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Consumption-Based CAPM and Option Pricing under Jump-Diffusion Uncertainty

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Discussion Paper Series

**Consumption-Based CAPM and Option Pricing
under Jump-Diffusion Uncertainty**

by

Koji Kusuda

Discussion Paper No. 317, April 2003

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Consumption-Based CAPM and Option Pricing under Jump-Diffusion Uncertainty

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Abstract. In Kusuda [45], we developed equilibrium analysis in security market economy with jump-Wiener information where no finite number of securities can complete markets. Assuming *approximately complete markets* (Björk *et al.* [11] [12]) in which a continuum of bonds are traded and any contingent claim can be replicated with an arbitrary precision, we have shown sufficient conditions for the existence of *approximate security market equilibrium*, in which every agent is allowed to choose any consumption plan that can be supported with any prescribed precision. In this paper, we derive the Consumption-Based Capital Asset Pricing Model (CCAPM) using the framework in case of heterogeneous with additively separable utilities (ASUs) and of homogeneous agents with a common stochastic differential utility (SDU). The CCAPM says that the risk premium between a risky security and the nominal-risk-free security can be decomposed into two groups of terms. One is related to the price fluctuation of the risky security, and the other is related to that of commodity. Each group can be further decomposed into two terms related to consumption volatility and consumption jump in case of ASUs, and into three terms related to consumption volatility, continuation utility volatility, and jumps of consumption and continuation utility in case of SDU. Next, we present a general equilibrium framework of jump-diffusion option pricing models in each case of heterogeneous agents with CRRA utilities and of homogeneous agents with a common Kreps-Porteus utility. Finally, we construct a general equilibrium version of an affine jump-diffusion model with jump-diffusion volatility for option pricing using the framework.

JEL No. C61, C63, D51, D58, G11, G13.

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1. INTRODUCTION

The strong evidence that most financial processes (equity prices, interest rates, exchange rates *etc.*) are better described by a combination of diffusion and jump processes (Akgiray and Booth [2], Andersen, Benzoni, and Lund [3], Bakshi, Cao, and Chen [7], Bates [9] [10], Jorion [42] *etc.*) has lead researchers to study jump-diffusion security market models (Back [6], Bakshi, Cao, and Chen [7], Bates [8] [9], Björk, Kabanov, and Runggaldier [11], Duffie, Pan, and Singleton [22], Naik and Lee [52] *etc.*), in particular, in the context of option pricing. In most jump-diffusion option pricing model, jump is specified by a continuously distributed random variable at each jump time. Then, the “number of sources of uncertainty” is infinite, and no finite set of securities can complete markets. However, in incomplete market economy with heterogeneous agents, it is difficult not only to show the existence of general equilibria, but also to construct analytically tractable market price of jump risk. Since market price of jump risk is a component of risk premia in Consumption-Based Capital Asset Pricing Model (CCAPM), analytically intractable market price of jump risk implies analytically and empirically intractable CCAPM.¹ Analytically intractable market price of jump risk also makes it difficult to construct a jump-diffusion option pricing model for which analytic or quasianalytic equilibrium price formulas for European options are available.² Some researchers assume homogeneous agents with a common additively separable utility (ASU) instead of heterogeneous agents (Attari [5], Bates [8], Das and Foresi [18], Heston [40], Naik and Lee [52] *etc.*).³ Needless to say, the assumption of homogeneous agents is restrictive. In addition, even in incomplete market economy with homogeneous agents with a common ASU, it is still difficult to show the existence of general equilibria, and the CCAPM with ASU agents have been often rejected (Epstein and Zin [27], Finn, Hoffman, and Schlagenhauf [29], Hansen and Singleton [36] *etc.*), and leads us the *equity premium puzzle* of Mehra and Prescott [50] and the *risk-free rate puzzle* of Weil [62]. One possible explanation of these puzzles is a possible deficiency of an ASU, that is, both risk aversion and intertemporal substitution depend on the curvature of von-Neumann Morgenstern utility function. We can disentangle these two characteristics in the class of *stochastic differential utilities* (SDUs) which is a continuous-time version of the class of Epstein-Zin utilities (Epstein and Zin [26]) and a generalization of the class of standard ASUs, and then expect that SDUs may contribute to solve the equity premium puzzle and/or the risk-free rate puzzle. The purpose of this paper is to derive an empirically tractable CCAPM and

¹We can derive a version of CCAPM even in such incomplete markets as in Madan [48] and Back [6], but this incomplete market version of CCAPM is empirically intractable in the sense that the market price of jump risk in the CCAPM depends on every agent’s consumption plan. If the jump risk premium were sufficiently small, then we could disregard market price of jump risk. However, an empirical analysis in Pan [54] suggests that the jump risk premium is not as small as to be ignored.

²The reason is as follows. In deriving the arbitrage-free price of a European option, we compute the expectation of the option’s payoff under an appropriate equivalent martingale measure. However, the distribution of option’s payoff under the equivalent martingale measure depends on the market price of jump risk, so if the market price of jump risk were analytically intractable, then the computation for the arbitrage-free price of option under would be difficult, too.

³Some of them assume representative agent instead of homogeneous agents. However, this assumption is inappropriate since representative agent can be constructed only if the associated equilibrium consumption allocation is Pareto optimal, which is generally unattainable in incomplete market economy.

give a general equilibrium framework of option pricing models under jump-Wiener uncertainty in each case of ASUs and of SDUs.

Björk, Kabanov, and Runggaldier [11] introduce *approximately complete security markets* with an infinite dimensional martingale generator consisting of a jump process given by the *marked point process* (see Appendix A.1) and a Wiener process. In approximately complete markets, a continuum of bonds are traded and any contingent claim can be approximately replicated with an arbitrary precision. In our previous paper (Kusuda [45]), we introduced the notion of *approximate security market equilibrium* in which every agent is allowed to choose any consumption plan that can be approximately supported by admissible portfolio with any prescribed precision, and present sufficient conditions for the existence of approximate security market equilibria in approximately complete markets in each case of additively separable utilities (ASUs) and of stochastic differential utilities (SDUs) with Inada conditions. In this paper, using the framework, we derive an empirically tractable CCAPM and give a general equilibrium framework of option pricing models under jump-diffusion uncertainty in each case of heterogeneous agents with ASUs and of homogeneous agents with a common SDU. In subsequent two papers (Kusuda [46] [47]), we have proposed jump-diffusion LIBOR rate models using the framework of this paper.

First, we derive the CCAPM in each case of heterogeneous agents with ASUs and of homogeneous agents with a common SDU exploiting the result that the product of the state price and the gain of any security is a martingale in equilibrium. The CCAPM says that the risk premium between any risky security and the nominal-risk-free security can be decomposed into two groups of terms. One is related to the price fluctuation of the security, and the other is related to that of commodity. Each group can be further decomposed into two terms related to consumption volatility and consumption jump in the case of ASUs, and into three terms related to consumption volatility, *continuation utility*⁴ volatility, and jumps of consumption and continuation utility in the case of SDU. We also present empirically tractable versions of these CCAPMs assuming heterogeneous agents with CRRA utilities and homogeneous agents with a common Kreps-Porteus utility (Kreps and Porteus [44]) which is a generalization of CRRA utility. We think that the CCAPM with a common Kreps-Porteus utility may contribute to solve the equity premium puzzle and risk-free rate puzzle since compared to the existing CCAPM with CRRA utilities under Wiener information, our CCAPM with a common Kreps-Porteus utility under jump-Wiener information contains two additional risks, and the resultant equity premium includes the two associated risks. One is the risk related to jump information, which is a generalization of possible very rare market crash risk pointed out in Rietz [56] since all magnitudes of upward and downward jumps are considered in our jump risk while only a very big magnitude of downward jump is considered in the market crash risk of Rietz [56].⁵ The other is the risk related to fluctuation of continuation utility. Note that this risk is zero in the case of CCAPM with standard ASUs since the associated equilibrium state price does not depend on the continuation utility.

⁴For definition of continuation utility, see a footnote in Assumption 3.

⁵For arguments of Rietz [56], see Remark 7.

Then, we show a general equilibrium framework of jump-diffusion option pricing models in each case of heterogeneous agents with CRRA utilities and of homogeneous agents with a common Kreps-Porteus utility. In concrete, we show analytically tractable formulas for equilibrium market prices of diffusive risk and of jump risk, for the Wiener process under the risk-neutral measure, and for the jump intensity under the risk-neutral measure, and derive the dynamics of equilibrium gain, forward rate, and nominal-risk-free rate processes. These formulas enable us to construct jump-diffusion option pricing models for which both of analytic (quasianalytic) equilibrium price formulas for European options and an efficient estimation method are available. Then, we construct a general equilibrium version of jump-diffusion stock index model with jump-diffusion volatility, which was proposed in Duffie, Pan, and Singleton [22] as a promising example in the class of affine jump-diffusion (ADJ) models developed by Heston [39], Bates [9], and Duffie, Pan, and Singleton [22]. For every ADJ model, both of quasi analytic European option price formulas and efficient estimation methods are available.

Our CCAPM with ASUs is a generalization of the CCAPM with heterogeneous agents under Wiener and non-Markovian information (Duffie and Zame [25]) and of the CCAPM with homogeneous agents under jump-Wiener and Markovian information (Ahn and Thompson [1]). Also, our CCAPM with homogeneous agents' common SDU is a generalization of the CCAPM with homogeneous agents' common SDU under Wiener and Markovian information (Duffie and Epstein [20]) and of the CCAPM with homogeneous agents' common Kreps-Porteus utility under Wiener and non-Markovian information (Fisher and Gilles [30]).

For option pricing, some researchers exploit *approximate arbitrage-free pricing approach* in incomplete markets (Duffie and Richardson [23], Föllmer and Schweizer [31], Schweizer [58] [59], *etc.*). In this approach, an appropriate approximate replicating criterion is chosen, and then an contingent claim outside the asset span is priced as the value of an approximate replicating portfolio. However, the market price of risk remains unknown in this approach.

This paper is organized as follows. In Sections 2 and 3, we review the framework of our previous paper (Kusuda [45]). In Section 4, we show some preliminary results for deriving CCAPM. In Sections 5 and 6, we derive the CCAPM in each case of heterogeneous agents with ASUs and of homogeneous agents with a common SDU. In Section 7, we present a general equilibrium framework of jump-diffusion option pricing models and construct a general equilibrium version of the jump-diffusion stock index model with jump-diffusion volatility.

2. APPROXIMATELY COMPLETE SECURITY MARKET ECONOMY

In this section, we introduce a specification of security market economy with jump-Wiener uncertainty provided in our previous paper (Kusuda [45]), and briefly review approximately complete markets given in Björk *et al.* [11] [12].

2.1. Security Market Economy with Jump-Wiener Uncertainty. We consider a continuous-time frictionless pure exchange security market economy with time span $\mathbf{T} \stackrel{\text{def}}{=} [0, T^+]$ for a fixed horizon time $T^+ > 0$. Agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbf{T}}$ is the natural filtration generated by a d -dimensional Wiener process W and a jump process called *marked point process* $\nu(dt \times dz)$ on a Lusin space $(\mathcal{Z}, \mathcal{Z})$ (in usual applications, $\mathcal{Z} = \mathbb{R}^d$, or \mathbb{N}^d , or a

finite set) with the \mathbb{P} -intensity kernel $\lambda_t(dz)$ (for marked point process, see Appendix A.1). Note that Martingale Representation Theorem (see Chapter III Corollary 4.31 in Jacod and Shiryaev [41]) shows that the martingale generator in this economy is $(W, (\nu(dt \times \{z\}) - \lambda_t(\{z\}))_{z \in \mathbb{Z}})$. Thus, if the mark set \mathbb{Z} is infinite, then the martingale generator is infinite dimensional.

There is a single perishable consumption commodity. The commodity space is a Banach space $\mathbf{L}^2 \stackrel{\text{def}}{=} \mathbf{L}^2(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where \mathcal{P} is the predictable σ -algebra on $\Omega \times \mathbf{T}$, μ is the product measure of \mathbb{P} and Lebesgue measure on \mathbf{T} . There are I agents, and each of them is represented by (U^i, \bar{c}^i) , where U^i is a strictly increasing and continuous utility on the positive cone \mathbf{L}_+^2 of consumption process and $\bar{c}^i \in \mathbf{L}_+^2$ is an endowment process which is assumed to be nonzero, for $i \in \mathbf{I} \stackrel{\text{def}}{=} \{1, 2, \dots, I\}$. The economy is described by a collection

$$\mathbf{E} \stackrel{\text{def}}{=} ((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), (U^i, \bar{c}^i)_{i \in \mathbf{I}}).$$

There are markets for the consumption commodity and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and a continuum of zero-coupon bonds whose maturity dates are $(0, T^\dagger]$, each of which has \$1 payoff (NOT one unit payoff of the commodity) at its maturity date. Let p , B , and $(B^T)_{T \in (0, T^\dagger]}$ denote the consumption commodity price process, nominal money market account price process and nominal bond price processes, respectively. We write $\mathbf{B} = (B, (B^T)_{T \in (0, T^\dagger]})$ and call it *bond price family*.

2.2. Approximately Complete Markets. We allow each agent to hold a portfolio of the money market account and continuum of bonds, so we set the portfolio component of continuum of bonds a signed finite Borel measure on $[t, T^\dagger]$ for every event $\omega \in \Omega$ and time $t \in \mathbf{T}$.

Definition 1. A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies:

1. The component ϑ^0 is a real-valued \mathcal{P} -measurable process.
2. The component ϑ^1 is such that:
 - (a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta_t^1(\omega, \cdot)$ is a signed finite Borel measure on $[t, T^\dagger]$.
 - (b) For every Borel set A , the process $\vartheta^1(A)$ is \mathcal{P} -measurable.

Let $n \in \mathbb{N}$. Let \mathcal{L}^n denote the set of real-valued \mathcal{P} -measurable process X satisfying the integrability condition $\int_0^{T^\dagger} |X_s|^n ds < \infty$ \mathbb{P} -almost surely. Also let $\mathcal{L}^n(\lambda_t(dz) \times dt)$ denote the set of real-valued $\mathcal{P} \otimes \mathcal{Z}$ -measurable process H satisfying the integrability condition $\int_0^{T^\dagger} \int_{\mathbb{Z}} |H_s(z)|^n \lambda_s(dz) ds < \infty$ \mathbb{P} -a.s. We introduce the notion of *viable bond price family*.

Definition 2. A bond price family \mathbf{B} is *viable* if and only if the following conditions hold:

1. (a) For every $T \in (0, T^\dagger]$, the dynamics of nominal bond price process B^T satisfies the following stochastic differential-difference equation (SDDE)

$$\frac{dB_t^T}{B_{t-}^T} = r_t^T dt + v_t^T \cdot dW_t + \int_{\mathbb{Z}} H_t^T(z) \{ \nu(dt \times dz) - \lambda_t(dz) dt \} \quad \forall t \in [0, T)$$

with $B_T^T = 1$ and $B_t^T = 0$ for every $t \in (T, T^\dagger]$ for some $r^T \in \mathcal{L}^1$, $v^T \in \prod_{j=1}^d \mathcal{L}^2$, and $H^T \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Moreover, it follows that:

- (i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, $r_t^i(\omega), v_t^i(\omega) \in \mathbf{C}^1((t, T^\dagger])$ and for every $(\omega, t, z) \in \Omega \times \mathbf{T} \times \mathbb{Z}$, $H_t^i(\omega, z) \in \mathbf{C}^1((t, T^\dagger])$.
- (ii) For every $T \in (0, T^\dagger]$, $H_t^T(\omega, z)$ is bounded.
- (iii) The processes $(B^T)_{T \in \mathbf{T}}$ are regular enough to allow for the differentiation under the integral sign and the interchange of integration order.⁶

- (b) The dynamics of nominal money market account price process B satisfies the following SDDE

$$\frac{dB_t}{B_t} = r_t^B dt \quad \forall t \in \mathbf{T}$$

with $B_0 = 1$ where $r_t^B = -\frac{\partial \ln B_t^T}{\partial T} \Big|_{T=t}$.

- 2. There exists a martingale process $A^{\mathbf{B}}$ such that

$$(2.1) \quad \frac{dA_t^{\mathbf{B}}}{A_{t-}^{\mathbf{B}}} = -v_t^{\mathbf{B}} \cdot dW_t - \int_{\mathbb{Z}} H_t^{\mathbf{B}}(z) \{ \nu(dt \times dz) - \lambda_t(dz) dt \} \quad \forall t \in \mathbf{T}$$

with $A_0^{\mathbf{B}} = 1$ where $(v^{\mathbf{B}}, H^{\mathbf{B}}) \in (\prod_{j=1}^d \mathcal{L}^2) \times \mathcal{L}^1(\lambda_t(dz) \times dt)$ satisfies the following equation

$$(2.2) \quad r_t^T = r_t^B + v_t^T \cdot v_t^{\mathbf{B}} + \int_{\mathbb{Z}} H_t^T(z) H_t^{\mathbf{B}}(z) \lambda_t(dz).$$

Let \mathcal{B} denote the class of viable bond price families.

