

Esser, Angelika; Mönch, Burkart

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Modeling Feedback Effects with Stochastic Liquidity ^{*}

Angelika Esser and Burkart Mönch

Department of Finance, Goethe University, Frankfurt am Main, Germany

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Abstract. We model the interactions between the trading activities of a large investor, the stock price, and the market liquidity. Our framework generalizes the model of Frey (2000), where liquidity is constant by introducing a stochastic liquidity factor. This innovation has two implications. First, we can analyze trading strategies for the large investor that are affected by a changing market depth. Second, the sensitivity of stock prices to the trading strategy of the large investor can vary due to changes in liquidity. Features of our model are demonstrated using Monte Carlo simulation for different scenarios. The flexibility of our framework is illustrated by an application that deals with the pricing of a liquidity derivative. The claim under consideration compensates a large investor who follows a stop loss strategy for the liquidity risk that is associated with a stop loss order. The derivative matures when the asset price falls below a stop loss limit for the first time and then pays the price difference between the asset price immediately before and after the execution of the stop loss order. The setup to price the liquidity derivative is calibrated for one example using real world limit order book data so that one gets an impression about the order of magnitude of the liquidity effect. Furthermore, we present a pragmatic approach to determine the market price of liquidity risk from traded European put options.

JEL classification: G12

Keywords: stochastic liquidity, large trader, liquidity derivative

1. Introduction

Aspects of market liquidity include the time involved in acquiring or liquidating a position and the price impact of this action. In this paper we focus on the second issue. Especially for large institutional investors many of the existing pricing models are only of limited use since they assume perfect elastic supply and demand functions for the asset under consideration. This assumption is often violated if the trading activity

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of a large investor accounts for a significant fraction of the overall turnover in an asset. In this case, ignoring liquidity issues can result in a serious underestimation of the risk that is inherent in a certain investment strategy.

We present a framework that incorporates the liquidity risk arising for a large investor, whose trading volume cannot be absorbed by the market without a price change. Our model has two main ingredients. On the one hand the stock price process is influenced by the trading activity of the large investor, whereas the impact of the trading strategy on the stock price is modeled using a stochastic liquidity factor. On the other hand the dynamics of the stock price and the liquidity factor can have an impact on the trading strategy of the large investor.

In principle one could describe this scenario in two ways. First one can build an equilibrium setup to explain the machinery of the market. Such an approach is sensible if one intends to analyze the motivation for trading or to investigate strategies that the large investor can use in order to exploit the power to move prices in a certain direction. However, for pricing purposes such complex frameworks are often unsuitable, as they are difficult to calibrate. In this paper we follow a second approach by directly modeling the asset price dynamics that result if the large trader follows a certain trading strategy. We assume that the liquidity in the market is given exogenously.

There is a growing theoretical literature that investigates the interaction of liquidity and trading strategies of large investors. Part of this literature considers optimal liquidation strategies for large portfolios. Dubil (2002), Almgren and Chriss (2000), Bertsimas and Lo (1998) are just a few examples. Another branch of this literature investigates how large traders can manipulate stock prices. Jarrow (1992), Allen and Gale (1992), Schönbucher and Wilmott (2000) can be mentioned in this line. Recently research focuses more and more on the modeling and hedging aspects that are introduced by illiquidity and the presence of one or more large traders. Cvitanić and Ma (1996), Cuoco and Cvitanić (1998), Papanicolaou (1998), Frey (1998; 2000), Schönbucher and Wilmott (2000), Kampovsky and Trautmann (2000), Frey and Patie (2001), Liu and Yong (2001), Bank and Baum (2002) are some prominent examples.

Our approach generalizes the model of Frey (2000), where liquidity is constant and an extension by Frey and Patie (2001), where liquidity is a deterministic function of the stock price. Modeling liquidity as a stochastic factor first of all enables us to incorporate random changes in market depth. Furthermore, we can significantly generalize existing models by introducing the concept of liquidity feedback effects. The presence of liquidity feedback effects implies that (i) trading strategies

of large investors are affected by the degree of illiquidity and (ii) the sensitivity of stock prices to the trading strategies of large investors can vary due to changes in liquidity.

The objective of our model is to provide a large investor with a flexible framework that allows to evaluate the liquidity risk associated with different types of trading strategies. We use a stop loss order as an example and analyze the effects of market liquidity for such a trading strategy in detail. To get a flavour of the problem, imagine a pension fund. The fund management has to limit the downside risk of the fund, and in order to do so assume that it follows a simple stop loss strategy. When the price of the security falls below a certain level the position in this security is liquidated completely and immediately by placing a market order. If the market is perfectly liquid the fund will always receive the stop loss price for the asset. However, if the market becomes illiquid, the investor will receive a price less than the stop loss limit. If the degree of the illiquidity in the market does not change or can be interpreted as a function of the asset price, this discount due to illiquidity is deterministic. This means that the investor can adjust the stop loss limit in advance, so that he or she will always receive a certain amount for the assets. However, if the order book depth is stochastic, a large investor faces liquidity risk.

We propose a liquidity derivative compensating for this liquidity discount. We show how the setup can be calibrated to market data and present a simple approach to determine the market price of liquidity risk from traded European plain vanilla put options.

The paper is organized as follows: Section 2 summarizes the main ideas of Frey's model. In section 3 we briefly discuss economic and empirical aspects of the question whether market liquidity can be modeled as a function of the asset price or if liquidity need to be modeled as an autonomous source of risk. Our general framework is presented in section 4. In section 5 we derive the effective dynamics for the underlying asset with stochastic liquidity. We use the results in order to compare the feedback effects of the large investor's trading strategy on the stock price dynamics for the Black-Scholes (1973) model (henceforth BS) and the model with stochastic liquidity. In section 6 we exemplify the effects of stochastic liquidity by simulating sample paths of the asset price. Section 7 describes the general setup for a liquidity derivative, in order to illustrate how the stochastic liquidity model can be used to build pricing tools. The setup to price the liquidity derivative is calibrated for one example using real world limit order book data so that one gets an impression about the order of magnitude of the liquidity effect. Furthermore, we present an innovative approach to estimate the market

price of liquidity risk. The paper concludes in section 8 with a brief summary and a discussion of issues for further research.

2. The Deterministic Liquidity Model

Frey (2000) proposes the following model setup. There exists a risky asset S (the stock) and a risk-free investment earning a zero interest rate (the bond). There are no liquidity effects on the bond, only the underlying S is affected by this source of risk. Further, there is a single large investor whose trading strategy influences the price process of the underlying. The underlying follows the stochastic differential equation

$$dS_t = \sigma S_{t-} dW_t^S + \rho S_{t-} d\phi_t^+.$$

ϕ denotes the trading strategy of the large investor, i.e. the number of stocks held by the large trader, whereas ϕ^+ denotes the right-continuous version of ϕ , and $\rho \geq 0$ is a constant liquidity parameter. An increase in ρ means a declining liquidity in the market. For $\rho = 0$ the model represents the standard BS setup with zero drift. $1/(\rho S)$ is called the market depth, i.e. the order size that moves the price by one unit. Furthermore, we need the assumption that $\rho d\phi_t^+ > -1$ in order to ensure non-negativity of the asset price.

The impact of the trading strategy on the price process is discussed in the case of a smooth strategy $\phi \equiv \phi(t, S) \in C^{1,2}$. The partial derivatives of ϕ are denoted by subscripts for ease of notation. This yields the effective dynamics for the underlying

$$dS_t = b(t, S)dt + Sv(t, S)dW_t^S,$$

where

$$\begin{aligned} v(t, S) &= \frac{\sigma}{1 - \rho S \phi_S} \\ b(t, S) &= \frac{\rho S}{1 - \rho S \phi_S} \left(\phi_t + \frac{1}{2} \phi_{SS} S^2 v^2 \right) \end{aligned}$$

assuming $\rho S \phi_S < 1$. Note that volatility has changed from σ to $\frac{\sigma}{1 - \rho S \phi_S}$ compared to a perfectly liquid market due to the trading strategy ϕ and the liquidity parameter ρ .

As discussed in Frey (2000), there are two basic types of trading strategies to consider: On the one hand the large investor can trade using a *positive feedback* strategy, i.e. $\phi_S > 0$. That means he or she buys the risky asset when the price is increasing, and he or she sells when the price is declining which reinforces the effect of rising or falling

prices, since the true volatility increases in this case. For example in a standard BS model one would use such a strategy to duplicate a convex payoff like a long call. On the other hand the large trader can use a *contrarian feedback* strategy, i.e. $\phi_S < 0$, which means buying stocks when prices drop and vice versa. This would be the strategy used to duplicate a concave payoff, like a short call.

The basic model of Frey (2000) is extended in the paper of Frey and Patie (2001) by introducing a deterministic liquidity function $\rho(S)$.

3. Is Market Liquidity Indeed Stochastic?

In this section we briefly discuss, whether variations in market liquidity can be explained empirically by variations in the asset price or as a function of time as proposed in Frey and Patie (2001) or if they need to be modeled as an autonomous source of risk.

