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## Robust optimization of consumption with random endowment

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# Robust Optimization of Consumption with Random Endowment

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# Robust Optimization of Consumption with Random Endowment

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## Abstract

We consider the problem of optimal consumption for an investor who is risk and uncertainty averse. We model these preferences of the investor with the help of a convex risk-measure. Apart from consumption the agent has the possibility to invest initial capital and random endowment in a market where stock-prices are semimartingales. We formulate this as a maximin problem that will be solved by duality methods.

keywords: duality theory; risk measures; optimal consumption; model uncertainty

AMS classification 91B16, 62C20

JEL classification D11, D81

## 1 Introduction

Starting in 1969, 1971 with the works by Merton [19], [20] the problem of optimal investment has become one of the major research areas in mathematical finance. Merton and many of the following authors determined the optimal financial position with respect to expected utility but expected utility does not reflect preferences correctly in situations where model uncertainty is present. Therefore, Gilboa and Schmeidler [9] extended the model by incorporating Knightian

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uncertainty which resulted in a robust approach. One possible interpretation of this approach is that the agent takes a class of possible models into consideration and decides for the worst-case outcome. This interpretation also explains why it is useful to extend the robust approach to an evaluation via risk measures: the agent will typically have some assessment of the different models (for instance she might use estimated data for the model specification and while she does not assume her estimates to be correct she believes values close to the estimates to be more likely than values far off). It is possible to include this assessment by the introduction of a penalty function, which allows to weight the impact of a model according to its plausibility. In Maccheroni et al. [18] a set of axioms is given yielding these preferences.

The non-robust problem of optimal terminal wealth was considered by several authors who discussed different aspects of the problem. For example Kramkov and Schachermayer [16], [17] characterized solutions of this problem in a general setting using duality methods. There exist many extensions of this problem, e.g. Cvitanić et al. [5], Karatzas and Zitkovic [15], or Hugonnier and Kramkov [14] consider the case where the agent optimizes the consumption process and also receives some kind of additional random endowment.

In [23], [25] and [10] the robust problem (without penalty term) was solved, also via the duality or martingale method. Furthermore, there are papers where the authors assume more specific market models and preferences and are therefore able to compute not just the optimal financial position but also the optimal investment strategy: See [7] for a stochastic control approach for a model with random endowment or [22] for computation of the optimal strategy in a robust setting. In [2] or [21] a optimal investment problem is solved using BSDE's. Also the robust problem with a penalty term was considered, among others by Hansen and Sargent [11], and Bordigoni et al.[3] for an entropic penalty term and by Hernández-Hernández and Schied [13] for a general penalty term.

In this paper we will investigate the problem of optimal consumption in a general semimartingale framework where the agent may invest in the stock market and receives additional random endowment. She evaluates positions by a robust utility functional described through a utility function  $U$  and a penalty term  $\gamma$ . More specifically the agent tries to maximize

$$\inf_{Q \ll P} \left( E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q) \right)$$

over all possible consumption rate processes  $c$ . First the dual function for this problem is given and then we examine properties and relations between primal and dual problem, e.g. we will show

that both problems have a solution and are conjugate to one another. Next we will verify that the above maximin problem and the corresponding minimax problem are equivalent.

In the long run we would like to compute explicitly the optimal consumption plan and investment strategy with the help of stochastic control methods (compare [13] or [12]). These techniques are - even in the nonrobust case - easier to apply to the dual problem than to the primal problem. In the robust case the primal problem becomes a maximin problem, i.e. we need to find a saddle-point whereas the dual problem can still be solved through pure minimization. Hence, if we want to use stochastic control techniques it is sensible to examine the dual problem first. Furthermore, the duality results are also valid for preferences that are not dynamically consistent.

The problem is motivated by [24] where Schied considers the problem of optimal terminal wealth with respect to a convex risk measure in a setting without random endowment. We extend this problem by introducing a concept of consumption, general enough to include also the maximization of terminal wealth, and additional random endowment (compare [15]). In contrast to the problem of optimal terminal wealth, the problem of optimal consumption does not admit an equivalent static version. Hence, we cannot restrict ourselves to random variables to solve the primal problem, but need to work with stochastic processes. Furthermore, our dual problem is different from the dual problem in [24] or [15] due to random endowment on the one hand and robustification on the other. These differences imply for instance that there are cases where, given the dual solution  $(\widehat{Q}, \widehat{R})$ ,  $\widehat{R}$  is no longer a solution to the associated dual problem under the model  $\widehat{Q}$ . Aspects regarding the dual function will be discussed in more detail in the third section. Robustification changes the problem of optimal consumption also significantly with respect to the optimization procedure since we consider a maximin instead of just a maximization problem. This extension yields some new results and partly we give new proofs for known results.

In [4] Burgert and Rüschendorf also consider a robust version of [15]. The most obvious difference to our setting is that we deal with preferences given by convex instead of just coherent risk measures. Furthermore Burgert and Rüschendorf work under the serious restriction that the set  $\mathcal{Q}$  should only consist of measures that are equivalent to  $P$  and have a uniformly bounded density. The first assumption rules out risk measures such as AVaR, the second standard dynamic consistent coherent utility functionals in a Brownian setting such as used in [13].

In the next section we will introduce the market setting and describe the agent's preferences in more detail before we state our main theorem and give an example. In the third section we will derive our dual problem and prove the statements of the theorem related to it. Finally, in the

fourth section we will finish the proof of the theorem.

## 2 Notations and main results

We consider an agent who wants to maximize her utility from consumption between time zero and some finite time horizon  $T$ . She is endowed with an initial capital and receives additional random endowment over time, which she may invest into  $d$  assets. To formalize this problem we use the same market model as Karatzas and Žitković [15]. That means we model the price process of the assets as a  $d$ -dimensional RCLL semi-martingale on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$  where the filtration satisfies the usual conditions. The financial market is assumed to be arbitrage-free in the sense that the set  $\mathcal{M}$  of supermartingale measures equivalent to  $P$  is not empty. The portfolio process is denoted by  $\theta = (\theta_t)_{0 \leq t \leq T}$ , and we allow only those strategies for which  $\int_0^t \theta_u dS_u$  is bounded from below by some constant.<sup>1</sup> The initial capital is denoted by  $x$  and the random endowment is described as a nondecreasing, adapted, RCLL process  $\mathcal{E} = (\mathcal{E}_t)_{0 \leq t \leq T}$  with  $\mathcal{E}_T \in L^\infty(P)$ . The consumption process  $C = (C_t)_{0 \leq t \leq T}$  is assumed to be a nonnegative, nondecreasing and adapted RCLL process. More specifically we will only deal with consumption processes that can be written as  $C_t = \int_0^t c_s \mu(ds)$  where  $\mu$  is a probability measure on  $[0, 1]$  which is diffuse on  $[0, 1)$ . This means we assume that the agent consumes in a continuous way except for the final time  $T$  where we also allow for lump consumption. In particular we can choose  $\mu = \delta_{\{T\}}$  which accords to the problem of optimizing terminal wealth.

The terminal wealth is required to be non-negative:

$$x + \mathcal{E}_T + \int_0^T \theta_t dS_t - C_T \geq 0 \quad P - \text{a.s.} \quad (1)$$

We denote by  $\mathcal{A}(x, \mu)$  the set of consumption process densities  $(c_t)_{0 \leq t \leq T}$ , for which an admissible strategy exists, such that condition (1) is fulfilled.

The utility from consumption is measured in terms of a robust utility functional of the form

$$c \longmapsto \inf_{Q \ll P} \left( E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q) \right). \quad (2)$$

This approach is called robust since such utility functionals are useful to deal with model uncertainty [9, 18]. The penalty function  $\gamma$  enables us to influence how serious a model should be taken. The

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<sup>1</sup>This condition will correspond to our admissibility condition for strategies, since the additional income of the agent is bounded.

robust functional is closely linked to the convex risk measure

$$\rho(Y) := \sup_{Q \ll P} (E_Q[-Y] - \gamma(Q)), \quad Y \in L^\infty(P). \quad (3)$$

More precisely the penalty function  $\gamma$  is supposed to be bounded from below and equal to the minimal penalty function of the convex risk measure above, which means, that it satisfies the biduality relation

$$\gamma(Q) = \sup_{Y \in L^\infty(P)} (E_Q[-Y] - \rho(Y)). \quad (4)$$

Furthermore we need the following conditions on  $\gamma$ ; compare [24, Assumption 2.1].