Remark 1. We call $v_t^{\mathbf{B}}$ and $H_t^{\mathbf{B}}(z) \lambda_t(dz)$ *market price of (nominal) diffusive risk* and *market price of (nominal) jump risk*, respectively.

Remark 2. Suppose that bond price family \mathbf{B} satisfies condition 1. Then, condition 2 is necessary and sufficient for the existence of *risk-neutral measures* (or also called *spot martingale measures*), and it implies that markets are *arbitrage-free* (for definitions of risk-neutral measure and arbitrage-free, see Appendix C.2).

Lemma 1. *Let $\mathbf{B} \in \mathcal{B}$ and $A^{\mathbf{B}}$ be the martingale process in Definition 2.*

- 1. *The probability measure $\tilde{\mathbb{P}}^{\mathbf{B}}$ given by the Radon-Nikodym derivative*

$$d\tilde{\mathbb{P}}^{\mathbf{B}} = A_{T^\dagger}^{\mathbf{B}} d\mathbb{P}$$

is a risk-neutral measure at \mathbf{B} .

- 2. *The process $\tilde{W}^{\mathbf{B}}$ given by*

$$(2.3) \quad \tilde{W}_t^{\mathbf{B}} = W_t + \int_0^t v_s^{\mathbf{B}} ds \quad \forall t \in \mathbf{T}$$

is a $\tilde{\mathbb{P}}^{\mathbf{B}}$ -Wiener process.

- 3. *The marked point process $\nu(dt \times dz)$ has the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel $\tilde{\lambda}_t(dz)$ such that*

$$(2.4) \quad \tilde{\lambda}_t^{\mathbf{B}}(dz) = (1 - H_t^{\mathbf{B}}(z)) \lambda_t(dz) \quad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}.$$

⁶For the marked point process integrals, we can apply the ordinary Fubini Theorem, and for the interchange of integration with respect to dW_t and dt , we can apply the Stochastic Fubini Theorem (see Protter [55]).

As shown in Section 5, in incomplete markets, it is difficult to obtain analytically tractable market prices of risks, in particular, market price of jump risk. Suppose that the market price of jump risk $H_t^{\mathbf{B}}(z) \lambda_t(dz)$ is analytically intractable. Then, the arbitrage-free condition (2.2) shows that the CCAPM is analytically and empirically intractable, and the formula (2.4) implies that $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel is analytically intractable, which makes it difficult to price derivative assets.

Björk, Masi, Kabanov, and Runggaldier [12]) prove that for every $\mathbf{B} \in \mathcal{B}$, if the process $\Lambda^{\mathbf{B}}$ in Definition 2 is unique,⁷, then markets are *approximately complete*, which is defined in the following.

Definition 3. Markets are *approximately complete at \mathbf{B}* if and only if for any $T \in (0, T^\dagger]$ and any T -contingent claim X_T there exists a sequence of replicable claims $(X_{T_n})_{n \in \mathbb{N}}$ converging to X_T in $\mathbf{L}^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}})$ for some risk-neutral measure $\tilde{\mathbb{P}}^{\mathbf{B}}$ (For definitions of *contingent claim* and *replicable claim*, see Appendix C.3).

3. APPROXIMATE SECURITY MARKET EQUILIBRIUM

In this section, we introduce the notion of approximate security market equilibrium, and show sufficient conditions for the existence of approximate security market equilibria in each case of ASUs and of SDUs with Inada conditions following our previous paper (Kusuda [45]).

3.1. Approximate Security Market Equilibrium. We give the definition of *approximate security market equilibrium* in which an agent is allowed to choose any consumption plan that is approximately supported by admissible portfolio with any prescribed precision.

Definition 4. A collection $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B}) \in \prod_{i \in \mathbf{I}} \mathbf{L}_+^2 \times \mathbf{L}^2 \times \mathcal{B}$ constitutes an *approximate security market equilibrium for \mathbf{E}* if and only if the following hold:

1. For every $i \in \mathbf{I}$, \hat{c}^i solves the problem

$$\max_{c^i \in \tilde{\mathcal{C}}^i(p, \mathbf{B})} U^i(c^i)$$

where

$$\tilde{\mathcal{C}}^i(p, \mathbf{B}) = \left\{ c^i \in \mathbf{L}_+^2 : \exists (\vartheta_n^i)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \underline{\mathcal{Q}}(\tilde{\mathbf{B}}) \quad \text{s.t.} \right.$$

$$\mathcal{V}_t^{\mathbf{B}}(\vartheta_n^i) = \int_0^t \vartheta_{ns}^{i0} dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_{ns}^{i1}(dT) dB_s^T + \int_0^t p_s(\bar{c}_s^i - c_s^i) ds \quad \forall (n, t) \in \mathbb{N} \times \mathbf{T},$$

$$\left. \lim_{n \rightarrow \infty} \mathcal{V}_{T^\dagger}^{\mathbf{B}}(\vartheta_n^i) = 0 \right\}$$

where $\underline{\mathcal{Q}}(\tilde{\mathbf{B}})$ is the class of *admissible portfolios at \mathbf{B}* (for definition, see Appendix C.1), and $\mathcal{V}_t^{\mathbf{B}}(\vartheta_n^i)$ is the value process of ϑ_n^i at \mathbf{B} given by

$$\mathcal{V}_t^{\mathbf{B}}(\vartheta_n^i) = B_t \vartheta_{nt}^{i0} + \int_t^{T^\dagger} B_t^T \vartheta_{nt}^{i1}(dT).$$

2. The commodity market is cleared as $\sum_{i \in \mathbf{I}} \hat{c}^i = \sum_{i \in \mathbf{I}} \bar{c}^i$.

⁷For some necessary and sufficient conditions for the uniqueness of $\Lambda^{\mathbf{B}}$, see Björk, Kabanov, and Runggaldier [11].

We refer to approximate security market equilibrium as ASM equilibrium, hereafter. We introduce the notion of *implementable bond price family* to show an equivalence of ASM equilibrium and Arrow-Debreu equilibrium.

Definition 5. A viable bond price family $\mathbf{B} \in \mathcal{B}$ is *implementable* if and only if the following two conditions hold:

1. The $\tilde{\mathbb{P}}^{\mathbf{B}}$ -density process $\Lambda^{\mathbf{B}}$ is unique.
2. The discounted $\tilde{\mathbb{P}}^{\mathbf{B}}$ -density process $\frac{\Lambda^{\mathbf{B}}}{B}$ is bounded above and bounded away from zero μ -a.e.

Let $\bar{\mathcal{B}}$ denote the class of implementable bond price families. Kusuda [45] proves that $((\tilde{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ with $\mathbf{B} \in \bar{\mathcal{B}}$ is an ASM equilibrium if and only if $((\tilde{c}^i)_{i \in \mathbf{I}}, \pi)$ is an Arrow-Debreu equilibrium under the relation $p = \frac{B}{\Lambda^{\mathbf{B}}} \pi$. Thus, in order to show sufficient conditions for the existence of ASM equilibria, it is enough to present ones for the existence of Arrow-Debreu equilibria.

3.2. Sufficient Conditions for Existence of Equilibria in Case of ASUs.

First, we suppose that every agent has an ASU (Additively Separable Utility).

Assumption 1. For every $i \in \mathbf{I}$, the utility U^i is an ASU of the form

$$U^i(c) = \mathbb{E} \left[\int_0^{T^\dagger} u^i(t, c_t^i) dt \right]$$

where the von Neumann-Morgenstern (VNM) utility function u^i is a real-valued $\mathbf{C}^{1,2}$ function on $\mathbf{T} \times \mathbb{R}_+$ such that $u^i(t, \cdot)$ is strictly increasing and strictly concave on \mathbb{R}_+ for every $t \in \mathbf{T}$.

We consider the aggregate utility in order to construct the representative agent. Let $\alpha \in \Delta_+^I$ where $\Delta_+^I = \{\alpha \in \mathbb{R}_+^I \mid \sum_{i \in \mathbf{I}} \alpha_i = 1\}$ and define the aggregate utility $U^\alpha : \mathbf{L}_+^2 \rightarrow \mathbb{R}$ by

$$U^\alpha(c) = \max_{(c^1, c^2, \dots, c^I) \in \prod_{i \in \mathbf{I}} \mathbf{L}_+^2} \sum_{i \in \mathbf{I}} \alpha_i U^i(c^i) \quad \text{s.t.} \quad \sum_{i \in \mathbf{I}} c^i \leq c.$$

We also define a function $c^* : \mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I \rightarrow \mathbb{R}_+^I$ by

$$(c_i^*(t, x, \alpha))_{i \in \mathbf{I}} = \operatorname{argmax}_{\{(x_1, x_2, \dots, x_I) \in \mathbb{R}_+^I : \sum_{i \in \mathbf{I}} x_i \leq x\}} \sum_{i \in \mathbf{I}} \alpha_i u^i(t, x_i).$$

Then, it follows that under Assumption 1, the aggregate utility U^α has an additively separable expected utility representation

$$U^\alpha(c) = \mathbb{E} \left[\int_0^{T^\dagger} u^\alpha(t, c_t) dt \right] \quad \text{where} \quad u^\alpha(t, x) = \sum_{i \in \mathbf{I}} \alpha_i u^i(t, c_i^*(t, x, \alpha)).$$

One can show that the following assumption is a sufficient condition for the existence of Arrow-Debreu equilibria (for proof, see Kusuda [45]).⁸

Assumption 2. The following condition holds:

$$(3.1) \quad \max_{\alpha \in \Delta^I} u_c^\alpha(t, \bar{c}_t(\omega)) \stackrel{\text{def}}{=} \bar{\pi}_t(\omega) \in \mathbf{L}_+^2$$

⁸The condition (3.1) is shown in Dana [17]. It is easy to see that this condition is weaker than the one that the aggregate endowment is bounded away from zero.

The CCAPM with ASU agents cannot explain the high equity premium (Mehra and Prescott [50]) and/or the low risk-free rate (Weil [62]). One of promising explanations for these puzzles is a deficiency of an ASU, that is, both of risk aversion and intertemporal substitution depend on the curvature of VNM utility function in an ASU. However, as argued in Hall [32], this linkage is inappropriate because risk aversion concerns the preference of an agent among events while elasticity of intertemporal substitution concerns the preference of an agent among time. We can independently handle these two characteristics in the class of stochastic differential utilities (SDUs), which is a generalization of the class of standard ASUs, and then expect that SDUs might contribute to solve the equity premium puzzle and/or the risk-free rate puzzle. In the following subsection, we show sufficient conditions for existence of equilibria in case of SDUs with Inada conditions.

3.3. Sufficient Conditions for Existence of Equilibria in Case of SDUs with Inada Conditions. An SDU is a continuous-time version of Epstein-Zin utility (Epstein and Zin [26]), and was introduced in Duffie and Epstein [19], which is a utility with expected recursive utility representation, and a generalization of the standard ASU. An SDU is not necessarily analytically tractable under jump-Wiener information, but Kusuda [45] shows a subclass of SDUs in which every SDU can be *normalized* as shown in Assumption 3 (for details, see Kusuda [45]). This subclass is still wide, and includes a class of SDUs in which each SDU is characterized by an *expected-utility certainty equivalent* (for definition, see Duffie and Epstein [19]). For instance, *Uzawa utility* (Uzawa [61]) and *Kreps-Porteus utility* which was introduced in Kreps and Porteus [44], and developed in Epstein and Zin [26] and Weil [63]. So we assume that every agent has a normalized SDU.

Assumption 3. For every $i \in \mathbf{I}$, the utility U^i is a normalized SDU, i.e. $U^i(c^i) = Y_0^i$ for every $c^i \in \mathbf{L}_+^2$ where Y^i is the unique solution in \mathbf{L}^2 for the following recursive equation⁹

$$(3.2) \quad Y_t^i = \mathbb{E}_t \left[\int_t^{T^\dagger} f^i(c_s^i, Y_s^i) ds \right] \quad \forall t \in \mathbf{T}$$

where the aggregator f^i is a $\mathbf{C}^{1,1}$ function on $\mathbb{R}_+ \times \mathbb{R}$ such that $f^i(\cdot, y)$ is strictly increasing for every $y \in \mathbb{R}$ and that f^i is concave.¹⁰

Duffie, Geoffard, and Skiadas [21] shows that the following assumption is sufficient for the existence of Arrow-Debreu equilibria.

Assumption 4. 1. For every $i \in \mathbf{I}$, the following two conditions hold:

- (a) $\sup_{y \in \mathbb{R}} f_c^i(x, y) < \infty$ for every $x > 0$.
 - (b) The aggregator f^i satisfies $\lim_{x \downarrow 0} \inf_{y \in \mathbb{R}} f_c^i(x, y) = \infty$.
2. The aggregate endowment is bounded away from zero μ -a.e.

⁹The process Y_t^i is called the *continuation utility process* since it is thought of as the continuation utility for c at time t conditional on the information up to the time t .

¹⁰A sufficient condition for the existence and uniqueness of the recursive equation (3.2) is that f^i satisfies the following two conditions (for proof, see Duffie and Epstein [19]):

1. A *growth condition in consumption*, i.e. there exist constants k_0 and k_1 such that for every $x \in \mathbb{R}_+$, we have $|f^i(x, 0)| \leq k_0 + k_1|x|$.
2. A *uniform Lipschitz condition in utility*, i.e. there exists a constant k such that for every $x \in \mathbb{R}_+$ and every $(y_1, y_2) \in \mathbb{R}^2$, we have $|f^i(x, y_1) - f^i(x, y_2)| \leq k|y_1 - y_2|$.

Remark 3. Consider a standard ASU of the form $U(c) = \mathbb{E} \left[\int_0^{T^\dagger} e^{-\rho s} u(c_s) ds \right]$. Then, it follows from Ito's formula that U can be interpreted as an SDU of the form

$$(3.3) \quad Y_t = \mathbb{E}_t \left[\int_t^{T^\dagger} (u(c_s) - \rho Y_s) ds \right] \quad \forall t \in \mathbf{T}.$$

It is straightforward to see that Assumption 4.1.(b) is equivalent to the Inada condition in the case of ASU.

4. PRELIMINARIES TO CCAPM

In this section, we show some preliminary results for deriving CCAPM. We consider a fixed ASM equilibrium in our approximately complete security market economy. First, we show that the equilibrium price of any security such that cumulative dividend process is expressed as a finite sum of dividends satisfies Euler equation, and then define the equilibrium price of a security with more general cumulative dividend process by Euler equation. Next, we see that the product of the equilibrium state price process and the real equilibrium gain process is a martingale under the agents' common belief \mathbb{P} , or equivalently that the discounted nominal equilibrium gain process is a martingale under risk-neutral measure. Finally, we assume that the aggregate endowment process has the SDDE representation, and show that the equilibrium state price process, real equilibrium gain process, and the equilibrium commodity price process have SDDE representations.

Hereafter, let $\mathbf{B} \in \tilde{\mathcal{B}}$ and consider a fixed ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} . Then, since the Arrow-Debreu equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, \pi)$ for \mathbf{E} with $\pi = \frac{\Lambda^{\mathbf{B}}}{B} p$ can be identified with the ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$, we call π as the associated equilibrium state price.

4.1. Extended Security Price System. We first consider a security with nominal cumulative dividend process D^0 given by

$$(4.1) \quad D_t^0 = \sum_{0 < s_n \leq t} d_{s_n} \quad \forall t \in \mathbf{T}$$

where $d_{s_n} \in \mathbf{L}^\infty(\Omega, \mathcal{F}_{s_n})$ for every $n \in \{1, 2, \dots, N\}$, and $(s_n)_{n \in \{1, 2, \dots, N\}}$ is a sequence of times satisfying $0 < s_1 < s_2 < \dots < s_N \leq T^\dagger$. Then, since markets are approximately complete, for every dividend d_{s_n} , there exists a sequence of replicable contingent s_n -claims $(X_{m s_n})_{m \in \mathbb{N}}$ converging to d_{s_n} in $\mathbf{L}^2(\Omega, \mathcal{F}_{s_n}, \tilde{\mathbb{P}}^{\mathbf{B}})$. Thus, it follows from arbitrage-free pricing theory that values of dividends d_{s_n} is the limit of values of contingent s_n -claims $(X_{m s_n})_{m \in \mathbb{N}}$. Therefore, we can derive the security's equilibrium price $S_t(D^0)$ at time t as the sum of the value of dividend d_{s_n} in the following:

$$(4.2) \quad S_t(D^0) = B_t \tilde{\mathbb{E}}_t^{\mathbf{B}} \left[\sum_{t < s_n \leq T^\dagger} \frac{d_{s_n}}{B_{s_n}} \right] = \frac{B_t}{\Lambda_t^{\mathbf{B}}} \mathbb{E}_t \left[\sum_{t < s_n \leq T^\dagger} \frac{\Lambda_{s_n}^{\mathbf{B}} d_{s_n}}{B_{s_n}} \right] = \frac{p_t}{\pi_t} \mathbb{E}_t \left[\sum_{t < s_n \leq T^\dagger} \pi_{s_n} \frac{d_{s_n}}{p_{s_n}} \right].$$

Hereafter, we write the real price process $\check{X} = \frac{X}{p}$ for every nominal price process X . Then, the asset pricing formula (4.3) is rewritten as the following Euler equation.

$$(4.3) \quad \check{S}_t(D^0) = \frac{1}{\pi_t} \mathbb{E}_t \left[\sum_{t < s_n \leq T^\dagger} \pi_{s_n} \check{d}_{s_n} \right].$$

Next, we consider a security with more general nominal cumulative dividend process D given by

$$(4.4) \quad \frac{dD_t}{D_t} = r_t^D dt + v_t^D \cdot dW_t + \int_0^t H_t^D(z) \{ \nu(dt \times dz) - \lambda_t(dz) dt \} \quad \forall t \in \mathbf{T}$$

for every $t \in \mathbf{T}$, for some $r^D \in \mathcal{L}^1$, $v^D \in \prod_{j=1}^d \mathcal{L}^2$, and $H^D \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. We define the equilibrium price of the security by Euler equation.

