

Schürger, Klaus

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BONN ECON DISCUSSION PAPERS

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by

Klaus Schürger

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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Maximal Arbitrage

Klaus Schürger

Department of Statistics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.
(e-mail: schuerg@finasto.uni-bonn.de)

Abstract. Let $S = (S_t)$, $t = 0, 1, \dots, T$ (T being finite), be an adapted \mathbb{R}^d -valued process. Each component process of S might be interpreted as the price process of a certain security. A trading strategy $H = (H_t)$, $t = 1, \dots, T$, is a predictable \mathbb{R}^d -valued process. A strategy H is called extreme if it represents a maximal arbitrage opportunity. By this we mean that H generates at time T a nonnegative portfolio value which is positive with maximal probability. Let F^e denote the set of all states of the world at which the portfolio value at time T , generated by an extreme strategy (which is shown to exist), is equal to zero. We characterize those subsets of F^e , on which no arbitrage opportunities exist.

Key words: Arbitrage, martingale measure

JEL Classification: G12, G13, D40

Mathematics Subject Classification (1991): 60G42

1 Introduction

A remarkable result due to Dalang, Morton and Willinger (cf.[1]) says that a finite-dimensional price process $S = (S_t)$ ($t = 0, 1, \dots, T$) defined on some probability space (Ω, \mathcal{F}, P) admits no arbitrage opportunities iff there exists a probability measure Q equivalent to P , under which (S_t) is a martingale. This remarkable result (the “first fundamental asset pricing theorem”), sometimes augmented by additional equivalent conditions, has been proved in many different ways (cf. [6], [3], [5], [2], [7], [4]).

In order to formulate one version of the first fundamental asset pricing theorem (cf.[4]), let (Ω, \mathcal{F}, P) be a probability space equipped with a filtration (\mathcal{F}_t) , $t = 0, 1, \dots, T$ (T being finite) such that $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)$ be an adapted d -dimensional process, i.e. $S_t = (S_t^1, \dots, S_t^d)$ is \mathcal{F}_t -measurable. If all components are positive, then S_t^i might be interpreted as the price of a certain stock i at time t . Let \mathcal{P} be the set of all predictable \mathbb{R}^d -valued processes $H = (H_t)(1 \leq t \leq T)$ (“trading strategies”), i.e. $H_t = (H_t^1, \dots, H_t^d)$ is \mathcal{F}_{t-1} -measurable. For $H \in \mathcal{P}$ put

$$H \bullet S_t = \sum_{k=1}^t H_k \Delta S_k, \quad 0 \leq t \leq T \quad (\Delta S_k := S_k - S_{k-1}).$$

The scalar product $H_k \Delta S_k$ might be interpreted as the increment (at time k) of the value of a portfolio consisting during the time period $]k-1, k]$ of H_k^i shares of stock i . Let \mathcal{P}_{\geq} denote the set of all $H \in \mathcal{P}$ such that $H \bullet S_T \geq 0$ a.s. Let L_+^0 denote the set of random variables which are a.s. nonnegative, and put

$$A_T = \{H \bullet S_T - \phi : H \in \mathcal{P}, \phi \in L_+^0\},$$

$$L_+^{(\delta)} = \{\xi : \xi \in L_+^0, \mathcal{P}(\xi > 0) \leq \delta\} \quad (0 \leq \delta \leq 1).$$

(S_t) is said to satisfy the *no-arbitrage condition* if

$$H \bullet S_T = 0 \text{ a.s.} \quad \text{for all } H \in \mathcal{P}_{\geq}.$$

This is equivalent to the condition

$$(NA) \quad A_T \cap L_+^0 = L_+^{(0)}.$$

By \bar{A}_T we denote the closure of A_T with respect to convergence in probability.

We will need the following version of the first fundamental asset pricing theorem (cf. [4]):

1.1 Theorem. *The following conditions are equivalent:*

- (a) (NA).
- (b) $A_T \cap L_+^0 = L_+^{(0)}$ and $A_T = \bar{A}_T$.
- (c) $\bar{A}_T \cap L_+^0 = L_+^{(0)}$.
- (d) There exists a probability measure $Q \sim P$ with $dQ/dP \in L^\infty$ such that (S_t) is a Q -martingale.

Here, $Q \sim P$ means that Q and P are *equivalent* (i.e. have the same null sets).