**Assumption 2.1**

We assume that the risk measure  $\rho$  is continuous from below, i.e. for a sequence  $(Y_n) \subset L^\infty$  increasing a.s. to some  $Y \in L^\infty$ , we have  $\rho(Y_n) \searrow \rho(Y)$ . Furthermore,  $\rho$  needs to be sensitive in the sense that  $\rho(Y)$  is strictly positive for all  $Y \in L^\infty \setminus \{0\}$ .

We work with a utility function  $U: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  with the properties assumed in Definition 3.1 of [15].

**Assumption 2.2**

More precisely, for fixed  $t \in [0, T]$  we request  $U(t, \cdot): \mathbb{R}^+ \rightarrow \mathbb{R}$  to be a utility function, i.e.  $U(t, \cdot)$  is strictly concave, increasing, continuously differentiable and satisfies the Inada conditions.

The marginal utility is assumed to be bounded by the strictly decreasing continuous functions  $K_1$  and  $K_2$ , such that  $K_1(x) \leq U_x(t, x) \leq K_2(x)$  and

$$\limsup_{x \rightarrow \infty} K_2(x)/K_1(x) < \infty.$$

The map  $t \mapsto U(t, 1)$  is bounded and

$$\lim_{x \rightarrow \infty} \inf_{t \in [0, T]} U(t, x) > 0.$$

Additionally  $U$  is supposed to be of reasonable asymptotic elasticity, i.e.

$$\limsup_{x \rightarrow \infty} \left( \sup_t \frac{x U_x(t, x)}{U(t, x)} \right) < 1.$$

See [15] for a discussion of these assumptions.

To avoid problems of evaluating  $E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q)$  for the case where  $U$  is unbounded from below it is sensible to restrict the set of measures that enter the optimization problem to  $\mathcal{Q} = \{Q \ll P \mid \gamma(Q) < \infty\}$ . Hence, our optimization problem is now

$$\text{maximize } \inf_{Q \in \mathcal{Q}} \left( E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q) \right) \text{ over all } c \in \mathcal{A}(x, \mu).$$

Furthermore, to circumvent difficulties when integrating we follow [24] in setting

$$E_Q[F] := \sup_n E_Q[F \wedge n] = \lim_n E_Q[F \wedge n] \quad \text{for arbitrary } F \in L^0. \quad (5)$$

In doing so we keep the functional  $(c_t)_{0 \leq t \leq T} \mapsto E_Q[\int_0^T U(t, c_t) \mu(dt)]$  concave.

In the following we will use  $\mathcal{U}_Q(c)$  to abbreviate  $E_Q[\int_0^T U(t, c_t) \mu(dt)]$ . We define the value function of the maximization problem as follows

$$u(x) = \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Q \in \mathcal{Q}} (\mathcal{U}_Q(c) + \gamma(Q)). \quad (6)$$

Hence, the investor needs to solve a maximin problem in order to find  $u$ . Another approach to the optimization problem is to solve the problem in each possible model, i.e. compute  $u_Q(x) = \sup_{c \in \mathcal{A}(x, \mu)} \mathcal{U}_Q(c)$  for each  $Q \in \mathcal{Q}$  and then robustify the result by taking the infimum. We will show that both methods lead to the same result, namely

$$u(x) = \inf_{Q \in \mathcal{Q}} (u_Q(x) + \gamma(Q)).$$

Sometimes it will be more convenient to work with densities than measures. We denote the density of  $Q$  with respect to  $P$  on  $\mathcal{F}_T$  by  $Z^Q = dQ/dP$  and the set  $\{Z^Q \mid Q \in \mathcal{Q}\}$  by  $\mathcal{Z}$ . We will identify  $Q$  and  $Z^Q$ , and thus write  $\gamma(Z^Q)$   $\gamma(Q)$  and  $\gamma(Z^Q)$  or  $u_Q$  and  $u_{Z^Q}$  denote the same object. While we write  $Z^Q$  for  $dQ/dP|_{\mathcal{F}_T}$  we will denote the corresponding density process by  $(Z_t^Q)_{0 \leq t \leq T}$ .

We will use the dual (or martingale) approach to characterize the solution to our optimization problem. A downside of the dual method is that it does not directly give the investment strategy  $\theta$  which is necessary in order to realize the optimal consumption plan. In some cases this strategy can be determined using stochastic control techniques, compare e.g. [13] or [22]. These authors also argue that the dual problem is easier to treat than the primal problem which shows that our results are still useful. Furthermore, in our case we did not restrict the form of the penalty function, therefore our approach covers also the cases where time consistency is lacking and hence optimal control techniques cannot be applied (see [24]).

In general the dual problem is an associated minimization problem where the dual domain is related to the set of equivalent martingale measures. As was shown in [5] and [15] we need to use  $\mathcal{D}$ , the weak\*-closure of the set of equivalent supermartingale measures  $\mathcal{M}$ , as dual domain. More precisely, we identify the set  $\mathcal{M}$  with its embedding in the dual of  $L^\infty(P)$ ,  $(L^\infty(P))^*$ . Then  $\mathcal{D}$  is the  $\sigma((L^\infty(P))^*, L^\infty(P))$ -closure of  $\mathcal{M}$ . The set  $\mathcal{D}$  contains also finitely-additive measures to which we cannot directly associate a density process. Therefore, we use that each  $R$  in  $\mathcal{D}$  has a unique



Hewitt-Yosida decomposition  $R^r + R^s$  where the regular part  $R^r$  is the maximal countable measure on  $\mathcal{F}$  that is dominated by  $R$ . Hence, we can for each  $R \in \mathcal{D}$  define a supermartingale  $L^R$ , where  $L_t^R$  is the density of the regular part  $(R|_{\mathcal{F}_t})^r$  of  $R|_{\mathcal{F}_t}$  with respect to  $P|_{\mathcal{F}_t}$ . In the following we will work with the RCLL supermartingale  $Y^R$  that coincides with  $L^R$  for all  $t \in \mathbb{Q} \cap [0, T]$ . For a proof of the existence of  $Y^R$  and further properties of  $Y$  and  $\mathcal{D}$  see [15]. We need as further notation  $\langle R, \mathcal{E}_T \rangle$  which gives the canonical pairing. Observe that in particular  $\langle R, I_\Omega \rangle = 1$ .

Using  $\mathcal{D}$  we consider as dual problem

$$v(y) = \inf_{Z \in \mathcal{Z}} \left( \inf_{R \in \mathcal{D}} \left( E \left[ Z \int_0^T V \left( t, y \frac{Y_t^R}{Z_t} \right) \mu(dt) \right] + y \langle R, \mathcal{E}_T \rangle \right) + \gamma(Z) \right) \quad (7)$$

where  $V$  is the convex conjugate of  $U$ , i.e.  $V(t, y) = \sup_{x \geq 0} (U(t, x) - xy)$ . This problem results from the capital constraint (1). This link will be explained more explicitly in section 3. To simplify notations,  $E_Q \left[ \int_0^T V(t, Y_t^R) \mu(dt) \right]$  will be denoted by  $\mathcal{V}_Q(Y^R)$ . We defined the function  $u_Q$  as the solution to the optimization problem under the subjective probability  $Q$ . We can now define an associated dual value function  $v_Q$  as

$$v_Q(y) = \inf_{R \in \mathcal{D}} (\mathcal{V}_Q(yY^R/Z^Q) + y \langle R, \mathcal{E}_T \rangle).$$

### Remark 2.3

In the introduction convex risk measures were interpreted as a worst-case approach for different scenarios  $Q$ . Using this interpretation it seems sensible to formulate the market restrictions under each measure  $Q$  but we formulated the assumptions for the market and the trading strategy with respect to the measure  $P$ . (In particular the dual domain depends on the measure we use to give the capital constraint (1).) One reason why we use  $P$  is that we use the convex risk measure as a model for the agent's preferences, not as a model of the "real" market. Furthermore, the worst-case measure may allow for arbitrage, compare [24, Example 3.2]. Hence, to exclude arbitrage opportunities it is necessary to formulate the admissibility condition under  $P$ .

Nevertheless, for all  $Q$  out of  $\mathcal{Q}_e := \{Q \in \mathcal{Q} \mid Q \sim P\}$  we have the conditions necessary to apply standard duality results. (Lemma 3.3 will guarantee that  $\mathcal{Q}_e$  is non-empty.)

### Assumption 2.4

*In the following we will assume that there exists  $Q_0 \in \mathcal{Q}_e$  that satisfies*

$$u_{Q_0}(x) < \infty \text{ for some } x > 0.$$

This is a similar assumption as is needed in [15] to guarantee the existence of solutions to both the primal and the dual problems under the subjective probability measure  $Q_0$ . Furthermore we

can conclude that  $u_{Q_0}$  and  $v_{Q_0}$  are dual functions [15, Theorem 3.10]. We will show that we have similar results in our robust setting.