Definition 6. Let $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ be an ASM equilibrium for \mathbf{E} with $\mathbf{B} \in \bar{\mathcal{B}}$ and the associated equilibrium state price π . Then, the equilibrium real security price process $\check{S}_t(D)$ is defined by

$$(4.5) \quad \check{S}_t(D) = \frac{1}{\pi_t} \mathbb{E}_t \left[\int_t^{T^\dagger} \pi_s \frac{dD_s}{p_s} \right] \quad \forall t \in \mathbf{T}.$$

It is straightforward to see that introducing the security with the price (4.5) into our economy does not change the equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$.

4.2. Martingale Properties. We introduce the notion of *nominal gain process of a security* in the following.

Definition 7. The *nominal gain process of a security with nominal dividend price process D* given by (4.4) is

$$(4.6) \quad G_t(D) = S_t(D) + B_t \int_0^t \frac{dD_s}{B_s} \quad \forall t \in \mathbf{T}.$$

We write G as $G(D)$, hereafter. It immediately follows from (4.5) and (4.6) that the product of the equilibrium state price process π and the real equilibrium gain process \check{G} is a \mathbb{P} -martingale, or equivalently that the discounted nominal equilibrium gain process $\tilde{G} \stackrel{\text{def}}{=} \frac{G}{B}$ is a $\tilde{\mathbb{P}}^{\mathbf{B}}$ -martingale.¹¹

Lemma 2. For a given ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathcal{B}}$ and the associated equilibrium state price π , the following two properties hold:

1. The process $\pi \check{G}$ is a \mathbb{P} -martingale.
2. The discounted nominal gain process \tilde{G} is a $\tilde{\mathbb{P}}^{\mathbf{B}}$ -martingale.

Proof. See Appendix D.1. □

¹¹The proposition that the discounted nominal equilibrium gain process is a $\tilde{\mathbb{P}}^{\mathbf{B}}$ -martingale is originated with Harrison and Kreps [37], and the one that the product of the equilibrium state price process and the real equilibrium gain process is a \mathbb{P} -martingale is generally attributed to Hansen and Richard [35].

4.3. SDDE Representations. We assume that the aggregate endowment has the SDDE representation (Assumption 5.1 is put just for making CCAPM equations empirically tractable).

- Assumption 5.** 1. (a) *The Lusin space $(\mathbb{Z}, \mathcal{Z})$ is the d' -dimensional Euclidean space, i.e. $(\mathbb{Z}, \mathcal{Z}) = (\mathbb{R}^{d'}, \mathcal{B}(\mathbb{R}^{d'}))$.*
 (b) *The \mathbb{P} -intensity kernel of the marked point process ν is given by*

$$\lambda_t(dz) = \lambda_t \phi(z) dz$$

where λ_t is a \mathcal{P} -measurable process and ϕ is a p.d.f. on $\mathbb{R}^{d'}$.

2. *The dynamics of the aggregate endowment process is given by the SDDE of the form*

$$(4.7) \quad \frac{d\bar{c}_t}{\bar{c}_{t-}} = r_t^{\bar{c}} dt + v_t^{\bar{c}} \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \} \quad \forall t \in \mathbf{T}$$

for some $r^{\bar{c}} \in \mathcal{L}^1$, $v^{\bar{c}} \in \prod_{j=1}^d \mathcal{L}^2$, and $H^{\bar{c}} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$.

Then, we can show that the equilibrium state price process, real equilibrium gain process, and the equilibrium commodity price process have SDDE representations. The associated equilibrium state price π_t at time t is a function of (t, \bar{c}_t) in case of ASUs, and of (t, \bar{c}_t, Y_t) in case of SDUs as shown in Lemma 4 in Section 6. Since the dynamics of \bar{c} and Y have SDDE representations, it follows from Ito's formula that the dynamics of π also has an SDDE representation of the form

$$(4.8) \quad \frac{d\pi_t}{\pi_{t-}} = r_t^{\pi} dt + v_t^{\pi} \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^{\pi}(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}$$

for every $t \in \mathbf{T}$, for some $r^{\pi} \in \mathcal{L}^1$, $v^{\pi} \in \prod_{j=1}^d \mathcal{L}^2$, and $H^{\pi} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Moreover, since $\pi\check{G}$ is a martingale, by Martingale Representation Theorem $\pi\check{G}$ has an SDDE representation, too. Thus, by Ito's formula again, the dynamics of equilibrium gain process $\check{G} = \frac{\pi\check{G}}{\pi}$ also has an SDDE representation of the form

$$(4.9) \quad \frac{d\check{G}_t}{\check{G}_{t-}} = r_t^{\check{G}} dt + v_t^{\check{G}} \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^{\check{G}}(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}$$

for every $t \in \mathbf{T}$, for some $r^{\check{G}} \in \mathcal{L}^1$, $v^{\check{G}} \in \prod_{j=1}^d \mathcal{L}^2$, and $H^{\check{G}} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Finally, since the equilibrium commodity price process p is the product of processes with SDDE representations, i.e., $p = B(\Lambda^{\mathbf{B}})^{-1}\pi$, Ito's formula implies that p also satisfies an SDDE representation of the form

$$(4.10) \quad \frac{dp_t}{p_{t-}} = r_t^p dt + v_t^p \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^p(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}$$

for every $t \in \mathbf{T}$, where

$$(4.11) \quad \begin{aligned} r_t^p &= r_t^B + r_t^{\pi} + \frac{1}{2} \left(\|v_t^p\|^2 - \|v_t^{\mathbf{B}}\|^2 - \|v_t^{\pi}\|^2 \right) + \lambda_t \int_{\mathbb{R}^{d'}} H_t^p(z) \phi(z) dz, \\ v_t^p &= v_t^{\mathbf{B}} + v_t^{\pi}, \\ H_t^p(z) &= \frac{1 + H_t^{\pi}(z)}{1 - H_t^{\mathbf{B}}(z)} - 1. \end{aligned}$$

5. CCAPM WITH ASUs

In this section, we derive CCAPM with ASUs. We first derive the ordinary risk premium, *i.e.*, the risk premium between the risky security with the nominal dividend process D and the risk-free security. Second, we derive the risk premium between a risky security and the nominal-risk-free security. Finally, we show an empirically tractable CCAPM assuming CRRA utilities with a common relative risk aversion.

5.1. CCAPM with respect to Risk-Free Security. First, we consider the risk premium between the risky security and the risk-free (Not nominal risk-free) security. Using the explicit formula of equilibrium state price and the result that the product of the state price process and the real gain process of any security is a martingale in ASM equilibrium, we obtain explicit formulas of risk-free security and of the risk premium between the risky security and the risk-free security.

Proposition 1. *In addition to Assumptions 1, 2, and 5, if $u^i \in \mathbf{C}^{1,3}$ for every $i \in \mathbf{I}$, then for a given ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathbf{B}}$ and the associated equilibrium state price $\pi_t = u_c^{\hat{c}}(t, \bar{c}_t)$, the following hold:*

1. *The real expected instantaneous interest rate (the real risk-free rate) is*

$$(5.1) \quad r_t^{\hat{F}} = -\frac{u_{ct}^{\hat{c}}(t, \bar{c}_t)}{u_c^{\hat{c}}(t, \bar{c}_t)} + \gamma_t^{\hat{c}} r_t^{\bar{c}} - \frac{1}{2} \gamma_t^{\hat{c}} \left(-\frac{\bar{c}_t u_{ccc}^{\hat{c}}(t, \bar{c}_t)}{u_{cc}^{\hat{c}}(t, \bar{c}_t)} \right) \|v_t^{\bar{c}}\|^2 \\ - \gamma_t^{\hat{c}} \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \phi(z) dz + \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{c}}(z) \phi(z) dz$$

for every $t \in \mathbf{T}$, where

$$(5.2) \quad \gamma_t^{\hat{c}} = -\frac{\bar{c}_t u_{cc}^{\hat{c}}(t, \bar{c}_t)}{u_c^{\hat{c}}(t, \bar{c}_t)}, \\ H_t^{\hat{c}}(z) = -\frac{u_c^{\hat{c}}(t, (1 + H_t^{\bar{c}}(z))\bar{c}_{t-}) - u_c^{\hat{c}}(t-, \bar{c}_{t-})}{u_c^{\hat{c}}(t-, \bar{c}_{t-})}.$$

2. *The real risk premium between the risky security with its gain process G and the risk-free security is*

$$(5.3) \quad r_t^{\hat{G}} - r_t^{\hat{F}} = \gamma_t^{\hat{c}} v_t^{\bar{c}} \cdot v_t^{\hat{G}} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{c}}(z) H_t^{\hat{G}}(z) \phi(z) dz$$

for every $t \in \mathbf{T}$.

Proof. See Appendix D.2. □

Remark 4. In the right-hand side of (5.3), we call the first term $\gamma_t^{\hat{c}} v_t^{\bar{c}} \cdot v_t^{\hat{G}}$ and the second term $\int_{\mathbb{R}^{d'}} H_t^{\hat{c}}(z) H_t^{\hat{G}}(z) \lambda_t(dz)$, the *consumption-gain correlated diffusive risk premium* and the *consumption-gain correlated jump risk premium*, respectively. Equation (5.3) shows that the risk premium under jump-Wiener information can be decomposed into the consumption-gain correlated diffusive risk premium and the consumption-gain correlated jump risk premium.

Remark 5. Related theoretical works on CCAPM with ASUs are as follows. Breen [13] first derived a CCAPM assuming homogeneous agents under Wiener and

Markovian information. Ahn and Thompson [1] extended the result to jump-Wiener and Markovian information. Neither Breeden [13] nor Ahn and Thompson [1] proved the existence of security market equilibria, and both of them assumed homogeneous agents. Duffie and Zame [25] proved the existence of security market equilibria and derived CCAPM with heterogeneous agents under Wiener and non-Markovian information. Proposition 1 is a generalization of their results. Madan [48] and Back [8] derived CCAPM with heterogeneous agents under semimartingale and non-Markovian information, although they did not prove the existence of security market equilibria. However, since we cannot construct the representative agent because of market incompleteness, they have derived the CCAPM as follows. Since the equilibrium state price can be expressed as each agent's marginal utility multiplied by a constant, we can replace the representative agent's marginal utility in (5.3) with each agent's marginal utility.

$$(5.4) \quad r_t^{\check{G}} - r_t^{\check{F}} = -\frac{\hat{c}_t^i u_{cc}^i(t, \hat{c}_t^i)}{u_c^i(t, \hat{c}_t^i)} v_t^{\check{c}^i} \cdot v_t^{\check{G}} + \lambda_t \int_{\mathbb{R}^d} \left(-\frac{u_c^i(t, (1 + H_t^{\check{c}^i}(z))\hat{c}_{t-}^i) - u_c^i(t-, \hat{c}_{t-}^i)}{u_c^i(t-, \hat{c}_{t-}^i)} \right) H_t^{\check{G}}(z) \phi(z) dz \quad \forall t \in \mathbf{T}.$$

Dividing both sides of (5.4) by $(-\frac{u_{cc}^i(t, \hat{c}_t^i)}{u_c^i(t, \hat{c}_t^i)})$ and then summing the resultant equations over I agents, we obtain

$$(5.5) \quad r_t^{\check{G}} - r_t^{\check{F}} = \gamma_t^{\mathbf{I}} v_t^{\bar{c}} \cdot v_t^{\check{G}} + \lambda_t \int_{\mathbb{R}^d} H_t^{\mathbf{I}}(z) H_t^{\check{G}}(z) \phi(z) dz \quad \forall t \in \mathbf{T}$$

where

$$\gamma_t^{\mathbf{I}} = \left\{ \sum_{i \in \mathbf{I}} \left(-\frac{u_{cc}^i(t, \hat{c}_t^i)}{u_c^i(t, \hat{c}_t^i)} \right)^{-1} \right\}^{-1} \bar{c}_t, \\ H_t^{\mathbf{I}}(z) = \sum_{i \in \mathbf{I}} \frac{\gamma_t^{\mathbf{I}}}{\left(-\frac{u_{cc}^i(t, \hat{c}_t^i)}{u_c^i(t, \hat{c}_t^i)} \bar{c}_t \right)} \left(-\frac{u_c^i(t, (1 + H_t^{\check{c}^i}(z))\hat{c}_{t-}^i) - u_c^i(t-, \hat{c}_{t-}^i)}{u_c^i(t-, \hat{c}_{t-}^i)} \right).$$

To make that return rates of gain processes are scale-invariant for the aggregate consumption and that the coefficient $\gamma_t^{\mathbf{I}}$ is empirically tractable, we need to assume CRRA (Constant Relative Risk Aversion) utilities with a common relative risk aversion. However, the coefficient $H_t^{\mathbf{I}}(z)$ with CRRA utilities with a common relative risk aversion γ becomes empirically intractable as follows

$$H_t^{\mathbf{I}}(z) = \sum_{i \in \mathbf{I}} \frac{\hat{c}_t^i}{\bar{c}_t} \left(1 - (1 + H_t^{\check{c}^i}(z))^{-\gamma} \right).$$

5.2. CCAPM with respect to Nominal-Risk-Free Security. We established CCAPM equations (5.1)-(5.3) with ASUs. However, this expression is inconvenient to test, since the risk-free security is not traded or very limitedly traded in the real economy. Thus, we replace the risk-free rate with the nominal-risk-free rate in CCAPM equations (5.1) and (5.3).

Theorem 1. *In addition to Assumptions 1, 2, and 5, if $u^i \in \mathbf{C}^{1,3}$, then for a given ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathbf{B}}$ and the associated equilibrium state price $\pi_t = u_c^{\hat{c}}(t, \bar{c}_t)$, the following hold:*

1. For real terms, the following hold:

(a) The real expected instantaneous interest rate of the money market account (real nominal-risk-free rate) is

$$(5.6) \quad r_t^{\tilde{B}} = r_t^{\tilde{F}} + \left(-\gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^p \right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \right)$$

for every $t \in \mathbf{T}$ where $r_t^{\tilde{F}}$ satisfies (5.1).

(b) The real risk premium between the risky security with its gain process G and the nominal-risk-free security is

$$(5.7) \quad r_t^{\tilde{G}} - r_t^{\tilde{B}} = \gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^{\tilde{G}} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) H_t^{\tilde{G}}(z) \phi(z) dz \\ - \left\{ \left(-\gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^p \right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \right) \right\}$$

for every $t \in \mathbf{T}$.

2. For nominal terms, the following hold:

(a) The nominal nominal-risk-free rate is

$$(5.8) \quad r_t^B = r_t^{\tilde{B}} + r_t^p - \|v_t^p\|^2 - \lambda_t \int_{\mathbb{R}^{d'}} H_t^p(z) \phi(z) dz$$

for every $t \in \mathbf{T}$.

(b) The nominal risk premium between the risky security with its gain process G and the nominal-risk-free security is

$$(5.9) \quad r_t^G - r_t^B = \gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^{\hat{\alpha}}(z)}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz \\ + v_t^p \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^p(z)}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz$$

for every $t \in \mathbf{T}$.

Proof. See Appendix D.3. □

Remark 6. Equation (5.7) can be rewritten as

$$(5.10) \quad r_t^{\tilde{G}} - r_t^{\tilde{B}} = (r_t^{\tilde{G}} - r_t^{\tilde{F}}) - (r_t^{\tilde{B}} - r_t^{\tilde{F}})$$

where

$$r_t^{\tilde{G}} - r_t^{\tilde{F}} = \gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^{\tilde{G}} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) H_t^{\tilde{G}}(z) \phi(z) dz, \\ r_t^{\tilde{B}} - r_t^{\tilde{F}} = \left(-\gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^p \right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \right).$$

We call terms $r_t^{\tilde{G}} - r_t^{\tilde{F}}$ and $r_t^{\tilde{B}} - r_t^{\tilde{F}}$, the *gain fluctuation risk premium* and the *commodity price fluctuation risk premium*, respectively. In the commodity price fluctuation risk premium, we call terms $(-\gamma_t^{\hat{\alpha}} v_t^{\tilde{c}} \cdot v_t^p)$ and $(-\lambda_t \int_{\mathbb{R}^{d'}} H_t^{\hat{\alpha}}(z) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz)$, the *consumption-commodity-price correlated diffusive risk premium* and the *consumption-commodity-price correlated jump risk premium*, respectively. Equation (5.10) shows that the risk premium between a risky security and the nominal-risk-free security equals the security price fluctuation risk premium minus the commodity price fluctuation risk premium. We do not know whether the commodity price fluctuation

risk premium is positive or not, and it is possible that one of two terms or both terms in the commodity price fluctuation risk premium are negative.