From an economic point of view one may argue that investors have different motivations to submit orders to the stock market. On the one hand there may be a close relationship between the trading activity and the dynamics of the stock prices, for example if investors follow feedback strategies and trade for speculative reasons. On the other hand so called liquidity or noise traders buy and sell assets to invest cash not needed for consumption or to meet cash needs in unforeseen situations. Imagine that a lucky retail investors who has won the national lottery might want to buy stocks or that an insurance company has to sell shares after a major damaging event to compensate clients for suffered losses. Individual liquidity shocks that occur independently of the stock price dynamics induce stochastic changes in market liquidity. As a theoretical reference serves for example Ericsson and Renault (2000). The authors model explicitly individual liquidity shocks to investors in a market for defaultable bonds and investigate optimal liquidation strategies if market liquidity is stochastic in a tree setup .

From an empirical perspective one can state that a growing branch of literature provides empirical evidence that market liquidity exhibits an intraday U-shaped or a J-shaped pattern stable over time. A review of related literature can be found in Coughenour and Shastri (1999) or Ranaldo (2000). For a discussion of a functional form that is able to reproduce an intraday U-shape pattern see for example Mönch (2003).

However, the intraday patterns can explain variations in market liquidity only to a certain extent, otherwise a time series of daily data should not exhibit any variations in market liquidity. We analyzed daily limit order book data collected at 12.00 a.m. every trading day from January 03 to March 28, 2002 for Medion AG, which belongs to the

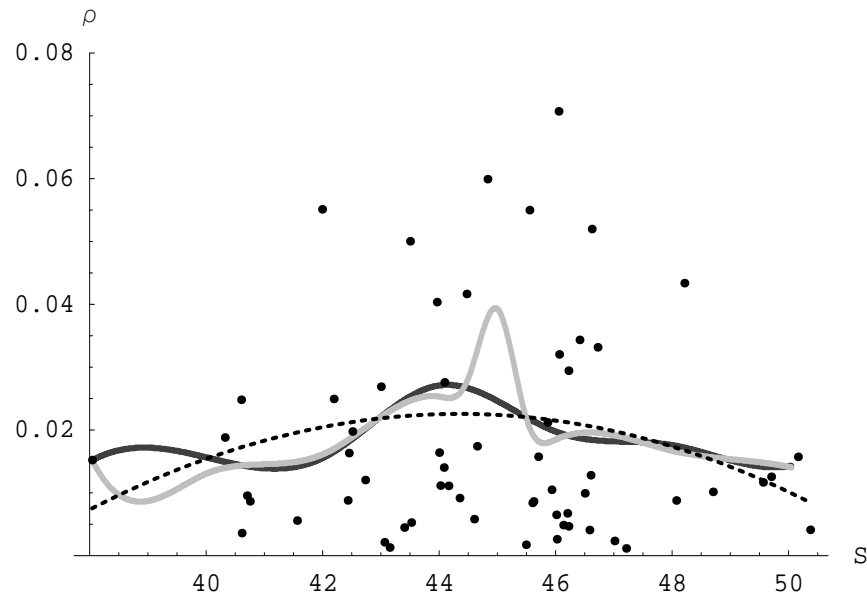


Figure 1. Scatterplot for best bid prices S and percentage liquidity discounts ρ if 5,000 Medion shares are sold in a single trade (daily data collected at 12.00 a.m. CET for every trading day from January 03 to March 28, 2002). The dotted line represents the fitted function, proposed by Frey and Patie (2001). The dark solid line shows a fitted cubic spline for 6 intervals. The bright solid line represents a fitted cubic spline for 10 intervals.

most heavily traded shares at Neuer Markt, the market segment for growing technology companies at the German stock exchange. Figure 1 shows a scatterplot for best bid prices S and percentage liquidity discounts ρ if 5,000 Medion shares are sold in a single trade. At first sight it is evident that daily liquidity discounts are not constant over time. Furthermore, one can state the relationship between the stock price and the relative liquidity discounts seems to be tiny. One can not reveal a clear pattern that would motivate a certain functional form between asset price and liquidity risk. For example, the fit of the function proposed in Frey and Patie (2001) given by

$$\rho(S) = \rho^{\text{const}} * \left[1 - (S - S_0)^2 \left(a_1 I_{\{S \leq S_0\}} + a_2 I_{\{S > S_0\}} \right) \right]$$

provides only a moderate R^2 of 0.03. The fitted function is plotted as a dotted line in figure 1. One may argue that other specifications of the function might better explain variations in market liquidity. However, even cubic splines that allow for a high degree of flexibility of the regression curve can not represent the arrangement of the data adequately for a reasonable number of knots. For example, for 6 intervals (5 interior

knots) we obtain a R^2 of 0.05, for 10 intervals (9 interior knots) the R^2 is 0.06. The calibrated splines are plotted as a dark and a bright solid line in figure 1. Of course, one could increase the number of knots excessively to improve the fit of the underlying dataset. In the limit all data points may be taken as spline knots. In this case the spline interpolates the data points. However, such a model is not robust concerning the time window of the sample data used for calibration. Thus, its use would be questionable from an economic point of view.

4. The Stochastic Liquidity Model

In our model the underlying asset price process is assumed to follow the stochastic differential equation

$$dS_t = \mu_t S_{t-} dt + \sigma S_{t-} dW_t^S + \rho_t S_{t-} d\phi_t^+ \quad (1)$$

In order to be slightly more general compared to Frey (2000) we relax the assumption of zero drift and interest rate. Furthermore, we now assume that ρ is stochastic with dynamics given by the continuous process

$$d\rho_t = \beta(t, \rho) dt + \nu(t, \rho) dW_t^\rho \quad (2)$$

with

$$dW_t^S \bullet dW_t^\rho = \gamma dt. \quad (3)$$

A sensible specification for ρ could be a mean-reversion process with a natural long-run level of liquidity in the market. We further assume that the process stays strictly positive for $\rho_0 > 0$. This restricts the choices for the volatility function $\nu(t, \rho)$. For example one might use functions of the type $\nu(t, \rho) = \zeta \sqrt{\rho}$.

Rewriting the above dynamics, using a Cholesky-decomposition we obtain

$$dS_t = \mu_t S_{t-} dt + \sigma S_{t-} dW_t + \rho_t S_{t-} d\phi_t^+ \quad (4)$$

$$d\rho_t = \beta(t, \rho) dt + \nu(t, \rho) \gamma dW_t + \nu(t, \rho) \sqrt{1 - \gamma^2} d\bar{W}_t \quad (5)$$

with a two-dimensional Brownian motion (W_t, \bar{W}_t) . We now assume that ϕ - the number of shares held by the large investor - does not depend on S and t only as in the model with deterministic liquidity. In our setup it can also depend on the stochastic liquidity factor ρ . The effects on the trading strategy are now twofold: First, ϕ is influenced by changes in S . Second, ϕ varies with changing liquidity. The impact of S

on ϕ can be modeled by the two basic types of trading strategies. For a positive feedback strategy ϕ is an increasing function of S for all ρ , for a contrarian feedback strategy ϕ is a decreasing function of S for all ρ . To characterize the impact of ρ on ϕ consider the following scenario. The more illiquid the market the fewer shares the large trader will hold due to external or internal regulations, no matter whether a positive or contrarian feedback strategy is considered. Thus, a reasonable choice would be a decreasing absolute ϕ -value with respect to ρ (for all S).

5. The Effective Price Process in the Stochastic Liquidity Model

In this section the effective price process for the stochastic liquidity model is derived and analyzed in detail. Consider a smooth trading strategy $\phi \equiv \phi(t, S_t, \rho_t) \in C^{1,2,2}$. An application of Itô's formula leads to the following proposition.

PROPOSITION 1. (Dynamics of the State Variables) *Suppose the trading strategy of the large trader is given by $\phi(t, S, \rho) \in C^{1,2,2}$. Then, under the assumption that $\rho S \phi_S < 1$ holds for any point in time, the solution to the system of stochastic differential equations (4) and (5) satisfies*

$$dS_t = b(t, S, \rho)dt + v(t, S, \rho)SdW_t + \bar{v}(t, S, \rho)Sd\bar{W}_t \quad (6)$$

$$d\rho_t = \beta(t, \rho)dt + \nu(t, \rho)\gamma dW_t + \nu(t, \rho)\sqrt{1 - \gamma^2}d\bar{W}_t, \quad (7)$$

where

$$v(t, S, \rho) = \frac{\sigma}{1 - \rho S \phi_S} + \gamma \frac{\rho \phi_\rho \nu}{1 - \rho S \phi_S} \quad (8)$$

$$\bar{v}(t, S, \rho) = \sqrt{1 - \gamma^2} \frac{\nu \rho \phi_\rho}{1 - \rho S \phi_S} \quad (9)$$

$$b(t, S, \rho) = \frac{\rho S}{1 - \rho S \phi_S} \left[\left(\frac{\mu_t}{\rho} + \phi_t + \beta(\rho_t, t) \phi_\rho + \frac{1}{2} \phi_{\rho\rho} \nu^2 \right) + \frac{1}{2} \phi_{SS} S^2 (v^2 + \bar{v}^2) + \nu S \phi_{\rho S} \left(\gamma v + \sqrt{1 - \gamma^2} \bar{v} \right) \right]. \quad (10)$$

The proof is given in Appendix I.

In the case of $\rho \equiv 0$ or $\phi \equiv \text{constant}$ we are in the classical BS scenario with zero drift. If $\rho \neq 0$ and $\phi \neq 0$ the trading strategy of the large trader has an effect on the instantaneous volatilities v and \bar{v} , as well as on the total volatility and the correlation between the two processes.

