same for all firms and is given by  $MC_t(i) = W_t/P_t$ .

The price-setting decision for profit-maximization is constrained by a standard Calvo mechanism. In each period, the intermediate goods producer faces the constant probability  $1 - \theta$  of being allowed to re-optimize his price  $P_t(i)$ . We follow Smets and Wouters (2003) by assuming that a firm which cannot re-optimize its price, resets the price according to  $P_t(i) = P_{t-1}(i)\pi_{t-1}^\gamma$ , where  $\gamma$  is the degree of price indexation. The firm chooses  $P_t(i)$  in order to maximize

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Delta_{t,t+k} \left( \frac{P_t(i) \Pi_{t,t+k-1}}{P_{t+k}} Y_{t+k}(i) - MC_{t+k} Y_{t+k}(i) \right), \quad (78)$$

subjected to the sequence of demand functions

$$Y_{t+k}(i) = \left( \frac{P_t(i) \Pi_{t,t+k-1}}{P_{t+k}} \right)^{-\frac{(1+\lambda_p)}{\lambda_p}} Y_{t+k} \quad \text{for } k = 0, 1, 2, \dots, \quad (79)$$

where  $\Delta_{t,t+k}$  denotes the stochastic discount factor for real payoffs and

$$\Pi_{t,t+k-1} = \pi_t^\gamma \pi_{t+1}^\gamma \dots \pi_{t+k-1}^\gamma = (P_{t+k-1}/P_{t-1})^\gamma. \quad (80)$$

The first-order condition for the price-setting problem yields

$$P_t^*(i) = (1 + \lambda_p) \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Delta_{t,t+k} MC_{t+k} (P_{t+k}/\Pi_{t,t+k-1})^{(1+\lambda_p)/\lambda_p} Y_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Delta_{t,t+k} (P_{t+k}/\Pi_{t,t+k-1})^{-1/\lambda_p} Y_{t+k}}. \quad (81)$$

Dividing equation (81) by  $P_t$  yields

$$\frac{P_t^*(i)}{P_t} = \mu_p \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Delta_{t,t+k} MC_{t+k} \left( \frac{P_{t+k}}{P_t} \right)^{\frac{1+\lambda_p}{\lambda_p}} \left( \frac{P_{t+k-1}}{P_{t-1}} \right)^{-\frac{\gamma(1+\lambda_p)}{\lambda_p}} Y_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Delta_{t,t+k} \left( \frac{P_{t+k}}{P_t} \right)^{\frac{1}{\lambda_p}} \left( \frac{P_{t+k-1}}{P_{t-1}} \right)^{-\frac{\gamma}{\lambda_p}} Y_{t+k}}, \quad (82)$$

where  $\mu_p = 1 + \lambda_p$ .

Since all firms which are allowed to re-optimize their price will choose the same price  $P_t^*(i) = P_t^*$ , the price index (70) can be rewritten as

$$1 = \theta \left( \frac{\pi_{t-1}^\gamma}{\pi_t} \right)^{-\lambda_p} + (1 - \theta) \left( \frac{P_t^*}{P_t} \right)^{-\lambda_p}. \quad (83)$$

Log-linearizing equation (83) yields

$$\hat{P}_t^* - \hat{P}_t = \frac{\theta}{1-\theta}(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}) . \quad (84)$$

We use the convention that a hat above a variable denotes the percentage deviation from its steady-state value.

By combining the latter equation with the log-linearized price-setting condition (82), we finally obtain

$$\hat{\pi}_t = \frac{\gamma}{1+\beta\gamma}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta\gamma}\text{E}_t\hat{\pi}_{t+1} + \Theta\hat{MC}_t , \quad (85)$$

where  $\Theta = \frac{(1-\beta\theta)(1-\theta)}{\theta(1+\beta\gamma)}$ . By log-linearizing the optimality condition (77), using the log-linearized overall resource constraint  $\hat{Y}_t = \hat{C}_t$  and using the fact that  $\widehat{W_t/P_t} = \widehat{MC}_t$  and  $\hat{Y}_t = \hat{N}_t$ , we obtain