5.3. CCAPM with CRRA Utilities. CCAPM equations (5.6)-(5.7) are not still empirically tractable. To obtain more empirically tractable CCAPM equations, we assume that agents have CRRA utilities with a common relative risk aversion coefficient since it has various desirable properties including the following: (1) CRRA utility is analytically tractable. (2) The representative agent also has the CRRA utility with the same relative risk aversion coefficient as shown in Lemma 3. (3) Return rates of securities are scale-invariant for the aggregate consumption.

Assumption 6. For every $i \in \mathbf{I}$, the agent's VNM utility function is given by

$$(5.11) \quad u^i(t, x) = \begin{cases} e^{-\rho_i t} \frac{\gamma}{1-\gamma} \left\{ \left(\frac{x}{\gamma} \right)^{1-\gamma} - 1 \right\} & \text{for } \gamma \neq 1 \\ e^{-\rho_i t} \ln x & \text{for } \gamma = 1 \end{cases}$$

for every $(t, x) \in \mathbf{T} \times \mathbb{R}_+$ and for some $\rho_i \in \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}_{++}$.

If agents have CRRA utilities with a common relative risk aversion coefficient, then the representative agent also has the CRRA utility with the same relative risk aversion coefficient.

Lemma 3. Under Assumption 6, 2, and 5, the VNM aggregate utility function is

$$(5.12) \quad u^\alpha(t, x) = \begin{cases} e^{-\rho(\alpha)t} \frac{\gamma}{1-\gamma} \left\{ \frac{\left(\sum_{i \in \mathbf{I}} (\alpha_i e^{\rho_i t})^{\frac{1}{\gamma}} \right)^\gamma}{\sum_{i \in \mathbf{I}} \alpha_i e^{\rho_i t}} \left(\frac{x}{\gamma} \right)^{1-\gamma} - 1 \right\} & \text{for } \gamma \neq 1 \\ e^{-\rho(\alpha)t} \ln x & \text{for } \gamma = 1 \end{cases}$$

for every $(t, x) \in \mathbf{T} \times \mathbb{R}_+$ where $\rho(\alpha) = -\frac{1}{t} \ln \left(\sum_{i \in \mathbf{I}} \alpha_i e^{-\rho_i t} \right)$. In addition, if $\rho_i = \rho$ for some $\rho \in \mathbb{R}_{++}$ for every $i \in \mathbf{I}$, then

$$(5.13) \quad u^\alpha(t, x) = \begin{cases} e^{-\rho t} \frac{\gamma}{1-\gamma} \left\{ \left(\frac{x}{\gamma} \right)^{1-\gamma} - 1 \right\} & \text{for } \gamma \neq 1 \\ e^{-\rho t} \ln x & \text{for } \gamma = 1 \end{cases}$$

for every $(t, x) \in \mathbf{T} \times \mathbb{R}_+$.

Proof. See Appendix D.4. □

It follows from Theorem 1 and Lemma 3 that an empirically tractable CCAPM with CRRA utilities is obtained.

Corollary 1.1. In addition to Assumptions 6, 2, 5, and 9, if $\rho_i = \rho$ for some $\rho \in \mathbb{R}_{++}$ for every $i \in \mathbf{I}$, then for a given ASM equilibrium $((\bar{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathbf{B}}$ and the associated equilibrium state price $\pi_t = u_c^{\hat{\alpha}}(t, \bar{c}_t)$, processes $\gamma_t^{\hat{\alpha}}$ and $H_t^{\hat{\alpha}}(z)$ in Proposition 1 and Theorem 1 satisfy

$$(5.14) \quad \begin{aligned} \gamma_t^{\hat{\alpha}} &= \gamma, \\ H_t^{\hat{\alpha}}(z) &= 1 - (1 + H_t^{\bar{c}}(z))^{-\gamma} \quad \forall z \in \mathbb{R}^{d'}, \end{aligned}$$

for every $t \in \mathbf{T}$, and the following hold:

1. For real terms, the following holds:

(a) *The real nominal-risk-free rate is*

(5.15)

$$r_t^{\tilde{B}} = r_t^{\tilde{F}} + \left(-\gamma v_t^{\tilde{c}} \cdot v_t^p\right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\tilde{c}}(z))^{-\gamma}\right) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz\right)$$

for every $t \in \mathbf{T}$, where

(5.16)

$$r_t^{\tilde{F}} = \rho + \gamma r_t^{\tilde{c}} - \frac{1}{2} \gamma (\gamma + 1) \|v_t^{\tilde{c}}\|^2 - \gamma \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\tilde{c}}(z) \phi(z) dz + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\tilde{c}}(z))^{-\gamma}\right) \phi(z) dz.$$

(b) *The real risk premium between the risky security with its gain process G and the nominal-risk-free security is*

(5.17)

$$r_t^{\tilde{G}} - r_t^{\tilde{B}} = \gamma v_t^{\tilde{c}} \cdot v_t^{\tilde{G}} + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\tilde{c}}(z))^{-\gamma}\right) H_t^{\tilde{G}}(z) \phi(z) dz - \left\{ \left(-\gamma v_t^{\tilde{c}} \cdot v_t^p\right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\tilde{c}}(z))^{-\gamma}\right) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz\right) \right\}$$

for every $t \in \mathbf{T}$.

2. *For nominal terms, the following hold:*

(a) *The nominal nominal-risk-free rate satisfies (5.8).*

(b) *The nominal risk premium between the risky security with its gain process G and the nominal-risk-free security is*

(5.18)

$$r_t^G - r_t^B = \gamma v_t^{\tilde{c}} \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{1 - (1 + H_t^{\tilde{c}}(z))^{-\gamma}}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz + v_t^p \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^p(z)}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz$$

for every $t \in \mathbf{T}$.

Remark 7. Euler equations with a common CRRA utility have been often rejected by tests of Generalized Method of Moments (GMM) developed in Hansen [34] and Cumby, Huizinga, and Obstfeld [16] (Epstein and Zin [27], Finn, Hoffman, and Schlagenhauf [29], and Hansen and Singleton [36] for US data), and lead us the equity premium puzzle of Mehra and Prescott [50] and the risk-free rate puzzle of Weil [62]. Rietz [56] pointed out that one possible explanation of such puzzles is to disregard effects of possible very rare market crashes. He respecified Euler equations derived by Mehra and Prescott [50], which is a discrete-time finite dimensional Markovian state model, to capture the crash risk, and claimed that the respecified Euler equations can explain high equity risk premia and low risk-free returns with reasonable degrees of time preference and risk aversion as long as crashes are plausibly severe and not too improbable. The jump risk premium in Ahn and Thompson [1]'s CCAPM equations and ours (5.15)-(5.17) is a generalization of the market crash risk premium in Euler equations of Rietz [56] since all magnitudes of upward and downward jumps are considered in the jump risk premium while only a very big magnitude of downward jump is considered in the market crash risk premium. However, as Campbell [15] argued, CCAPM equations with jump risk premia could not completely solve the equity premium puzzle and

the risk-free rate puzzle. Another possible explanation of such puzzles is possible deficiencies of ASU.¹² One of such possible deficiencies¹³ is that both risk aversion and intertemporal substitution depend on the curvature of VNM utility function.¹⁴ These characteristics can be disentangled in the class of SDUs (Stochastic Differential Utilities). Another rationalization for SDUs is given in Anderson, Hansen, and Sargent [4], Maenhout [49], and Skiadas [60]. To keep analytical tractability, we usually assume that every agent does not suspect that his or her belief is misspecified, but it is obvious that he or she does in the real economy. Anderson, Hansen, and Sargent [4] assumed that every agent suspects that his or her belief is misspecified and uses robust control theory to make decisions. Maenhout [49] showed that under Wiener and Markovian information if an agent with a CRRA utility uses a robust control criterion to make decisions, then the agent's utility becomes a Kreps-Porteus utility, and Skiadas [60] extended this result to the one that under Wiener and non-Markovian information if an agent with a standard ASU uses a robust control criterion to make decisions, then the agent's utility becomes an SDU. In next section, we consider the CCAPM with SDUs.

6. CCAPM WITH SDU

In this section, we derive the CCAPM assuming homogeneous agents with a common SDU. We first derive the CCAPM with general SDU, and then present more empirically tractable CCAPM equations assuming Kreps-Porteus utility (Kreps and Porteus [44]) which is a generalization of CRRA utility and have various desirable properties as explained later.

We first consider an equilibrium state price. Duffie and Skiadas [24] give the explicit formula of the equilibrium state price process using a *utility gradient approach* (see Duffie and Skiadas [24]).

Lemma 4. *Under Assumptions 3 and 4, for a given ASM equilibrium $((\bar{c})_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathcal{B}}$, the associated equilibrium state price process satisfies for every $i \in \mathbf{I}$,*

$$(6.1) \quad \pi_t = \hat{\alpha}_i \exp \left(\int_0^t f_y^i(\hat{c}_s^i, Y_s^i) ds \right) f_c^i(\hat{c}_t^i, Y_t^i) \quad \forall t \in \mathbf{T}$$

for some $\hat{\alpha} \in \Delta_{++}^I$.

Proof. See Duffie and Skiadas [24]. □

6.1. CCAPM with General SDU. Since it is difficult to construct the representative agent's utility in the case of SDUs, we introduce the following assumption.

Assumption 7. *Agents are homogeneous and each of them has a common normalized SDU with an aggregator f and the common endowment \bar{c} .*

¹²For other possible explanations, we could give survival bias, market incompleteness, market imperfections, limited participants of agents in the stock market, and problems of temporal aggregation (for surveys on these explanations, see Campbell [15], and Mehra and Prescott [51]).

¹³For other alternative utilities for other possible deficiencies of an ASU, we could give some classes of internal and external habit formation utilities (for surveys on asset pricing with habit formation utilities, see Campbell [15]).

¹⁴The standard ASU has another possible deficiency that agent is indifferent toward the *timing of resolution of uncertainty* (for definition, see Kreps and Porteus [44]).

Then, we can obtain the CCAPM with the common SDU in the same way as in the previous section.

Theorem 2. *In addition to Assumptions 7, 3, 4, and 5, if $f \in \mathbf{C}^{3,2}$, then for a given ASM equilibrium $((\bar{c})_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \mathcal{B}$ and the associated equilibrium state price $\pi_t = \exp\left(\int_0^t f_y(\bar{c}_s, Y_s) ds\right) f_c(\bar{c}_t, Y_t)$, the following hold:*

1. *For real terms, the following hold:*

(a) *The real nominal-risk-free rate is*

(6.2)

$$r_t^{\tilde{B}} = r_t^{\tilde{F}} + \left(-\bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^p\right) + \left(-\bar{\zeta}_t v_t^Y \cdot v_t^p\right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) \frac{H_t^P(z)}{1 + H_t^P(z)} \phi(z) dz\right)$$

for every $t \in \mathbf{T}$, where v_t^Y and $H_t^Y(z)$ are volatility and jump magnitude of continuation utility, and

(6.3)

$$r_t^{\tilde{F}} = -f_y(\bar{c}_t, Y_t) + \bar{\gamma}_t r_t^{\bar{c}} + \bar{\zeta}_t r_t^Y \\ - \frac{1}{2} \bar{\gamma}_t \left(-\frac{\bar{c}_t f_{ccc}(\bar{c}_t, Y_t)}{f_{cc}(\bar{c}_t, Y_t)} \right) \|v_t^{\bar{c}}\|^2 - \frac{1}{2} \bar{\zeta}_t \left(-\frac{Y_t f_{cyy}(\bar{c}_t, Y_t)}{f_{cy}(\bar{c}_t, Y_t)} \right) \|v_t^Y\|^2 - \frac{\bar{c}_t Y_t f_{ccy}(\bar{c}_t, Y_t)}{f_c(\bar{c}_t, Y_t)} v_t^{\bar{c}} \cdot v_t^Y \\ - \bar{\gamma}_t \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \phi(z) dz - \bar{\zeta}_t \lambda_t \int_{\mathbb{R}^{d'}} H_t^Y(z) \phi(z) dz - \lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) \phi(z) dz,$$

$$\bar{\gamma}_t = -\frac{\bar{c}_t f_{cc}(\bar{c}_t, Y_t)}{f_c(\bar{c}_t, Y_t)}, \quad \bar{\zeta}_t = -\frac{Y_t f_{cy}(\bar{c}_t, Y_t)}{f_c(\bar{c}_t, Y_t)},$$

$$\bar{H}_t(z) = -\frac{f_c((1 + H_t^{\bar{c}}(z))\bar{c}_{t-}, (1 + H_t^Y(z))Y_{t-}) - f_c(\bar{c}_{t-}, Y_{t-})}{f_c(\bar{c}_{t-}, Y_{t-})}.$$

(b) *The real risk premium between the risky security with its gain process G and the nominal-risk-free security is*

$$(6.4) \quad r_t^{\tilde{G}} - r_t^{\tilde{B}} = \bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^{\tilde{G}} + \bar{\zeta}_t v_t^Y \cdot v_t^{\tilde{G}} + \lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) H_t^{\tilde{G}}(z) \phi(z) dz \\ - \left\{ \left(-\bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^p\right) + \left(-\bar{\zeta}_t v_t^Y \cdot v_t^p\right) + \left(-\lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) \frac{H_t^P(z)}{1 + H_t^P(z)} \phi(z) dz\right) \right\}$$

for every $t \in \mathbf{T}$.

2. *For nominal terms, the following hold:*

(a) *The nominal nominal-risk-free rate satisfies (5.8).*

(b) *The nominal risk premium between the risky security with its gain process G and the nominal-risk-free security is*

$$(6.5) \quad r_t^G - r_t^B = \bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^G + \bar{\zeta}_t v_t^Y \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{\bar{H}_t(z)}{1 + H_t^P(z)} H_t^G(z) \phi(z) dz \\ + v_t^p \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^P(z)}{1 + H_t^P(z)} H_t^G(z) \phi(z) dz$$

for every $t \in \mathbf{T}$.

Proof. See Appendix D.5. □

Remark 8. We call each term in (6.4) as follows:

$\bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^{\tilde{G}}$: *consumption-gain correlated diffusive risk premium.*

$\bar{\zeta}_t v_t^Y \cdot v_t^{\tilde{G}}$: continuation-utility-gain correlated diffusive risk premium.

$\lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) H_t^{\tilde{G}}(z) \phi(z) dz$: consumption-continuation-utility-gain correlated jump risk premium.

$-\bar{\gamma}_t v_t^{\bar{c}} \cdot v_t^p$: consumption-commodity-price correlated diffusive risk premium.

$-\bar{\zeta}_t v_t^Y \cdot v_t^p$: continuation-utility-commodity-price correlated diffusive risk premium.

$-\lambda_t \int_{\mathbb{R}^{d'}} \bar{H}_t(z) \frac{H_t^p(z)}{1+H_t^p(z)} \phi(z) dz$: consumption-continuation-utility-commodity-price correlated jump risk premium.

Remark 9. If the agents' common utility is a standard ASU of the form (3.3), then $f_c = u'$ and $f_y = -\rho$. Thus, it is straightforward to see that CCAPM equations (6.2)-(6.4) are reduced to the ones (5.6) and (5.7) with ASUs.

Remark 10. Related theoretical works on CCAPM with SDU is in the following. Duffie and Epstein [20] derived a CCAPM assuming homogeneous agents with a common SDU under Wiener and Markovian information. In their CCAPM, the risk premium between a risky security and the risk-free security is a linear combination of the consumption volatility and the market portfolio volatility. Fisher and Gilles [30] derived a CCAPM assuming homogeneous agents with a common Kreps-Porteus utility under Wiener and non-Markovian information. Their CCAPM says that the risk premium between a risky security and the risk-free security is a linear combination of the consumption volatility and the volatility of some process that summarizes all information about future opportunities. Theorem 2 is a generalization of their results.

6.2. CCAPM with Kreps-Porteus Utility. To obtain more empirically tractable CCAPM equations than (6.2)-(6.4), we assume that the common SDU is a Kreps-Porteus utility (Kreps and Porteus [44]) which is a generalization of CRRA utility since the class of Kreps-Porteus utilities has various desirable properties including the following: (1) Return rates of securities are scale-invariant for the aggregate consumption for every Kreps-Porteus utility. (2) The Arrow-Pratt coefficient of relative risk aversion and the elasticity of intertemporal substitution are specified as independent constants in the class. (3) Agent can prefer early or late resolution of uncertainty in the class.

