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by

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Parametric Contest Success Functions

Birendra K. Rai* and Rajiv Sarin†

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Abstract

The key element of models of contest is the Contest Success Function (CSF) which specifies the winning probabilities of agents. We provide an axiomatization of two parametric families of CSF's. In the first, the winning probability of each agent depends on the investments and a vector of parameters, where each parameter is specific to one of the contestants. In the second, the winning probabilities depend on investments and a scalar parameter common to all contestants.

Keywords and Phrases: Contest success function, Axiomatization, Rent seeking.

JEL Classification Numbers: C70, D72, D74.

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1. Introduction

Over the last few decades models of contest have been used to study rent seeking (Nitzan, 1994) and conflict (Skaperdas and Garfinkel, 2006) by political economists, reward structure in firms (Rosen, 1986, Prendergast, 1999) by labor economists, electoral competition (Snyder, 1989) and design of optimal voluntary contribution mechanisms by public economists (Kolmar and Wagener, 2005), and individual and team sports (Szymanski, 2003) by sports economists. A contest is modeled as a non-cooperative game between multiple agents. Agents make irreversible investments, which can be effort, money, or any other valuable resource depending on the context, to increase their probability of winning the contest and obtaining a private prize. The key element of all models of contest is the Contest Success Function (CSF) which specifies the winning probabilities of the contestants.

In order to analyze contests it is important to take into account the heterogeneity that exists among the contestants for two main reasons. First, if agents differ in their contest-relevant inherent characteristics then two agents with the same *actual* investments can have different winning probabilities as they can still have different levels of *effective* investments (Clark and Riis, 1998). One can think of the actual investment and the contest-relevant inherent characteristics of an agent as inputs that he combines using his technology to produce effective investments. The effective investments then determine the winning probabilities. Second, different agents might have different *technologies* of combining actual investments with their inherent characteristics to produce the effective investments.

The contest-relevant inherent characteristics of agents can be treated as parameters that influence the winning probabilities. Skaperdas (1996) axiomatizes several non-parametric classes of CSF's in which the winning probability of each contestant depends *solely* on the investments, and all agents are implicitly assumed to have the same level of contest-relevant inherent characteristics and the same technology to produce effective investments. Clark and Riis (1998) allow contestants to differ in their characteristics but not in their technologies of converting actual investments into effective investments.

In this paper we first provide an axiomatization of *vector-parametric* CSF's where the scalar parameter associated with a contestant reflects his contest-relevant inherent characteristics. In addition, our formulation allows agents to possess different technologies of converting actual investments into effective investments. The vector-parametric CSF's are relevant for analyzing *unmediated* contests wherein each contestant uses his own technology to combine his actual investment and inherent characteristics and produce the effective investment. Elections, rent seeking, wars, sports, and patent races are some of the many real world examples of unmediated contests.

We shall use the term *mediated contests* to describe those contests in which the technology for converting actual investments of contestants into effective investments is determined by the contest administrator, and the exact CSF is known to the contestants before they invest.¹ Kolmar and Wagener (2005) suggest to modify public good games such that the contributions by various agents are used by the public agency to finance the public good and a private prize. The administrator informs the agents about the exact specification of the CSF. But, which CSF should the agency employ to ensure that only agents with high marginal productivity in the production of the public good contribute? They show that the *noisy-CSF* proposed by Amegashie (2006), in which the probability with which a contributor wins the private prize depends on the vector of contributions and a scalar parameter that reflects noise, might be useful in achieving this objective. We provide an axiomatization for such *scalar-parametric CSF's* as well.

The paper is organized as follows. Section 2 provides a brief discussion of the required axioms. Section 3 axiomatizes the two types of parametric CSF's. We show that the CSF axiomatized by Clark and Riis (1998) is a special case of the vector-parametric CSF, and all the CSF's axiomatized by Skaperdas are special cases of the scalar-parametric CSF. Section 4 concludes. The appendix contains the proofs.

2. The Axioms

Let N represent the set of agents $(1, 2, \dots, i, \dots, n)$ who are competing for a prize. Define the probability that agent- i wins the contest as

$$p_i(\mathbf{x}; \alpha) = p_i(x_1, x_2, \dots, x_i, \dots, x_n; \alpha_1, \alpha_2, \dots, \alpha_i, \dots, \alpha_n) \in [0, 1], \quad \forall i \in N.$$

The function $p_i(\mathbf{x}; \alpha)$ is the vector-parametric CSF. The non-negative choice variable $x_i \in \mathfrak{R}_+$ is the *actual* investment by agent- i . The strictly positive parameter $\alpha_i \in \mathfrak{R}_{++}$ is a measure of all contest-relevant inherent characteristics of contestant i that affect the winning probabilities of all agents.

We shall refer to the contest involving all the N agents as the *global* contest; and any contest involving agents in $M \subseteq N$ as a *sub-contest*. $p_i^M(\mathbf{x}; \alpha)$ will henceforth be used to denote the winning probabilities of agents in the sub-contest among agents contained in $M \subseteq N$. Similarly, the vector $x^M \in \mathfrak{R}_+^M$ will be used to represent the investment by agents in $M \subseteq N$. Whenever unnecessary, we will denote the winning probabilities in the global contest as $p_i(\mathbf{x}; \alpha)$, instead of $p_i^N(\mathbf{x}; \alpha)$. We first list the five basic axioms.²

¹All-pay auctions are probably the best examples of mediated contests.