Now consider the special case where liquidity has no impact on the strategy of the large trader, i.e. $\phi_\rho \equiv 0$. Then, \bar{v} and the second summand of v will vanish. This means we get close to the scenario of Frey, where feedback effects are only incorporated due to the term $\rho S \phi_S$. However, since the parameter ρ is stochastic in the proposed model, the sensitivity of the stock price on the trading activity of the large investor will vary in contrast to the deterministic liquidity model.

The analysis is more complex for $\phi_\rho \neq 0$. We assume $\rho > 0$, in order to discuss how variations in liquidity influence the trading strategy of the large investor. As illiquidity increases the large trader has to reduce the position in the stock. Thus, the investor sells stocks if he or she has a long position or he or she buys back shares if a short position is considered. In the first case ϕ is monotonically decreasing in ρ , starting with a positive ϕ . In the latter case ϕ is monotonically increasing in ρ , starting with a negative ϕ . Thus, ϕ is approaching zero in absolute value (the large investor has closed the position in the stock almost completely) as ρ tends to infinity.

We assume a positive ϕ in the following so that ϕ_ρ should be negative, no matter if a positive feedback or a contrarian trading strategy is considered. From (8) we can see that the sign of $v(t, S, \rho)$ depends on γ . For $\gamma \equiv 0$ the value of v is the same as in the deterministic liquidity model. Nevertheless, in the stochastic liquidity model we have an additional volatility parameter \bar{v} contributing to total volatility. The parameter \bar{v} is negative since ϕ_ρ is negative. The instantaneous quadratic variation $d[S]$ is given by (see Appendix I)

$$d[S]_t = S^2 \left(\frac{\sigma^2 + \nu^2 \rho^2 \phi_\rho^2 + 2\gamma \sigma \nu \rho \phi_\rho}{(1 - \rho S \phi_S)^2} \right) dt$$

so that the total instantaneous volatility v_{tot} is equal to

$$v_{tot} = \sqrt{v^2 + \bar{v}^2} = \frac{\sqrt{\sigma^2 + \nu^2 \rho^2 \phi_\rho^2 + 2\gamma \sigma \nu \rho \phi_\rho}}{1 - \rho S \phi_S}. \quad (11)$$

Note that the argument of the square root is non-negative since $|\gamma| \leq 1$. The instantaneous covariation of the two processes $d[S, \rho]$ is given by

$$\begin{aligned} d[S, \rho]_t &= S\nu \left(\gamma v + \sqrt{1 - \gamma^2 \bar{v}} \right) dt \\ &= S\nu \left(\frac{\nu \rho \phi_\rho}{1 - \rho S \phi_S} + \gamma \frac{\sigma}{1 - \rho S \phi_S} \right) dt. \end{aligned} \quad (12)$$

Thus, the instantaneous correlation η equals

$$\eta = \frac{d[S, \rho]}{\nu S v_{tot} dt} = \frac{\gamma v + \sqrt{1 - \gamma^2 \bar{v}}}{\sqrt{v^2 + \bar{v}^2}}$$

$$= \frac{\nu\rho\phi_\rho + \gamma\sigma}{\sqrt{\sigma^2 + \nu^2\rho^2\phi_\rho^2 + 2\gamma\sigma\nu\rho\phi_\rho}}. \quad (13)$$

In order to compare the formal setup of Frey (2000) and Frey and Patie (2001) to the stochastic liquidity model (henceforth SLM) we analyze the volatility and correlation structure for different specifications of the respective liquidity related parameters. An overview is given in table I.

Table I. Liquidity related parameters in different models

Model	γ	v	\bar{v}	v_{tot}	η
BS	0	σ	0	v	0
Frey	0	$\frac{\sigma}{1-\rho S\phi_S}$	0	v	0
Frey/Patie	1	$\frac{\sigma}{1-\rho S\phi_S}$	0	v	1
SLM	0	$\frac{\sigma}{1-\rho S\phi_S}$	$\frac{\nu\rho\phi_\rho}{1-\rho S\phi_S}$	$\sqrt{v^2 + \bar{v}^2} > v$	$\frac{\bar{v}}{\sqrt{v^2 + \bar{v}^2}} < 0$
SLM	$\neq 0$	$\frac{\sigma + \gamma\nu\rho\phi_\rho}{1-\rho S\phi_S}$	$\frac{\sqrt{1-\gamma^2}\nu\rho\phi_\rho}{1-\rho S\phi_S}$	$\sqrt{v^2 + \bar{v}^2}$	$\frac{\sqrt{1-\gamma^2}v + \gamma\bar{v}}{\sqrt{v^2 + \bar{v}^2}}$

For $\rho \equiv 0$ or $\phi \equiv \text{constant}$, respectively, we are in the standard BS model with drift μ , resulting in $\bar{v} \equiv 0$ and $v = \sigma$. The first additional feature is included in Frey's approach where liquidity is represented by a constant ρ (i.e. $d\rho \equiv 0$) implying $\beta \equiv \nu \equiv 0$. This yields $\bar{v} \equiv 0$ and

$$v \equiv v_{tot} = \frac{\sigma}{1 - \rho S\phi_S}.$$

The correlation parameter η must be zero in this scenario, since all terms containing ϕ_ρ vanish so that the dependence of the strategy on ρ is of no interest for the effective stock price dynamics.

Now, we take a closer look at the model of Frey and Patie (2001), where ρ is a deterministic function of S . Therefore, the dynamics of ρ are only driven by the first component W of the two-dimensional Brownian motion, which implies $\bar{v} \equiv 0$ and $\gamma = 1$ yielding $\eta \equiv \gamma = 1$. Since ϕ only depends on S , the volatility is the same as in the Frey setup. In formula (13) we can observe that $\eta \equiv 1$ in this case. Thus, the approach of Frey and Patie (2001) is a special case of the general

framework with $\gamma = 1$ allowing the coefficients in the dynamics of ρ to depend explicitly on S .

We now consider the stochastic liquidity model and start with the correlation structure.

5.1. IMPACT ON CORRELATION

It is important to note that $\gamma \equiv 0$ does not imply that the increments of the effective stock price process and the liquidity process are uncorrelated, see equation (13). In the stochastic liquidity setup the situation where $\eta = 0$ cannot be obtained for a deterministic choice of γ . Even for $\gamma \equiv 0$, there is still some correlation $\eta < 0$ induced by $\rho\phi_\rho$, which is negative since $\phi_\rho < 0$.

For $\gamma = 0$ there is a negative correlation between S and ρ , given by

$$\eta = \frac{\nu\rho\phi_\rho}{\sqrt{\sigma^2 + \nu^2\rho^2\phi_\rho^2}} < 0.$$

This term ultimately models liquidity feedback effects. It vanishes for a trading strategy independent of ρ , i.e. for $\phi_\rho \equiv 0$. The negative value of η can be interpreted in the following way: if liquidity is low over a longer period the trader will be forced to close the position which will cause the stock price to drop. For $\gamma \neq 0$ the numerator of the correlation η in equation (13) carries an additional summand $\gamma\sigma$. Thus, η is an increasing function of γ . For $\phi_\rho \equiv 0$ this correlation is equal to γ , the correlation between the increments of the components of the Brownian motion. Thus, the difference between γ and η is a result of the liquidity feedback effect, exclusively.

Next, we discuss the changes in volatility caused by stochastic liquidity.

5.2. IMPACT ON VOLATILITY

First, consider the case $\gamma = 0$. Then, v is the same as in the deterministic liquidity model, but in our framework there is the additional volatility parameter

$$\bar{v} = \frac{\nu\rho\phi_\rho}{1 - \rho S\phi_S} < 0.$$

The variable \bar{v} incorporates the liquidity feedback effect. It contributes to the total volatility if and only if the trading strategy depends on the liquidity parameter ρ .

In this case the total volatility increases, compared to the deterministic liquidity model, and we obtain a total volatility of

$$v_{tot} = \frac{\sqrt{\sigma^2 + \nu^2 \rho^2 \phi_\rho^2}}{1 - \rho S \phi_S}.$$

Here we see a key result of our approach: For $\gamma = 0$ the total volatility of the stochastic liquidity model is greater than the volatility in the constant liquidity setup, no matter which strategy is used.

For $\gamma \neq 0$ the direction of the changes in total volatility (see equation (11)) depends on the sign of γ . The parameter ν carries an additional summand describing the liquidity feedback effect, and \bar{v} is reduced in absolute values due to $\gamma \neq 0$ (see equations (8) and (9)). The total volatility in (11) is a decreasing function of γ , since $\phi_\rho < 0$. Thus, even for $\gamma \leq 0$ the total volatility is greater than the volatility in Frey's setup. Only for large positive values of γ it is possible that the total volatility is lower than the volatility in Frey's model. This will be the case if

$$1 \geq \gamma > \frac{1}{2\sigma} \nu \rho |\phi_\rho|. \quad (14)$$

For a strategy ϕ independent of ρ the volatility in the stochastic liquidity model is equal to the volatility in the constant liquidity model, again reflecting the lack of the liquidity feedback effect in this case.

Finally, we look at the two basic types of trading strategies with respect to S .

5.2.1. Positive Feedback Strategy

A positive feedback strategy, i.e. $\phi_S > 0$, leads to

$$v_{tot} > \sqrt{\sigma^2 + \rho^2 \nu^2 \phi_\rho^2 + 2\gamma \sigma \nu \rho \phi_\rho}.$$

The expression on the right hand side is greater than or equal to σ if and only if

$$\gamma \leq \frac{1}{2\sigma} \nu \rho |\phi_\rho|. \quad (15)$$

Especially, v_{tot} is greater than the Black Scholes volatility σ for $\gamma \leq 0$. This is similar to the result derived in Frey's model.