First, we need to assume that there exists a unique square-integrable continuation process Y satisfying (6.6) since the aggregator of a Kreps-Porteus utility does not satisfy the growth and uniform Lipschitz conditions in Assumption 3.¹⁵

Assumption 8. A utility U is given by $U(c) = Y_0$ for every $c \in \mathbf{L}_+^2$ where Y is the unique solution in \mathbf{L}^2 for the following recursive equation

$$(6.6) \quad Y_t = \mathbb{E}_t \left[\int_t^{T^*} f(c_s, Y_s) ds \right] \quad \forall t \in \mathbf{T}$$

¹⁵For Wiener information, Schroder and Skiadas [57] prove the existence and uniqueness of the square-integrable continuation utility process satisfying (6.6).

where f satisfies the Kreps-Porteus utility of the form

$$(6.7) \quad f(x, y) = \begin{cases} \rho \frac{(1-\gamma)y}{1-\frac{1}{\eta}} \left[\left\{ \frac{x}{((1-\gamma)y)^{\frac{1}{1-\gamma}}} \right\}^{1-\frac{1}{\eta}} - 1 \right] & \text{for } \eta \neq 1 \\ \rho(1-\gamma)y \left[\ln x - \frac{1}{1-\gamma} \ln(1-\gamma)y \right] & \text{for } \eta = 1 \end{cases}$$

for every $(x, y) \in \mathbb{R}_+^2$ and for some constants $\rho \in \mathbb{R}_{++}$, $1 > \gamma \geq 0$, $\eta \in \mathbb{R}_{++}$.

Remark 11. Coefficients in the aggregator f are interpreted as follows. γ is the Arrow-Pratt coefficient of relative risk aversion in static stochastic setting, and ρ and η are the rate of time preference and the elasticity of intertemporal substitution in dynamic nonstochastic setting, respectively. The preference for early versus late resolution of uncertainty is characterized by the sign of $\gamma - \frac{1}{\eta}$. The agent prefers early resolution for $\gamma > \frac{1}{\eta}$, late resolution for $\gamma < \frac{1}{\eta}$, and indifferent toward the timing of resolution of uncertainty for $\gamma = \frac{1}{\eta}$, *i.e.* the case of CRRA utility.

It follows from Theorem 1 that the CCAPM with Kreps-Porteus utility is obtained.

Corollary 2.1. *Under Assumptions 7, 8, 4.2, and 5, for a given ASM equilibrium $((\bar{c})_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathbf{B}}$ and the associated equilibrium state price $\pi_t = \exp\left(\int_0^t f_y(\bar{c}_s, Y_s) ds\right) f_c(\bar{c}_t, Y_t)$, processes $\bar{\gamma}_t$, $\bar{\zeta}_t$, and $\bar{H}_t(z)$ in Theorem 2 satisfy*

$$(6.8) \quad \begin{aligned} \bar{\gamma}_t &= \frac{1}{\eta}, & \bar{\zeta}_t &= \frac{\gamma - \frac{1}{\eta}}{1 - \gamma}, \\ \bar{H}_t(z) &= 1 - (1 + H_t^{\bar{c}}(z))^{-\frac{1}{\eta}} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}} & \forall z \in \mathbb{R}^{d'}, \end{aligned}$$

for every $t \in \mathbf{T}$, and the following hold:

1. For real terms, the following hold:
 - (a) The real nominal-risk-free rate is

$$(6.9) \quad \begin{aligned} r_t^{\bar{\mathbf{B}}} &= r_t^{\bar{F}} + \left(-\frac{1}{\eta} v_t^{\bar{c}} \cdot v_t^p \right) + \left(-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y \cdot v_t^p \right) \\ &+ \left(-\lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\bar{c}}(z))^{-\gamma} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}} \right) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \right) \end{aligned}$$

for every $t \in \mathbf{T}$, where

(6.10)

$$\begin{aligned} r_t^{\bar{F}} &= \rho(\bar{c}_t, Y_t) + \frac{1}{\eta} r_t^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} r_t^Y \\ &\quad - \frac{1}{2} \frac{1}{\eta} \left(\frac{1}{\eta} + 1 \right) \|v_t^{\bar{c}}\|^2 - \frac{1}{2} \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} \left(\frac{\gamma - \frac{1}{\eta}}{1 - \gamma} + 1 \right) \|v_t^Y\|^2 - \frac{1}{\eta} \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^{\bar{c}} \cdot v_t^Y \\ &\quad - \frac{1}{\eta} \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \phi(z) dz - \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} \lambda_t \int_{\mathbb{R}^{d'}} H_t^Y(z) \phi(z) dz \\ &\quad + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\bar{c}}(z))^{-\frac{1}{\eta}} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}} \right) \phi(z) dz \end{aligned}$$

where

$$\rho(\bar{c}_t, Y_t) = \frac{1 - \gamma}{1 - \frac{1}{\eta}} \left\{ 1 + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} \left(\frac{\bar{c}_t}{((1 - \gamma)Y_t)^{\frac{1}{1 - \gamma}}} \right)^{1 - \frac{1}{\eta}} \right\} \rho.$$

(b) The real risk premium between the risky security with its gain process G and the nominal-risk-free security is

(6.11)

$$\begin{aligned} r_t^{\bar{G}} - r_t^{\bar{B}} &= \frac{1}{\eta} v_t^{\bar{c}} \cdot v_t^{\bar{G}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y \cdot v_t^{\bar{G}} \\ &\quad + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\bar{c}}(z))^{-\frac{1}{\eta}} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}} \right) H_t^{\bar{G}}(z) \phi(z) dz \\ &\quad - \left\{ \left(-\frac{1}{\eta} v_t^{\bar{c}} \cdot v_t^p \right) + \left(-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y \cdot v_t^p \right) \right. \\ &\quad \left. + \left(-\lambda_t \int_{\mathbb{R}^{d'}} \left(1 - (1 + H_t^{\bar{c}}(z))^{-\frac{1}{\eta}} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}} \right) \frac{H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \right) \right\} \end{aligned}$$

for every $t \in \mathbf{T}$.

2. For nominal terms, the following hold:

- (a) The nominal nominal-risk-free rate satisfies (5.8).
- (b) The nominal risk premium between the risky security with its gain process G and the nominal-risk-free security is

(6.12)

$$\begin{aligned} r_t^G - r_t^B &= \frac{1}{\eta} v_t^{\bar{c}} \cdot v_t^G + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{1 - (1 + H_t^{\bar{c}}(z))^{-\frac{1}{\eta}} (1 + H_t^Y(z))^{-\frac{\gamma - \frac{1}{\eta}}{1 - \gamma}}}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz \\ &\quad + v_t^p \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^p(z)}{1 + H_t^p(z)} H_t^G(z) \phi(z) dz \end{aligned}$$

for every $t \in \mathbf{T}$.

Remark 12. If the common utility is a CRRA utility, *i.e.* $\eta = \frac{1}{\gamma}$, then it follows that $\bar{\gamma}_t = \gamma$, $\bar{c}_t = 0$, $\bar{H}_t(z) = 1 - (1 + H_t^{\bar{c}}(z))^{-\gamma}$, and $\rho(\bar{c}_t, Y_t) = \rho$, and CCAPM equations (6.9)-(6.11) are reduced to the ones (5.15)-(5.17) with the CRRA utility. If the common utility is such that the elasticity of intertemporal substitution is

infinity, *i.e.* $\eta = \infty$, then it follows that $\bar{\gamma}_t = 0$, $\bar{\zeta}_t = \frac{\gamma}{1-\gamma}$, and $\bar{H}_t(z) = 1 - (1 + H_t^Y(z))^{-\frac{\gamma}{1-\gamma}}$.

Remark 13. Epstein and Zin [27], Jorion and Giovannini [43], and Hamori [33] tested Euler equations with a common Kreps-Porteus utility in discrete-time model using the GMM. However, a GMM estimator is inefficient, and the result of a GMM test often depends on selected instrument set. Test results for U.S. data in Epstein and Zin [27] and in Jorion and Giovannini [43] depend on selected instrument sets, while test results for Japanese data in Hamori [33] do not reject the CCAPM. These results suggest that to test the CCAPM with SDU, it might be better than to use the GMM to specify processes of consumption and asset prices and to use an efficient estimation method such as a likelihood-based method or a Bayesian method. Jorion and Giovannini [43] also assumed that the growth rate of consumption and return rates of assets are jointly lognormal and homoscedastic, and tested the CRRA utility hypothesis $\gamma = \frac{1}{\eta}$ using the likelihood ratio test. The test result showed that the CRRA utility hypothesis is not rejected. Note that their assumption that return rates of assets are jointly lognormal and that they are homoscedastic correspond to the one that return rates of assets are subject to Ito processes and that volatilities of assets are constants in continuous-time model. However, recent empirical analyses have shown that return rates of assets are subject to jump-Wiener processes with stochastic volatilities, and that the class of affine jump-diffusion models with stochastic volatilities is promising to explain dynamics of asset return rates. We can exploit an efficient estimation and test method such as a characteristic-function-based method or a Markov Chain Monte Carlo (MCMC) method, using the time-series data of G , \bar{c} , and p as long as we specify these processes as an affine jump-diffusion model with stochastic volatilities.

7. GENERAL EQUILIBRIUM FRAMEWORK FOR JUMP-DIFFUSION OPTION PRICING MODELS

In this section, we present a general equilibrium (GE) framework for jump-diffusion option pricing models in each case of heterogeneous agents with CRRA utilities and of homogeneous agents with a common Kreps-Porteus utility. In concrete, we show analytically tractable formulas for equilibrium market prices of diffusive risk and of jump risk, for the Wiener process under the risk-neutral measure, and for the jump intensity under the risk-neutral measure, and derive the dynamics of equilibrium nominal gain, forward rate, and nominal-risk-free rate processes. These formulas enable us to construct jump-diffusion option pricing models for which both of analytic (quasianalytic) equilibrium price formulas for European options and an efficient estimation method are available. Then, we construct a GE version of jump-diffusion stock index model with jump-diffusion volatility, which was proposed in Duffie, Pan, and Singleton [22] as a promising example in the class of *affine jump-diffusion (AJD) models*¹⁶ developed by Heston [39], Bates [9], and Duffie, Pan, and Singleton [22]. For every AJD model, both of quasianalytic European option price formulas and efficient estimation methods are available (see Duffie, Pan, and Singleton [22]).

¹⁶For definition of AJD model, see Duffie, Pan, and Singleton [22].

7.1. GE Framework for Jump-Diffusion Option Pricing Models. We assume that jump magnitudes of the aggregate endowment, of the utility process, and of the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -density $\Lambda^{\mathbf{B}}$ are analytically tractable as follows.

Assumption 9. *Jump magnitudes of the aggregate endowment \bar{c} , of the utility Y , and of the density $\Lambda^{\mathbf{B}}$ of \mathbb{P} relative to $\tilde{\mathbb{P}}^{\mathbf{B}}$ satisfy*

$$(7.1) \quad \begin{aligned} H_t^{\bar{c}}(z) &= e^{J_{\bar{c}}(z)} - 1, \\ H_t^Y(z) &= e^{J_Y(z)} - 1, \\ H_t^{\mathbf{B}}(z) &= 1 - e^{J_{\mathbf{B}}(z)}, \end{aligned}$$

for every $(t, z) \in \mathbf{T} \times \mathbb{R}^{d'}$ where $J_{\bar{c}}, J_Y, J_{\mathbf{B}} : \mathbb{R}^{d'} \rightarrow \mathbb{R}$.

Then, we obtain the following analytically tractable formulas for the equilibrium market prices of diffusive risk and of jump risk, for the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -Wiener process, and for the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel of marked point process ν , and the dynamics of equilibrium nominal gain, forward rate, and nominal-risk-free rate processes, in each case of heterogeneous agents with CRRA utilities and of homogeneous agents with a common Kreps-Porteus utility.

Theorem 3. 1. *Under Assumptions 6, 2, 5, and 9, for a given ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \mathcal{B}$ and the associated equilibrium state price $\pi_t = u_{\bar{c}}^{\hat{c}}(t, \bar{c}_t)$, the following hold:*

(a) *Market prices of nominal diffusive risk $v_t^{\mathbf{B}}$ and of nominal jump risk $H_t^{\mathbf{B}}(z)\lambda_t(dz)$ satisfy*

$$(7.2) \quad \begin{aligned} v_t^{\mathbf{B}} &= \gamma_t^{\hat{c}} v_t^{\bar{c}} + v_t^p = \gamma v_t^{\bar{c}} + v_t^p, \\ H_t^{\mathbf{B}}(z)\lambda_t(dz) &= \lambda_t \frac{H_t^{\hat{c}}(z) + H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \\ &= \lambda_t \left(1 - e^{-\left(\gamma J_{\bar{c}}(z) + J_p(z)\right)} \right) \phi(z) dz \quad \forall z \in \mathbb{R}^{d'}, \end{aligned}$$

for every $t \in \mathbf{T}$, where $J_p(z) = -(\gamma J_{\bar{c}}(z) + J_{\mathbf{B}}(z))$ and satisfies $H_t^p(z) = e^{J_p(z)} - 1$ for every $(t, z) \in \mathbf{T} \times \mathbb{R}^{d'}$.

(b) *The $\tilde{\mathbb{P}}^{\mathbf{B}}$ -Wiener process $\tilde{W}_t^{\mathbf{B}}$ and the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel $\tilde{\lambda}_t^{\mathbf{B}} \tilde{\phi}^{\mathbf{B}}(z) dz$ of marked point process ν satisfy*

$$(7.3) \quad \begin{aligned} \tilde{W}_t^{\mathbf{B}} &= W_t + \int_0^t (\gamma v_s^{\bar{c}} + v_s^p) ds, \\ \tilde{\lambda}_t^{\mathbf{B}} &= \iota \lambda_t, \quad \tilde{\phi}^{\mathbf{B}}(z) = \iota^{-1} e^{-\left(\gamma J_{\bar{c}}(z) + J_p(z)\right)} \phi(z) \quad \forall z \in \mathbb{R}^{d'}, \end{aligned}$$

for every $t \in \mathbf{T}$ where $\iota = \int_{\mathbb{R}^{d'}} e^{-\left(\gamma J_{\bar{c}}(z) + J_p(z)\right)} \phi(z) dz$.

(c) *The dynamics of nominal gain process G satisfies*

$$(7.4) \quad \begin{aligned} \frac{dG_t}{G_{t-}} &= r_t^G dt + v_t^G \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^G(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \} \\ &= r_t^B dt + v_t^G \cdot d\tilde{W}_t^{\mathbf{B}} + \int_{\mathbb{R}^{d'}} H_t^G(z) \{ \nu(dt \times dz) - \tilde{\lambda}_t^{\mathbf{B}} \tilde{\phi}^{\mathbf{B}}(z) dz dt \} \end{aligned}$$

for every $t \in \mathbf{T}$, where

$$r_t^G = r_t^B + (\gamma v_t^{\bar{c}} + v_t^p) \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - e^{-\left(\gamma J_{\bar{c}}(z) + J_p(z)\right)} \right) H_t^G(z) \phi(z) dz,$$

or equivalently

$$\begin{aligned}
(7.5) \quad d \ln G_t &= \left\{ r_t^B + \left(\gamma v_t^{\bar{c}} + v_t^p - \frac{1}{2} v_t^G \right) \cdot v_t^G \right. \\
&\quad \left. - \lambda_t \int_{\mathbb{R}^{d'}} e^{-(\gamma J_{\bar{c}}(z) + J_p(z))} H_t^G(z) \phi(z) dz \right\} dt \\
&\quad + v_t^G \cdot dW_t + \int_{\mathbb{R}^{d'}} \ln(1 + H_t^G(z)) \nu(dt \times dz) \\
&= \left\{ r_t^B - \frac{1}{2} \|v_t^G\|^2 - \tilde{\lambda}_t^{\mathbf{B}} \int_{\mathbb{R}^{d'}} H_t^G(z) \tilde{\phi}^{\mathbf{B}}(z) dz \right\} dt \\
&\quad + v_t^G \cdot d\tilde{W}_t^{\mathbf{B}} + \int_{\mathbb{R}^{d'}} \ln(1 + H_t^G(z)) \nu(dt \times dz)
\end{aligned}$$

for every $t \in \mathbf{T}$.

(d) The dynamics of nominal T -forward rate process f_t^T satisfies

$$\begin{aligned}
(7.6) \quad df_t^T &= -a_t^T dt - \frac{\partial v_t^T}{\partial T} \cdot dW_t - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^T}{\partial T}(z)}{1 + H_t^T(z)} \nu(dt \times dz) \\
&= \left(v_t^T \cdot \frac{\partial v_t^T}{\partial T} + \tilde{\lambda}_t^{\mathbf{B}} \int_{\mathbb{R}^{d'}} \frac{\partial H_t^T}{\partial T}(z) \tilde{\phi}^{\mathbf{B}}(z) dz \right) dt \\
&\quad - \frac{\partial v_t^T}{\partial T} \cdot d\tilde{W}_t^{\mathbf{B}} - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^T}{\partial T}(z)}{1 + H_t^T(z)} \nu(dt \times dz)
\end{aligned}$$

for every $t \in [0, T)$, where

$$a_t^T = (\gamma v_t^{\bar{c}} + v_t^p - v_t^T) \cdot \frac{\partial v_t^T}{\partial T} - \lambda_t \int_{\mathbb{R}^{d'}} e^{-(\gamma J_{\bar{c}}(z) + J_p(z))} \frac{\partial H_t^T}{\partial T}(z) \phi(z) dz.$$

(e) The dynamics of nominal nominal-risk-free rate process r_t^B satisfies

$$\begin{aligned}
(7.7) \quad dr_t^B &= \left(\frac{\partial f_t^t}{\partial T} - a_t^t \right) dt - \frac{\partial v_t^t}{\partial T} \cdot dW_t - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^t}{\partial T}(z)}{1 + H_t^t(z)} \nu(dt \times dz) \\
&= \left(\frac{\partial f_t^t}{\partial T} + v_t^t \cdot \frac{\partial v_t^t}{\partial T} + \tilde{\lambda}_t^{\mathbf{B}} \int_{\mathbb{R}^{d'}} \frac{\partial H_t^t}{\partial T}(z) \tilde{\phi}^{\mathbf{B}}(z) dz \right) dt \\
&\quad - \frac{\partial v_t^t}{\partial T} \cdot d\tilde{W}_t^{\mathbf{B}} - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^t}{\partial T}(z)}{1 + H_t^t(z)} \nu(dt \times dz)
\end{aligned}$$

for every $t \in [0, T)$.