²The axioms A1, A2, A3, and the first part of axiom A4 were introduced by Skaperdas (1996).

- [A1] **Probability.** $p_i(\mathbf{x}; \alpha) \in [0, 1]$, $\forall i \in N$, and $\sum_{i \in N} p_i(\mathbf{x}; \alpha) = 1$.
- [A2] **Consistency.** $p_i^M(\mathbf{x}; \alpha) = \frac{p_i^N(x; \alpha)}{\sum_{k \in M} p_k^N(x; \alpha)}$, $\forall i \in M$, and $\forall M \subseteq N$.
- [A3] **IIAs.** $p_i^M(x^N; \alpha^N) = p_i^M(x^M; \alpha^M) \forall i \in M$, and $\forall M \subseteq N$,
 where $x^N = (x^M, x^{N \setminus M})$, and $\alpha^N = (\alpha^M, \alpha^{N \setminus M})$.
- [A4] **Marginal Effects.** $\frac{\partial p_i(x; \alpha)}{\partial x_i}, \frac{\partial p_i(x; \alpha)}{\partial \alpha_i} \geq 0$, $\forall i \in N$.
- [A5] **Continuity.** For each sequence $(x^\nu, \alpha^\nu) \in \mathfrak{R}_+^N \times \mathfrak{R}_{++}^N$ that converges to some $(x, \alpha) \in \mathfrak{R}_+^N \times \mathfrak{R}_{++}^N$, $p_i(x^\nu; \alpha^\nu)$ converges to $p_i(\mathbf{x}; \alpha)$, $\forall i \in N$.

The axiom of probability requires that each agent's winning probability should lie between zero and one; and the winning probabilities of agents should sum to unity. For a given vector of investments $x \in \mathfrak{R}_+^N$, consider the sub-contest among agents in $M \subseteq N$. Consistency requires the probability that $i \in M$ wins this *sub*-contest ($p_i^M(\mathbf{x}; \alpha)$) to be the same as the probability that i would have won the *global* contest among all the N agents, given that one of the agents in M wins it. Consistency does not, however, rule out the possibility that the winning probability of an agent in the sub-contest might depend on the investments and characteristics of agents in $N \setminus M$. The axiom of Independence of Irrelevant Alternatives (IIAs) is introduced to eliminate this possibility. We shall refer to axioms A1, A2, and A3 as the *within axioms* as they are stated for a given vector of $\alpha \in \mathfrak{R}_{++}^N$, and do not impose any restriction on the relationship between $p_i(\mathbf{x}; \alpha)$ and $p_i(x; \tilde{\alpha})$ for any $\alpha, \tilde{\alpha} \in \mathfrak{R}_{++}^N$.

The axioms A4 and A5 are *mixed axioms* as they impose restrictions on the form of the CSF by considering variations in both the actual investments and the inherent characteristics of the contestants. The axiom of marginal effects prescribes that the winning probability of a contestant be non-decreasing in his own investment, and his contest-relevant inherent characteristics. Several CSF's that have been used to analyze contests suffer from a discontinuity when all agents invest zero resources (Corchon, 2000). We wish to come up with continuous contest success functions and therefore we have introduced the axiom of continuity.

Theorem 1. *The axioms A1 – A5 are satisfied if and only if*

$$p_i(\mathbf{x}; \alpha) = \frac{f_i(x_i; \alpha_i)}{\sum_{k \in N} f_k(x_k; \alpha_k)}, \quad \forall i \in N, \quad (1)$$

where $f_i(x_i; \alpha_i)$ is a continuously differentiable strictly positive function with $\frac{\partial f_i(x_i; \alpha_i)}{\partial x_i} > 0$, $\frac{\partial f_i(x_i; \alpha_i)}{\partial \alpha_i} \geq 0$, $\forall (x_i, \alpha_i) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, $\forall i \in N$.

Proof: In the Appendix.

We interpret the value of $f_i(x_i; \alpha_i)$ as the level of effective investment by contestant- i . The effective investment is the output determined by the levels of the two inputs (inherent characteristics, and the actual investment), and the technology of combining these inputs. The technology of producing effective investments ($f_i(\cdot)$) can differ across agents as reflected by the subscript i . The ratio of winning probabilities of any two agents is thus equal to the ratio of their effective investments (Cornes and Hartley, 2005).

The winning probability of each agent can be expressed using n parametric functions, with the function corresponding to agent- i depending only on x_i and α_i . This suggests that each contestant produces his effective investment independently of the others.