$$\widehat{MC}_t = \hat{\lambda}_{w,t} + (\eta + \delta_1)\hat{Y}_t - \delta_2\hat{Y}_{t-1} - \beta\delta_2\text{E}_t\hat{Y}_{t+1} , \quad (86)$$

where  $\delta_1 = \frac{\sigma(1+\beta h^2)}{(1-h)(1-\beta h)}$ ,  $\delta_2 = \frac{h\sigma}{(1-h)(1-\beta h)}$ . The log-linearized mark-up  $\hat{\lambda}_{w,t}$  is described by the AR(1) process

$$\hat{\lambda}_{w,t} = \xi_w\hat{\lambda}_{w,t-1} + e_t . \quad (87)$$

By inserting the latter equation into equation (85), we obtain a hybrid Phillips curve that follows

$$\hat{\pi}_t = \omega_1\text{E}_t\hat{\pi}_{t+1} + \omega_2\hat{\pi}_{t-1} + \omega_3\hat{Y}_t - \omega_4\hat{Y}_{t-1} - \beta\omega_4\text{E}_t\hat{Y}_{t+1} + \Theta\hat{\lambda}_{w,t} , \quad (88)$$

where  $\omega_1 = \frac{\beta}{1+\beta\gamma}$ ,  $\omega_2 = \frac{\gamma}{1+\beta\gamma}$ ,  $\omega_3 = \Theta(\eta + \delta_1)$ , and  $\omega_4 = \Theta\delta_2$ .

Note that in our model the level of output in the absence of nominal rigidities (the natural level)  $Y_t^n$  is constant. Thus, the linearized output  $\hat{Y}_t$  coincides with the linearized output gap  $\hat{Y}_t^g = \hat{Y}_t - \hat{Y}_t^n$ , where  $\hat{Y}_t^n = 0$ . Further note that for  $\gamma = 0$ , equation (88)

collapses into the purely forward-looking New Keynesian Phillips curve.

By log-linearizing the optimality condition (76) and using  $\hat{Y}_t = \hat{C}_t$ , we obtain

$$\hat{Y}_t = \kappa_1 \hat{Y}_{t-1} + \kappa_2 E_t \hat{Y}_{t+1} - \kappa_3 E_t \hat{Y}_{t+2} - \kappa_4 (\hat{R}_t - E_t \hat{\pi}_{t+1}), \quad (89)$$

where  $\kappa_1 = \frac{h}{1+h+\beta h^2}$ ,  $\kappa_2 = \frac{1+\beta h+\beta h^2}{1+h+\beta h^2}$ ,  $\kappa_3 = \frac{\beta h}{1+h+\beta h^2}$ , and  $\kappa_4 = \frac{(1-h)(1-\beta h)}{\sigma(1+h+\beta h^2)}$ . Note that for  $h = 0$ , we obtain the purely forward-looking New Keynesian IS curve.

Following Woodford (2003, Ch. 6) and Giannoni and Woodford (2004), a second-order approximation to the households' utility yields a loss function of the form

$$J_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left( (\hat{\pi}_t - \gamma \hat{\pi}_{t-1})^2 + \alpha_Y (\hat{Y}_t - \delta \hat{Y}_{t-1})^2 \right), \quad (90)$$

where  $\alpha_Y = \frac{\Theta h \sigma \lambda_p}{(1+\lambda_p) \delta (1-\beta h)(1-h)}$  and  $\delta$  is the smaller root of the quadratic equation

$$\frac{h\sigma}{(1-\beta h)(1-h)}(1+\beta\delta^2) = \left( \eta + \frac{\sigma}{(1-\beta h)(1-h)}(1+\beta h^2) \right) \delta. \quad (91)$$

We follow Giannoni and Woodford (2004) and Casares (2006) by assuming that the monetary authority is concerned about the volatility of the nominal interest rate. Therefore, we augment the welfare-theoretic loss function by the additional term  $\alpha_R \hat{R}_t^2$ , where  $\alpha_R$  measures the weight on interest rate stabilization.<sup>5</sup>

The monetary authority then seeks to minimize the loss function

$$J_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left( (\hat{\pi}_t - \gamma \hat{\pi}_{t-1})^2 + \alpha_Y (\hat{Y}_t - \delta \hat{Y}_{t-1})^2 + \alpha_R \hat{R}_t^2 \right), \quad (92)$$

subjected to the equations (87), (88), and (89). Note that in our model, the discount factor for the policy-maker,  $\lambda$ , is equal to the household's discount factor  $\beta$ .