2. Under Assumptions 8, 4, 2, 5, 7, and 9, for a given ASM equilibrium $((\bar{c})_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \mathcal{B}$ and the associated equilibrium state price, the following hold:

(a) Market prices of nominal diffusive risk $v_t^{\mathbf{B}}$ and of nominal jump risk $H_t^{\mathbf{B}}(z) \lambda_t(dz)$ satisfy

$$\begin{aligned}
(7.8) \quad v_t^{\mathbf{B}} &= \bar{\gamma}_t v_t^{\bar{c}} + \bar{\zeta}_t v_t^Y + v_t^p = \frac{1}{\eta} v_t^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y + v_t^p, \\
H_t^{\mathbf{B}}(z) \lambda_t(dz) &= \lambda_t \frac{\bar{H}_t(z) + H_t^p(z)}{1 + H_t^p(z)} \phi(z) dz \\
&= \lambda_t \left(1 - e^{-\left(\frac{1}{\eta} J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z) \right)} \right) \phi(z) dz \quad \forall z \in \mathbb{R}^{d'},
\end{aligned}$$

- for every $t \in \mathbf{T}$, where $J_p(z) = -\left(\frac{1}{\eta}J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma}J_Y(z) + J_{\mathbf{B}}(z)\right)$ and satisfies $H_t^p(z) = e^{J_p(z)} - 1$ for every $(t, z) \in \mathbf{T} \times \mathbb{R}^{d'}$.
- (b) The $\tilde{\mathbb{P}}^{\mathbf{B}}$ -Wiener process $\tilde{W}_t^{\mathbf{B}}$ and the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel $\tilde{\lambda}_t^{\mathbf{B}} \tilde{\phi}^{\mathbf{B}}(z) dz$ of marked point process ν satisfy

(7.9)

$$\tilde{W}_t^{\mathbf{B}} = W_t + \int_0^t \left(\gamma v_s^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_s^Y + v_s^p \right) ds,$$

$$\tilde{\lambda}_t^{\mathbf{B}} = \iota \lambda_t, \quad \tilde{\phi}^{\mathbf{B}}(z) = \iota^{-1} e^{-\left(\gamma J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z)\right)} \phi(z) \quad \forall z \in \mathbb{R}^{d'},$$

for every $t \in \mathbf{T}$ where $\iota = \int_{\mathbb{R}^{d'}} e^{-\left(\gamma J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z)\right)} \phi(z) dz$.

- (c) The dynamics of nominal gain process
- G
- satisfies

$$(7.10) \quad \frac{dG_t}{G_{t-}} = r_t^G dt + v_t^G \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^G(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}$$

for every $t \in \mathbf{T}$, where

$$r_t^G = r_t^B + \left(\gamma v_t^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y + v_t^p \right) \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - e^{-\left(\gamma J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z)\right)} \right) H_t^G(z) \phi(z) dz,$$

or equivalently

$$(7.11) \quad d \ln G_t = \left\{ r_t^G + \left(\gamma v_t^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y + v_t^p - \frac{1}{2} v_t^G \right) \cdot v_t^G \right. \\ \left. - \lambda_t \int_{\mathbb{R}^{d'}} e^{-\left(\gamma J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z)\right)} H_t^G(z) \phi(z) dz \right\} dt \\ + v_t^G \cdot dW_t + \int_{\mathbb{R}^{d'}} \ln(1 + H_t^G(z)) \nu(dt \times dz)$$

for every $t \in \mathbf{T}$.

- (d) The dynamics of nominal
- T
- forward rate process
- f_t^T
- satisfies

$$(7.12) \quad df_t^T = -a_t^T dt - \frac{\partial v_t^T}{\partial T} \cdot dW_t - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^T}{\partial T}(z)}{1 + H_t^T(z)} \nu(dt \times dz)$$

for every $t \in [0, T)$, where

$$a_t^T = \left(\gamma v_t^{\bar{c}} + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} v_t^Y + v_t^p - v_t^T \right) \cdot \frac{\partial v_t^T}{\partial T} - \lambda_t \int_{\mathbb{R}^{d'}} e^{-\left(\gamma J_{\bar{c}}(z) + \frac{\gamma - \frac{1}{\eta}}{1 - \gamma} J_Y(z) + J_p(z)\right)} \frac{\partial H_t^T}{\partial T}(z) \phi(z) dz.$$

- (e) The dynamics of nominal nominal-risk-free rate process
- r_t^B
- satisfies

$$(7.13) \quad dr_t^B = \left(\frac{\partial f_t^t}{\partial T} - a_t^t \right) dt - \frac{\partial v_t^t}{\partial T} \cdot dW_t - \int_{\mathbb{R}^{d'}} \frac{\frac{\partial H_t^t}{\partial T}(z)}{1 + H_t^t(z)} \nu(dt \times dz)$$

for every $t \in [0, T)$.

Remark 14. Heath, Jarrow, and Morton [38] derived the dynamics of arbitrage-free forward rate processes under Wiener information. Björk, Kabanov, and Runggaldier [11] extended the result to jump-Wiener information. SDDs (7.6) and (7.12) of equilibrium forward rate processes are GE versions of SDDs of arbitrage-free forward rate processes given in Björk, Kabanov, and Runggaldier [11].

Proof. Proofs of 1 and 2 are essentially same, so we prove just 1. Comparing the arbitrage-free equation (2.2) with the CCAPM equation (5.9), we have

$$(7.14) \quad \begin{aligned} v_t^{\mathbf{B}} &= \gamma_t^{\hat{\alpha}} v_t^{\bar{c}} + v_t^p, \\ H_t^{\mathbf{B}}(z) &= \frac{H_t^{\hat{\alpha}}(z) + H_t^p(z)}{1 + H_t^p(z)} = 1 - e^{-(\gamma J_{\bar{c}}(z) + J_p(z))} \quad \forall z \in \mathbb{R}^{d'}, \end{aligned}$$

for every $t \in \mathbf{T}$. Substituting (5.14) and (7.1) into (7.14) yields (7.2). Substituting (7.2) into (2.3) and (2.4), we obtain (7.3). Then, it is straightforward to have SDDEs (7.4) and (7.5) of gain process. Finally, we obtain SDDEs (7.6) and (7.7) of forward rate and nominal-risk-free rate processes using Proposition 2.2 in Björk, Kabanov, and Runggaldier [11]. \square

7.2. Affine Jump-Diffusion Model with Jump-Diffusion Volatility. Bakshi, Cao, and Chen [7], and Bates [10] studied jump-diffusion stock index models with diffusion volatility and found that models do not explain the level of skewness implied by the volatility smirk observed in market data. Duffie, Pan, and Singleton [22] proposed a jump-diffusion stock index model (Duffie-Pan-Singleton model, hereafter) with jump-diffusion volatility as an example in the class of AJD models, and showed that the model could explain the level of skewness. Eraker, Johannes, and Polson [28] estimated a simplified version of the Duffie-Pan-Singleton model using the MCMC method, and the test result showed that their version of the Duffie-Pan-Singleton model can capture the dynamics of the return rate of U.S. stock index better than nested models such as Heston's diffusion model with diffusion volatility (Heston [39]) and Bakshi-Cao-Chen's jump-diffusion model with diffusion volatility (Bakshi, Cao, and Chen [7]). The Duffie-Pan-Singleton model is an arbitrage-free model in which the dynamics of a stock index process is exogenously given under a risk-neutral measure such that the market price of diffusive risk is unknown, and that the market price of jump risk is exogenously given. We present a GE model in which equilibrium prices conform to the Duffie-Pan-Singleton model's specification.

The Duffie-Pan-Singleton model is a 2-dimensional AJD model of the return rate of a stock and volatility with 4-dimensional space of jump source's magnitude. Our GE version of the Duffie-Pan-Singleton model is a 4-dimensional AJD model of the return rate of a stock, the growth rate of consumption, the inflation rate, and volatility, with 8-dimensional space of jump source's magnitude.

- Assumption 10.**
1. *The dimensionality of Wiener process W is four, i.e. $d = 4$.*
 2. *The dimensionality of space of jump source's magnitude is eight, i.e. $d' = 8$, and the \mathbb{P} -intensity kernel $\lambda_t \phi(z) dz$ of marked point process ν is given by*

$$(7.15) \quad \begin{aligned} \lambda_t &= \lambda \quad \forall t \in \mathbf{T}, \\ \phi(z) &= q^{(1)} \phi^{(1)}(z) + q^{(V)} \phi^{(V)}(z) + q^{(2)} \phi^{(2)}(z) \quad \forall z \in \mathbf{R}^8, \end{aligned}$$

where $\lambda \in \mathbb{R}_+$, $q = (q^{(1)}, q^{(V)}, q^{(2)})' \in \Delta_+^3$, $z = (z^{(1)}, z^{(V)}, z^{(2)})' \in \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^4$, and

$$(7.16) \quad \begin{aligned} \phi^{(1)}(z) &= \delta_{\{(z^{(V)}, z^{(2)})=(0,0)\}}(z^{(V)}, z^{(2)}) (2\pi)^{-\frac{3}{2}} \exp\left[-\frac{1}{2} \sum_{j=1}^3 (z_j^{(1)})^2\right], \\ \phi^{(V)}(z) &= \delta_{\{(z^{(1)}, z^{(2)})=(0,0)\}}(z^{(1)}, z^{(2)}) 1_{\{z^{(V)} \geq 0\}}(z^{(V)}) e^{-z^{(V)}}, \\ \phi^{(2)}(z) &= \delta_{\{(z^{(1)}, z^{(V)})=(0,0)\}}(z^{(1)}, z^{(V)}) \\ &\quad \times (2\pi)^{-\frac{3}{2}} \exp\left[-\frac{1}{2} \sum_{j=1}^3 (z_j^{(2)} - \rho_j' z_4^{(2)})^2\right] 1_{\{z_4^{(2)} \geq 0\}}(z_4^{(2)}) e^{-z_4^{(2)}} \end{aligned}$$

where δ is the Dirac's delta function.

For simplicity, we suppose the case of ASUs and consider a security with a payoff only at time T^\dagger . Consider a fixed ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with $\mathbf{B} \in \bar{\mathbf{B}}$ and the associated equilibrium state price $\pi_t = u_c^{\hat{\alpha}}(t, \bar{c}_t)$. Then, the equilibrium nominal price S of the security coincides with its nominal gain process, and has the following SDDE representation of the form

$$(7.17) \quad \frac{dS_t}{S_{t-}} = r_t^S dt + v_t^S \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^S(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dt \} \quad \forall t \in \mathbf{T}$$

where $v^S \in \prod_{j=1}^d \mathcal{L}^2$, $H^S \in \mathcal{L}^1(\lambda_t(dz) \times dt)$, and r^S satisfies

$$r_t^S = r_t^B + (\gamma v_t^{\bar{c}} + v_t^p) \cdot v_t^S + \lambda_t \int_{\mathbb{R}^{d'}} \left(1 - e^{-(\gamma J_{\bar{c}}(z) + J_p(z))}\right) H_t^S(z) \phi(z) dz.$$

Then, our GE version of the Duffie-Pan-Singleton model is shown in the following corollary. Note that it is straightforward to derive the quasianalytic equilibrium price formula for a European option on the security applying formulas given in Duffie, Pan, and Singleton [22] to the risk-neutral equilibrium dynamics (7.18) of $(\ln S, V)'$.

Corollary 3.1. *Suppose that Assumptions 6, 2, 5, 9, and 10 are satisfied, and that the bond price family $\mathbf{B} \in \bar{\mathbf{B}}$ is such that the nominal-risk-free rate is constant, i.e. $r_t^B = r$ ν -a.e for some $r \in \mathbb{R}_{++}$. For a given ASM equilibrium $((\hat{c}^i)_{i \in \mathbf{I}}, p, \mathbf{B})$ for \mathbf{E} with the associated equilibrium state price $\pi_t = u_c^{\hat{\alpha}}(t, \bar{c}_t)$, assume that the following conditions hold:*

1. *Volatilities satisfy*

$$\begin{pmatrix} v_t^{S'} \\ v_t^{\bar{c}'} \\ v_t^{p'} \end{pmatrix} = \sqrt{V_t} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varrho_{\bar{c}1} \varsigma_{\bar{c}} & \varrho_{\bar{c}2} \varsigma_{\bar{c}} & \varrho_{\bar{c}3} \varsigma_{\bar{c}} & 0 \\ \varrho_{p1} \varsigma_p & \varrho_{p2} \varsigma_p & \varrho_{p3} \varsigma_p & \varrho_{p4} \varsigma_p \end{pmatrix}$$

for every $t \in \mathbf{T}$, where $(\varrho_{\bar{c}1}, \varrho_{\bar{c}2}, \varrho_{\bar{c}3})' \in \{x \in \mathbb{R}^3 : \|x\| = 1\}$, $(\varrho_{\bar{c}1}, \varrho_{\bar{c}2}, \varrho_{\bar{c}3}, \varrho_{\bar{c}4}) \in \{x \in \mathbb{R}^4 : \|x\| = 1\}$, $\varsigma_{\bar{c}}, \varsigma_p \in \mathbb{R}_{++}$, and V is a positive real-valued process satisfying

$$dV_t = \kappa(\bar{V} - V_t) dt + \sqrt{V_t} (\varrho_{V1} \varsigma_V, \varrho_{V2} \varsigma_V, 0, 0) dW_t + \int_{\mathbb{R}^8} J_V(z) \{ \nu(dt \times dz) - \lambda \phi(z) dt \}$$

where κ, \bar{V} are positive constants, $(\varrho_{V1}, \varrho_{V2}) \in \{x \in \mathbb{R}^2 : \|x\| = 1\}$, $\varsigma_V \in \mathbb{R}_{++}$, and J_V is a function on \mathbb{R}^8 satisfying

$$J_V(z) = 1_{\{z^{(V)} \neq 0\}}(z) \sigma_V^{(V)} z^{(V)} + 1_{\{z_4^{(2)} \neq 0\}}(z) \sigma_V^{(2)} z_4^{(2)}$$

where $(\sigma_V^{(V)}, \sigma_V^{(2)}) \in \mathbb{R}_{++}^2$.

2. Jump magnitudes satisfy

$$\begin{pmatrix} J_S(z) \\ J_{\bar{c}}(z) \\ J_p(z) \end{pmatrix} = 1_{\{z^{(1)} \neq 0\}}(z) \left\{ \begin{pmatrix} \sigma_S^{(1)} & 0 & 0 \\ \rho_{\bar{c}1}^{(1)} \sigma_{\bar{c}}^{(1)} & \rho_{\bar{c}2}^{(1)} \sigma_{\bar{c}}^{(1)} & 0 \\ \rho_{p1}^{(1)} \sigma_p^{(1)} & \rho_{p2}^{(1)} \sigma_p^{(1)} & \rho_{p3}^{(1)} \sigma_p^{(1)} \end{pmatrix} z^{(1)} + \begin{pmatrix} \mu_S^{(1)} \\ \mu_{\bar{c}}^{(1)} \\ \mu_p^{(1)} \end{pmatrix} \right\} \\ + 1_{\{z^{(2)} \neq 0\}}(z) \left\{ \begin{pmatrix} \sigma_S^{(2)} & 0 & 0 \\ \rho_{\bar{c}1}^{(2)} \sigma_{\bar{c}}^{(2)} & \rho_{\bar{c}2}^{(2)} \sigma_{\bar{c}}^{(2)} & 0 \\ \rho_{p1}^{(2)} \sigma_p^{(2)} & \rho_{p2}^{(2)} \sigma_p^{(2)} & \rho_{p3}^{(2)} \sigma_p^{(2)} \end{pmatrix} z^{(2)} + \begin{pmatrix} \mu_S^{(2)} \\ \mu_{\bar{c}}^{(2)} \\ \mu_p^{(2)} \end{pmatrix} \right\}$$

for every $t \in \mathbf{T}$ where $J_S(z) = \ln(1 + H_t^S(z))$ for every $(t, z) \in \mathbf{T} \times R^d$, and $(\rho_{\bar{c}1}^{(j)}, \rho_{\bar{c}2}^{(j)}) \in \{x \in \mathbb{R}^2 : \|x\| = 1\}$, $(\rho_{p1}^{(j)}, \rho_{p2}^{(j)}, \rho_{p3}^{(j)}) \in \{x \in \mathbb{R}^3 : \|x\| = 1\}$, $\mu_S^{(j)}, \mu_{\bar{c}}^{(j)}, \mu_p^{(j)} \in \mathbb{R}$ for $j = 1, 2$.