### Theorem 2.5

Under the above assumption the following assertions are valid.

1. Both value functions  $u$  and  $v$  take only finite values and satisfy

$$u'(\infty-) = 0 \quad \text{and} \quad v'(0+) = -\infty, \quad (8)$$

$u$  is strictly concave and  $v$  is continuously differentiable.

2. The value function  $u$  satisfies

$$u(x) = \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Q \in \mathcal{Q}} (\mathcal{U}_Q(c) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}} \left( \sup_{c \in \mathcal{A}(x, \mu)} \mathcal{U}_Q(c) + \gamma(Q) \right).$$

3. The two value functions  $u$  and  $v$  are conjugate to another:

$$u(x) = \inf_{y > 0} (v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x > 0} (u(x) - xy). \quad (9)$$

In particular,  $v$  is convex.

4. The derivative of  $v$  satisfies

$$v'(\infty-) \in \left[ \inf_{R \in \mathcal{D}} \langle R, \mathcal{E}_T \rangle, \sup_{R \in \mathcal{D}} \langle R, \mathcal{E}_T \rangle \right]. \quad (10)$$

If  $\mathcal{E}_T \equiv 0$  the derivatives of  $v$  and  $u$  satisfy

$$u'(0+) = \infty \quad \text{and} \quad v'(\infty-) = 0. \quad (11)$$

5. There exists a solution  $(\widehat{Q}, \widehat{R}) \in \mathcal{Q} \times \mathcal{D}$  to the dual problem, i.e.

$$v(y) = \mathcal{V}_{\widehat{Q}, \mu} \left( y \frac{Y^{\widehat{R}}}{Z^{\widehat{Q}}} \right) + y \langle \widehat{R}, \mathcal{E}_T \rangle + \gamma(\widehat{Q}). \quad (12)$$

6. For any  $x > 0$  there exists an optimal consumption strategy  $\hat{c} \in \mathcal{A}(x, \mu)$ . If  $(\widehat{Q}, Y^{\widehat{R}})$  is a solution to the dual problem for  $y > 0$  such that  $x = -v'(y)$  then

$$u(x) = \inf_{Q \in \mathcal{Q}} (\mathcal{U}_Q(\hat{c}) + \gamma(Q)) = \mathcal{U}_{\widehat{Q}, \mu}(\hat{c}) + \gamma(\widehat{Q}) = u_{\widehat{Q}}(x) + \gamma(\widehat{Q})$$

and

$$\hat{c} = I \left( \cdot, \frac{\widehat{y} Y^{\widehat{R}}}{\widehat{Z}} \right) \quad \widehat{Q} \otimes \mu\text{-a.s.},$$

where  $\widehat{Z} = d\widehat{Q}/dP$  and  $I(t, \cdot) = \left( \frac{\partial U}{\partial x}(t, \cdot) \right)^{-1}$ .

Obviously the problem is easier to treat if  $\mathcal{Q}$  contains only measures that are equivalent to  $P$ . As in Schied [24] we get additional results in this case:

**Corollary 2.6**

If Assumption 2.4 and  $\mathcal{Q} = \mathcal{Q}_e$  are satisfied and  $\gamma$  is strictly convex on  $\mathcal{Q}$  then the value function  $u$  is continuously differentiable, the dual value function  $v$  is strictly convex, and for each  $y > 0$  there exist  $\widehat{Q} \in \mathcal{Q}$  and  $\widehat{R} \in \mathcal{D}$  such that

$$v(y) = \mathcal{V}_{\widehat{Q}, \mu} \left( y \frac{Y^{\widehat{R}}}{Z^{\widehat{Q}}} \right) + y \langle \widehat{R}, \mathcal{E}_T \rangle + \gamma(\widehat{Q}).$$

Moreover,  $Y^{\widehat{R}}$  is unique. For any  $x > 0$ , the optimal solution  $\hat{c} \in \mathcal{A}(x, \mu)$  is  $P$ -a.s. unique.

(The condition of strict convexity for  $\gamma$  is crucial for the differentiability of  $u$ .)

The following example considers the case of uncertain drift in a Black-Scholes model. Here one can see the advantage of a penalty function - if we used a purely robust approach the least absolute drift would always give the optimum but when a penalty is introduced the result is dependent on the drift under the measure  $P$ . For simplicity we consider a complete market. For computations in a more complex setting considerable additional effort is needed (compare e.g. [13]). Therefore we will treat these problems later. Nevertheless, we will then need the results from duality theory which we developed in this paper.

**Example 2.7**

We consider a Black-Scholes model, i.e.  $S$  is given through  $dS_t = S_t(b_t dt + \sigma dW_t)$  for constant  $\sigma > 0$  and As penalty function we take the relative entropy between  $Q$  and  $P$  and we look at the non time-dependent utility functional  $U(t, x) = \log(x)$ . If we do not restrict the market, we do not need to consider different processes  $Y^R$ ,  $R \in \mathcal{D}$  but only

$$Y_t = \exp \left( - \int_0^t \frac{b_s}{\sigma} dW_s - \frac{1}{2} \int_0^t \frac{b_s^2}{\sigma^2} ds \right)$$

(Compare [15, Proposition 4.1]). The convex conjugate of  $U$  is given by  $V(t, y) = -1 - \log(y)$ .

In [12, Lemma 3.1] it is proved that for each measure  $Q \ll P$  we can write the density

$$\frac{dQ}{dP} = \mathcal{E} \left( \int_0^T \eta_s dW_s \right) \quad Q - \text{a.s.}$$

for a progressive  $\eta$ . We set  $\tilde{b}_s = \sigma \eta_s + b_s$  which corresponds to the drift under  $Q$ .

With this we have

$$\begin{aligned}
v(y) &= \inf_{Z \in \mathcal{Z}} \left( E \left[ Z \int_0^T V(t, y Y_t / Z_t) \mu(dt) \right] + \gamma(Z) \right) \\
&= \inf_{\tilde{b}} \left( E_Q \left[ \int_0^T \log(1/y) + \int_0^t \frac{\tilde{b}_s}{\sigma} dW_s^Q + \frac{1}{2} \int_0^t \frac{\tilde{b}_s^2}{\sigma^2} ds \mu(dt) \right] - 1 \right. \\
&\quad \left. + E_Q \left[ - \int_0^T \frac{b_t - \tilde{b}_t}{\sigma} dW_t - \frac{1}{2} \int_0^T \frac{(b_t - \tilde{b}_t)^2}{\sigma^2} dt \right] \right).
\end{aligned}$$

If we consider the case where  $\mu = \delta_{\{T\}}$  we get

$$v(y) = - E_Q \left[ \int_0^T \frac{b_t}{\sigma} dW_t^Q \right] + 2E_Q \left[ \int_0^T \frac{\tilde{b}_t}{\sigma} dW_t^Q \right] + E_Q \left[ \int_0^T \frac{(b_t - \tilde{b}_t)^2 + b_t \tilde{b}_t - \frac{1}{2} b_t^2}{\sigma^2} dt \right].$$

Hence, here the infimum is achieved for  $\tilde{b}_t^* = b_t/2$ . Furthermore, one can calculate that

$$u(x) = \log x + \frac{Tb^2}{4\sigma^2}.$$

If we choose  $\mu(dt) = dt$ , take  $b$  to be constant, and restrict the set  $\mathcal{Z}$  to processes for which  $\eta$  is also constant we get

$$v(y) = \inf_{\tilde{b}} \left( E_Q \left[ \int_0^T \log(1/y) + \frac{1}{2} \int_0^t \frac{\tilde{b}_s^2}{\sigma^2} ds \mu(dt) \right] - 1 + E_Q \left[ \frac{1}{2} \int_0^T \frac{(b_t - \tilde{b}_t)^2}{\sigma^2} dt \right] \right).$$

Which yields that the optimal  $\tilde{b}$  is given by  $\tilde{b}^* = 2b/(T+2)$ . Obviously the drift that gives the infimum depends on the original drift  $b$ .