In order to solve the model by using the methods outlined in Section 2, we define the policy objective parameters  $\hat{Y}_t^o = \hat{Y}_t - \delta \hat{Y}_{t-1}$  and  $\hat{\pi}_t^o = \hat{\pi}_t - \gamma \hat{\pi}_{t-1}$ . Furthermore, we define the auxiliary variables  $\tilde{\pi}_t = \hat{\pi}_{t-1}$ ,  $\tilde{Y}_t = \hat{Y}_{t-1}$ , and  $s_t = E_t \hat{\pi}_{t+1}$ . If we add the definition of

<sup>5</sup>Note that our results are not changed if we assume  $\alpha_R = 0$ . We set  $\alpha_R > 0$  mainly because we want to demonstrate our solution method which allows the consideration of the volatility of policy instruments in the policy-maker's objective function.

the real interest rate  $\hat{r}_t = \hat{R}_t - E_t \hat{\pi}_{t+1}$ , we finally obtain a  $3 \times 1$  vector  $w_t$  of predetermined variables given by  $w_t = (\hat{\lambda}_{w,t}, \tilde{\pi}_t, \tilde{Y}_t)'$ , a  $6 \times 1$  vector  $v_t$  of non-predetermined variables given by  $v_t = (\hat{\pi}_t, \hat{Y}_t, s_t, \hat{r}_t, \hat{\pi}_t^o, \hat{Y}_t^o)'$ , the vector of policy instruments  $u_t$  which is simply the scalar  $u_t = \hat{R}_t$ , and the  $1 \times 1$  shock vector  $\nu_t = e_t$ . The  $9 \times 9$  matrices  $A$  and  $B$  are given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta}{1+\beta\gamma} & -\beta\omega_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_4 & \kappa_2 & -\kappa_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \xi_w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\Theta & -\frac{\gamma}{1+\beta\gamma} & \omega_4 & 1 & -\omega_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \gamma & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \delta & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

while the  $9 \times 1$  matrices  $C$  and  $D$  are

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \kappa_4 & 0 & 1 & 0 & 0 \end{pmatrix}',$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}'.$$



Finally, the matrices  $\tilde{W}$ ,  $P$ , and  $R$  are given by  $P = \mathbf{0}_{9 \times 9}$ ,  $R = \alpha_R$ , and

$$\tilde{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_Y \end{pmatrix}.$$

We complete the description of the model by presenting the calibration. The time unit is one quarter. The discount rate is equal to  $\beta = 0.99$ , implying a quarterly steady-state real interest rate of approximately one percent. The intertemporal elasticity of substitution is assumed to be  $\sigma = 2$ . We follow Casares (2006) and set the habit formation parameter to  $h = 0.85$  implying that the weight on lagged output in the IS equation is  $1/3$ . The calibrated  $\eta = 3$  implies a labor supply elasticity with respect to the real wage of  $1/3$ .  $\lambda_p$  is set to  $8/7$  which implies a steady-state mark-up in the goods market of approximately 14 percent. We assume the linearized wage mark-up  $\hat{\lambda}_{w,t}$  to be persistent and choose  $\xi_w$  equal to 0.8. In our baseline scenario, the Calvo parameter  $\theta$  is set to 0.75 implying an average duration of price contracts of one year. In order to check whether our welfare results will also hold in the case of flexible price adjustment, we set  $\theta = 0.05$ . The price indexation parameter  $\gamma$  is set to 0.45 which is roughly equal to the value reported by Smets and Wouters (2003). This implies that the weight on lagged inflation in the Phillips curve equation is 0.31.

The parameter values chosen for our model imply a weight on output in the policy-makers' objective function of approximately  $\alpha_Y = 0.69$ . Following Casares (2006), we set  $\alpha_R = 0.0088$  implying a small preference for interest rate smoothing.