We put

$$(1.1.1) \quad \alpha(S) = \sup_{H \in \mathcal{P}_{\geq}} P(H \bullet S_t > 0).$$

Clearly (S_t) satisfies the no-arbitrage condition iff $\alpha(S) = 0$.

We will be interested in strategies which are extreme according to

1.2 Definition. We say that a strategy $H \in \mathcal{P}_{\geq}$ is *extreme* if

$$(1.2.1) \quad P(H \bullet S_T > 0) = \alpha(S).$$

If $\alpha(S) > 0$, then we might say that an extreme strategy represents a *maximal arbitrage opportunity*. It will be shown later (cf. Theorem 2.1) that an extreme strategy always exists, and that the set $\{H^e \bullet S_T > 0\}$ does (a.s.) *not* depend on the choice of the extreme strategy H^e . In the sequel H^e will always denote an extreme strategy.

This paper is devoted to characterizing all subsets F of the set $F^e = \{H^e \bullet S_T = 0\}$ such that (S_t) satisfies the *no-arbitrage condition on F* (this terminology will be explained below). In particular we shall determine the largest subset of F^e with this property. In general one cannot expect that (S_t) satisfies the no-arbitrage condition on F^e (assuming that $0 < P(F^e) < 1$). In fact, suppose e.g. that (S_t) is a P -martingale. Knowing in advance that the event F^e occurs may provide “too much information” on the evolution of (S_t) . In that case the restriction of the process (S_t) to F^e is not a martingale. We will derive conditions which are necessary and sufficient for (S_t) to satisfy the no-arbitrage condition on F^e (cf. Theorem 2.14).

We shall close this section with explaining the terminology used in the last paragraph. If $\mathcal{G} \subset \mathcal{F}$ is any σ -algebra and $\emptyset \neq F \in \mathcal{F}$, then $F \cap \mathcal{G} := \{F \cap G : G \in \mathcal{G}\}$ denotes the *trace* of \mathcal{G} on F ; $(F \cap \mathcal{F}_t)$ is the *trace* of the filtration (\mathcal{F}_t) on F . Let $\mathcal{P}(F)$ denote the set of \mathbb{R}^d -valued processes $H = (H_t)$ ($1 \leq t \leq T$) defined on F , which are predictable with respect to $(F \cap \mathcal{F}_t)$.

If $P(F) > 0$, then $\mathcal{P}_{\geq}(F)$ is the set of all $H \in \mathcal{P}(F)$ such that

$$H \bullet (S|F)_T \geq 0 \quad P(\cdot|F) - \text{a.s.}$$

$(S|F = (S_t|F)$ denoting the process S restricted to F). Note that $S|F$ is adapted to $(F \cap \mathcal{F}_t)$. Correspondingly we say that S satisfies the *no-arbitrage condition on F* or (for short): *NA holds on F* if

$$H \bullet (S|F)_T = 0 \quad P(\cdot|F) - \text{a.s. for all } H \in \mathcal{P}_{\geq}(F).$$

For sets $A, B \in \mathcal{F}$ we write $A = B$ a.s. ($A \subset B$ a.s.) if, for their indicator functions I_A, I_B , we have $I_A = I_B$ a.s. ($I_A \leq I_B$ a.s.).

2 Characterization of sets on which NA holds

We show first that an extreme strategy always exists:

2.1 Theorem. (a) *An extreme strategy always exists.*

(b) *If H^* and H^{**} are extreme strategies, then*

$$(2.1.1) \quad \{H^* \bullet S_T > 0\} = \{H^{**} \bullet S_T > 0\} \text{ a.s.}$$

(c) *For any extreme strategy H^e we have*

$$(2.1.2) \quad \{H \bullet S_t > 0\} \subset \{H^e \bullet S_t > 0\} \text{ a.s., } H \in \mathcal{P}_{\geq}.$$

Proof. (a) Let $H^{(n)} \in \mathcal{P}_{\geq}$ be such that

$$(2.1.3) \quad P\left(H^{(n)} \bullet S_t > 0\right) \rightarrow \alpha(S) \quad (n \rightarrow \infty).$$

Put ($|\cdot|$ denoting the Euclidean norm)

$$\xi_n = \sum_{k=1}^T |H_k^{(n)}|$$

and choose numbers $c_n > 0$ such that $P(\xi_n > c_n) \leq 2^{-n}$. The Borel-Cantelli lemma implies that

$$\eta_t := \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} |H_t^{(n)}| < \infty \text{ a.s., } 1 \leq t \leq T.$$

Let $A_t := \{\eta_t < \infty\}$. Then $A_t \in \mathcal{F}_{t-1}$,

$$H_t^e := I_{A_t} \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} H_t^{(n)}$$

is \mathcal{F}_{t-1} -measurable, and $H^e = (H_t^e) \in \mathcal{P}_{\geq}$. Since $P(H^e \bullet S_t > 0) \geq P(H^{(n)} \bullet S_t > 0)$ for all n , H^e is extreme by (2.1.3).