Then, the following hold:

1. Under the agents' common belief \mathbb{P} , the dynamics of system of processes $(S, V, \bar{c}, p)'$ satisfies (E.1) in Appendix E.
2. Under the risk-neutral measure $\mathbb{P}^{\mathbf{B}}$, the dynamics of system of processes $(S, V, \bar{c}, p)'$ satisfies the following system of SDEs:

$$(7.18) \quad d \begin{pmatrix} \ln S_t \\ V_t \\ \ln \bar{c}_t \\ \ln p_t \end{pmatrix} = \begin{pmatrix} r - \tilde{\lambda} \tilde{\beta}_S - \frac{1}{2} V_t \\ \kappa \bar{V} - \lambda \beta_V + \left(\sum_{j=1}^2 (\gamma \varrho_{Vj} \varrho_{\bar{c}j} \varsigma_V \varsigma_{\bar{c}} + \varrho_{Vj} \varrho_{pj} \varsigma_V \varsigma_{\bar{c}}) - \kappa \right) V_t \\ r_{\bar{c}}^{\bar{c}} - \lambda \beta_{\bar{c}} + \left(\gamma \varsigma_{\bar{c}}^2 + \sum_{j=1}^3 (\varrho_{\bar{c}j} \varrho_{pj} \varsigma_{\bar{c}} \varsigma_p) - \frac{1}{2} \varsigma_{\bar{c}}^2 \right) V_t \\ r_t^p - \lambda \beta_p + \left(\varsigma_p^2 - \sum_{j=1}^3 (\varrho_{\bar{c}j} \varrho_{pj} \gamma \varsigma_{\bar{c}} \varsigma_p) - \frac{1}{2} \varsigma_p^2 \right) V_t \end{pmatrix} dt + \\ \sqrt{V_t} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varrho_{V1} \varsigma_V & \varrho_{V2} \varsigma_V & 0 & 0 \\ \varrho_{\bar{c}1} \varsigma_{\bar{c}} & \varrho_{\bar{c}2} \varsigma_{\bar{c}} & \varrho_{\bar{c}3} \varsigma_{\bar{c}} & 0 \\ \varrho_{p1} \varsigma_p & \varrho_{p2} \varsigma_p & \varrho_{p3} \varsigma_p & \varrho_{p4} \varsigma_p \end{pmatrix} d\tilde{W}_t^{\mathbf{B}} + \int_{\mathbb{R}^8} \begin{pmatrix} J_S(z) \\ J_V(z) \\ J_{\bar{c}}(z) \\ J_p(z) \end{pmatrix} \nu(dt \times dz)$$

for every $t \in \mathbf{T}$, where $(\tilde{\beta}_S, \beta_V, \beta_{\bar{c}}, \beta_p)'$ satisfies (E.2) in Appendix E, and the $\tilde{\mathbb{P}}^{\mathbf{B}}$ -intensity kernel $\tilde{\lambda}_t^{\mathbf{B}} \tilde{\phi}^{\mathbf{B}}(z) dz$ of marked point process ν satisfies

$$\begin{aligned} \tilde{\lambda}_t^{\mathbf{B}} &= \tilde{\lambda}^{\mathbf{B}} \stackrel{\text{def}}{=} (\iota^{(1)} q^{(1)} + q^{(V)} + \iota^{(2)} q^{(2)}) \lambda, \\ \tilde{\phi}^{\mathbf{B}}(z) &= \tilde{q}^{(1)} \tilde{\phi}^{(1)}(z) + \tilde{q}^{(V)} \phi^{(V)}(z) \tilde{q}^{(2)} \tilde{\phi}^{(2)}(z) \quad \forall z \in \mathbb{R}^8, \end{aligned}$$

where

$$\begin{aligned}
\iota^{(1)} &= \exp \left[-(\gamma \mu_{\bar{c}}^{(1)} + \mu_p^{(1)}) + \frac{1}{2} \left((\gamma \sigma_{\bar{c}}^{(1)})^2 + (\sigma_p^{(1)})^2 \right) + \sum_{j=1}^3 (\gamma \rho_{\bar{c}j}^{(1)} \rho_{pj}^{(1)} \sigma_{\bar{c}}^{(1)} \sigma_p^{(1)}) \right], \\
\iota^{(2)} &= \frac{1}{1 + \sum_{j=1}^3 \rho'_j (\gamma \rho_{\bar{c}j}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{pj}^{(2)} \sigma_p^{(2)})} \\
&\quad \times \exp \left[-(\gamma \mu_{\bar{c}}^{(2)} + \mu_p^{(2)}) + \frac{1}{2} \left((\gamma \sigma_{\bar{c}}^{(2)})^2 + (\sigma_p^{(2)})^2 \right) + \sum_{j=1}^3 (\gamma \rho_{\bar{c}j}^{(2)} \rho_{pj}^{(2)} \sigma_{\bar{c}}^{(2)} \sigma_p^{(2)}) - \sum_{j=1}^3 \rho'_j (\gamma \rho_{\bar{c}j}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{pj}^{(2)} \sigma_p^{(2)}) \right], \\
\tilde{q}^{(1)} &= \frac{\iota^{(1)}}{\iota^{(1)} q^{(1)} + q^{(V)} + \iota^{(2)} q^{(2)}} q^{(1)}, \\
\tilde{q}^{(V)} &= \frac{1}{\iota^{(1)} q^{(1)} + q^{(V)} + \iota^{(2)} q^{(2)}} q^{(V)}, \\
\tilde{q}^{(2)} &= \frac{\iota^{(2)}}{\iota^{(1)} q^{(1)} + q^{(V)} + \iota^{(2)} q^{(2)}} q^{(2)}, \\
\tilde{\phi}^{(1)}(z) &= \delta_{\{(z^{(V)}, z^{(2)}) = (0,0)\}} (z^{(V)}, z^{(2)}) (2\pi)^{-\frac{3}{2}} \exp \left[-\frac{1}{2} \sum_{j=1}^3 (z_j^{(1)} + \gamma \rho_{\bar{c}j}^{(1)} \sigma_{\bar{c}}^{(1)} + \rho_{pj}^{(1)} \sigma_p^{(1)})^2 \right], \\
\tilde{\phi}^{(2)}(z) &= \delta_{\{(z^{(1)}, z^{(V)}) = (0,0)\}} (z^{(1)}, z^{(V)}) (2\pi)^{-\frac{3}{2}} \exp \left[-\frac{1}{2} \sum_{j=1}^3 \left\{ z_j^{(2)} - \left(\rho'_j z_4^{(2)} - (\gamma \rho_{\bar{c}j}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{pj}^{(2)} \sigma_p^{(2)}) \right) \right\}^2 \right] \\
&\quad \times \mathbf{1}_{\{z_4^{(2)} \geq 0\}} (z_4^{(2)}) \left\{ 1 - \sum_{j=1}^3 \rho'_j (\gamma \rho_{\bar{c}j}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{pj}^{(2)} \sigma_p^{(2)}) \right\}^{-1} \exp \left[-\left\{ 1 - \sum_{j=1}^3 \rho'_j (\gamma \rho_{\bar{c}j}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{pj}^{(2)} \sigma_p^{(2)}) \right\} z_4^{(2)} \right].
\end{aligned}$$

Remark 15. In this model, the following three types of jumps are incorporated:

1. Jumps in $(\ln S, \ln \bar{c}, \ln p)'$ with arrival intensity $q^{(1)}\lambda$ and multivariate normally distributed jump size with mean vector $(\mu_S^{(1)}, \mu_{\bar{c}}^{(1)}, \mu_p^{(1)})'$ and variance-covariance matrix

$$\Sigma^{(1)} = \begin{pmatrix} \left(\sigma_S^{(1)} \right)^2 & \rho_{\bar{c}1}^{(1)} \sigma_S^{(1)} \sigma_{\bar{c}}^{(1)} & \rho_{p1}^{(1)} \sigma_S^{(1)} \sigma_p^{(1)} \\ \rho_{\bar{c}1}^{(1)} \sigma_S^{(1)} \sigma_{\bar{c}}^{(1)} & \left(\rho_{\bar{c}2}^{(1)} \sigma_{\bar{c}}^{(1)} \right)^2 & \rho_{\bar{c}2}^{(1)} \rho_{p2}^{(1)} \sigma_{\bar{c}}^{(1)} \sigma_p^{(1)} \\ \rho_{p1}^{(1)} \sigma_S^{(1)} \sigma_p^{(1)} & \rho_{\bar{c}2}^{(1)} \rho_{p2}^{(1)} \sigma_{\bar{c}}^{(1)} \sigma_p^{(1)} & \left(\rho_{p2}^{(1)} \sigma_p^{(1)} \right)^2 \end{pmatrix}.$$

2. Jumps in V with arrival intensity $q^{(V)}\lambda$ and exponentially distributed jump size with mean $\sigma_V^{(V)}$.
3. Simultaneous correlated jumps in $\ln S$ and V with arrival intensity $q^{(2)}\lambda$. The marginal distribution of the jump size in V is exponential with mean $\sigma_V^{(2)}$. Conditional on a realization $\sigma_V^{(2)} z_4^{(2)}$ of the jump size in V , the jump size in $(\ln S, \ln \bar{c}, \ln p)'$ is multivariate normally distributed with mean vector

$(\mu_S^{(2)} + \rho'_1 z_4^{(2)}, \mu_{\bar{c}}^{(2)} + \rho'_2 z_4^{(2)}, \mu_p^{(2)} + \rho'_3 z_4^{(2)})'$ and variance-covariance matrix

$$\Sigma^{(2)} = \begin{pmatrix} \left(\sigma_S^{(2)}\right)^2 & \rho_{\bar{c}1}^{(2)} \sigma_S^{(2)} \sigma_{\bar{c}}^{(2)} & \rho_{p1}^{(2)} \sigma_S^{(2)} \sigma_p^{(2)} \\ \rho_{\bar{c}1}^{(2)} \sigma_S^{(2)} \sigma_{\bar{c}}^{(2)} & \left(\rho_{\bar{c}2}^{(2)} \sigma_{\bar{c}}^{(2)}\right)^2 & \rho_{\bar{c}2}^{(2)} \rho_{p2}^{(2)} \sigma_{\bar{c}}^{(2)} \sigma_p^{(2)} \\ \rho_{p1}^{(2)} \sigma_S^{(2)} \sigma_p^{(2)} & \rho_{\bar{c}2}^{(2)} \rho_{p2}^{(2)} \sigma_{\bar{c}}^{(2)} \sigma_p^{(2)} & \left(\rho_{p2}^{(2)} \sigma_p^{(2)}\right)^2 \end{pmatrix}.$$

APPENDIX A. MARKED POINT PROCESS

A.1. Definitions. We consider a double sequence $(s_n, Z_n)_{n \in \mathbb{N}}$ where s_n is the occurrence time of n th jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n . Define the random counting measure $\nu(dt \times dz)$ by

$$\nu([0, t] \times A) = \sum_{n \in \mathbb{N}} 1_{\{s_n \leq t, Z_n \in A\}} \quad \forall (t, A) \in [0, T^\dagger] \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the \mathbb{Z} -marked point process.

Let λ be such that

1. For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on \mathbb{Z} .
2. For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is \mathcal{P} -measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

If the equation

$$\mathbb{E} \left[\int_0^{T^\dagger} Y_s \nu(ds \times A) \right] = \mathbb{E} \left[\int_0^{T^\dagger} Y_s \lambda_s(A) ds \right] \quad \forall A \in \mathcal{Z}$$

holds for any nonnegative \mathcal{P} -measurable process Y , then we say that the marked point process $\nu(dt \times dz)$ has the \mathbb{P} -intensity kernel $\lambda_t(dz)$.

A.2. Integration Theorem. Let $\nu(dt \times dz)$ be a \mathbb{Z} -marked point process with the \mathbb{P} -intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P} \otimes \mathcal{Z}$ -measurable function. It follows that:

1. If we have

$$\mathbb{E} \left[\int_0^{T^\dagger} \int_{\mathbb{Z}} |H_s(z)| \lambda_s(z) ds \right] < \infty,$$

then the process $\int_0^t \int_{\mathbb{Z}} H_s(z) \{ \nu(ds \times dz) - \lambda_s(dz) ds \}$ is a \mathbb{P} -martingale.

2. If $H \in \mathcal{L}(\lambda_t(dz) \times dt)$, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z) \{ \nu(ds \times dz) - \lambda_s(dz) ds \}$ is a local \mathbb{P} -martingale.

Proof. See p. 235 in Brémaud [14]. □

APPENDIX B. ITO'S FORMULA AND GIRSANOV'S THEOREM

B.1. Ito's Formula. Let $X = (X^1, \dots, X^d)'$ be a d -dimensional semimartingales, and g be a real-valued \mathbf{C}^2 -function on \mathbb{R}^d . Then, $g(X)$ is a semimartingale of the

form

$$g(X_t) = g(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} g(X_{s-}) dX_s^i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-}) d\langle X^{ic}, X^{jc} \rangle$$

$$+ \sum_{0 \leq s \leq t} \left\{ g(X_s) - g(X_{s-}) + \sum_{i=1}^d \frac{\partial}{\partial x_i} g(X_{s-}) \Delta X_s^i \right\}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc} \rangle$ is the quadratic covariation of X^{ic} and X^{jc} .

B.2. Girsanov's Theorem.

1. Let $v \in \prod_{j=1}^d \mathcal{L}^2$ and $H \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Define the process A by

$$\frac{dA_t}{A_{t-}} = -v_t \cdot dW_t - \int_{\mathbb{Z}} H_t(z) \{ \nu(dt \times dz) - \lambda_t(dz) dt \} \quad \forall t \in \mathbf{T}$$

with $A_0 = 1$, and suppose $\mathbb{E}[A_{T^+}] = 1$. Then there exists a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ given by the Radon-Nikodym derivative

$$d\tilde{\mathbb{P}} = A_{T^+} d\mathbb{P}$$

such that:

- (a) The measure $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} .
- (b) The process given by

$$\tilde{W}_t = W_t + \int_0^t v_s ds \quad \forall t \in \mathbf{T}$$

is a $\tilde{\mathbb{P}}$ -Wiener process.

- (c) The marked point process $\nu(dt \times dz)$ has the $\tilde{\mathbb{P}}$ -intensity kernel such that

$$\tilde{\lambda}_t(dz) = (1 - H_t(z)) \lambda_t(dz) \quad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}.$$

2. Every probability measure equivalent to \mathbb{P} has the structure above.

APPENDIX C. DEFINITIONS ON ARBITRAGE-FREE PRICING THEORY

C.1. Feasible, Self-Financing, and Admissible Portfolios. Let X denote a real-valued \mathcal{P} -measurable process. The *discounted process of X* is denoted by \tilde{X} . Thus $\tilde{X} = \frac{X}{B}$. We write $\tilde{\mathbf{B}} = (B^*, (B^{T*})_{T \in \mathbf{T}})$. We introduce notions of the *feasible*, *self-financing*, and *admissible portfolios*.

Definition 8. Let $\mathbf{B} \in \mathcal{B}$.

1. A portfolio ϑ is a *feasible portfolio at \mathbf{B}* if and only if it follows that:

$$\int_t^{T^\dagger} |B_t^T| |\vartheta_t^1(dT)| < \infty \quad \mathbb{P}\text{-a.s.} \quad \forall t \in \mathbf{T},$$

$$B_t r_t^B \vartheta_t^0, \int_t^{T^\dagger} |B_t^T r_t^T| |\vartheta_t^1(dT)| \in \mathcal{L}^1, \quad \int_t^{T^\dagger} \|B_t^T v_t^T\| |\vartheta_t^1(dT)| \in \mathcal{L}^2,$$

$$\int_t^{T^\dagger} |B_t^T H_t^T(z)| |\vartheta_t^1(dT)| \in \mathcal{L}^1(\lambda_t(dz) \times dt).$$

Let $\Theta(\mathbf{B})$ denote the class of feasible portfolios at \mathbf{B} .

2. A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at \mathbf{B} is a *self-financing portfolio at \mathbf{B}* if and only if its value process satisfies

$$\mathcal{V}_t^{\mathbf{B}}(\vartheta) = \mathcal{V}_t^{\mathbf{B}}(\vartheta) + \int_0^t \vartheta_s^0 dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_s^1(dT) dB_s^T \quad \forall t \in \mathbf{T}.$$

3. A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at \mathbf{B} is an *admissible portfolio at \mathbf{B}* if and only if $\tilde{\mathcal{V}}^{\mathbf{B}}(\vartheta) \stackrel{\text{def}}{=} \frac{\mathcal{V}^{\mathbf{B}}(\vartheta)}{B}$ is bounded below \mathbb{P} -a.s. Let $\underline{\mathcal{Q}}(\tilde{\mathbf{B}})$ denote the class of admissible portfolios at \mathbf{B} .

C.2. Arbitrage-Free Markets and Risk-Neutral Measure. Definitions of *arbitrage portfolio*, *arbitrage-free*, and *risk-neutral measure* are given in the following.

Definition 9. Let $\mathbf{B} \in \mathcal{B}$.