If each $f_i(x_i; \alpha_i)$ is continuous at all $(x_i, \alpha_i) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, then the denominator of the above expression will be continuous as it is a sum of continuous functions. However, $p_i(\mathbf{x}; \alpha)$ is a ratio of two continuous functions. If $f_i(x_i; \alpha_i)$ and $f_j(x_j; \alpha_j)$ are zero at some $(\tilde{x}_i, \tilde{x}_j; \tilde{\alpha}_i, \tilde{\alpha}_j) \in \mathbb{R}_+^2 \times \mathbb{R}_{++}^2$ for any $i, j \in N$, then the winning probabilities will not be well defined in the sub-contest involving these two contestants at $(\tilde{x}_i, \tilde{x}_j; \tilde{\alpha}_i, \tilde{\alpha}_j)$. The conditions on $f_i(x_i; \alpha_i)$ given in Theorem 2 ensure that the winning probabilities will be well defined even if the actual investment by all agents is *zero*. It is difficult to imagine a contest during which no contestant makes an investment. Following Corchon (2000), we shall interpret $p_i(0; \alpha)$ as the *prior* winning probabilities of the contestants. It seems reasonable to suggest that different agents with different levels of inherent characteristics may have different ex-ante winning probabilities.

3. Axiomatization

Our aim is to come up with the general expression for parametric CSF's which will include most of the CSF's used by applied researchers as special cases. In particular, we will show that all the axiomatization results of Skaperdas (1996) can be obtained as special cases of a parametric family of CSF's with a scalar parameter. The CSF axiomatized by Clark and Riis (1998) will be shown to be a special case of the vector-parametric CSF's.

Skaperdas (1996) considers contests in which only investments matter and proves that the only CSF satisfying the within axioms of probability, consistency, independence of irrelevant alternatives, marginal effects with respect to own investments, homogeneity of order zero with respect to investments ($p_i(\lambda \mathbf{x}) = p_i(\mathbf{x})$), and anonymity ($p_i(\mathbf{x}) = p_j(\mathbf{x})$, if $x_i = x_j$) is the Tullock-CSF

$$p_i(\mathbf{x}) = \frac{x_i^r}{\sum_{k \in N} x_k^r} \quad \left(= \frac{f(x_i)}{\sum_{k \in N} f(x_k)} \right).$$

The axiom of anonymity, as used by Skaperdas (1996), requires that the winning probabilities of two contestants that invest equally must be the same. His framework neither allows for differences in the inherent characteristics of agents, nor in the technology of producing effective investments.

Clark and Riis (1998) argue that the anonymity axiom is inappropriate in several contests as the winning probabilities depend not only on the investments but also on the personal characteristics of the contestants. They go on to prove that a CSF satisfies the within axioms of probability, marginal effects with respect to own investments, IIA_{CR} , and homogeneity with respect to investments if and only if it is given by³

$$p_i(\mathbf{x}; \alpha) = \frac{\alpha_i x_i^r}{\sum_{k \in N} \alpha_k x_k^r} \quad \left(= \frac{f(x_i; \alpha_i)}{\sum_{k \in N} f(x_k; \alpha_k)} \right).$$

Clark and Riis use *within* axioms. For example, their homogeneity axiom can be formally expressed as

$$p_i(\lambda \mathbf{x}; \alpha) = p_i(\mathbf{x}; \alpha), \quad \forall \lambda > 0, \quad \forall i \in N,$$

which implies that if the investments of all agents change by the same multiplicative factor the winning probabilities remain unaltered even though different agents have different levels of inherent characteristics.⁴ Another feature of their axiomatization is that even after dropping the anonymity axiom used by Skaperdas, their framework still implies that the technology of producing effective investments is the same for all contestants.

3.1. Mixed homogeneity

We now provide an axiomatization for the vector-parametric CSF's using the five axioms discussed in the previous section and the following mixed homogeneity axiom.

³Clark and Riis (1998) combine the consistency and IIA_S axioms of Skaperdas (1996) and propose IIA_{CR} which requires that $p_i^N(\mathbf{x}; \alpha | x_j = 0) = \frac{p_i^N(\mathbf{x}; \alpha | x_j > 0)}{\sum_{k \in N \setminus \{j\}} p_k^N(\mathbf{x}; \alpha | x_j > 0)}$. However, IIA_{CR} is equivalent to consistency and IIA_S only for those CSF's that imply a win probability of zero for a participating agent when his investment is zero. For example, the logit-CSF ($p_i(\mathbf{x}) = \frac{e^{\theta x_i}}{\sum_{k \in N} e^{\theta x_k}}$) axiomatized by Skaperdas (1996) satisfies consistency and IIA_S but not IIA_{CR} .

⁴Another within version of the homogeneity axiom is: $p_i(\mathbf{x}; \lambda \alpha) = p_i(\mathbf{x}; \alpha)$.

[A6] **Mixed Homogeneity.** $p_i(\lambda \mathbf{x}; \lambda \alpha) = p_i(\mathbf{x}; \alpha), \forall \lambda > 0, \forall i \in N.$

This formulation says that the winning probabilities should remain unchanged if each agent's investment and inherent characteristics change by the same multiplicative factor.⁵ The main result of the paper follows.