(b) If (2.1.1) does not hold, then

$$P\left((H^* + H^{**}) \bullet S_T > 0\right) > \alpha(S)$$

which is impossible since $H^* + H^{**} = (H_t^* + H_t^{**}) \in \mathcal{P}_{\geq}$. (c) is proved in the same way as (b). \square

In the sequel H^e will always denote an extreme strategy. Note that, according to Theorem 2.1(b), the set $\{H^e \bullet S_t = 0\}$ does (a.s.) not depend on the choice of H^e .

The following example shows that, in general, an extreme strategy is not uniquely determined (up to multiplication by positive constants).

2.2 Example. ($T = d = 1$). Let $\Omega = \{\omega_1, \dots, \omega_4\}$, $\mathcal{F} = \{0, 1\}^\Omega$ (the power set of Ω), and suppose that $P\{\omega\} > 0$, $\omega \in \Omega$. The filtration is given by $\mathcal{F}_0 = \sigma\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, $\mathcal{F}_1 = \mathcal{F}$. Let $S_0 \equiv 0$, $S_1(\omega_1) = 1$, $S_1(\omega_2) = S_1(\omega_3) = 0$, $S_1(\omega_4) = -1$. Then the strategies H^* , H^{**} given by

$$\begin{aligned} H_1^*(\omega_1) = H_1^*(\omega_2) = 1 & \quad , \quad H_1^*(\omega_3) = H_1^*(\omega_4) = -1, \\ H_1^{**}(\omega_1) = H_1^{**}(\omega_2) = 2 & \quad , \quad H_1^{**}(\omega_3) = H_1^{**}(\omega_4) = -1 \end{aligned}$$

are extreme, and $\{H^* \bullet S_1 > 0\} = \{H^{**} \bullet S_1 > 0\} = \{\omega_1, \omega_4\}$.

Our first characterization of sets on which NA holds is given by

2.3 Theorem. *Let $F \subset \{H^e \bullet S_T = 0\}$ be such that $P(F) > 0$. Then the following conditions are equivalent:*

- (a) *NA holds on F .*
- (b) *For every strategy $\tilde{H} \in \mathcal{P}_{\geq}(F)$ there exists a strategy $H \in \mathcal{P}_{\geq}$ such that*

$$(2.3.1) \quad \tilde{H} \bullet (S|F)_T = H \bullet S_T \text{ a.s. on } F.$$

Proof. (b) \Rightarrow (a): Let $\tilde{H} \in \mathcal{P}_{\geq}(F)$ be given, and let $H \in \mathcal{P}_{\geq}$ satisfy (2.3.1). Since H^e is extreme, we have $\tilde{H} \bullet S_T = 0$ a.s. on F , which, by (2.3.1), implies $\tilde{H} \bullet (S|F)_T = 0$ a.s. on F .

(a) \Rightarrow (b): Suppose that NA holds on F . This implies that, for any $\tilde{H} \in \mathcal{P}_{\geq}(F)$, we have $\tilde{H} \bullet (S|F)_T = 0$ a.s. on F . Hence (2.3.1) holds for $H = H^e$. \square

2.4 Remark. Suppose that NA holds on F . Then, in general, it is not true that for any strategy $\tilde{H} \in \mathcal{P}_{\geq}(F)$ there exists an extension $H \in \mathcal{P}_{\geq}$ of \tilde{H} . In fact, in Example 2.2 we have that NA holds on $F^e = \{H^e \bullet S_T = 0\} = \{\omega_2, \omega_3\}$. Let $\tilde{H} \in \mathcal{P}_{\geq}(F^e)$ given by $\tilde{H}_1(\omega_2) = \tilde{H}_1(\omega_3) = 1$. The only \mathcal{F}_0 -measurable extension H of \tilde{H} is given by $H_1 \equiv 1$, and $H \notin \mathcal{P}_{\geq}$.