### 3 Dual Problem

We now develop the dual problem. The general approach is to use the capital constraint to bound the result of the primal problem from above which gives a related minimization problem. Karatzas and Žitković prove (Proposition 2.13 in [15]) that the capital constraint (1) can equivalently be formulated as follows, using the set  $\mathcal{D}$  introduced above. The set  $\mathcal{A}(x, \mu)$  of admissible consumption rates consists of all processes  $c$  such that  $(C_t := \int_0^t c_u \mu(du))_{0 \leq t \leq T}$  is a nonnegative, nondecreasing, right-continuous and adapted process satisfying

$$E \left[ \int_0^T Y_t^R c_t \mu(dt) \right] \leq x + \langle R, \mathcal{E}_T \rangle \quad \text{for all } R \in \mathcal{D}. \quad (13)$$

With the help of this characterization we can justify our dual problem by the next calculation. Due to (13) we have for all  $y > 0$  and all  $R \in \mathcal{D}$

$$\begin{aligned} \mathcal{U}_Q(c) &\leq E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + y \left( x + \langle R, \mathcal{E}_T \rangle - E \left[ \int_0^T c_t Y_t^R \mu(dt) \right] \right) \\ &\leq E \left[ Z^Q \int_0^T U(t, c_t) - y c_t \frac{Y_t^R}{Z_t^Q} \mu(dt) \right] + y \left( x + \langle R, \mathcal{E}_T \rangle \right) \\ &\leq E \left[ Z^Q \int_0^T V \left( t, y \frac{Y_t^R}{Z_t^Q} \right) \mu(dt) \right] + y \left( x + \langle R, \mathcal{E}_T \rangle \right). \end{aligned}$$

Hence, we arrive at the dual problem

$$v(y) = \inf_{Z \in \mathcal{Z}} \left( \inf_{R \in \mathcal{D}} \left( E \left[ Z \int_0^T V \left( t, y \frac{Y_t^R}{Z_t} \right) \mu(dt) \right] + y \langle R, \mathcal{E}_T \rangle \right) + \gamma(Z) \right). \quad (14)$$

### Remark 3.1

When considering the inner problem we could also define

$$\tilde{v}_Q(y) = \inf_{R \in \mathcal{D}_Q} E_Q \left[ \int_0^T V \left( t, y Y_t^{R,Q} \right) \mu(dt) \right] + y \langle R, \mathcal{E}_T \rangle$$

where  $\mathcal{D}_Q$  is the weak\*-closure of the set of supermartingale measures equivalent to  $Q$  in  $(L^\infty(P))^*$  and  $Y^{R,Q}$  is again defined as a version of the density process of the regular part of  $R$  but here with respect to  $Q$ . Obviously  $\mathcal{D}_Q$  can be empty if the market given by the measure  $Q$  admits arbitrage. In this case we set the infimum to infinity. With the help of the functions  $(\tilde{v}_Q)_{Q \in \mathcal{Q}}$  we can define an alternative dual function  $\tilde{v}$ :

$$\tilde{v}(y) = \inf_{Q \in \mathcal{Q}} (\tilde{v}_Q(y) + \gamma(Q)).$$

Schied defines his dual problem in [24] in this way. In contrast to our setting he is able to show that there is a one-to-one correspondence between the two kinds of dual sets [24, Lemma 4.2]. In our case no similar result holds, i.e. there is no way to decide whether  $R$  belongs to  $\mathcal{D}$  based on the knowledge of  $cD_Q$ . This is due to the random endowment. Nevertheless, we can still show that  $v(y) = \tilde{v}(y)$  for all  $y > 0$ . If we define the dual problem via the function  $\tilde{v}$  there are cases where the infima are not attained in contrast to assertion 5 of Theorem 2.5. This will be illustrated by the example below.

### Example 3.2

In this example we will have a unique solution  $(\hat{R}, \hat{Q})$  to the dual problem for which  $\hat{R} \notin \mathcal{D}_{\hat{Q}}$ . We consider all convex combinations of two measures  $Q^*$  and  $P$ , i.e.  $\mathcal{Q} = \{Q^\alpha = \alpha P + (1 - \alpha)Q^* | 0 \leq \alpha \leq 1\}$ . The penalty function is given by  $\gamma(Q^\alpha) = \alpha n$ , and  $\gamma(Q) = \infty$  for all  $Q \notin \mathcal{Q}$ . Under

$P$ ,  $S_0 = 10$  and  $S_T$  takes the values 5 and 15 each with probability 0.5. Under  $Q^*$ ,  $S_0 = 10$  and  $S_T = 5$   $Q^*$ -a.s. In order to achieve a continuous time model we set  $S_t = S_0$  for all  $t < T$ . Obviously  $Q^* \ll P$ . Assume that  $\mathcal{E}_T = mI_{\{S_T=5\}}$ ,  $\mu = \delta_T$  and  $U(t, x) = -e^{-x}$ . Hence,  $V(t, y)$  equals  $y(\ln y - 1)$  for  $y \leq 1$  and  $-1$  otherwise. Observe that  $\mathcal{D}$  contains all measures  $R$  that satisfy  $\beta := R[S_T = 5] \geq 0.5$  and  $\mathcal{D}_{Q^*}$  contains only  $Q^*$ . Then

$$\begin{aligned} v(y) &= \inf_{\alpha \in [0,1]} \inf_{R \in \mathcal{D}} \left( E_{Q^\alpha} \left[ \int_0^T V(t, y Y_t^R / Z_t^\alpha) \mu(dt) \right] + \gamma(Q) + \langle R, \mathcal{E}_T \rangle \right) \\ &= \inf_{\alpha \in [0,1]} \inf_{\beta \in [1/2,1]} \left( \left(1 - \frac{\alpha}{2}\right) V\left(T, y \frac{\beta}{1 - \alpha/2}\right) + \frac{\alpha}{2} V\left(T, y \frac{1 - \beta}{\alpha/2}\right) + \alpha n + \beta m \right). \end{aligned}$$

We want to compute  $v(1)$ , hence we get different situations for  $\beta < 1 - \alpha/2$  and  $\beta \geq 1 - \alpha/2$ . First we compute the infima for the case  $\beta < 1 - \alpha/2$ . Here we need to consider

$$g(\alpha, \beta) = \left(1 - \frac{\alpha}{2}\right) \frac{\beta}{1 - \alpha/2} \left(\ln \frac{\beta}{1 - \alpha/2} - 1\right) - \frac{\alpha}{2} + \alpha n + \beta m.$$

The partial derivatives of  $g$  are

$$g_\alpha(\alpha, \beta) = n - \frac{\alpha}{2} + \frac{\beta}{2(1 - \alpha/2)}$$

and

$$g_\beta(\alpha, \beta) = m + \ln\left(\frac{\beta}{1 - \alpha/2}\right).$$

If we choose  $m = 1$  and  $n = 2$  both derivatives are positive and therefore the infimum for this case is reached for  $\alpha = 0$  and  $\beta = 1/2$ , and takes the value  $1/2(\ln 1/2 - 1) + 1/2 < 0$ . In the case  $\beta \geq 1 - \alpha/2$  we need to compute the infimum of the function

$$h(\alpha, \beta) = -(1 - \alpha/2) + \frac{\alpha}{2} \frac{1 - \beta}{\alpha/2} \left(\ln \frac{1 - \beta}{\alpha/2} - 1\right) + \alpha n + \beta m.$$

We have that

$$\begin{aligned} h(\alpha, \beta) &= -1 + \alpha/2 + \frac{\alpha}{2} \frac{1 - \beta}{\alpha/2} \ln \frac{1 - \beta}{\alpha/2} - (1 - \beta) + 2\alpha + \beta \\ &\geq -2 + 2\beta + (2 + 1/2 - 1/2e^{-1})\alpha \\ &\geq -2 + 2(1 - \alpha/2) + (2 + 1/2 - 1/2e^{-1})\alpha > 0. \end{aligned}$$

For the first inequality we used the fact that  $x \ln x \geq -e^{-1}$ .

Hence, the solution to the dual problem is given by the first case, and equals  $(Q^0, P) = (Q^*, P)$ .

Obviously  $P \notin \mathcal{D}_{Q^*}$ .

In the rest of this section we will prove that in contrast to the example the solution to our dual problem exists, and that  $v$  equals  $\tilde{v}$ . First we observe that for  $Q_0 \in \mathcal{Q}_e$  of Assumption 2.4 [15, Theorem 3.10] guarantees that  $v_{Q_0}(y) < \infty$  for all  $y > 0$  and consequently also  $v(y) < \infty$  for all  $y > 0$ . We will repeatedly make use of a version of Komlós principle of convergence, compare e.g. [6, Lemma A1.1]. To control the behavior of the penalty function we will need the following lemma, which is taken from Schied [24, Lemma 4.1].