Theorem 2. *The axioms A1 – A6 are satisfied if and only if*

$$p_i(\mathbf{x}; \alpha) = \frac{f_i(x_i; \alpha_i)}{\sum_{k \in N} f_k(x_k; \alpha_k)} = \frac{\alpha_i^r \cdot h_i\left(\frac{x_i}{\alpha_i}\right)}{\sum_{k \in N} \alpha_k^r \cdot h_k\left(\frac{x_k}{\alpha_k}\right)}, \quad (2)$$

where $h_i(\cdot)$ is any continuously differentiable polynomial-order⁶ function such that $h_i(0) > 0, h'_i \geq 0 \forall x_i \in \mathfrak{R}_+, h'_i(0)$ is finite, and $r \geq r_c = \max_i [\max_{x_i} \left(\frac{\frac{\partial f_i(x_i; \alpha_i)}{\partial x_i}}{f_i(x_i; \alpha_i)} \right) \cdot x_i]$ is any finite real number such that $\frac{\partial p_i}{\partial \alpha_i} \geq 0 \forall x_i \in \mathfrak{R}_+, \forall i \in N.$

Proof. In the Appendix.⁷

The first thing to note is that homogeneity implies

$$f_i(x_i; \alpha_i) = \alpha_i^r \cdot h_i\left(\frac{x_i}{\alpha_i}\right) = g(\alpha_i) \cdot h_i\left(\frac{x_i}{\alpha_i}\right). \quad (3)$$

In line with the interpretation of a production function, the total output of effective investment by an agent is a product of two terms. The first term is a function of the agent's inherent characteristics, and the second term is a function of the agent's investment per

⁵Like Skaperdas (1996) and Clark and Riis (1998), we are unable to provide a genuine justification for the mixed homogeneity axiom.

⁶A polynomial-order function is bounded from above by an $O(x^T)$ function, for some finite $T > 0.$

⁷It can be easily shown that (a) If the vector-parametric CSF satisfies homogeneity with respect to investments only, then $f_i(x_i; \alpha_i) = h_i(\alpha_i) \cdot x_i^r.$ This is precisely the result obtained by Clark and Riis (1998). However, the resulting CSF is not continuous when all contestants invest zero. (b) If the CSF satisfies homogeneity with respect to inherent characteristics only, then $f_i(x_i; \alpha_i) = \alpha_i^r \cdot h_i(x_i).$ The class of CSF's satisfying A1 – A5 and homogeneity with respect inherent characteristics only is given by $p_i(x; \alpha) = \frac{\alpha_i^r \cdot h_i(x_i)}{\sum_{k \in N} \alpha_k^r \cdot h_k(x_k)},$ where $h_i(\cdot)$ is continuously differentiable with $h_i(0) > 0, h'_i \geq 0, \forall i \in N,$ and r is any strictly positive real number.

unit of inherent characteristics. Moreover, homogeneity restricts $g_i(\cdot)$, but not $h_i(\cdot)$, to be same for all $i \in N$.

The detailed proof of Theorem 2 is provided in the appendix. We briefly discuss the significance of the involved restrictions. The effective investments by all agents must be strictly greater than zero when the actual investment by all agents are zero so that the winning probabilities are well defined. The axiom of marginal effects requires that the winning probability of a contestant should be non-decreasing in his own investment. The exponent r must be greater than a critical value r_c for the winning probability of each contestant to be non-decreasing in his level of contest relevant inherent characteristics. But, the CSF is well defined only if r_c is finite. This is ensured if the h_i 's are continuously differentiable polynomial-order functions with finite first order derivatives at zero. The critical value r_c is the maximum value of the ratio of marginal effective investment to the average effective investment over all levels of actual investments across all contestants.

We present two examples that will help us distinguish the CSF's that belong to the parametric family axiomatized in this theorem, from those that do not.

Example 1. $h_i(\frac{x_i}{\alpha_i}) = d_{i0} + \sum_{t=1}^T d_{it} \cdot (\frac{x_i}{\alpha_i})^t$, where T is a finite integer, $d_{i0} > 0$, and d_{it} is non-negative for all $i \in N$, for all $t \in \{1, 2, \dots, T\}$.

Define $z_i = \frac{x_i}{\alpha_i}$, for all $i \in N$. Note that $h_i(z_i)$ is a continuously differentiable polynomial-order function such that $h_i(0) = d_{i0} > 0$, $h'_i \geq 0 \forall x_i \in \mathfrak{R}_+$, and $h'_i(0) = d_{i1}$ is finite. Also,

$$\frac{\partial h_i(z_i)}{\partial z_i} = h'_i(z_i) = \sum_{t=1}^T t \cdot d_{it} \cdot z_i^{t-1} \geq 0, \quad \forall z_i \in \mathfrak{R}_+, \quad \forall i \in N.$$

This helps us show that

$$y_i(z_i) = \frac{\frac{\partial f_i(x_i; \alpha_i)}{\partial x_i}}{\frac{f_i(x_i, \alpha_i)}{x_i}} = \frac{z_i \cdot h'_i(z_i)}{h_i(z_i)} = \frac{\sum_{t=1}^T t \cdot d_{it} \cdot z_{it}}{d_{i0} + \sum_{t=1}^T d_{it} \cdot z_{it}},$$

which in turn implies

$$\lim_{z_i \rightarrow 0} [y_i(z_i)] = 0,$$

and

$$\lim_{z_i \rightarrow \infty} [y_i(z_i)] = T, \quad \forall i \in N.$$

Therefore, the extreme value theorem implies that the maximum value of $y_i(z_i)$ for all $z_i \in \mathfrak{R}_+$, for all $i \in N$, exists and is finite. Let us denote this maximum value of $y_i(z_i)$ by r_c . The resulting CSF is

$$p_i(\mathbf{x}; \alpha) = \frac{\alpha_i^r [d_{i0} + \sum_{t=1}^T d_{it} \cdot (\frac{x_i}{\alpha_i})^t]}{\sum_{k \in N} \alpha_k^r [d_{k0} + \sum_{t=1}^T d_{kt} \cdot (\frac{x_k}{\alpha_k})^t]}, \quad (4)$$

where $r \geq r_c$. Let us look at some particular cases.