5.2.2. Contrarian Feedback Strategy

A contrarian feedback strategy (i.e. $\phi_S < 0$) implies that the total volatility satisfies

$$v_{tot} < \sqrt{\sigma^2 + \rho^2 \nu^2 \phi_\rho^2 + 2\gamma\sigma\nu\rho\phi_\rho}$$

which is less or equal to σ if and only (14) holds. Thus, for non-positive values of γ the instantaneous volatility in the stochastic liquidity model is not lower than the Black Scholes volatility in this case in contrast to the result in Frey's model.

6. Numerical Results

In order to visualize the formal analysis of the previous section we present some simulation-based results. We compare a sample path of stock prices in the BS model to stock prices in the stochastic liquidity setting for both a positive and a contrarian feedback strategy.

6.1. PARAMETER SPECIFICATION

In order to ensure non-negativity and a stationary behaviour of the liquidity process we specify the dynamics for ρ in equation (2) as a square-root-process with a mean-reverting drift component (see Cox, Ingersoll and Ross (1985)):

$$\beta(t, \rho) = \kappa(\theta - \rho), \quad \nu(t, \rho) = \zeta\sqrt{\rho}.$$

To create paths of the underlying and the liquidity parameter Monte Carlo Simulation techniques are used. For a fixed realization of (W_t, \bar{W}_t) we have plotted the dynamics of equations (4) and (5) for $S_0 = 80.0$, $\sigma = 0.1$ and

1. for the BS setting: $\rho \equiv 0$ and
2. for the stochastic liquidity framework:

$$\begin{aligned} \kappa &= 0.35 & \theta = \rho_0 &= 0.05 & \zeta &= 0.2 \\ \gamma &= 0.0 \end{aligned}$$

The stochastic processes are discretized with an Euler scheme with $N = 4000$ steps and $\Delta t = 1/360$.

Figure 2 shows the stock holdings of the large investor as a function of ρ and S for a positive (left graph) and a contrarian feedback strategy (right graph).

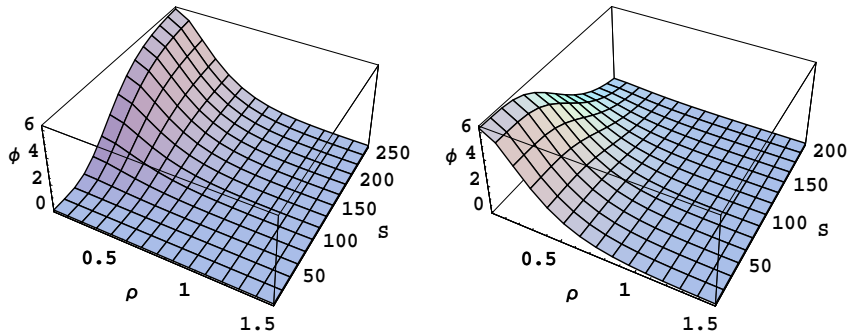


Figure 2. Positive and Contrarian Feedback Strategy

The shape of the functions can be explained by the following intuition. If liquidity drops, i.e. ρ increases, the large trader is forced to sell shares. Therefore, ϕ is assumed to be monotonically decreasing in ρ for all S and for any feedback strategy, i.e. the derivative of ϕ with respect to ρ is negative. It seems reasonable to assume that for very small and for very large values of ρ the value of ϕ_ρ is small in absolute values. In the first case the asset still has a sufficient market depth. In the latter case the large trader has already sold almost all of his holdings in the stock. Thus, in both scenarios, the large trader adjusts the position in the stock only by a small amount.

In order to characterize the relationship between ϕ and S we have to distinguish between the positive feedback and the contrarian feedback strategy. In the first (second) case, the large trader buys (sells) assets as the stock price increases and sells (buys) when the stock price declines. Thus, ϕ is monotonically increasing (decreasing) in S for all ρ in the case of the positive (contrarian) feedback strategy. For very small and very large values of S the changes of the stock holdings of the large trader are negligible when the asset prices vary (similar to the relationship between ρ and ϕ). However, for asset prices in between the absolute value of ϕ_S increases when S increases and a positive (contrarian) feedback strategy is considered.

There is a variety of functional forms for $\phi = \phi(S, \rho)$ that are able to reproduce the features described above. We used the incomplete gamma function to model this scenario. The exact functional form can be found in Appendix II.

6.2. POSITIVE FEEDBACK STRATEGY

In figure 3 we compare BS to a positive feedback strategy in the stochastic liquidity model.

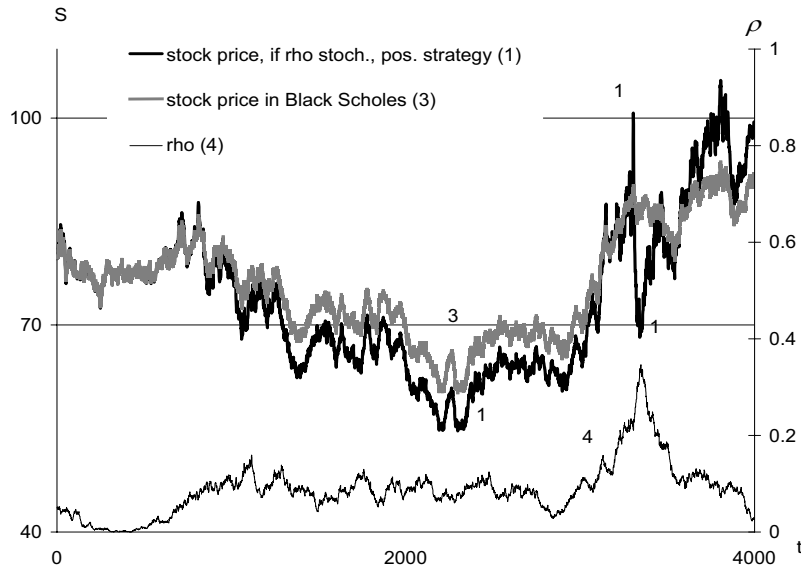


Figure 3. Sample paths for the stock price in the BS model and the stochastic liquidity setup if large trader follows a positive feedback strategy.

As shown theoretically (see equation (15)) one can see that for our choice of parameters the volatility of the stock price in the stochastic liquidity model is increased compared to BS. Path 1 exceeds path 3 when the BS price is increasing over a longer period. Rising stock prices motivate the large investor to buy additional stocks which will cause the stock price to grow even further. One can notice the opposite effect for a decreasing S since in this case the large investor wants to get rid of the holdings which will accelerate the decline in the stock price.

The role of the liquidity parameter ρ is more subtle. In fact it can have two different implications: First, all else equal, the trader has to sell stocks if they become more and more illiquid. Second, if ρ is very high, the stock becomes more volatile so that a large trader who follows a positive feedback strategy can cause the stock price to rise to tremendously high values in bullish markets. However, when illiquidity exceeds a certain threshold ($\rho \approx 0.3$ in figure 3), the large trader is forced to close the position and the market collapses. Those features that distinguish the stochastic liquidity model from the BS and the deterministic liquidity model, become obvious around the time step 3,500 in figure 3.

If ρ approaches zero path 1 runs parallel to the BS path 3, as one can observe around time step 500.

6.3. CONTRARIAN FEEDBACK STRATEGY

In figure 4 we contrast stock prices simulated in the BS model to stock prices generated in the stochastic liquidity model for a contrarian feedback strategy.

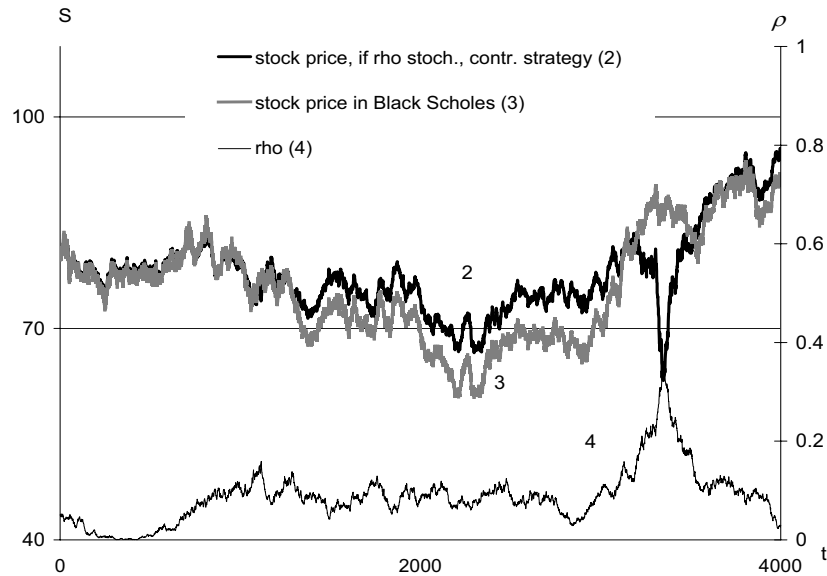


Figure 4. Contrarian Feedback Strategy versus Black Scholes

In general volatility is reduced compared to BS. However, as one can observe around time step 3,500, there are exceptions. If the asset becomes very illiquid the trading activities of the large trader can dominate the stock price dynamics and so have a destabilizing effect. Again, this is a unique feature of the stochastic liquidity setup that cannot be modeled in the constant liquidity framework.