We shall now derive conditions for S to satisfy the no-arbitrage condition on a given set $F \subset \{H^e \bullet S_T = 0\}$, that involve martingale measures for S .

By \mathcal{M}_S we denote the set of all probability measures Q on \mathcal{F} such that $Q \ll P$ (i.e. Q is absolutely continuous with respect to P), and Q is a martingale measure for S , i.e. each component process (S_i^t) is a Q -martingale. If $\mathcal{M}_S \neq \emptyset$, then we put

$$(2.4.1) \quad \mu(S) = \sup_{Q \in \mathcal{M}_S} P\left(\frac{dQ}{dP} > 0\right).$$

2.5 Definition. A probability measure $Q \in \mathcal{M}_S$ is called *extreme* if

$$(2.5.1) \quad P\left(\frac{dQ}{dP} > 0\right) = \mu(S).$$

2.6 Theorem. *Suppose that $\mathcal{M}_S \neq \emptyset$.*

- (a) *An extreme probability measure always exists.*
(b) *If Q^* and Q^{**} are extreme probability measures, then*

$$(2.6.1) \quad \left\{ \frac{dQ^*}{dP} > 0 \right\} = \left\{ \frac{dQ^{**}}{dP} > 0 \right\} \quad \text{a.s.}$$

- (c) *For any extreme probability measure Q^e we have*

$$(2.6.2) \quad \left\{ \frac{dQ}{dP} > 0 \right\} \subset \left\{ \frac{dQ^e}{dP} > 0 \right\} \quad \text{a.s.,} \quad Q \in \mathcal{M}_S.$$

Proof. (a) Let $Q_1, Q_2, \dots \in \mathcal{M}_S$ be such that

$$(2.6.3) \quad P\left(\frac{dQ_n}{dP} > 0\right) \rightarrow \mu(S) \quad (n \rightarrow \infty).$$

Let $c_1 > 0, c_2 > 0, \dots$ be real numbers such that $c_1 + c_2 + \dots = 1$. It is easy to check that $Q := c_1 Q_1 + c_2 Q_2 + \dots$ is a probability measure such that $Q \ll P$ and

$$\frac{dQ}{dP} = \sum_{n=1}^{\infty} c_n \frac{dQ_n}{dP}$$

which, by (2.6.3), implies

$$(2.6.4) \quad P\left(\frac{dQ}{dP} > 0\right) \geq \mu(S).$$

Let

$$\xi := 1 + \sum_{t=0}^T \sum_{i=1}^d |S_t^i|.$$

Then $1 \leq \mathbb{E}_{Q_n}[\xi] < \infty$ ($\mathbb{E}_{Q_n}[\xi]$ denoting the expectation of ξ , taken with respect to Q_n). Hence, for

$$c := \sum_{n=1}^{\infty} \frac{1}{2^n \mathbb{E}_{Q_n}[\xi]}$$

we have $0 < c \leq 1$. Therefore, choosing

$$c_n := (c 2^n \mathbb{E}_{Q_n}[\xi])^{-1}$$

implies $Q \in \mathcal{M}_S$. By (2.6.4), Q is extreme. It is clear that (b) and (c) hold. \square

In the sequel, Q^e will always denote an extreme probability measure. Note that the set $\{dQ^e/dP > 0\}$ does (a.s.) not depend on the choice of Q^e .

Finally, we consider a third kind of extreme objects. Let

$$(2.6.5) \quad \beta(S) = \sup_{\zeta \in \bar{A}_T \cap L_+^0} P(\zeta > 0).$$

Clearly,

$$(2.6.6) \quad 0 \leq \alpha(S) \leq \beta(S) \leq 1.$$

Note that, by Theorem 1.1,

$$(2.6.7) \quad \alpha(S) = 0 \quad \text{implies} \quad \beta(S) = 0.$$

2.7 Definition. A random variable $\zeta \in \bar{A}_T \cap L_+^0$ is called *extreme* if

$$(2.7.1) \quad P(\zeta > 0) = \beta(S).$$

2.8 Theorem. (a) *There always exists an extreme random variable of $\bar{A}_T \cap L_+^0$.*
(b) *If ζ^* and ζ^{**} are extreme, then*

$$(2.8.1) \quad \{\zeta^* > 0\} = \{\zeta^{**} > 0\} \quad \text{a.s.}$$

(c) *For any extreme random variable ζ^e we have*

$$(2.8.2) \quad \{\zeta > 0\} \subset \{\zeta^e > 0\} \quad \text{a.s.}, \quad \zeta \in \bar{A}_T \cap L_+^0.$$