(i) When $T = 1$, r_c equals unity. If we take $r = r_c = 1$ the above CSF reduces to

$$p_i(\mathbf{x}; \alpha) = \frac{\alpha_i \cdot (d_{i0} + \frac{d_{i1}x_i}{\alpha_i})}{\sum_{k \in N} \alpha_k \cdot (d_{k0} + \frac{d_{k1}x_k}{\alpha_k})} = \frac{d_{i0}\alpha_i + d_{i1}x_i}{\sum_{k \in N} d_{i0}\alpha_k + d_{k1}x_k}. \quad (5)$$

(ii) When d_{it} is zero for all $i \in N$, for all $t \neq (r+1) \in \{1, 2, \dots, T\}$, where $T \geq r+1 \geq r_c$ then the above CSF reduces to

$$p_i(\mathbf{x}; \alpha) = \frac{\alpha_i^{r+1}d_{i0} + \alpha_i d_{ir}x_i^r}{\sum_{k \in N} \alpha_k^{r+1}d_{k0} + \alpha_k d_{kr}x_k^r} = \frac{\hat{\alpha}_i + \tilde{\alpha}_i x_i^r}{\sum_{k \in N} \hat{\alpha}_k + \tilde{\alpha}_k x_k^r}. \quad (6)$$

This is a continuous extension of the CSF axiomatized by Clark and Riis (1998).

Example 2. $h_i(\frac{x_i}{\alpha_i}) = h_i(z_i) = e^{z_i} = 1 + \sum_{t=1}^{\infty} \frac{z_i^t}{t!}$, for all $i \in N$.

Note that $h_i(z_i)$ is a continuously differentiable *exponential-order*⁸ function such that $h_i(0) = 1 > 0$, $h'_i(z_i) = e^{z_i} \geq 0 \forall x_i \in \mathfrak{R}_+$, and $h'_i(0) = 1$ is finite. However, the fact that the exponential function is an infinite series implies that

$$\lim_{z_i \rightarrow \infty} [y_i(z_i)] = \infty, \quad \forall i \in N.$$

Hence, r_c is infinite, which in turn implies that any function $h_i(\cdot)$ which grows faster than polynomial-order functions can not be used to generate a CSF that belongs to the class of CSF's axiomatized in Theorem 2. In particular, the logit-CSF does not belong to this family.

⁸Any function that is not bounded from above by any polynomial-order function is termed an exponential-order function.

3.2. Conditional Anonymity

We now introduce the (mixed) axiom of *conditional anonymity* which requires that two agents with same level of investment must have same winning probabilities if they have the same level of contest-relevant inherent characteristics. It formalizes the idea that the technology of producing effective investments is same across all the contestants even though they differ in their contest-relevant inherent characteristics.

[A7] Conditional Anonymity. $p_i(x; \alpha) = p_j(\mathbf{x}; \alpha)$ if $x_i = x_j$ and $\alpha_i = \alpha_j$.

This axiom is meaningful only if each agent's technology for producing effective investments is determined by some factors common to all agents, but unrelated to their contest relevant inherent characteristics.⁹ Conditional anonymity imposes the restriction that $f_i(\cdot) = f_j(\cdot)$ for all $i, j \in N$. This in turn implies that $h_i(\cdot) = h_j(\cdot)$ for all $i, j \in N$. Thus, a CSF satisfies axioms A1 – A7 if and only if

$$p_i(\mathbf{x}; \alpha) = \frac{f(x_i; \alpha_i)}{\sum_{k \in N} f(x_k; \alpha_k)} = \frac{\alpha_i^r \cdot h\left(\frac{x_i}{\alpha_i}\right)}{\sum_{k \in N} \alpha_k^r \cdot h\left(\frac{x_k}{\alpha_k}\right)} = \frac{g(\alpha_i) \cdot h\left(\frac{x_i}{\alpha_i}\right)}{\sum_{k \in N} g(\alpha_k) \cdot h\left(\frac{x_k}{\alpha_k}\right)}. \quad (7)$$

With the additional axiom of conditional anonymity the example presented in equation (5) becomes

$$p_i(\mathbf{x}; \alpha) = \frac{\alpha_i \cdot (d_0 + d_1 \frac{x_i}{\alpha_i})}{\sum_{k \in N} \alpha_k \cdot (d_0 + \frac{d_1 x_k}{\alpha_k})} = \frac{\tilde{\alpha}_i + x_i}{\sum_{k \in N} \tilde{\alpha}_k + x_k}, \quad (8)$$

where $\tilde{\alpha}_i = \frac{d_0 \alpha_i}{d_1}$, for all $i \in N$. This is the exact CSF employed by Nti (2004).