For the analysis concerning anticipated and unanticipated shocks, we assume that

the economy is in a deterministic steady-state until period  $t = 0$ . In the case of an unanticipated shock, the mark-up  $\hat{\lambda}_{w,t}$  jumps by one percent in period  $t = 0$  and begins to fall thereafter. In the case of an anticipated shock, the agents anticipate in period  $t = 0$  that a one percent increase in the mark-up will take place at some future date  $\tau > 0$ . They also know that the mark-up will subsequently decline according to the autoregressive process (87), where now  $e_t = 1$  for  $t = \tau$  and  $e_t = 0$  for  $t \neq \tau$ . Note that  $\tau$  also defines the length of the anticipation period or the time interval between  $t = 0$  and  $t = \tau$ . In order to obtain impulse response functions and welfare results, we simulate the dynamic adjustment paths and the welfare loss function by using the methods outlined in Section 2.<sup>6</sup>

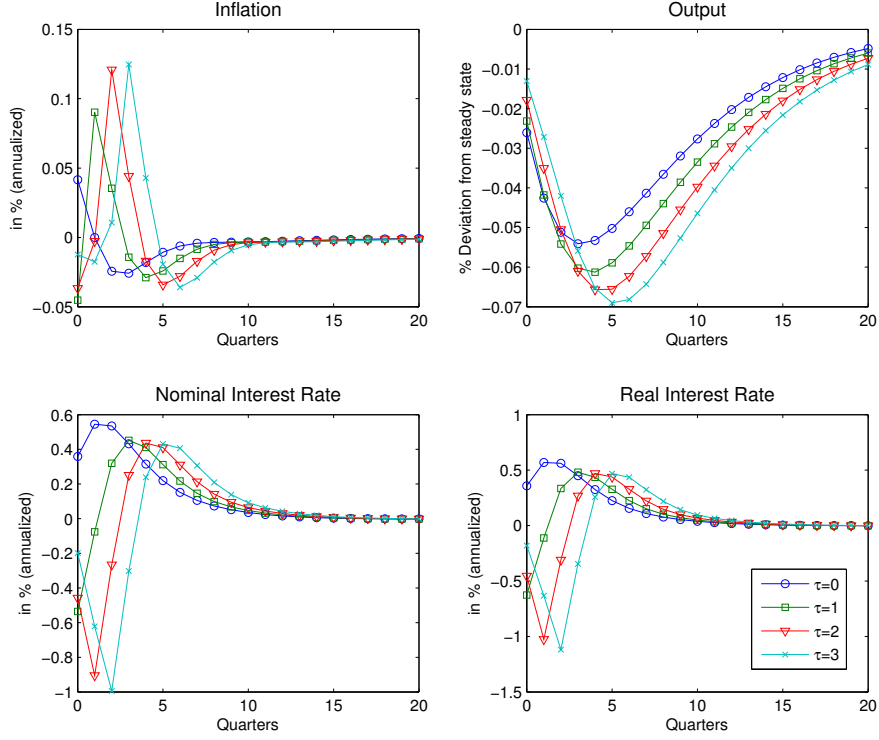
Figure 1 depicts the impulse response functions of inflation, output, nominal, and real interest rates under the unrestricted optimal monetary policy and under the baseline calibration. The solid lines with circles represent the responses to an unforeseen cost-push shock that emerged in period  $t = 0$ . The solid lines with squares, triangles, and stars represent responses to a cost-push shock whose realization in period  $\tau = 1$ ,  $\tau = 2$ , or  $\tau = 3$  is anticipated in period  $t = 0$ .

An *unanticipated* rise in the wage mark-up puts upward pressure on the prices of intermediate goods and hence on inflation. Despite the instantaneous jump in inflation, the real interest rate rises due to the sharp increase in the nominal interest rate. The increase in the real interest rate induces households to postpone consumption implying an abrupt drop in output. Subsequently, the nominal interest rate continues to rise. This leads – in conjunction with the decline in inflation – to hump-shaped response functions of the real interest rate and output.