7. Application: Liquidity Derivative

After having introduced the general framework of our model, we now present an application that deals with the pricing of a liquidity derivative on an underlying that is traded in an illiquid market. To simplify the explanation we take a step backward and restrict the general framework so that the trading strategy of the large investor only depends on the asset price S . The derivative under consideration compensates the large investor who follows a stop loss strategy for the liquidity discount if the stop loss order is executed. We provide an example of

such a derivative that is priced with our sample data from the German electronic limit order book XETRA for Medion.

In contrast to retail investors whose individual trading volume is too small to affect prices adversely large traders have to consider liquidity aspects before an investment is made. If the degree of illiquidity in a certain market is rapidly changing over time one may hesitate to invest substantial amounts although the risk reward profile of the investment might be promising. A long position in the proposed liquidity derivative hedges against the liquidity risk and enables a large trader to act like a small retail investor. At the moment such liquidity derivatives are not actively traded in the market. The lack of appropriate pricing tools and orderbook data needed for calibration may explain this situation. However, as competition among electronic trading platforms sharpens many exchanges may provide clients with real time data to access market liquidity.

Trading strategies that limit the downside risk of a portfolio and that only depend on the asset price are heavily used by institutional investors. However, considering a simple stop loss strategy one may argue that typically a large trade in the stock market is broken down into smaller packages to minimize the adverse impact on the overall transaction price. However, we focus on this application since although the calculus is simple, it already incorporates a basic structure that can serve as a guideline to construct more elaborate pricing tools that are suitable for a variety of purposes.

7.1. PRICING FORMULAS

Assume that ϕ is a stop loss trading strategy. The random variable $\tau := \inf\{t \mid S_t < \bar{S}\}$ denotes the stopping time when the underlying falls below a certain level \bar{S} for the first time, implying $S_{\tau-} = \bar{S}$. The initial price of the underlying is assumed to be $S_0 > \bar{S}$. Up to the hitting time τ the large trader does not trade, and at τ he or she sells all assets. Assuming a constant, positive initial value $\phi_0 > 0$ this implies

$$d\phi_t^+ = \begin{cases} 0 & \text{for } t < \tau \\ \phi_t^+ - \phi_t = -\phi_0 & \text{for } t = \tau \end{cases}$$

and

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^S \\ d\rho_t &= \beta(t, \rho) dt + \nu(t, \rho) d\bar{W}_t^p \end{aligned}$$

for $t < \tau$. Equivalently using the uncorrelated Brownian motion (W, \bar{W}) this writes as

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$d\rho_t = \beta(t, \rho)dt + \gamma\nu(t, \rho)dW_t + \sqrt{1 - \gamma^2}\nu(t, \rho)d\bar{W}_t$$

for $t < \tau$. Under a risk-neutral measure \hat{P} the dynamics are given by

$$dS_t = rS_tdt + \sigma S_t d\hat{W}_t \quad (16)$$

$$d\rho_t = \beta^*(t, \rho)dt + \gamma\nu(t, \rho)d\hat{W}_t + \nu(t, \rho)\sqrt{1 - \gamma^2}d\hat{\bar{W}}_t \quad (17)$$

for $t < \tau$, where β^* denotes the risk-adjusted drift. Using

$$\begin{aligned} d\hat{W}_t &= \lambda_t^{(S)}dt + dW_t \\ d\hat{\bar{W}}_t &= \lambda_t^{(\rho)}dt + d\bar{W}_t \end{aligned}$$

it holds as usual that $\lambda_t^{(S)} = \frac{\mu - r}{\sigma}$ and $\lambda_t^{(\rho)}$ arbitrary such that

$$\beta^*(t, \rho) = \beta(t, \rho) - \gamma\nu(t, \rho)\frac{\mu - r}{\sigma} - \sqrt{1 - \gamma^2}\lambda_t^{(\rho)}\nu(t, \rho).$$

In total, the market price of liquidity risk is given by

$$\lambda^{\text{liq}} = \gamma\lambda_t^{(S)} + \sqrt{1 - \gamma^2}\lambda_t^{(\rho)}$$

in this framework. Since the market is incomplete as liquidity is not traded, this leads to one degree of freedom for the market price of liquidity risk which cannot be eliminated in this setup. The market price of liquidity risk manifests in the difference between dynamics under P and \hat{P} . To calculate the drift under \hat{P} a unique price of at least one derivative depending explicitly on liquidity is needed. We provide a theoretical discussion how the market price of liquidity risk can be calculated from traded European options subsection 7.4.

This provides a perfect analogy to the stochastic volatility model where the market price of volatility risk can only be computed when the price of a derivative is known.

At $\tau-$ the threshold is hit. This yields a jump in the stock price at τ given by $S_{\tau-} = \bar{S}$ and the reduced price at τ :

$$S_\tau = \bar{S}(1 - \rho_{\tau-}\phi_0) = \bar{S}(1 - \rho_\tau\phi_0),$$

assuming a continuous version of ρ in τ . We propose a derivative contract that compensates for the price difference between the reduced price at τ and the threshold \bar{S} by paying $\bar{S}\rho_\tau\phi_0 I(\tau \leq T)$ at τ for one unit of the underlying.

To model a more realistic scenario we additionally introduce a floor $F \geq 0$ which represents a deductible for the investor. In this case the contract would compensate only for a critical liquidity discount. If

small settlement payments between both parties of the contract cause significant transaction costs such an agreement would be appropriate. Thus the payoff at τ for one unit of the underlying is

$$\begin{aligned} Z_\tau &= \max[(\bar{S}\phi_0\rho_\tau - F)I(\tau \leq T); 0] \\ &= (\bar{S}\phi_0\rho_\tau - F)I(\tau \leq T; \rho_\tau > \frac{F}{\bar{S}\phi_0}) \end{aligned}$$

The price of the contract is given by the following proposition:

PROPOSITION 2. *The price at $t = 0$ of a derivative paying $Z_\tau = \max[(\bar{S}\phi_0\rho_\tau - F)I(\tau \leq T); 0]$ when the price process falls below the threshold \bar{S} for the first time is given by*

$$\begin{aligned} Z_0 &= \hat{\mathbb{E}} [e^{-r\tau} Z_\tau] \\ &= \bar{S}\phi_0 \hat{\mathbb{E}} \left[e^{-r\tau} (\rho_\tau - \frac{F}{\bar{S}\phi_0}) I(\tau \leq T; \rho_\tau > \frac{F}{\bar{S}\phi_0}) \right] \\ &= \bar{S}\phi_0 \int_0^T \int_{F/\bar{S}\phi_0}^{\bar{\rho}} e^{-rt} (\rho - \frac{F}{\bar{S}\phi_0}) g(t, \rho) d\rho dt \end{aligned} \quad (18)$$

where $\hat{\mathbb{E}}$ denotes the expectation under the risk-neutral measure, and g represents the joint (risk-neutral) density of τ and ρ_τ . The parameter $\bar{\rho} \leq \infty$ denotes some upper bound for the liquidity process. In general, the price of the derivative cannot be calculated explicitly, but for some special cases we are able to derive explicit solutions of (18). One of them is presented in the following.

First of all, we need the distribution of the hitting time. From (16) we know that the process for the underlying S under a risk-neutral measure \hat{P} up to τ is a geometric Brownian motion such that the log of the process is an arithmetic Brownian motion. The distribution of the first hitting time of a Brownian motion with drift is well-known (see, for example, Borodin (1996)). In this scenario we ask for the distribution of the first hitting time of $\ln \bar{S}$, starting at $\ln S_0 > \ln \bar{S}$ under a risk-neutral measure. This is given by the risk-neutral density

$$f_{\ln S_0, \ln \bar{S}}(t) = \frac{|\ln(\bar{S}/S_0)|}{\sqrt{2\pi t^3}} \exp\left(\frac{-(\ln(\bar{S}/S_0) - (r - \sigma^2/2)t)^2}{2t}\right).$$

The specification of the diffusion process for liquidity is arbitrary so far. In the following we consider some scenarios in which the price of the liquidity derivative can be calculated explicitly under certain assumptions on the liquidity process.

If liquidity is constant or a time-independent stochastic variable, uncorrelated with the underlying asset, equation (18) simplifies to

$$Z_0 = \bar{S}\phi_0 \left(\hat{\mathbb{E}}[\rho] - \frac{F}{\bar{S}\phi_0} \right) \hat{\mathbb{E}} [e^{-r\tau} I(\tau \leq T)]$$

$$= \bar{S}\phi_0 \left(\hat{\mathbb{E}}[\rho] - \frac{F}{\bar{S}\phi_0} \right) \int_0^T e^{-rt} f_{\ln S_0, \ln \bar{S}}(t) dt.$$

If the Brownian motions exhibit zero correlation, i.e. $\gamma = 0$, the hitting time τ and the process ρ are independent. In this case we can calculate the expectation taking the product of the corresponding risk-neutral densities, i.e. $g(t, \rho) = f(t)h_t(\rho)$, where h_t denotes the risk-neutral density of the process ρ at time t and we get

$$\begin{aligned} Z_0 &= \bar{S}\phi_0 \hat{\mathbb{E}}[e^{-r\tau} \rho_\tau I(\tau \leq T)] \\ &= \bar{S}\phi_0 \int_0^T \left(\int_{F/\bar{S}\phi_0}^{\bar{\rho}} \left(\rho - \frac{F}{\bar{S}\phi_0} \right) h_t(\rho) d\rho \right) e^{-rt} f_{\ln S_0, \ln \bar{S}}(t) dt \end{aligned} \quad (19)$$

For $F \equiv 0$ this expression simplifies to

$$Z_0 = \bar{S}\phi_0 \int_0^T \left(\hat{\mathbb{E}}[\rho_t] \right) e^{-rt} f_{\ln S_0, \ln \bar{S}}(t) dt. \quad (20)$$

Now, we are free to choose an appropriate process for ρ . To our knowledge the question which specification of the liquidity process is empirically adequate, is still unanswered in the literature.