3.3. Scalar-parametric CSF's

In order to axiomatize the scalar-parametric CSF's we will assume that the winning probabilities depend on the vector of investments and a scalar parameter that is common to all agents. It is hard to justify that all contestants will share a common belief regarding the level of the scalar parameter unless the nature of the contest allows the contestants to

⁹We believe the axiom of anonymity is more suited to mediated contests, where the administrator determines the technology for converting actual investments into effective investments.

do so. Therefore, we believe, scalar-parametric CSF's are especially suited for the analysis of contests mediated by an administrator. Following Kolmar and Wagener (2005), we shall interpret the scalar parameter (noise) as a design variable in hands of the administrator. The administrator announces the value of this parameter before agents make their investments in order to induce different responses from different contestants.

We denote by $A1_s$ the axiom $A1$ stated with $\alpha_s \in \mathfrak{R}_{++}$, instead of $\alpha \in \mathfrak{R}_{++}^N$. All axioms except the axiom of marginal effects and the axiom of anonymity remain unaltered in essence. In the previous section it was reasonable to assume that if the level of an agent's contest-relevant inherent characteristics (α_i) increases then his winning probability should not decrease. However, in this case, we find no reason to introduce an axiom regarding how changes in the value of the scalar parameter should affect the winning probabilities. The effect of of this parameter on will be a feature of the CSF resulting from the other axioms. Further, since this parameter is common to all agents, conditional anonymity collapses to anonymity. In this context anonymity reflects the idea that the contest administrator uses the same technology to convert the actual investments of agents into effective investments. Of course, the contest administrator can choose to discriminate among the contestants by assigning different technologies of producing effective investments to different contestants.

Theorem 3. *The axioms $A1_s - A7_s$ are satisfied if and only if*

$$p_i(\mathbf{x}; \alpha_s) = \frac{f(x_i; \alpha_s)}{\sum_{k \in N} f(x_k; \alpha_s)} = \frac{\alpha_s^r \cdot h\left(\frac{x_i}{\alpha_s}\right)}{\sum_{k \in N} \alpha_s^r \cdot h\left(\frac{x_k}{\alpha_s}\right)} = \frac{h\left(\frac{x_i}{\alpha_s}\right)}{\sum_{k \in N} h\left(\frac{x_k}{\alpha_s}\right)}, \quad (9)$$

where $h(\cdot)$ is any continuously differentiable function such that $h(0) > 0$, and $h' \geq 0$.

Proof. The proof is very similar to that of Theorem 2, and hence omitted.

It should again be noted that

$$f_i(x_i; \alpha_i) = \alpha_s^r \cdot h\left(\frac{x_i}{\alpha_s}\right) = g(\alpha_s) \cdot h\left(\frac{x_i}{\alpha_s}\right). \quad (10)$$

As in the previous section, homogeneity implies $g_i(\cdot)$ is the same for all $i \in N$, and anonymity implies $h_i(\cdot)$ is the same for all $i \in N$. In addition, as the scalar parameter is common to all contestants, the term $g(\alpha_s)$ cancels out in the CSF.¹⁰

¹⁰The function $g(\cdot)$ will also cancel out in the vector-parametric CSF if all contestants have the same

Example 3. $h(\frac{x_i}{\alpha_s}) = d_0 + d_1 \cdot \frac{x_i}{\alpha_s}$, such that $h(0) = d_0 > 0$, and $h' = d_1 > 0$.

We thus obtain

$$p_i(\mathbf{x}; \alpha) = \frac{d_0\alpha_s + d_1x_i}{\sum_{k \in N} d_0\alpha_s + d_1x_k} = \frac{\tilde{\alpha}_s + x_i}{\sum_{k \in N} \tilde{\alpha}_s + x_k}, \quad (11)$$

where $\tilde{\alpha}_s = \frac{d_0\alpha_s}{d_1}$. One can interpret this CSF as follows. The administrator of a mediated contest invites agents to invest and explicitly announces that (i) an *actual* investment of x_i will be treated as an *effective* investment of $(\alpha'_s + x_i)$, and (ii) the winning probability of each agent will be the ratio of his effective investment to the total effective investment. By suitably choosing α'_s , the administrator can hope to manipulate the investments made by the agents if they have different costs of investing. It should also be noted that this CSF is precisely the noisy-CSF introduced by Amegashie (2006). It can also be interpreted as a continuous extension of the Tullock-CSF. If we were to drop the axiom of continuity then all the CSF's that can be axiomatized using the framework of Skaperdas (1996) will belong to this scalar-parametric family axiomatized in Theorem 3.

Example 4. $h(\frac{x_i}{\alpha_s}) = e^{\frac{x_i}{\alpha_s}}$, such that $h(0) = 1 > 0$, and $h' = e^{\frac{x_i}{\alpha_s}} > 0$.

Clearly,

$$p_i(\mathbf{x}; \alpha) = \frac{e^{\theta x_i}}{\sum_{k \in N} e^{\theta x_k}}, \quad (12)$$

where $\theta = \alpha_s^{-1}$ is strictly positive. This is the logit-CSF axiomatized by Skaperdas (1996).