Proof. (a) Let $\zeta_n \in \bar{A}_T \cap L_+^0$ be such that

$$(2.8.3) \quad P(\zeta_n > 0) \rightarrow \beta(S) \quad (n \rightarrow \infty).$$

Choose numbers $c_n > 0$ such that $P(\zeta_n > c_n) \leq 2^{-n}$. The Borel-Cantelli lemma implies that

$$\zeta^e := \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} \zeta_n < \infty \quad \text{a.s.}$$

Clearly, by (2.8.3), $P(\zeta^e > 0) \geq \beta(S)$. Since $\bar{A}_T \cap L_+^0$ is a closed convex cone, $\zeta^e \in \bar{A}_T \cap L_+^0$, and ζ^e is extreme. (b) and (c) are obvious. \square

In the sequel, ζ^e will always denote an extreme random variable of $\bar{A}_T \cap L_+^0$. Note that the set $\{\zeta^e = 0\}$ does (a.s.) not depend on the choice of ζ^e . Theorem 2.8(c) implies that

$$(2.8.4) \quad \{\zeta^e = 0\} \subset \{H^e \bullet S_T = 0\} \quad \text{a.s.}$$

2.9 Theorem. *Let $F \in \mathcal{F}$ be such that $P(F) > 0$. Then there is equivalence between:*

(a) *NA holds on F .*

(b) *There exists a probability measure $Q \in \mathcal{M}_S$ such that $F = \{dQ/dP > 0\}$ a.s.*

Proof. (a) \Rightarrow (b): Assume that NA holds on F . By Theorem 1.1 there exists a probability measure \tilde{Q} on $F \cap \mathcal{F}$ such that $\tilde{Q} \sim P(\cdot|F)$, and $(S_t|F)$ is a \tilde{Q} -martingale (the filtration being $(F \cap \mathcal{F}_t)$). Let Q denote the probability measure which is defined on \mathcal{F} by $Q(A) = \tilde{Q}(F \cap A)$, $A \in \mathcal{F}$. It is easy to see that $Q \in \mathcal{M}_S$, and $F = \{dQ/dP > 0\}$ a.s.

(b) \Rightarrow (a): Let $Q \in \mathcal{M}_S$ and put $F = \{dQ/dP > 0\}$. Let \tilde{Q} be the restriction of Q to $F \cap \mathcal{F}$. Then $\tilde{Q} \sim P(\cdot|F)$, and $(S_t|F)$ is a \tilde{Q} -martingale. Hence, by Theorem 1.1, NA holds on F . \square

2.10 Corollary. *Let $Q^e \in \mathcal{M}_S$ be extreme. Then, for every F on which NA holds, we have*

$$(2.10.1) \quad F = \left\{ \frac{dQ^e(\cdot|F)}{dP(\cdot|F)} > 0 \right\} \quad \text{a.s.}$$

Proof. Let $F \in \mathcal{F}$ be such that $P(F) > 0$. Clearly

$$\frac{dQ^e(\cdot|F)}{dP(\cdot|F)} = \frac{P(F)}{Q^e(F)} \cdot \frac{dQ^e}{dP} I_F \quad \text{a.s.}$$

which implies that

$$\left\{ \frac{dQ^e(\cdot|F)}{dP(\cdot|F)} > 0 \right\} = F \cap \left\{ \frac{dQ^e}{dP} > 0 \right\} \quad \text{a.s.}$$

Hence if NA holds on F , then (2.10.1) follows from Theorem 2.9 and Theorem 2.6(c). \square

2.11 Corollary. (a) *Suppose that $\mathcal{M}_S \neq \emptyset$. Then we have*

$$(2.11.1) \quad \left\{ \frac{dQ}{dP} > 0 \right\} \subset \{\zeta^e = 0\} \quad \text{a.s., } Q \in \mathcal{M}_S.$$

(b) *If NA holds on F , then $F \subset \{\zeta^e = 0\}$ a.s.*

Proof. (a) Let $Q \in \mathcal{M}_S$ and put $F = \{dQ/dP > 0\}$. It follows from Theorem 2.9 that NA holds on F . Let $H^{(n)} \in \mathcal{P}$ and $\phi_n \in L_+^0$ be such that

$$\xi_n := H^{(n)} \bullet S_T - \phi_n \rightarrow \zeta^e \quad \text{in probability.}$$

Applying Theorem 1.1 to the restriction of (ξ_n) to F , we obtain that $\zeta^e = 0$ a.s. on F . This proves (2.11.1).