**Lemma 3.3**

For  $d \geq 0$  denote the subsets of  $\mathcal{Z}$  corresponding to

$$\mathcal{Q}(d) := \{Q \in \mathcal{Q} \mid \gamma(Q) \leq d\} \quad \text{and} \quad \mathcal{Q}_e(d) := \{Q \in \mathcal{Q}(d) \mid Q \sim P\}$$

by  $\mathcal{Z}(d)$  and  $\mathcal{Z}_e(d)$ . Then for every  $d > 0$ , the level set  $\mathcal{Z}(d)$  is weakly compact, and  $\mathcal{Z}_e(d)$  is nonempty. Moreover,  $Z \mapsto \gamma(Z)$  is lower semicontinuous with respect to  $P$ -a.s. convergence on  $\mathcal{Z}(d)$ .

We will need the following technical result.

**Lemma 3.4**

For each constant  $d > 0$  the set,

$$\left\{ ZV^-\left(\cdot, \frac{yY^R}{Z}\right) \mid Z \in \mathcal{Z}(d), R \in \mathcal{D} \right\},$$

is uniformly integrable with respect to  $P \otimes \mu$ .

**Proof.** Proposition 3.5 in [15] guarantees the existence of a utility function  $\underline{U}$  such that  $\underline{U}(x) \leq U(t, x)$  for all  $x > 0$  and all  $t \in [0, T]$ . Furthermore, the convex conjugate  $\underline{V}$  to  $\underline{U}$  satisfies  $\underline{V}(\cdot) \leq V(t, \cdot)$  for all  $0 \leq t \leq T$ . Since  $\mathcal{Z}(d)$  is uniformly integrable (according to Lemma 3.3 and the Dunford-Pettis theorem) the claim follows immediately if  $\underline{V}$  is bounded from below. Assume  $\underline{V}$  is not bounded from below. Let  $\varphi$  denote the inverse function of  $-\underline{V}$  and  $y_0 = \varphi(0)$ . Then it follows that

$$\begin{aligned} E \left[ \int_0^T Z_t \varphi \left( V^-\left(y \frac{Y_t^R}{Z_t}\right) \right) \mu(dt) \right] &\leq E \left[ \int_0^T Z_t \varphi \left( -\underline{V}\left(y \frac{Y_t^R}{Z_t}\right) \right) \mu(dt) \right] + y_0 \\ &\leq E \left[ \int_0^T y Y_t^R \mu(dt) \right] + y_0 \\ &\leq y + y_0 = M. \end{aligned}$$

In [16] it was proved that  $\varphi(h)/h \rightarrow \infty$  as  $h \rightarrow \infty$ . Hence for each  $a$  there exists  $d(a)$  such that  $\varphi(h) \geq ah$  for all  $h \geq d(a)$ . Let  $\varepsilon > 0$  and take  $d = d(2M/\varepsilon)$ ,  $\eta = \varepsilon/2d$ . If  $A \in \mathcal{F} \otimes \mathcal{B}([0, T])$  with  $E[\int_0^T Z_t I_A \mu(dt)] < \eta$  then we obtain the following inequality.

$$\begin{aligned} & E\left[\int_0^T Z_t V^-\left(t, \frac{y Y_t^{R^*}}{Z_t}\right) I_A \mu(dt)\right] \\ & \leq E\left[\int_0^T Z_t \underline{V}^-\left(\frac{y Y_t^{R^*}}{Z_t}\right) \left(I_{A \cap \{\underline{V}^-(y Y_t^{R^*}/Z_t) \geq d\}} + I_{A \cap \{\underline{V}^-(y Y_t^{R^*}/Z_t) < d\}}\right) \mu(dt)\right] \\ & \leq E\left[\int_0^T Z_t \varphi\left(\underline{V}^-\left(\frac{y Y_t^{R^*}}{Z_t}\right)\right) I_{A \cap \{\underline{V}^-(y Y_t^{R^*}/Z_t) \geq d\}} \mu(dt)\right] \frac{\varepsilon}{2M} + dE\left[\int_0^T Z_t I_A \mu(dt)\right] \\ & < \varepsilon. \end{aligned}$$

Because of the uniform integrability of  $\mathcal{Z}(d)$  there exists  $\delta > 0$  such that  $E[\int_0^T Z_t I_A \mu(dt)] < \eta$  as soon as  $(P \otimes \mu)[A] < \delta$ . This finishes the proof.  $\square$

The next lemma shows assertion 5 of the main theorem. In this and some later proofs we will need that the map  $(x, y) \mapsto xV(t, y/x)$  is convex for all  $0 \leq t \leq T$ , and that

$$\alpha x_0 V(t, y_0/x_0) + (1 - \alpha)x_1 V(t, y_1/x_1) > (\alpha x_0 + (1 - \alpha)x_1) V\left(t, \frac{\alpha y_0 + (1 - \alpha)y_1}{\alpha x_0 + (1 - \alpha)x_1}\right)$$

for  $y_1/x_1 \neq y_0/x_0$  and  $\alpha \in (0, 1)$ ; see e.g. [24, equation (25)].

### Lemma 3.5

There exist  $\widehat{Z} \in \mathcal{Z}$  and  $\widehat{R} \in \mathcal{D}$  such that

$$v(y) = E\left[\widehat{Z} \int_0^T V\left(t, y \frac{Y_t^{\widehat{R}}}{\widehat{Z}_t}\right) \mu(dt)\right] + y\langle \widehat{R}, \mathcal{E}_T \rangle + \gamma(\widehat{Z}).$$

**Proof.** Observe that Assumption 2.4 guarantees that  $v(y) < \infty$  (compare Karatzas/Zitkovic [15, Theorem 3.10]). Let  $(Z^n, R^n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{Z} \times \mathcal{D}$  such that

$$E\left[Z^n \int_0^T V\left(t, y \frac{Y_t^{R^n}}{Z_t^n}\right) \mu(dt)\right] + y\langle R^n, \mathcal{E}_T \rangle + \gamma(Z^n) \rightarrow v(y).$$

Let  $dQ^n = Z^n dP$ , then Jensen's inequality and Fubini's theorem imply that

$$\begin{aligned} E\left[Z^n \int_0^T V\left(t, y \frac{Y_t^{R^n}}{Z_t^n}\right) \mu(dt)\right] & \geq E\left[Z^n \int_0^T \underline{V}\left(y \frac{Y_t^{R^n}}{Z_t^n}\right) \mu(dt)\right] \\ & \geq \int_0^T \underline{V}\left(E_{Q^n}\left[\frac{y Y_t^{R^n}}{Z_t^n}\right]\right) \mu(dt) \\ & = \int_0^T \underline{V}\left(E\left[y Y_t^{R^n}; Z_t^n > 0\right]\right) \mu(dt) \\ & \geq \underline{V}(y). \end{aligned} \tag{15}$$



Since  $\mathcal{E}_T \geq 0$  it follows that  $d := 1 + \limsup_{n \rightarrow \infty} \gamma(Z^n) < \infty$  and we may assume that  $Z^n \in \mathcal{Z}(d)$  for all  $n$ .

Now we want to construct a sequence converging to the optimal  $(\widehat{Z}, \widehat{R})$ . For this we apply repeatedly a Komlós-type argument. Observe that in our case convex combinations of a converging sequence are still convergent. First we obtain a sequence  $(\widetilde{Z}^n, \widetilde{R}^n)_n$  of convex combinations of  $(Z^n, R^n)_n$ , i.e.  $(\widetilde{Z}^n, \widetilde{R}^n) \in \text{conv}\{(Z^n, R^n), (Z^{n+1}, R^{n+1}), \dots\}$ , where  $\widetilde{Z}^n$  converges  $P$ -a.s. to some  $\widehat{Z} \in \mathcal{Z}(d)$  (compare Lemma 3.3). To get also convergence for the processes  $(Z_t)_{0 \leq t \leq T}$  and  $(Y_t^{R^n})_{0 \leq t \leq T}$  we will argue analogously to the proof of Proposition A.2 in [15]. Due to [8, Lemma 5.2] we can choose a sequence of convex combinations  $((\widehat{Z}_t^n)_{0 \leq t \leq T}, (Y_t^{\widehat{R}^n})_{0 \leq t \leq T})$  of elements in  $((\widetilde{Z}_t^m)_{0 \leq t \leq T}, (Y_t^{\widetilde{R}^m})_{0 \leq t \leq T})_{m \geq n}$  that Fatou-converges to some RCLL supermartingale  $(\widehat{Z}_t)_{0 \leq t \leq T}$ . Due to Lemma 3.3 and the fact that  $\mu$  is diffuse on  $[0, T)$  we can find a subsequence that converges also  $P \otimes \mu$ -a.s. to  $(\widehat{Z}_t)_{0 \leq t \leq T}$ . We know that  $\widehat{Z} = \widehat{Z}_T \in \mathcal{Z}(d)$  hence  $(\widehat{Z}_t)_{0 \leq t \leq T}$  is a density process. To get similarly  $Y^{\widehat{R}}$  we first extract a subsequence of  $(\widehat{R}^n, \widehat{Z}^n)_n$  also denoted by  $(\widetilde{R}^n, \widetilde{Z}^n)_n$  such that  $\langle \widehat{R}^n, \mathcal{E}_T \rangle$  converges in  $\mathbb{R}$ . Then we consider the corresponding series  $(Y^{\widehat{R}^n})_{n \in \mathbb{N}}$ . We have a sequence  $(\widetilde{Y}^{\widehat{R}^n})_{n \in \mathbb{N}}$  of convex combinations of  $(Y^{\widehat{R}^n})$  converging  $P \otimes \mu$ -a.s. to some  $Y^{\widehat{R}}$  where  $\widehat{R}$  is a weak\* cluster point of  $\widehat{R}^n \in \mathcal{D}$ .