4. Conclusion

We have provided the axiomatization of two parametric CSF's. In the first, the winning probability of each contestant depends on the investments made by all agents and a vector of parameters where each parameter captures the contest-relevant inherent characteristics of a contestant. Such CSF's are suitable for analyzing unmediated contests as

level of inherent characteristics. However, this does not imply that all CSF's that belong to the single parameter family also belong to the family with a vector of parameters when all elements of this vector are the same. This is because while axiomatizing the scalar-parametric CSF we use the within version of the axiom of marginal effects. Note that, the logit-CSF belongs to the scalar-parametric family, but not the vector-parametric family.

in these contests each agent is likely to possess his own technology of producing effective investments. The second type of parametric CSF's axiomatized in this paper give the winning probability of each contestant as a function of the investments made by all contestants and a common scalar parameter. These CSF's can be used by administrators who plan to design a contest in order to achieve a particular objective by suitably choosing the parameter. Moreover, the results provided in this paper contain the existing axiomatizations by Skaperdas (1996) and Clark and Riis (1998) as special cases.

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Appendix. Proofs

Proof of Theorem 1: This theorem is a straightforward extension of Theorem (1) in Skaperdas (1996). Note that Consistency and IIA_S imply

$$\frac{p_i^N(\mathbf{x}; \alpha)}{p_j^N(x; \alpha)} = \frac{p_i^M(x; \alpha)}{p_j^M(x; \alpha)} = \frac{p_i^M(x^M; \alpha^M)}{p_j^M(x^M; \alpha^M)}. \quad (13)$$

Let $p_i^N(\mathbf{x}; \alpha) = [i]_N$ represent the probability that agent- i wins the contest among all N agents. Similarly, let $p_i^M(\mathbf{x}; \alpha) = p_i^M(x^M; \alpha^M) = [i]_M$ represent the probability that agent- i wins the sub-contest among $M \subseteq N$ agents. For the sub-contest between agents in M ,

$$\sum_{i \in M} [i]_M = 1. \quad (14)$$

This equality can be rewritten as

$$\frac{[i]_M}{\sum_{k \neq i}^{k \in M} [k]_M} \cdot \frac{[j]_M}{[i]_M} \cdot \sum_{k \neq i}^{k \in M} \frac{[k]_M}{[j]_M} = 1. \quad (15)$$

Consider the two player contests in which one player is always agent- j . For such contests, let $[i]_{\{i,j\}}$ denote the probability that agent- i wins it. By consistency

$$\frac{[j]_M}{[i]_M} = \frac{[j]_{\{i,j\}}}{[i]_{\{i,j\}}} = \frac{1 - [i]_{\{i,j\}}}{[i]_{\{i,j\}}}, \quad \text{and} \quad \frac{[k]_M}{[j]_M} = \frac{[k]_{\{k,j\}}}{1 - [k]_{\{k,j\}}}. \quad (16)$$

Using the equalities in equation (16), equation (15) becomes

$$\frac{[i]_M}{1 - [i]_M} \cdot \frac{1 - [i]_{\{i,j\}}}{[i]_{\{i,j\}}} \cdot \sum_{k \neq i}^{k \in M} \frac{[k]_{\{k,j\}}}{1 - [k]_{\{k,j\}}} = 1. \quad (17)$$

Let the investment and other factors of agent- j be fixed at x_0 and α_0 . It follows from ΠA_S that we can define

$$\frac{[i]_{\{i,j\}}}{1 - [i]_{\{i,j\}}} = f_i(x_i, x_0; \alpha_i, \alpha_0) = f_i(x_i; \alpha_i), \quad \forall i \in M. \quad (18)$$

Equation (17) can now be written as

$$\frac{[i]_M}{1 - [i]_M} \cdot \frac{1}{f_i(x_i; \alpha_i)} \cdot \sum_{k \neq i}^{k \in M} f_k(x_k; \alpha_k) = 1. \quad (19)$$

Rearranging the above equation we obtain

$$[i]_M = p_i^M(x^M; \alpha^M) = \frac{f_i(x_i; \alpha_i)}{\sum_{k \in M} f_k(x_k; \alpha_k)}, \quad \forall i \in M, \forall M \subseteq N. \quad (20)$$

The necessity of conditions on $f_i(x_i, \alpha_i)$ is explained in the discussion following Theorem 1. *QED*.

Proof of Theorem 2: From Theorem 1,

$$p_i^N(\mathbf{x}; \alpha) = \frac{f_i(x_i; \alpha_i)}{\sum_{k \in N} f_k(x_k; \alpha_k)}, \quad \forall i \in N. \quad (21)$$

We will first show that if $p_i(\lambda x; \lambda \alpha) = p_i(\mathbf{x}; \alpha)$ for all $\lambda > 0$, then $f_i(x_i; \alpha_i) = \alpha_i^r \cdot h_i(\frac{x_i}{\alpha_i})$, where r is some real number. Homogeneity implies

$$p_i(\lambda x; \lambda \alpha) = \frac{f_i(\lambda x_i; \lambda \alpha_i)}{\sum_{k \in N} f_k(\lambda x_k; \lambda \alpha_k)} = \frac{f_i(x_i; \alpha_i)}{\sum_{k \in N} f_k(x_k; \alpha_k)} = p_i(\mathbf{x}; \alpha). \quad (22)$$

The equalities above hold for all $i \in N$. Thus

$$\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f_i(x_i; \alpha_i)} = \frac{f_j(\lambda x_j; \lambda \alpha_j)}{f_j(x_j; \alpha_j)} = \frac{\sum_{k \in N} f_k(\lambda x_k; \lambda \alpha_k)}{\sum_{k \in N} f_k(x_k; \alpha_k)}, \quad \forall i, j \in N. \quad (23)$$