One may argue that a mean reversion type is sensible, since there exists a natural level of liquidity in the market. Thus, we again assume a CIR process for liquidity given by the following specification of the risk-neutral parameters in equation (17):

$$\begin{aligned} \beta^*(t, \rho) &:= \kappa^*(\theta^* - \rho) \\ \nu(t, \rho) &:= \zeta \sqrt{\rho} \end{aligned}$$

The asterisks denote risk-neutral parameters. For zero correlation the dynamics are given by

$$\begin{aligned} d\rho_t &= \kappa(\theta - \rho_t)dt + \zeta \sqrt{\rho_t} d\bar{W}_t \text{ under } P \\ d\rho_t &= \kappa^*(\theta^* - \rho_t)dt + \zeta \sqrt{\rho_t} d\hat{W}_t \text{ under } \hat{P} \end{aligned}$$

with $d\hat{W}_t = \sqrt{\rho_t} \lambda_t dt + d\bar{W}_t$, where we set $\lambda_t^{(\rho)} = \sqrt{\rho_t} + \lambda_t$ such that the liquidity process is again a CIR-process after the change of measure. The parameter are then given by

$$\kappa^* = \kappa + \zeta \lambda, \quad \theta^* = \frac{\kappa}{\kappa^*} \theta. \quad (21)$$

In order to assure a strictly positive liquidity path we have to consider the constraint:

$$\theta^* > \frac{\zeta^2}{2\kappa^*} \quad (22)$$

The risk-neutral probability density of ρ at t with initial value ρ_0 is given by

$$h(t, \rho; \rho_0) = ce^{-u-v} \left(\frac{v}{u}\right)^{q/2} \mathcal{I}_q(2\sqrt{uv})$$

where

$$\begin{aligned} c &= \frac{2\kappa^*}{\zeta^2(1 - e^{-\kappa^*t})} \\ u &= c\rho_0 e^{-\kappa^*t} \\ v &= c\rho \\ q &= \frac{2\kappa^*\theta^* - \zeta^2}{\zeta^2}. \end{aligned}$$

\mathcal{I}_z denotes the modified Bessel function of the first kind of order z , i.e.

$$\mathcal{I}_z(y) = \left(\frac{y}{2}\right)^z \sum_{n=0}^{\infty} \frac{(y^2/4)^n}{\Gamma(z+n+1)n!}.$$

Note that if the floor F is equal to zero the expectation value of liquidity in equation (20) can be computed explicitly in this case:

$$\mathbf{E}[\rho_t] = \rho_0 e^{-\kappa^*t} + \theta^* (1 - e^{-\kappa^*t}), \quad (23)$$

Hence, equation (20) simplifies to

$$Z_0 = \bar{S}\phi_0 \int_0^T (\rho_0 e^{-\kappa^*t} + \theta^* (1 - e^{-\kappa^*t})) e^{-rt} f_{\ln S_0, \ln \bar{S}}(t) dt.$$

If one additionally assumes a zero market price of liquidity risk ($\theta^* = \theta$), then the price of the liquidity derivative does not depend on the volatility parameter of the liquidity process ζ . However, as soon as this restrictive assumptions are relaxed, for example by considering a non-zero market price of liquidity risk ($\lambda^l \neq 0$) or a non linear payoff function ($F > 0$), the volatility of the liquidity process has an impact on the price of the liquidity derivative. This property is visualized later on in figures (6) and (7) in subsection 7.2.

7.2. EXAMPLE

As an example we use the limit order book data for Medion already introduced in section 3 to calibrate the stochastic processes for the best bid price and the market liquidity. For numerical reasons we scaled the liquidity parameter with 5,000. Thus, it can be interpreted as

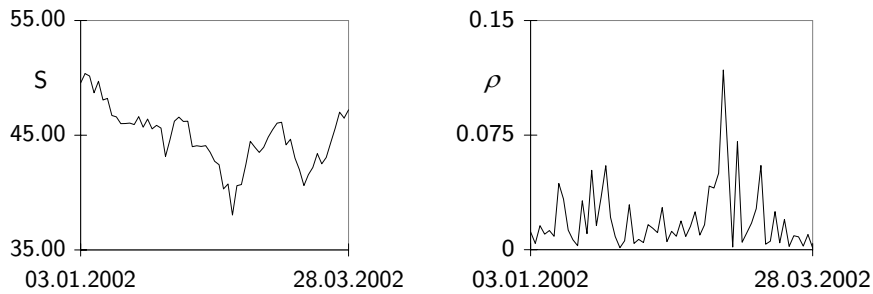


Figure 5. Best bid prices and percentage liquidity discounts if 5,000 Medion shares are sold in a single trade (daily data at 12.00 a.m. CET).

the relative price difference between the best bid price and the average execution price if a *hypothetical* market order of 5,000 shares is executed.

Applying the standard maximum likelihood estimation techniques and assuming a zero market price of liquidity risk the following parameter estimates were obtained:

$$\begin{aligned} \sigma &= 0.48 & \theta^* &= 0.020188 \\ \kappa^* &= 248.12 & \zeta &= 3.1906347 \end{aligned}$$

The correlation between the best bid prices and the illiquidity discounts in the sample is -0.0354. Thus, presuming zero correlation seems to be a sensible assumption.

We assume that the derivative under consideration compensates for the price difference between the stop loss limit of EUR 40.00 and the average execution price if a stop loss order of 5,000 shares will be executed within the next month.¹ Since the initial liquidity parameter ρ_0 is scaled with 5,000 we have $\phi_0 = 1.0$.

Furthermore, we set $S_0 = 47.61$, $\rho_0 = 0.05$ and $r = 0.05$ and in the model with deterministic liquidity $\rho = \theta^* = 0.020188$. Now we are able to calculate prices for the liquidity derivative. In the model with constant liquidity one obtains a price of 0.465449 whereas in the model with stochastic liquidity the price for one derivative is 0.489582. Thus the price difference for one contract (in our example 5,000 derivatives) between the stochastic and the deterministic liquidity model would be EUR 120.67.

¹ For the sake of simplicity we assume that a stop loss order is executed if the best bid price falls to the stop loss limit for the first time.

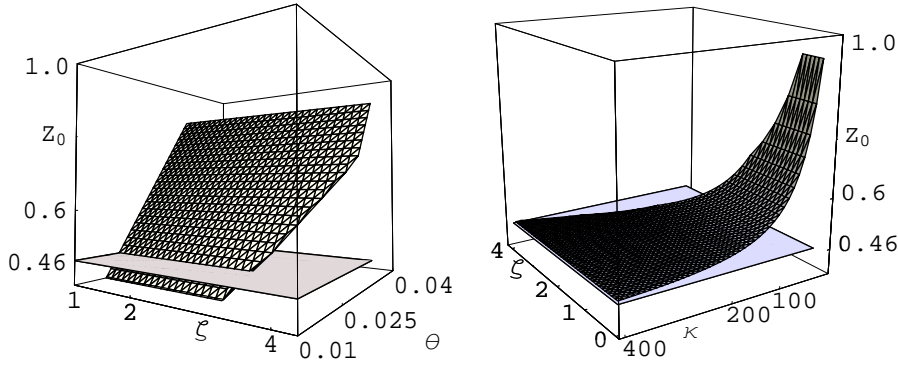


Figure 6. Price of the liquidity derivative Z_0 as a function of the parameters ζ , θ (in the left figure) and ζ , κ (in the right figure). All other parameters are chosen in accordance with the example, especially we assume $F = 0$ and $\lambda \equiv 0$. While the checked dark surfaces represent Z_0 in the stochastic liquidity framework, the bright flat surfaces represent Z_0 in the constant liquidity setup.

7.3. SENSITIVITY ANALYSIS

How does a variation of relevant parameters changes the results? Recall that the parameters of the CIR process have to meet the boundary condition $\theta^* > \zeta^2/(2\kappa^*)$ in order to assure positive outcomes. Thus, the choice of parameters is restricted. When the boundary condition becomes binding a reduction in κ^* or θ^* implies a reduction in ζ (for a given θ^* or κ^*) to get an admissible solution.