In the case of *anticipated* shocks, the optimal policy calls for a decline in nominal and real interest rates in response to the anticipation of a future rise in marginal costs. At the latest with the occurrence of the anticipated shock in period  $\tau$ , the nominal and real interest rates start to rise and display a hump-shaped development. Inflation declines in response to the anticipation of the future rise in marginal costs. After this initial decline,

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<sup>6</sup>Matlab codes can be downloaded from the author's webpage at [http://www.wiso.uni-kiel.de/vwl/institute/Wohltmann/REAS\\_solution.zip](http://www.wiso.uni-kiel.de/vwl/institute/Wohltmann/REAS_solution.zip).

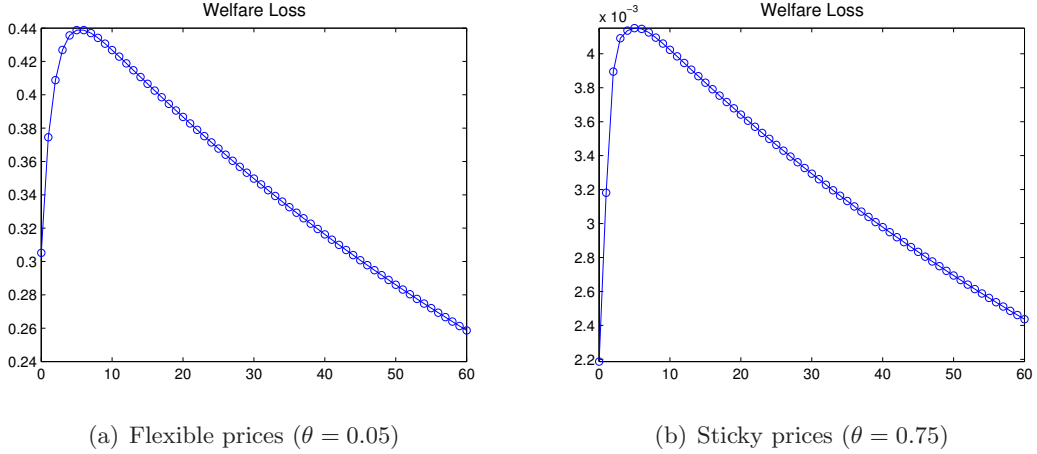


**Figure 1:** Impulse response functions under unrestricted optimal monetary policy.

Notes: Solid lines with circles denote responses to an unanticipated cost-push shock, solid lines with squares, triangles, and stars denote responses to an anticipated cost-push shock taking place in period  $\tau = 1$ ,  $\tau = 2$ , and  $\tau = 3$ .

inflation starts to rise and peaks in the period when the anticipated shock materializes. Output displays a hump-shaped downturn, starting at the point of anticipation,  $t = 0$ . The drop in output is thereby amplified by the lengths of the anticipation period  $\tau$ .

Notably, the anticipation of future shocks leads to an increase in the persistence of inflation, output as well as nominal and real interest rates which is increasing with the lead time  $\tau$ . Thereby, we depart from the usual approach of measuring persistence by the speed of dying out. Instead – and in the spirit of the measure of quantitative inertia proposed by Merkl and Snower (2009) – we measure persistence as the total variation of a variable over time, i.e. by its intertemporal deviation from its initial steady-state. However, the impact or anticipation effect is inversely related to the time span between the anticipation and the realization of the cost-push shock. It measures the initial jump of a variable taking place at the time of anticipation.



**Figure 2:** Welfare loss for different lengths of the anticipation period under unrestricted optimal monetary policy

The opposing effects of anticipations are shown in Figure 2 which displays the welfare loss as a function of the time span between the anticipation and the occurrence of the cost-push shock (a) in the case of highly flexible prices and (b) in the baseline case of sticky prices.

The welfare function exhibits – independent of the degree of price flexibility – a hump-shaped pattern implying that for a realistic time span between the anticipation and the realization of cost-push shocks, anticipated disturbances entail *higher* welfare losses than unanticipated disturbances of equal size. The rationale is that the anticipation effect is dominated by the persistence effect. A welfare gain from anticipating can only be achieved for very large (and unrealistic) values of  $\tau$ . Besides the anticipation effect, this can also be explained by discounting the realization impacts from period  $\tau$  to period  $t = 0$ .