Since the last term in the above equality is the same for all $i \in N$, we can conclude that $\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f_i(x_i; \alpha_i)}$ depends on λ but not on x_i and α_i . Hence, if the investment by agent- i were to be y_i , and his individual specific factors were β_i , we would have

$$\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f_i(x_i; \alpha_i)} = \frac{f_i(\lambda y_i; \lambda \beta_i)}{f_i(y_i; \beta_i)}, \text{ for all } \lambda > 0. \quad (24)$$

The above equation can be rewritten as

$$\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f_i(\lambda y_i; \lambda \beta_i)} = \frac{f_i(x_i; \alpha_i)}{f_i(y_i; \beta_i)}, \text{ for all } \lambda > 0. \quad (25)$$

Let us define

$$g_i^w(s) = f_i(sw, s), \text{ for any positive } w \text{ and } s. \quad (26)$$

For any positive real numbers u and v

$$\frac{g_i^w(uv)}{g_i^w(u)} = \frac{f_i(uvw, uv)}{f_i(uw, u)}. \quad (27)$$

Using equation (25), the above equation (by treating u as λ , and then substituting $u = 1$) gives

$$\frac{f_i(uvw, uv)}{f_i(uw, u)} = \frac{f_i(vw, v)}{f_i(w, 1)} = \frac{g_i^w(v)}{g_i^w(1)}. \quad (28)$$

From equations (27) and (28) we get

$$\frac{g_i^w(uv)}{g_i^w(1)} = \frac{g_i^w(u)}{g_i^w(1)} \cdot \frac{g_i^w(v)}{g_i^w(1)}. \quad (29)$$

Equation (29) is one of Cauchy's fundamental functional equations (Aczel, 1966) whose standard solution is given by

$$g_i^w(u) = f_i(uw, u) = f_i(w, 1) \cdot u^{e(w)}. \quad (30)$$

Let $x_i = uw$ and $u = \alpha_i$. This implies $w = \frac{x_i}{\alpha_i}$, which in turn gives

$$f_i(x_i; \alpha_i) = f_i\left(\frac{x_i}{\alpha_i}; 1\right) \cdot \alpha_i^{e\left(\frac{x_i}{\alpha_i}\right)} = h_i\left(\frac{x_i}{\alpha_i}\right) \cdot \alpha_i^{e\left(\frac{x_i}{\alpha_i}\right)}. \quad (31)$$

The above equation implies

$$\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f(x_i; \alpha_i)} = \lambda^{e(\frac{x_i}{\alpha_i})}. \quad (32)$$

Since $(\frac{f_i(\lambda x_i; \lambda \alpha_i)}{f(x_i; \alpha_i)})$ is the same for all $i \in N$, it is independent of x_i and α_i , and a function of λ only. Hence, $e(\frac{x_i}{\alpha_i})$ must be a constant, say, r . Thus,

$$f_i(x_i; \alpha_i) = h_i(\frac{x_i}{\alpha_i}) \cdot \alpha_i^r, \quad (33)$$

where r is a real number. This in turn gives us the CSF in equation (2). The derivatives of $p_i(x; \alpha)$ with respect to x_i and α_i are

$$\frac{\partial p_i}{\partial x_i} = \frac{\alpha_i^{r-1} \cdot h'_i \cdot [\sum_{k \neq i}^{k \in N} \alpha_k^r \cdot h_k(\frac{x_k}{\alpha_k})]}{[\sum_{k \in N} \alpha_k^r \cdot h_k(\frac{x_k}{\alpha_k})]^2}, \quad (34)$$

and

$$\frac{\partial p_i}{\partial \alpha_i} = \frac{(r - \frac{x_i \cdot h'_i}{\alpha_i \cdot h_i}) \cdot \alpha_i^{r-1} \cdot h_i(\frac{x_i}{\alpha_i}) \cdot [\sum_{k \neq i}^{k \in N} \alpha_k^r \cdot h_k(\frac{x_k}{\alpha_k})]}{[\sum_{k \in N} \alpha_k^r \cdot h_k(\frac{x_k}{\alpha_k})]^2}. \quad (35)$$

The first part of the axiom of marginal effects dictates that $h'_i \geq 0$ for all $i \in N$. The second part of this will be satisfied only if r is high enough to ensure that the marginal impact of α_i on $p_i(x, \alpha)$ is non-negative for all $i \in N$. This requires that r must not be smaller than a critical value. It is clear from equation (35) that r must be no less than

$$r_c = \max_i [\max_{x_i} (\frac{x_i \cdot h'_i}{\alpha_i \cdot h_i})] = \max_i [\max_{x_i} (\frac{\frac{\partial f_i(x_i; \alpha_i)}{\partial x_i}}{\frac{f_i(x_i; \alpha_i)}{x_i}})].$$

The existence of r_c is assured for all polynomial-order functions $h_i(\cdot)$ by the extreme value theorem as $(\frac{x_i \cdot h'_i}{\alpha_i \cdot h_i})$ is continuous and bounded for all $i \in N$, if $h_i(0) > 0$, and $h'_i(0)$ is bounded. *QED*.

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