First we stay for a while in the the rather unrealistic world of the previous example and consider a contract that is linear in ρ_t ($\Rightarrow F = 0$) and assume a zero market price of liquidity risk. Figure 6 illustrates in two three-dimensional plots how the price of the derivative Z_0 varies when altering combinations either of θ and ζ for fixed κ (left graph) or of κ and ζ for fixed θ (right graph) such that the restriction in inequality (22) is satisfied. The derivative in the stochastic liquidity framework is represented by the checked dark surfaces. The brighter surfaces represent the derivative in the constant liquidity setup that serves as a benchmark. Under the given assumptions ($F = 0$ and $\lambda^{(\rho)} = 0 \Rightarrow \theta^* = \theta, \kappa^* = \kappa$) the world is simple. Remember, in this case equation (19) simplifies to (20). In the stochastic liquidity setup the price of the liquidity derivative Z_0 is a linear increasing function of the long term mean θ and an exponentially decreasing function of the mean reversion parameter κ . The volatility parameter ζ has no impact on the

price of the derivative Z_0 . Reconsidering the definition of $\mathbb{E}[\rho_t]$ given by equation (23) one can state that the stochastic and the deterministic liquidity setup give almost identical results if ρ_0 and θ are close to $\rho^{const.}$ and if the parameters t and κ are large enough.

If the assumptions of the previous examples are relaxed and either a contract that is non-linear in ρ_t or a non-zero market price of liquidity risk is taken into consideration the volatility parameter ζ has an impact on the price of the liquidity derivative Z_0 . Figure 7 shows the price of the derivative Z_0 for admissible combinations of ζ and F (left graph) and of ζ and λ (right graph) for fixed κ and θ .

If the parameter F is increased, the derivative becomes less expensive. In this case the contract pays out in less states of the world. However, the price reduction is less pronounced if the volatility parameter ζ is large. To put it differently: an increase in ζ transfers weight to the tails of the distribution and raises the price of the derivative Z_0 for a given positive F . A similar phenomenon can be observed in a BS world. A derivative that is linear in the asset price, for example a forward contract is insensitive to the volatility of the underlying. In contrast, the volatility of the underlying influences the price of plain vanilla options that are non-linear contracts in the asset price. Furthermore, the Vega of a plain vanilla European call option is a positive function of the strike price which coincides with the fact that the greater the floor F the higher the sensitivity of the price with respect to changes in volatility.

The parameter λ influences the drift of the liquidity process under the risk-neutral measure. Additionally, if $\lambda \neq 0$, the volatility parameter ζ appears in the risk-neutral drift, see equation (21). If $\lambda > 0$, an increase in ζ causes an increase in κ^* and a reduction in θ^* . Both aspects reduce the price of the derivative Z_0 . The opposite effects can be observed for $\lambda < 0$.

7.4. MARKET PRICE OF LIQUIDITY RISK

7.4.1. Overview

Recent literature provides strong evidence that investors care about liquidity risk which implies that liquidity risk is priced into asset returns. For example, Acharya and Pedersen (2003) analyze daily return and volume data from 1962 until 1999 for all common shares listed on NYSE and AMEX. They corroborate the hypothesis that investors require a premium for a security that is illiquid when the market as a whole is illiquid and that investors are willing to pay a premium for a security that is liquid when stock returns are low. More over, they show that investors are willing to pay a premium for a security with a

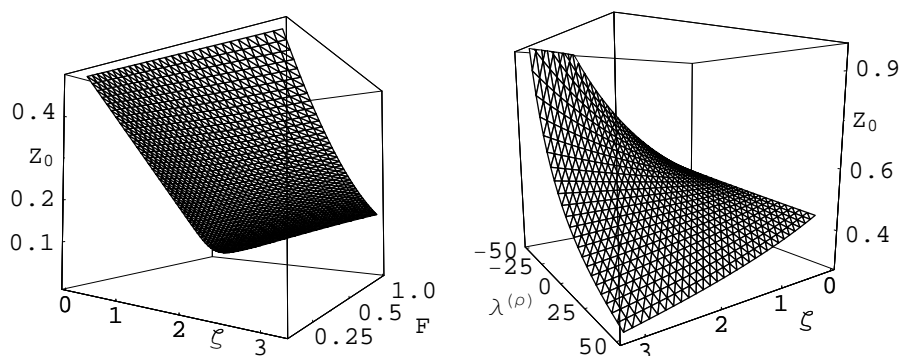


Figure 7. Price of the liquidity derivative Z_0 as a function of the parameters ζ , F (in the left figure) and ζ , λ (in the right figure). All other parameters are chosen in accordance with the example.

high return when the market is illiquid. This is empirically supported by Pástor and Stambaugh (2002) who use monthly data for 34 years of common stocks traded on NYSE, AMEX and NASDAQ. They find that stocks which are more sensitive to aggregated market liquidity have higher expected returns. A comprehensive review of theoretical and empirical approaches can be found in Pritsker (2002).

From this perspective and with the results of subsection 7.2 in mind it seems important to provide a reasonable approach to determine the market price of liquidity risk from traded instruments. As an example for such a traded claim we consider a plain vanilla European put option that gives the holder the right to sell ϕ_0 units of the stock for the strike price X at maturity T . In the following we make four assumptions to simplify the explanation, specifically we suppose that

- if the option is exercised the holder has to deliver the shares physically,
- the day of execution coincides with the day of delivery (Note that in practice there may be some exchange trading days in between.)
- the short seller of the put liquidates the ϕ_0 units of the stock immediately if the option is exercised,
- neither the long nor the short party trades the underlying before the maturity of the option.

The first assumption is met at many exchanges for equity options. The second and the third assumption are not critical and can be relaxed if

a more realistic framework is desired. The last assumption ensures that the underlying price process and the liquidity process are independent. Easing this restriction for example by assuming that the short seller hedges the option in the stock market results in more complex dynamics for the underlying as we have seen before when considering general trading strategies of a large investor.

7.4.2. *Intuition*

Assume, that the holder of the put has to liquidate a long position in the underlying at a certain time T . For this purpose the investor can either sell the securities at the exchange or execute the put option and deliver the stocks to the short seller of the put. Since the market for the underlying is illiquid, the terminal payoff of the put is given by $\mathcal{P}_T = \max[X - S_T(1 - \rho_T\phi_0), 0]$. This can be explained as follows:

The execution of the put is optimal if it provides higher revenues than the liquidation of the underlying in the market. In the latter case one will receive $S_T(1 - \rho_T\phi_0)$ due to the market illiquidity. Thus, it is optimal to exercise the put option if and only if $X > S_T(1 - \rho_T\phi_0)$. If the put is exercised the short party faces a liquidity risk. Assume that the underlying is liquidated immediately after the execution of the put. Then the short seller has to bear a liquidity discount amounting to $S_T\rho_T\phi_0$.

The story is similar to the liquidity derivative considered in the previous section for a floor $F \equiv 0$. If one holds the liquidity derivative and the stop loss order is executed at $\tau \leq T$, one would sell the underlying at the market for $\bar{S}(1 - \rho_\tau\phi_0)$ and receive the payoff of the derivative $\bar{S}\rho_\tau\phi_0$. In total one would obtain $\bar{S}\phi_0$. If one is hedged by the European put option one delivers at T the shares to the short seller and receives $X\rho_T\phi_0$.

Although the payoff of the European put option and the liquidity derivative seem closely related to each other, assuming $X = \bar{S}$, there are important differences. The European put option matures at time T . Therefore, it may be suitable for an investor who has to liquidate stocks at a fixed point in time. In contrast to that, the liquidity derivative matures at the stopping time $\tau \leq T$, when the stock price hits \bar{S} for the first time, in order to compensate an investor who follows a stop loss strategy for the liquidity discount. Considering this particular trading strategy an European put option would not hedge the large investor perfectly. Imagine for example a situation, where \bar{S} is hit at time $\tau < T$, but then the stock price rises again and we observe $S_T > \bar{S}$. In this case the put will not be exercised at T , although the stop loss order was executed. Only for $\tau = T$ (which is an event with zero probability) the investor would be indifferent between the two claims.

One may argue, that an American put option may better meet the needs of an investor who follows a stop loss strategy than an European put since it could be exercised at any stopping time $\tau \leq T$. However, early execution of an American put option following this particular strategy does not have to be optimal. Since there are infinitely many possible stopping strategies the American put option will be more expensive than the liquidity derivative. Thus, a long position in the liquidity derivative would be the most effective and cheapest way to hedge against the liquidity risk arising from a stop loss trading strategy.

7.4.3. Pricing Formula for the Put Written on an Illiquid Stock

The price of the European put option can be computed analytically in our framework. The put price is not unique, since the market is incomplete. As before, the underlying follows a geometric Brownian motion with drift r under the risk-neutral measure. For the liquidity process we again consider a CIR-process. It exhibits one degree of freedom since the unknown market price of liquidity risk shows up in the risk-neutral dynamics in equation (17). However, comparing the theoretical price with market data would give an explicit specification of the market price of liquidity risk.