For  $Z \in \mathcal{Z}(d), R \in \mathcal{D}$  the function  $(Z, Y^R) \mapsto E[Z \int_0^T V(t, y Y_t^R / Z_t) \mu(dt)]$  is lower semicontinuous with respect to  $P$ -a.s. convergence. This can be proved the same way as Lemma 3.7 in [25] using Lemma 3.4. Combining this lower semicontinuity with Lemma 3.3 and the fact that  $(x, y) \mapsto xV(t, y/x)$  is a convex function, results in

$$\begin{aligned} & E \left[ \widehat{Z} \int_0^T V \left( t, y \frac{Y_t^{\widehat{R}}}{\widehat{Z}_t} \right) \mu(dt) \right] + y \langle \widehat{R}, \mathcal{E}_T \rangle + \gamma(\widehat{Z}) \\ & \leq \liminf_{n \rightarrow \infty} \left( E \left[ Z^n \int_0^T V \left( t, y \frac{Y_t^{R^n}}{Z_t^n} \right) \mu(dt) \right] + y \langle R^n, \mathcal{E}_T \rangle + \gamma(Z^n) \right) \\ & = v(y). \end{aligned}$$

This proves the optimality of  $(\widehat{Z}, \widehat{R})$ . □

We will now show that we may replace the set  $\mathcal{Q}$  in the dual problem by  $\mathcal{Q}_e$  as well as  $\mathcal{Q}_e^f$ , and that the dual functions are equal. (Where  $\mathcal{Q}_e^f$  denotes the set of measures  $Q \in \mathcal{Q}_e$  where  $u_Q(x)$  is finite for some  $x > 0$ .)

**Lemma 3.6**

The dual value function of the robust problem satisfies

$$\tilde{v}(y) = v(y) = \inf_{Q \in \mathcal{Q}_e^f} (v_Q(y) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e} (v_Q(y) + \gamma(Q)).$$

**Proof.** Let  $(Z^1, R^1) \in \mathcal{Z} \times \mathcal{D}$  be such that

$$v(y) = E \left[ Z^1 \int_0^T V(t, y Y_t^{R^1} / Z_t^1) \mu(dt) \right] + \gamma(Z^1) + \langle R^1, \mathcal{E}_T \rangle < \infty.$$

Due to Assumption 2.4 and the assumptions on  $U$  we can choose  $Z^0 \in \mathcal{Z}_e$  and  $R^0 \in \mathcal{D}$  such that also  $E[Z^0 \int_0^T V(t, y Y_t^{R^0} / Z_t^0) \mu(dt)] + \gamma(Z^0) < \infty$ . Let  $Z^\alpha := \alpha Z^1 + (1 - \alpha)Z^0 \in \mathcal{Z}_e$  and  $R^\alpha = \alpha R^1 + (1 - \alpha)R^0$  for  $0 \leq \alpha < 1$ . Since

$$\alpha \mapsto E \left[ Z^\alpha \int_0^T V(t, y Y_t^{R^\alpha} / Z_t^\alpha) \mu(dt) \right]$$

is convex and takes only finite values it is upper semicontinuous. Moreover, with the same argument we can conclude that  $\alpha \mapsto \gamma(Z^\alpha)$  is upper semicontinuous. This yields together with Lemma 3.3 that  $\alpha \mapsto \gamma(Z^\alpha)$  is continuous<sup>2</sup>. The functional  $\alpha \mapsto \langle R^\alpha, \mathcal{E}_T \rangle$  is linear and bounded and hence continuous. Consequently, the function

$$\begin{aligned} \alpha \mapsto & \inf_{R \in \mathcal{D}} \left( E \left[ Z^\alpha \int_0^T V \left( t, \frac{y Y_t^R}{Z_t^\alpha} \right) \mu(dt) \right] + y \langle R, \mathcal{E}_T \rangle \right) + \gamma(Z^\alpha) \\ & = v_{Z^\alpha}(y) + \gamma(Z^\alpha), \end{aligned}$$

is also upper semicontinuous on  $[0, 1]$ , therefore we get

$$\begin{aligned} v(y) &= \inf_{R \in \mathcal{D}} \left( E \left[ Z^1 \int_0^T V \left( t, \frac{y Y_t^{R^1}}{Z_t^1} \right) \mu(dt) \right] + y \langle R^1, \mathcal{E}_T \rangle \right) + \gamma(Z^1) \\ &\geq \limsup_{\alpha \uparrow 1} (v_{Z^\alpha}(y) + \gamma(Z^\alpha)). \end{aligned}$$

This yields  $v(y) = \inf_{Q \in \mathcal{Q}_e} (v_Q(y) + \gamma(Q))$ . Furthermore, observe that  $v_Q(y) = \infty$  for  $Q \in \mathcal{Q}_e \setminus \mathcal{Q}_e^f$ . This follows again from [15, Lemma A.3] for  $Q \sim P$  since then  $v_Q$  and  $u_Q$  satisfy the duality relations. We have  $v(y) \leq \tilde{v}(y)$  as  $\mathcal{D}_Q \subset \mathcal{D}$  for  $Q \ll P$ . Since  $Z^\alpha \in \mathcal{Z}_e$  for  $\alpha \in (0, 1)$  we also get

$$v(y) \geq \limsup_{\alpha \uparrow 1} (v_{Z^\alpha}(y) + \gamma(Z^\alpha)) \geq \inf_{Z \in \mathcal{Z}} (v_Z(y) + \gamma(Z)) = \tilde{v}(y).$$

This proves the first identity. □

<sup>2</sup>For a more detailed argument compare [24, Remark 4.5].

## 4 Proofs for the primal problem

In this section we will prove the missing assertions. First we make some easy observations. Due to (13) we know that for all  $\alpha \in [0, 1]$  we have

$$\alpha \mathcal{A}(x_1, \mu) + (1 - \alpha) \mathcal{A}(x_2, \mu) \subset \mathcal{A}(\alpha x_1 + (1 - \alpha)x_2, \mu).$$

Furthermore, if Assumption 2.4 is satisfied it is easy to show that, under the convention (5),  $c \mapsto \mathcal{U}_Q(c)$  is a concave functional on  $\mathcal{A}(x, \mu)$  for each  $Q \in \mathcal{Q}$  and all  $x > 0$ . These facts yield the concavity of the value functions  $u_Q$  and  $u$  and therefore under Assumption 2.4 the finiteness of the value function  $u(x)$  for all  $x > 0$ . The concavity of  $u_Q$  implies in turn that

$$u_Q \equiv +\infty \text{ as soon as } E_Q \left[ \int_0^T U^+(t, c_t) \mu(dt) \right] = +\infty \text{ for some } c \in \bigcup_{x>0} \mathcal{A}(x, \mu). \quad (16)$$

Indeed, if  $E_Q \left[ \int_0^T U^+(t, c_t) \mu(dt) \right] = +\infty$  for  $c \in \mathcal{A}(x, \mu)$  then it follows that  $c + 1 \in \mathcal{A}(x + 1, \mu)$  and  $E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] = +\infty$ . Thus,  $u_Q(x + 1) = +\infty$ . As  $u_Q$  is concave this implies  $u_Q \equiv +\infty$ .

### Lemma 4.1

We have the following minimax identity.

$$\begin{aligned} u(x) &= \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Q \in \mathcal{Q}} (\mathcal{U}_Q(c) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}} (u_Q(x) + \gamma(Q)) \\ &= \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Q \in \mathcal{Q}_e} (\mathcal{U}_Q(c) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e} (u_Q(x) + \gamma(Q)). \end{aligned}$$

Hence, assertion 2 of Theorem 2.5 is valid.