(b) This follows from Theorem 2.9 and (a). \square

If the definition of $\bar{A}_T \cap L_+^0$ is based on a probability measure P^* (instead of P) we shall write $\bar{A}_T \cap L_+^0[P^*]$ instead of $\bar{A}_T \cap L_+^0$.

Our main result is

2.12 Theorem. *Suppose that $P\{\zeta^e = 0\} > 0$. Then:*

(a) *There exists a probability measure $Q^* \in \mathcal{M}_S$ such that*

$$(2.12.1) \quad \left\{ \frac{dQ^*}{dP} > 0 \right\} = \{\zeta^e = 0\} \quad \text{a.s.}$$

and

$$(2.12.2) \quad \frac{dQ^*}{dP} \in L^\infty.$$

(b) *If $Q^e \in \mathcal{M}_S$ is extreme, then*

$$\left\{ \frac{dQ^e}{dP} > 0 \right\} = \{\zeta^e = 0\} \quad \text{a.s.}$$

(c) *$\{\zeta^e = 0\}$ is (a.s.) the largest set on which NA holds.*

Proof. (a) Let $G^e := \{\zeta^e = 0\}$. In the sequel, we shall assume that $P(G^e) < 1$. (If $P(G^e) = 1$, then the desired result follows from Theorem 1.1.) First note that

$$(2.12.3) \quad \xi \in \bar{A}_T \cap L_+^0 \quad \text{implies that } \xi = 0 \text{ a.s. on } G^e.$$

Let us show that, for the probability measure $P^* = P(\cdot | G^e)$ (defined on \mathcal{F}), we have

$$(2.12.4) \quad \zeta \in \bar{A}_T \cap L_+^0 [P^*] \quad \text{implies that } \zeta = 0 \quad P^*\text{-a.s.}$$

In order to show this, let $\zeta \in \bar{A}_T \cap L_+^0 [P^*]$ be fixed. Then $\zeta \geq 0$ a.s. on G^e , and there exist $H^{(n)} \in \mathcal{P}$ and random variables ϕ_n such that $\phi_n \geq 0$ a.s. on G^e and, for any $\delta > 0$,

$$(2.12.5) \quad P \left(\left\{ |H^{(n)} \bullet S_T - \phi_n - \zeta| > \delta \right\} \cap G^e \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Choose numbers $a_n > 0$ such that

$$(2.12.6) \quad P \left(H^{(n)} \bullet S_T + a_n \zeta^e < -1/n \mid \Omega \setminus G^e \right) \leq 2^{-n}, \quad n \geq 1.$$

Note that, by (2.12.5), for any $\delta > 0$,

$$(2.12.7) \quad P \left(\left\{ |H^{(n)} \bullet S_T + a_n \zeta^e - \phi_n - \zeta| > \delta \right\} \cap G^e \right) \rightarrow 0 \quad (n \rightarrow \infty).$$

Applying the Borel-Cantelli lemma with respect to the probability measure $\tilde{P} = P(\cdot | \Omega \setminus G^e)$ (defined on \mathcal{F}), we obtain, by (2.12.6), that

$$(2.12.8) \quad \tilde{P} \left(H^{(n)} \bullet S_T + a_n \zeta^e \geq -1/n \quad \text{for all sufficiently large } n \right) = 1.$$

Put

$$\tilde{\phi}_n = \begin{cases} \phi_n & \text{on } G^e \\ (H^{(n)} \bullet S_T + a_n \zeta^e)^+ & \text{on } \Omega \setminus G^e \end{cases}$$

and

$$\tilde{\zeta} = \begin{cases} \zeta & \text{on } G^e \\ 0 & \text{on } \Omega \setminus G^e. \end{cases}$$