1. A self-financing portfolio $\vartheta \in \Theta(\mathbf{B})$ at \mathbf{B} is an *arbitrage portfolio at \mathbf{B}* if and only if either of the following condition holds:
 - (a) $\mathcal{V}_0^{\mathbf{B}}(\vartheta) \leq 0$, and $\mathcal{V}_{T^\dagger}^{\mathbf{B}}(\vartheta) > 0$, i.e. $\mathcal{V}_{T^\dagger}^{\mathbf{B}}(\vartheta) \geq 0$ \mathbb{P} -a.s. and $\mathbb{P}(\{\mathcal{V}_{T^\dagger}^{\mathbf{B}}(\vartheta) > 0\}) > 0$.
 - (b) $\mathcal{V}_0^{\mathbf{B}}(\vartheta) < 0$, and $\mathcal{V}_{T^\dagger}^{\mathbf{B}}(\vartheta) \geq 0$ \mathbb{P} -a.s.
2. Markets are *arbitrage-free at \mathbf{B}* if and only if there exists no arbitrage portfolio in the class of admissible portfolios at \mathbf{B} .
3. A probability measure $\tilde{\mathbb{P}}^{\mathbf{B}}$ on (Ω, \mathcal{F}) is a *risk-neutral measure at \mathbf{B}* if and only if $\tilde{\mathbb{P}}^{\mathbf{B}}$ is equivalent to \mathbb{P} , and the discounted bond price family $\tilde{\mathbf{B}}$ is a local $\tilde{\mathbb{P}}^{\mathbf{B}}$ -martingale.

C.3. Contingent and Replicable Claims. We give notions of *contingent* and *replicable claims*.

Definition 10. Let $\mathbf{B} \in \mathcal{B}$.

1. For every $T \in (0, T^\dagger]$, a *contingent T -claim at \mathbf{B}* is a \mathcal{F}_T -measurable random variable X_T such that $\tilde{X}_T \in \mathbf{L}_+^\infty(\Omega, \mathcal{F}_T)$ where $\mathbf{L}^\infty(\Omega, \mathcal{F}_T)$ is the space of almost surely bounded \mathcal{F}_T -measurable random variables.
2. A contingent T -claim X_T is *replicable at \mathbf{B}* if and only if there exists an admissible self-financing portfolio $\vartheta \in \underline{\mathcal{Q}}(\tilde{\mathbf{B}})$ such that its value process satisfies $\mathcal{V}_T^{\mathbf{B}}(\vartheta) = X_T$.

APPENDIX D. PROOFS

D.1. Proof of Lemma 2. Let $0 \leq t < T \leq T^\dagger$. Recall $\pi_t = \frac{\Lambda_t^{\mathbf{B}}}{B_t} p_t$. It follows from Definition 6 and the \mathbb{P} -martingale property of $\Lambda^{\mathbf{B}}$ that

$$\begin{aligned} \mathbb{E}_t[\pi_T \check{G}_T] &= \mathbb{E}_t \left[\pi_T \check{S}_T + \pi_T \check{B}_T \int_0^T \frac{dD_s}{B_s} \right] \\ &= \mathbb{E}_t \left[\int_T^{T^\dagger} \pi_s \frac{dD_s}{p_s} + \Lambda_T^{\mathbf{B}} \int_0^T \frac{\pi_s dD_s}{\Lambda_s^{\mathbf{B}} p_s} \right] \\ &= \mathbb{E}_t \left[\int_T^{T^\dagger} \pi_s \frac{dD_s}{p_s} + \int_t^T \mathbb{E}_s \left[\Lambda_T^{\mathbf{B}} \frac{\pi_s dD_s}{\Lambda_s^{\mathbf{B}} p_s} \right] \right] + \mathbb{E}_t \left[\Lambda_T^{\mathbf{B}} \int_0^t \frac{\pi_s dD_s}{\Lambda_s^{\mathbf{B}} p_s} \right] \\ &= \mathbb{E}_t \left[\int_t^{T^\dagger} \pi_s \frac{dD_s}{p_s} \right] + \Lambda_t^{\mathbf{B}} \int_0^t \frac{dD_s}{B_s} = \pi_t \check{S}_t + \pi_t \check{B}_t \int_0^t \frac{dD_s}{B_s} = \pi_t \check{G}_t. \end{aligned}$$

In the similar way, we can show 2.

D.2. Proof of Proposition 1. Applying Ito's formula to the equilibrium state price process $\pi_t = u_c^{\hat{\alpha}}(t, \bar{c}_t)$, we have

(D.1)

$$r_t^\pi = \frac{1}{u_c^{\hat{\alpha}}(t, \bar{c}_t)} \left\{ u_{ct}^{\hat{\alpha}}(t, \bar{c}_t) + u_{cc}^{\hat{\alpha}}(t, \bar{c}_t) \bar{c}_t \left(r_t^{\bar{c}} - \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \phi(z) dz \right) + \frac{1}{2} u_{ccc}^{\hat{\alpha}}(t, \bar{c}_t) \bar{c}_t^2 \|v_t^{\bar{c}}\|^2 \right\} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^\pi(z) \phi(z) dz,$$

$$v_t^\pi = -\gamma_t^{\hat{\alpha}} v_t^{\bar{c}},$$

$$H_t^\pi(z) = -H_t^{\hat{\alpha}}(z).$$

Thus, it follows from (4.9) and (4.8) that for every $t \in \mathbf{T}$,

$$(D.2) \quad \frac{d(\pi_t \check{G}_t)}{\pi_{t-} \check{G}_{t-}} = \left\{ r_t^\pi + r_t^{\check{G}} + v_t^\pi \cdot v_t^{\check{G}} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^\pi(z) H_t^{\check{G}}(z) \phi(z) dz \right\} dt + (v_t^\pi + v_t^{\check{G}}) \cdot dW_t + \int_{\mathbb{R}^{d'}} \{ (1 + H_t^\pi(z))(1 + H_t^{\check{G}}(z)) - 1 \} \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}.$$

Since the process $\pi \check{G}$ is martingale under \mathbb{P} by Lemma 2, it follows from (D.2) that

$$(D.3) \quad r_t^{\check{G}} = -r_t^\pi - v_t^\pi \cdot v_t^{\check{G}} - \lambda_t \int_{\mathbb{R}^{d'}} H_t^\pi(z) H_t^{\check{G}}(z) \phi(z) dz \quad \forall t \in \mathbf{T}.$$

Hence, the real risk-free rate is given by $r_t^{\check{F}} = -r_t^\pi$ by definition. Therefore, (5.1) follows from (D.1). Moreover, substituting $r_t^\pi = -r_t^{\check{F}}$, $v_t^\pi = -\gamma_t^{\hat{\alpha}} v_t^{\bar{c}}$, and $H_t^\pi(z) = -H_t^{\hat{\alpha}}(z)$ into (D.3), we have (5.3).

D.3. Proof of Theorem 1. Applying Ito's formula to $G = p \check{G}$ yields

(D.4)

$$\frac{dG_t}{G_{t-}} = r_t^G dt + v_t^G \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^G(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \} \quad \forall t \in \mathbf{T}$$

where

$$(D.5) \quad \begin{aligned} r_t^G &= r_t^p + r_t^{\check{G}} + v_t^p \cdot v_t^{\check{G}} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^p(z) H_t^{\check{G}}(z) \phi(z) dz, \\ v_t^G &= v_t^p + v_t^{\check{G}}, \\ H_t^G(z) &= (1 + H_t^p(z))(1 + H_t^{\check{G}}(z)) - 1. \end{aligned}$$

Thus, by definition of the nominal-risk-free security, we have $v_t^{\check{B}} = -v_t^p$ and $H_t^{\check{B}}(z) = -\frac{H_t^p(z)}{1 + H_t^p(z)}$. Substituting them into (D.3), we have (5.6). Combining (5.6) with (5.3) yields (5.7). Next, it follows from (D.5) that

(D.6)

$$\begin{aligned} r_t^G - r_t^{\check{B}} &= r_t^{\check{G}} - r_t^{\check{B}} + v_t^p \cdot (v_t^{\check{G}} - v_t^{\check{B}}) + \lambda_t \int_{\mathbb{R}^{d'}} H_t^p(z) (H_t^{\check{G}}(z) - H_t^{\check{B}}(z)) \phi(z) dz \\ &= r_t^{\check{G}} - r_t^{\check{B}} + v_t^p \cdot v_t^{\check{G}} + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^p(z)}{1 + H_t^p(z)} H_t^{\check{G}}(z) \phi(z) dz. \end{aligned}$$

On the other hand, substituting $v_t^{\check{G}} = v_t^G - v_t^p$ and $H_t^{\check{G}}(z) = \frac{1+H_t^G(z)}{1+H_t^p(z)} - 1$ into (5.7) yields

$$(D.7) \quad r_t^{\check{G}} - r_t^{\check{B}} = \gamma_t^{\hat{\alpha}} v_t^{\check{c}} \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^{\hat{\alpha}}(z)}{1+H_t^p(z)} H_t^G(z) \phi(z) dz.$$

Substituting (D.7) into (D.6), we have

$$(D.8) \quad r_t^G - r_t^B = (\gamma_t^{\hat{\alpha}} v_t^{\check{c}} + v_t^p) \cdot v_t^G + \lambda_t \int_{\mathbb{R}^{d'}} \frac{H_t^{\hat{\alpha}}(z) + H_t^p(z)}{1+H_t^p(z)} H_t^G(z) \phi(z) dz.$$

D.4. Proof of Lemma 3. Let u^i be given by (5.11) for every $i \in \mathbf{I}$. Let $x \in \mathbb{R}_{++}$. Then, it follows from definition of c^* that for every $i \in \mathbf{I}$

$$(D.9) \quad \alpha_i e^{-\rho_i t} \left(\frac{c_i^*}{\gamma} \right)^{-\gamma} = \pi_t \quad \mu\text{-a.e.}$$

for some $\pi_t > 0$ and

$$(D.10) \quad \sum_{i \in \mathbf{I}} c_i^* = x.$$

The equation (D.9) is rewritten as

$$(D.11) \quad \frac{c_i^*}{\gamma} = \left(\frac{\alpha_i e^{-\rho_i t}}{\pi_t} \right)^{\frac{1}{\gamma}}.$$

Also, it follows from (D.10) and (D.11) that

$$(D.12) \quad \pi_t = \left(\sum_{i \in \mathbf{I}} (\alpha_i e^{-\rho_i t})^{\frac{1}{\gamma}} \right)^{\gamma} \left(\frac{x}{\gamma} \right)^{-\gamma}.$$

Substituting (D.9) and (D.10) into

$$u^\alpha(t, x) = \sum_{i \in \mathbf{I}} \alpha_i e^{-\rho_i t} \frac{\gamma}{1-\gamma} \left(\left(\frac{c_i^*}{\gamma} \right)^{1-\gamma} - 1 \right),$$

we have

$$(D.13) \quad u^\alpha(t, x) = \frac{\gamma}{1-\gamma} \left(\frac{x}{\gamma} \pi_t - \sum_{i \in \mathbf{I}} \alpha_i e^{-\rho_i t} \right).$$

Substituting (D.12) into (D.13) yields (5.12).

D.5. Proof of Theorem 2. Applying Ito's formula to $\pi_t = \exp \left(\int_0^t f_y(\bar{c}_s, Y_s) ds \right) f_c(\bar{c}_t, Y_t)$, we have

$$(D.14) \quad \frac{d\pi_t}{\pi_{t-}} = r_t^\pi dt + v_t^\pi \cdot dW_t + \int_{\mathbb{R}^{d'}} H_t^\pi(z) \{ \nu(dt \times dz) - \lambda_t \phi(z) dz dt \}$$

where

(D.15)

$$\begin{aligned} r_t^\pi &= f_y(\bar{c}_t, Y_t) + \frac{1}{f_c(\bar{c}_t, Y_t)} \left\{ f_{cc}(\bar{c}_t, Y_t) \bar{c}_t \left(r_t^{\bar{c}} - \lambda_t \int_{\mathbb{R}^{d'}} H_t^{\bar{c}}(z) \phi(z) dz \right) \right. \\ &\quad + f_{cy}(\bar{c}_t, Y_t) Y_t \left(r_t^{\bar{c}} - \lambda_t \int_{\mathbb{R}^{d'}} H_t^Y(z) \phi(z) dz \right) + \frac{1}{2} f_{ccc}(\bar{c}_t, Y_t) \bar{c}_t^2 \|v_t^{\bar{c}}\|^2 \\ &\quad \left. + \frac{1}{2} f_{cyy}(\bar{c}_t, Y_t) Y_t^2 \|v_t^Y\|^2 + f_{cyy}(\bar{c}_t, Y_t) \bar{c}_t Y_t v_t^{\bar{c}} \cdot v_t^Y \right\} + \lambda_t \int_{\mathbb{R}^{d'}} H_t^\pi(z) \phi(z) dz, \end{aligned}$$

$$v_t^\pi = -\bar{\gamma}_t v_t^{\bar{c}} - \bar{\zeta}_t v_t^Y,$$

$$H_t^\pi(z) = -\bar{H}_t(z).$$

APPENDIX E. GE VERSION OF DUFFIE-PAN-SINGLETON MODEL

Under conditions in Corollary 3.1, the dynamics of system of processes $(S, V, \bar{c}, p)'$ satisfies the following system of SDDEs under the agents' common belief \mathbb{P} :

$$(E.1) \quad d \begin{pmatrix} \ln S_t \\ V_t \\ \ln \bar{c}_t \\ \ln p_t \end{pmatrix} = \begin{pmatrix} r - \bar{\lambda} \bar{\beta}_S + \left(\gamma \varrho_{\bar{c}1} \varsigma_{\bar{c}} + \varrho_{p1} \varsigma_p - \frac{1}{2} \right) V_t \\ \kappa(\bar{V} - V_t) - \lambda \beta_V \\ r_t^{\bar{c}} - \lambda \beta_{\bar{c}} - \frac{1}{2} \varsigma_{\bar{c}}^2 V_t \\ r_t^p - \lambda \beta_p - \frac{1}{2} \varsigma_p^2 V_t \end{pmatrix} dt + \sqrt{V_t} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varrho_{V1} \varsigma_V & \varrho_{V2} \varsigma_V & 0 & 0 \\ \varrho_{\bar{c}1} \varsigma_{\bar{c}} & \varrho_{\bar{c}2} \varsigma_{\bar{c}} & \varrho_{\bar{c}3} \varsigma_{\bar{c}} & 0 \\ \varrho_{p1} \varsigma_p & \varrho_{p2} \varsigma_p & \varrho_{p3} \varsigma_p & \varrho_{p4} \varsigma_p \end{pmatrix} dW_t + \int_{\mathbb{R}^8} \begin{pmatrix} J_S(z) \\ J_V(z) \\ J_{\bar{c}}(z) \\ J_p(z) \end{pmatrix} \nu(dt \times dz)$$

for every $t \in \mathbf{T}$, where

(E.2)

$$\begin{aligned} \tilde{\beta}_S &= \tilde{q}^{(1)} \left\{ \exp \left[\mu_S^{(1)} + \frac{1}{2} (\sigma_S^{(1)})^2 - (\gamma \rho_{\bar{c}1}^{(1)} \sigma_{\bar{c}}^{(1)} + \rho_{p1}^{(1)} \sigma_p^{(1)}) \sigma_S^{(1)} \right] - 1 \right\} \\ &\quad + \tilde{q}^{(2)} \left\{ \frac{1}{1 - \rho_{p1}^{(2)} \sigma_S^{(2)}} \exp \left[\mu_S^{(2)} + \frac{1}{2} (\sigma_S^{(2)})^2 - (\gamma \rho_{\bar{c}1}^{(2)} \sigma_{\bar{c}}^{(2)} + \rho_{p1}^{(2)} \sigma_p^{(2)}) \sigma_S^{(2)} \right] - 1 \right\}, \\ \beta_V &= q^{(V)} \sigma_V^{(V)} + q^{(2)} \sigma_V^{(2)}, \\ \beta_{\bar{c}} &= q^{(1)} \left\{ \exp \left[\mu_{\bar{c}}^{(1)} + \frac{1}{2} (\sigma_{\bar{c}}^{(1)})^2 \right] \right. \\ &\quad \left. + q^{(2)} \left\{ \frac{1}{1 - \sum_{j=1}^2 (\rho_j' \rho_{\bar{c}j}^{(2)}) \sigma_{\bar{c}}^{(2)}} \exp \left[\mu_{\bar{c}}^{(2)} + \frac{1}{2} (\sigma_{\bar{c}}^{(2)})^2 \right] - 1 \right\} \right\}, \\ \beta_p &= q^{(1)} \left\{ \exp \left[\mu_p^{(1)} + \frac{1}{2} (\sigma_p^{(1)})^2 \right] - 1 \right\} \\ &\quad + q^{(2)} \left\{ \frac{1}{1 - \sum_{j=1}^3 (\rho_j' \rho_{pj}^{(2)}) \sigma_p^{(2)}} \exp \left[\mu_p^{(2)} + \frac{1}{2} (\sigma_p^{(2)})^2 \right] - 1 \right\}. \end{aligned}$$

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