The results we obtained from our simulations show that – irrespective of the degree of price rigidity – the welfare loss of anticipated cost-shocks exceeds the welfare loss of an unanticipated cost-shock of equal magnitude for plausible lengths of the anticipation period. In a purely forward-looking version of our model, Wohltmann and Winkler (2009) show analytically that anticipated cost shocks do lead to higher welfare losses when compared to unanticipated shocks of equal magnitude only for a sufficiently high degree of nominal rigidity. In this study however, we show that the features of habit formation and partial indexation – which generate a strong internal propagation mechanism – give rise

to a welfare-reducing effect of anticipations even in the case of a low degree of nominal rigidity.

## 4 Conclusion

In this paper, we presented a method to solve linear dynamic rational expectations models with anticipated shocks and optimal policy by using the generalized Schur decomposition method. Furthermore, we determine the optimal unrestricted and restricted policy responses to anticipated shocks. Our approach also allows for the evaluation of the widely discussed case of unpredictable shocks and can therefore be seen as a generalization of the methods summarized by Söderlind (1999). We demonstrated our method by means of a calibrated New Keynesian model with internal habit formation in consumption preferences, a variant of Calvo price staggering with partial indexation to past inflation, a time-varying wage mark-up which represents a typical cost-push shock, and a utility-based loss function. We simulated the model economy's responses to unanticipated and anticipated cost-push shocks under the unrestricted optimal monetary policy. We showed that anticipated shocks amplify both, the stagflationary effects of cost-push shocks and the overall welfare loss.

## References

- Beaudry, Paul and Franck Portier (2006) Stock Prices, News, and Economic Fluctuations. *American Economic Review* 96, 1293–1307.
- Calvo, Guillermo A. (1983) Staggered Prices in a Utility Maximizing Framework. *Journal of Monetary Economics* 12, 383–398.
- Casares, Miguel (2006) Monetary Policy Rules in a New Keynesian Euro Area Model. *Journal of Money, Credit, and Banking* 39, 875–900.
- Currie, David and Paul Levine (1993) *Rules, Reputation and Macroeconomic Policy Coordination*. Cambridge University Press.

- Davis, Joshua M. (2007) News and the Term Structure in General Equilibrium. Unpublished Manuscript.
- Fujiwara, Ippei, Yasuo Hirose and Mototsugu Shintani (2008) Can News Be a Major Source of Aggregate Fluctuations? A Bayesian DSGE Approach. IMES Discussion Paper 2008-E-16. Institute for Monetary and Economic Studies, Bank of Japan.
- Giannoni, Marc P. and Michael Woodford (2004) Optimal Inflation-Targeting Rules. In *The Inflation Targeting Debate*, edited by B. S. Bernanke and M. Woodford. University of Chicago Press.
- Klein, Paul (2000) Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics & Control* 24, 1405–1423.
- Merkel, Christian and Dennis Snower (2009) Monetary Persistence, Imperfect Competition, and Staggering Complementarities. *Macroeconomic Dynamics* 13, 81–106.
- Rudebusch, Glenn D. and Lars E. O. Svensson (1999) Policy Rules for Inflation Targeting. In John B. Taylor (ed.), *Monetary Policy Rules*. University of Chicago Press.
- Schmitt-Grohé, Stephanie and Martin Uribe (2008) What's News in Business Cycles. NBER Working Paper 14215, National Bureau of Economic Research.
- Smets, Frank and Rafael Wouters (2003) An Estimated Dynamic Stochastic General Equilibrium Model for the Euro Area. *Journal of the European Economic Association* 1, 1123–1175.
- Söderlind, Paul (1999) Solution and Estimation of RE Macromodels with Optimal Policy. *European Economic Review* 43, 813–823.
- Wohltmann, Hans-Werner and Roland Winkler (2009) On the Non-Optimality of Information: An Analysis of the Welfare Effects of Anticipated Shocks in the New Keynesian Model. Kiel Working Paper 1497, Kiel Institute for the World Economy.
- Woodford, Michael (2003) *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.