**Proof.** Let  $\varepsilon \in (0, 1)$ . Proposition 3.5 in [15] gives the existence of a utility function  $\underline{U}$  such that  $\underline{U}(x) \leq U(t, x)$  for all  $x > 0$  and all  $t \in [0, T]$ . With the help of this utility function we define  $d = 1 + u(x + 1) - \underline{U}(\varepsilon) \wedge 0$ . Then we have

$$u(x + 1) \geq u(x + \varepsilon) \geq \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Q \in \mathcal{Q}} (\mathcal{U}_Q(c + \varepsilon) + \gamma(Q)) = \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Z \in \mathcal{Z}(d)} (\mathcal{U}_Q(c + \varepsilon) + \gamma(Z)).$$

Now  $U(t, \cdot + \varepsilon)$  is bounded from below and thus

$$Z \mapsto E \left[ Z \int_0^T U(t, c_t + \varepsilon) \mu(dt) \right] = \sup_n E \left[ Z \left( \int_0^T U(t, c_t + \varepsilon) \mu(dt) \wedge n \right) \right]$$

is a weakly lower semicontinuous affine functional. Lemma 3.3 states that  $Z \mapsto \gamma(Z)$  is weakly lower semi-continuous and  $\mathcal{Z}(d)$  is a weakly compact and convex set. Furthermore, for each  $Z \in \mathcal{Z}(d)$

$c \mapsto \mathcal{U}_Z(c)$  is a concave functional defined on the convex set  $\mathcal{A}(x, \mu)$ . Therefore we may use the lopsided minimax theorem [1] to obtain

$$\sup_{c \in \mathcal{A}(x, \mu)} \min_{Z \in \mathcal{Z}(d)} (\mathcal{U}_Z(c + \varepsilon) + \gamma(Z)) = \min_{Z \in \mathcal{Z}(d)} \sup_{c \in \mathcal{A}(x, \mu)} (\mathcal{U}_Z(c + \varepsilon) + \gamma(Z)).$$

We know that these expressions are bounded by  $u(x + \varepsilon) < d + \underline{U}(\varepsilon) \wedge 0$ . Thus, it does not matter whether we take the infimum over  $\mathcal{Z}$  or over  $\mathcal{Z}(d)$ . We obtain

$$u(x + \varepsilon) \geq \inf_{Z \in \mathcal{Z}} \sup_{c \in \mathcal{A}(x, \mu)} (\mathcal{U}_Z(c + \varepsilon) + \gamma(Z)) \quad (17)$$

$$\geq \inf_{Z \in \mathcal{Z}} \sup_{c \in \mathcal{A}(x, \mu)} (\mathcal{U}_Z(c) + \gamma(Z)) \quad (18)$$

$$\geq \sup_{c \in \mathcal{A}(x, \mu)} \inf_{Z \in \mathcal{Z}} (\mathcal{U}_Z(c) + \gamma(Z)) = u(x). \quad (19)$$

As  $\varepsilon \rightarrow 0$  the assertion follows since  $u$  is continuous.  $\square$

Observe now that Theorem 3.10 in [15] states that  $u_Q(x) = \inf_{y>0} (v_Q(y) + xy)$  for all  $Q \in \mathcal{Q}_e^f$ . Pooling our lemmas and using this result, we get

$$\begin{aligned} u(x) &= \inf_{Q \in \mathcal{Q}_e} (u_Q(x) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e^f} (u_Q(x) + \gamma(Q)) \\ &= \inf_{Q \in \mathcal{Q}_e^f} \inf_{y>0} (v_Q(y) + \gamma(Q) + xy) = \inf_{y>0} (v(y) + xy) \end{aligned}$$

which is assertion 3 of the theorem. Finiteness of  $v$  and general duality principles yield then

$$\lim_{x \rightarrow \infty} u(x)/x \rightarrow 0. \quad (20)$$

#### Lemma 4.2

For any  $x > 0$  there exist  $\hat{c} \in \mathcal{A}(x, \mu)$  such that

$$\inf_{Q \in \mathcal{Q}} \left( E_Q \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right] + \gamma(Q) \right) = u(x).$$

**Proof.** Let  $(c^n)$  be a maximizing sequence. Using again a Komlós-type argument we get a sequence  $(\tilde{c}^n)_{n \in \mathbb{N}}$  with  $\tilde{c}^n \in \text{conv}(c^n, c^{n+1}, \dots)$  converging  $P \otimes \mu$ -a.s. to some  $\hat{c} \in \mathcal{A}(x, \mu)$  since  $\mathcal{A}(x, \mu)$  is closed under convergence in probability. The following adaption of the argument in [17] shows that the positive parts of  $U(t, \tilde{c}_t^n)$  are uniformly integrable with respect to  $Q \otimes \mu$  for all  $Q \in \mathcal{Q}_e^f$ .

Assume  $(U^+(t, \tilde{c}_t^n))_n$  is not uniformly integrable. Then there is a constant  $\alpha$ , a subsequence which is also denoted  $(\tilde{c}_n)_n$  and a partition  $(A_n)$  such that  $E[\int_0^T U^+(t, \tilde{c}_t^n) I_{A_n} \mu(dt)] \geq \alpha$  for

$n \geq 1$ . For  $0 \leq t \leq T$  let  $x_t^0 = \inf\{x > 0 \mid U(t, x) \geq 0\}$  and define a process  $s^n = (s_t^n)_{0 \leq t \leq T}$  by  $s_t^n = x_t^0 + \sum_{k=1}^n \tilde{c}_t^k$ . Hence, for any  $R \in \mathcal{D}$  we have

$$E \left[ \int_0^T Y_t^R s_t^n \mu(dt) \right] \leq \int_0^T x_t^0 \mu(dt) + n(x + \langle R, \mathcal{E}_T \rangle).$$

This implies  $s^n \in \mathcal{A}(\int_0^T x_t^0 \mu(dt) + nx, \mu)$ . Furthermore

$$E \left[ \int_0^T U(t, s_t^n) \mu(dt) \right] \geq \sum_{k=1}^n E \left[ \int_0^T U^+(t, \tilde{c}_t^k) I_{A_k} \mu(dt) \right] \geq \alpha n.$$

Which yields

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{u(x)}{x} &\geq \limsup_{n \rightarrow \infty} \frac{E \left[ \int_0^T U(t, s_t^n) \mu(dt) \right]}{\int_0^T x_t^0 \mu(dt) + nx} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\alpha n}{\int_0^T x_t^0 \mu(dt) + nx} \\ &= \alpha > 0 \end{aligned}$$

in contradiction to (20).

Using the uniform integrability we can deduce that

$$c \mapsto E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q)$$

is upper semicontinuous with respect to almost sure convergence. From the concavity of

$$c \mapsto \inf_{Q \in \mathcal{Q}_e^f} (E_Q \left[ \int_0^T U(t, c_t) \mu(dt) \right] + \gamma(Q))$$

it follows that  $(\tilde{c}^n)$  is still a maximizing sequence. Then the upper semicontinuity yields that

$$\inf_{Q \in \mathcal{Q}_e} \left( E_Q \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right] + \gamma(Q) \right) \geq u(x).$$

Actually we also have

$$\inf_{Q \in \mathcal{Q}} \left( E_Q \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right] + \gamma(Q) \right) \geq u(x) :$$

First we have that  $\{Q \in \mathcal{Q} \mid E_Q \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right] = -\infty\} = \emptyset$  since otherwise this set would have a nonempty intersection with  $\mathcal{Q}_e$ . Hence, for  $Q \in \mathcal{Q} \setminus \mathcal{Q}_e$ ,  $Q_0 \in \mathcal{Q}_e^f$  and  $Q_\alpha := \alpha Q + (1 - \alpha)Q_0 \in \mathcal{Q}_e$ , the limit of  $E_{Q_\alpha} \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right]$  as  $\alpha \uparrow 1$  is  $E_Q \left[ \int_0^T U(t, \hat{c}_t) \mu(dt) \right]$ . Due to the convexity and lower semicontinuity of  $\gamma$ , we also have  $\gamma(Q_\alpha) \rightarrow \gamma(Q)$ .  $\square$

**Lemma 4.3**

The dual value function  $v$  is lower semicontinuous.