By fixing a risk-neutral measure \hat{P} the risk-neutral pricing formula is given by

$$\begin{aligned}
\mathcal{P}_t &= e^{-r(T-t)} \hat{\mathbb{E}} [\max[X - S_T(1 - \rho_T \phi_0), 0]] \\
&= e^{-r(T-t)} \hat{\mathbb{E}} [(X - S_T(1 - \rho_T \phi_0))I(X > S_T(1 - \rho_T \phi_0))] \\
&= e^{-r(T-t)} \hat{P}(X > S_T(1 - \rho_T \phi_0)) \\
&\quad - e^{-r(T-t)} \hat{\mathbb{E}} [S_T(1 - \rho_T \phi_0)I(X > S_T(1 - \rho_T \phi_0))] \\
&= e^{-r(T-t)} \hat{P}(X > S_T(1 - \rho_T \phi_0)) \\
&\quad - e^{-r(T-t)} \hat{\mathbb{E}} [S_T I(X > S_T(1 - \rho_T \phi_0))] \\
&\quad + e^{-r(T-t)} \hat{\mathbb{E}} [S_T \rho_T \phi_0 I(X > S_T(1 - \rho_T \phi_0))]
\end{aligned}$$

In integral representation - where f denotes the risk-neutral density of the asset price process at t and g the risk-neutral density of the CIR-process at t - this yields

$$\begin{aligned}
\mathcal{P}_t &= e^{-r(T-t)} \int_{X > s(1 - \rho \phi_0)} f(s)g(\rho) ds d\rho \\
&\quad - e^{-r(T-t)} \int_{X > s(1 - \rho \phi_0)} s f(s)g(\rho) ds d\rho \\
&\quad + e^{-r(T-t)} \phi_0 \int_{X > s(1 - \rho \phi_0)} s \rho f(s)g(\rho) ds d\rho
\end{aligned}$$

$$\begin{aligned}
&= e^{-r(T-t)} \int_0^X \left(\int_0^\infty \frac{1}{1-\rho\phi_0} f(z/(1-\rho\phi_0)) g(\rho) d\rho \right) dz \\
&\quad - e^{-r(T-t)} \int_0^X \left(\int_0^\infty \frac{z}{(1-\rho\phi_0)^2} f(z/(1-\rho\phi_0)) g(\rho) d\rho \right) dz \\
&\quad + e^{-r(T-t)} \phi_0 \int_0^X \left(\int_0^\infty \frac{z\rho}{(1-\rho\phi_0)^2} f(z/(1-\rho\phi_0)) g(\rho) d\rho \right) dz,
\end{aligned}$$

using the substitution $z = s(1 - \rho\phi_0)$ in the last step. The density of the CIR-process is non-central χ^2 with the adjusted drift containing the market price of liquidity risk. Thus, in order to determine the market price of liquidity risk one has to solve the following root finding problem

$$\mathcal{P}_0^{\text{obs}} - \mathcal{P}_0^{\text{th}}(\lambda^{(\rho)}) \stackrel{!}{=} 0$$

numerically by adjusting $\lambda^{(\rho)}$ iteratively.

8. Conclusion

Criteria such as consistency with empirical phenomena, flexibility and also computational aspects should be considered in the development of a liquidity model. We have presented a framework that we feel meets these requirements and has proved worthy of investigation.

This paper introduces a continuous-time model for an illiquid market, where the trading strategy of a large investor can move prices. The innovating features of our setup include on the one hand a time varying market depth and on the other hand the modeling of liquidity feedback effects.

We have analyzed two basic types of trading strategies. For positive feedback strategies and non-positively correlated Brownian motions volatility generally is increased compared to BS. For contrarian feedback strategies one can observe basically the opposite. The picture can change completely if the asset becomes highly illiquid and the asset price dynamics are completely dominated by the trading activities of the large investor. These features are discussed both analytically as well as numerically by using some simulation results.

Furthermore, an application of the general framework is proposed. A closed form expression to price a liquidity option, that's payoff depend on the price difference between a stop loss limit and the average execution price of a stop loss order, is provided. Furthermore, a pragmatic way to determine the market price of liquidity risk from traded European plain vanilla put options is presented.

Further research may focus on derivative pricing under the effective price processes or the application of hedging strategies from incomplete markets to this setting. An investigation in order to reveal optimal liquidation strategies in the proposed framework seems to be promising.

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Appendix

I. PROOF OF PROPOSITION 1

Using Itô for $\phi(t, S, \rho)$ where subscripts denote partial derivatives gives

$$d\phi = \phi_t dt + \phi_S dS + \phi_\rho d\rho + \phi_{S\rho} d[S, \rho]_t + \frac{1}{2} (\phi_{SS} d[S]_t + \phi_{\rho\rho} \nu^2 dt).$$

Plugging this into equation (4) we get

$$dS_t = (\sigma + \rho\phi_\rho\nu\gamma)S_t dW_t + \rho\phi_\rho\nu S d\bar{W}_t + \rho_t S_t ((\mu_t S_t + \phi_t)dt + \phi_S dS + \phi_\rho d\rho + \nu\phi_{S\rho} d[S, \bar{W}]_t + \frac{1}{2} (\phi_{SS} d[S]_t + \phi_{\rho\rho} \nu^2 dt))$$

which implies

$$dS_t = \frac{1}{1 - \rho S \phi_S} \left((\sigma + \rho\phi_\rho\nu\gamma)S_t dW_t + S_t (\rho\phi_\rho\nu\sqrt{1 - \gamma^2}) d\bar{W}_t + \rho S_t \left[\left(\frac{\mu_t}{\rho} + \phi_t + \beta(\rho_t, t)\phi_\rho + \frac{1}{2}\phi_{\rho\rho}\nu^2 \right) dt + \frac{1}{2}\phi_{SS} d[S]_t + \nu\phi_{S\rho} d[S, \bar{W}]_t \right] \right),$$

assuming $\rho_t S_t \phi_S < 1$. Using the trial solution

$$dS_t = b(t, S_t, \rho_t)dt + v(t, S_t, \rho_t)S_t dW_t + \bar{v}(t, S_t, \rho_t)S_t d\bar{W}_t$$

and comparing coefficients leads to

$$v(t, S, \rho) = \frac{\sigma + \rho\phi_\rho\nu\gamma}{1 - \rho S \phi_S}$$

$$\begin{aligned}
\bar{v}(t, S, \rho) &= \frac{\nu\rho\phi_\rho\sqrt{1-\gamma^2}}{1-\rho S\phi_S} \\
b(t, S, \rho)dt &= \frac{\rho S}{1-\rho S\phi_S} \left[\left(\frac{\mu_t}{\rho} + \phi_t + \beta(\rho_t, t)\phi_\rho + \frac{1}{2}\phi_{\rho\rho}\nu^2 \right) dt \right. \\
&\quad \left. + \frac{1}{2}\phi_{SS}d[S]_t + \nu\phi_{S\rho}d[S, \bar{W}]_t \right] \\
&= \frac{\rho S}{1-\rho S\phi_S} \left[\left(\frac{\mu_t}{\rho} + \phi_t + \beta(\rho_t, t)\phi_\rho + \frac{1}{2}\phi_{\rho\rho}\nu^2 \right) \right. \\
&\quad \left. + \frac{1}{2}\phi_{SS}S^2 \frac{\sigma^2 + \rho^2\nu^2\phi_\rho^2 + 2\gamma\rho\sigma\nu\phi_\rho}{(1-\rho S\phi_S)^2} \right. \\
&\quad \left. + \nu\phi_{S\rho}S \frac{\nu\rho\phi_\rho + \gamma\sigma}{1-\rho S\phi_S} \right] dt
\end{aligned}$$

since

$$\begin{aligned}
d[S]_t &= S^2(v^2 + \bar{v}^2)dt \\
&= S^2 \left(\frac{\sigma^2 + \rho^2\nu^2\phi_\rho^2 + 2\gamma\rho\sigma\nu\phi_\rho}{(1-\rho S\phi_S)^2} \right) dt
\end{aligned}$$

and

$$\begin{aligned}
d[S, \rho]_t &= S\nu(\gamma v + \sqrt{1-\gamma^2}\bar{v})dt \\
&= S\nu \left(\frac{\nu\rho\phi_\rho + \gamma\sigma}{1-\rho S\phi_S} \right) dt
\end{aligned}$$

II. FUNCTIONAL FORM OF $\phi(S, \rho)$

In order to incorporate the idea that ϕ is a function of both S and ρ we chose a product approach by separating the strategy with respect to S and ρ :

$$\phi(S, \rho) = a\psi(S)\chi(\rho)$$

A function that is able to model the features described in section 6 is the incomplete gamma function, defined by

$$\Gamma(x, z) = \int_z^\infty e^{-t}t^{x-1}dt.$$

We read $\Gamma(x, z)$ as a function of z for fixed x -values. It is monotonically decreasing to zero for $z \rightarrow \infty$. The incomplete gamma function captures the dependency of the trading strategy with respect to ρ for fixed S . For the contrarian feedback strategy the scenario is similar with respect to S for fixed ρ . Since the image is reversed for the positive feedback

strategy the functional form has to be adjusted. Therefore we multiply the gamma function by -1 and then shift it to positive quadrant. Thus, we chose the following representation for the dependency on ρ

$$\chi(\rho) = \Gamma(b_1, c_1\rho)$$

and for the dependence on S

$$\psi(S) = \begin{cases} \Gamma(b_2, c_2S) & \text{(contrarian strategy)} \\ \Gamma(b_2) - \Gamma(b_2, c_2S) & \text{(positive strategy)} \end{cases}$$

Consequently, the contrarian feedback strategy is given by

$$\phi^-(S) = a\Gamma(b_1, c_1\rho)\Gamma(b_2, c_2S) \quad (24)$$

whereas the positive feedback strategy is given by

$$\phi^+(S, \rho) = a\Gamma(b_1, c_1\rho) [\Gamma(b_2) - \Gamma(b_2, c_2S)]. \quad (25)$$

where $\Gamma(x)$ denotes the standard gamma function. For the simulation in section 6 we use the following parameter constellation:

$$\begin{array}{lll} b_1 = 2.1 & b_2 = 6.0 & \\ c_1 = 5.5 & c_2 = 0.05 & a = -0.05 \end{array}$$