**Proof.** Define  $v(0) = \int_0^T \lim_{\varepsilon \rightarrow 0} V(t, \varepsilon) \mu(dt)$  and let  $(y_n)_{n \in \mathbb{N}}$  be a sequence of positive reals converging to  $y \geq 0$ . If  $\liminf_{n \rightarrow \infty} v(y_n) = \infty$  we are done. Otherwise we may assume that  $\sup_n v(y_n) < \infty$ . We proved that we have  $\hat{R}^n \in \mathcal{D}$  and  $\hat{Z}^n \in \mathcal{Z}$  such that

$$v(y_n) = \mathcal{V}_{\hat{Z}^n, \mu} \left( \frac{y_n Y^{\hat{R}^n}}{\hat{Z}^n} \right) + y_n \langle \hat{R}^n, \mathcal{E}_T \rangle + \gamma(\hat{Z}^n).$$

And with an estimate similar to (15) and  $\mathcal{E}_T \geq 0$  we know that  $v(y_n) \geq \int_0^T V(t, y_n) \mu(dt) + \gamma(\hat{Z}^n)$ . Since  $\gamma$  is bounded from below we get that  $V(\cdot, y_n) < \infty \hat{Q}^n \otimes \mu$ -a.s., where  $d\hat{Q}^n = \hat{Z}^n dP$ . Furthermore we can find  $d$  such that  $\hat{Z}_n \in \mathcal{Z}(d)$ . This enables us to use once more a Komlós-type argument to get a sequence of convex combinations,  $((\tilde{R}^n, \tilde{Z}^n))_n$  where  $(\tilde{R}^n, \tilde{Z}^n) \in \text{conv}\{(R^n, Z^n), (R^{n+1}, Z^{n+1}), \dots\}$ , which converges  $P$ -a.s. to some  $(\hat{R}, \hat{Z})$ . Now we use Lemma 3.3 and the convexity of  $(Z, R) \mapsto Z \int_0^T V(t, y Y_t^R / Z_t) \mu(dt)$  to obtain

$$v(y) \leq \mathcal{V}_{\hat{Z}, \mu} \left( \frac{y Y^{\hat{R}}}{\hat{Z}} \right) + y \langle \hat{R}, \mathcal{E}_T \rangle + \gamma(\hat{Z}) \leq \liminf_{n \rightarrow \infty} v(y_n),$$

which is the desired lower semicontinuity. □

**Proof of Theorem 2.5.** Assertion 2 and 3, the finiteness of  $u$  and  $v$ , and the concavity of  $u$  were already proved. The convexity of  $v$  is an immediate consequence of the convexity of  $(Z, y Y^R) \mapsto \mathcal{V}_Z(Y^R/Z) + \gamma(Z)$ . Furthermore, the lower semicontinuity of  $v$  (see Lemma 4.3) implies assertion 3 of Theorem 2.5. Equation (10) follows as in [15, Lemma A.7]. If  $\mathcal{E}_T \equiv 0$  this implies  $v'(\infty-) = 0$  which yields by general duality results that  $u'(0) = \infty$  which is equation (11). Assertion 5 corresponds to Lemma 3.5. Furthermore, the existence of an optimal  $c$  (item 6) is the content of Lemma 4.2.

We will now deal with the existence of a saddle point. Let  $y > 0$  be such that  $v(y) + yx = u(x)$ , such a  $y$  exists due to the behavior of  $v'$ . Take then a solution  $(\hat{R}, \hat{Q})$  to the dual problem for  $y$  and let  $\hat{Z}$  be the density of  $\hat{Q}$  with respect to  $P$ . Let  $Z^1$  be in  $\mathcal{Z}_e^f$  and define  $Z^\alpha = \alpha Z^1 + (1 - \alpha)\hat{Z}$  for  $\alpha \in (0, 1]$ . Then  $v_{Z^\alpha}(y) + \gamma(Z^\alpha) \rightarrow v(y)$  as  $\alpha \rightarrow 0$ : Let  $R^1$  be such that

$$\mathcal{V}_{Z^1}(y Y^{R^1} / \hat{Z}^1) + \langle R^1, \mathcal{E}_T \rangle = v_{Z^1}(y).$$

Then define  $R^\alpha = \alpha R^1 + (1 - \alpha)\widehat{R}$ . We get

$$\begin{aligned} v(y) &\leq v_{Z^\alpha}(y) + \gamma(Z^\alpha) \\ &\leq \mathcal{V}_{Z^\alpha}(yY^{R^\alpha}/Z^\alpha) + \langle R^\alpha, \mathcal{E}_T \rangle + \gamma(Z^\alpha) \\ &\leq \alpha(v_{Z^1}(y) + \gamma(Z^1)) + (1 - \alpha)(\tilde{v}_{\widehat{Z}}(y) + \gamma(\widehat{Z})) \end{aligned}$$

because of convexity. Observe that the righthand side of the equation goes to  $v(y)$  as  $\alpha$  goes to 0.

Due to the duality relations we have that  $v_{Z^\alpha} + xy \geq u_{Z^\alpha}$  and  $u_{Z^\alpha} + \gamma(Z^\alpha) \rightarrow u_{\widehat{Z}} + \gamma(\widehat{Z})$ . Hence,

$$\begin{aligned} u(x) &= v(y) + xy = \lim_{\alpha \rightarrow 0} (v_{Z^\alpha}(y) + xy + \gamma(Z^\alpha)) \\ &\geq \lim_{\alpha \rightarrow 0} (u_{Z^\alpha}(x) + \gamma(Z^\alpha)) = u_{\widehat{Z}}(x) + \gamma(\widehat{Z}) \end{aligned}$$

With the minimax identity we get  $u(x) = u_{\widehat{Z}} + \gamma(\widehat{Z})$  and therefore we get for  $\hat{c}$  as in Lemma 4.2

$$u(x) = u_{\widehat{Z}}(x) + \gamma(\widehat{Z}) \geq \mathcal{U}_{\widehat{Z}, \mu}(\hat{c}) + \gamma(\widehat{Z}) \geq \inf_{Q \in \mathcal{Q}} \mathcal{U}_Q(\hat{c}) + \gamma(\widehat{Z}) = u(x).$$

Now we show that  $\hat{c} = I(\cdot, yY^{\widehat{R}}) \widehat{Q} \otimes \mu$ -a.s. We have

$$0 \leq V\left(t, \frac{\widehat{y} Y_t^{\widehat{R}}}{\widehat{Z}_t}\right) + \frac{\widehat{y} Y_t^{\widehat{R}}}{\widehat{Z}_t} \hat{c}_t - U(t, \hat{c}_t)$$

and therefore

$$\begin{aligned} 0 &\leq E_{\widehat{Q}} \left[ \int_0^T V\left(t, \frac{y Y_t^{\widehat{R}}}{\widehat{Z}_t}\right) \mu(dt) + \int_0^T \frac{y Y_t^{\widehat{R}}}{\widehat{Z}_t} \hat{c}_t \mu(dt) - \int_0^T U(t, \hat{c}_t) \mu(dt) \right] \\ &= v(y) + E \left[ \int_0^T y Y_t^{\widehat{R}} \hat{c}_t \mu(dt) \right] - y \langle \widehat{R}, \mathcal{E}_T \rangle - u(x) \\ &\leq v(y) + xy - u(x) = 0. \end{aligned}$$

Together this implies

$$0 = V\left(t, \frac{y Y_t^{\widehat{R}}}{\widehat{Z}_t}\right) + \frac{y Y_t^{\widehat{R}}}{\widehat{Z}_t} \hat{c}_t - U(t, \hat{c}_t) \quad \widehat{Q} \otimes \mu(dt)\text{-a.s.}$$

which means that  $\hat{c}_t = I(t, yY_t^{\widehat{R}}/\widehat{Z}_t)$ .

The fact that  $u'(\infty-) = 0$  follows from (20). Therefore we also have  $v'(0+) = -\infty$ . To obtain the strict concavity of the function  $u$  (and hence the differentiability of  $v$ ) assume that  $u$  is not strictly concave. Since  $u$  is increasing and because of the conditions for the derivatives we know there exist  $0 < x_0 < x_1$  and  $y > 0$  such that  $v(y) + yx_0 = u(x_0)$  and  $v(y) + yx_1 = u(x_1)$ . Let  $c^0, c^1$  be the corresponding optimal consumption processes and  $(\widehat{Q}, \widehat{R})$  the solution to the dual problem. Then we have  $c_t^0 = I(t, yY^{\widehat{R}}) = c_t^1 \widehat{Q}$ -a.s. and  $E_{\widehat{Q}}[\int_0^T c_t^0 y Y_t^{\widehat{R}} \mu(dt)] + y \langle \widehat{R}, \mathcal{E}_T \rangle = x_0 y < x_1 y = E_{\widehat{Q}}[\int_0^T c_t^1 y Y_t^{\widehat{R}} \mu(dt)] + y \langle \widehat{R}, \mathcal{E}_T \rangle$ . This is a contradiction.  $\square$

**Proof of Corollary 2.6.** This corollary can be proved by copying the arguments of the proof of Proposition 2.5 in [24]. □

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