Let us show that

$$(2.12.9) \quad \tilde{\zeta} \in \bar{A}_T \cap L_+^0.$$

In fact, (2.12.8) implies

$$H^{(n)} \bullet S_T + a_n \zeta^e - \tilde{\phi}_n \rightarrow \tilde{\zeta} \quad \tilde{P}\text{-a.s.}$$

which, in turn, gives for any $\delta > 0$,

$$P \left(\left\{ |H^{(n)} \bullet S_T + a_n \zeta^e - \tilde{\phi}_n - \tilde{\zeta}| > \delta \right\} \cap (\Omega \setminus G^e) \right) \rightarrow 0.$$

Combining this with (2.12.7) and passing to a subsequence (if necessary), we therefore arrive at

$$(2.12.10) \quad H^{(n)} \bullet S_T + a_n \zeta^e - \tilde{\phi}_n \rightarrow \tilde{\zeta} \quad \text{a.s.}$$

There exist $\hat{H}^{(n)} \in \mathcal{P}$ and $\hat{\phi}_n \in L_+^0$ such that

$$P \left(|\hat{H}^{(n)} \bullet S_T - \hat{\phi}_n - a_n \zeta^e| > 1/n \right) \leq 2^{-n}, \quad n \geq 1.$$

Using the Borel-Cantelli lemma once more, shows that (2.12.10) entails (2.12.9) which, combined with (2.12.3), implies $\zeta = 0$ a.s. on G^e . This proves (2.12.4). Hence, by Theorem 1.1, there exists a probability measure

$$(2.12.11) \quad Q^* \sim P^*$$

such that (S_t) is a Q^* -martingale, and

$$(2.12.12) \quad dQ^*/dP^* \in L^\infty.$$

Since $P^* \ll P$, it follows from (2.12.11) that $Q^* \in \mathcal{M}_S$. Finally, (2.12.1) and (2.12.2) are easily obtained from (2.12.11) and (2.12.12). (b) follows from (a) and Corollary 2.11. (c) is a consequence of (b) and Theorem 2.9. \square

At the end of this section we shall outline an alternative proof of Theorem 2.12(a) which is based on a generalization of a certain version of Yan's [8] theorem.

2.13 Corollary. *The following conditions are equivalent:*

- (a) $\mathcal{M}_S \neq \emptyset$.
- (b) $P(\zeta^e = 0) > 0$.

Proof. (a) \Rightarrow (b): This follows from Corollary 2.11.

(b) \Rightarrow (a): This follows from Theorem 2.12. \square

2.14 Theorem. *Suppose that $P(H^e \bullet S_T = 0) > 0$. Then there is equivalence between:*

- (a) NA holds on $\{H^e \bullet S_T = 0\}$.
- (b) $\{H^e \bullet S_t = 0\} = \{\zeta^e = 0\}$ a.s.
- (c) $\bar{A}_T \cap L_+^0 \subset L_+^{(\alpha(S))}$ ($\alpha(S)$ given by (1.1.1)).

Proof. (a) \Rightarrow (b): Assume that NA holds on $\{H^e \bullet S_T = 0\}$. It follows from Theorem 2.9 and Corollary 2.11 that

$$\{H^e \bullet S_T = 0\} = \{dQ^*/dP > 0\} \subset \{\zeta^e = 0\} \quad \text{a.s.}$$

for some $Q^* \in \mathcal{M}_S$. This implies (b).

(b) \Rightarrow (a): This follows from Theorem 2.12.

(c) \Leftrightarrow (b): Note that (c) is equivalent to

$$P(\zeta^e > 0) = \alpha(S) = P(H^e \bullet S_T > 0).$$

By Theorem 2.8 this is equivalent to (b). \square

The subsequent examples show that, in the case $0 < \alpha(S) < 1$, each of the relations $\alpha(S) = \beta(S)$, $\alpha(S) < \beta(S) < 1$ and $\beta(S) = 1$ is possible.

2.15 Example. ($T = 1, d = 3; 0 < \alpha(S) < \beta(S) < 1$). Let $\Omega = \Omega_1 \cup \dots \cup \Omega_4$ (a disjoint union) where $\Omega_1 = \{\omega_{1m} : m \geq 1\}$ and $\Omega_i = \{\omega_i\}$ ($i = 2, 3, 4$). The filtration is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = \{0, 1\}^\Omega$. Let $P\{\omega\} > 0, \omega \in \Omega$. (S_t) is given by $S_0 \equiv (0, 0, 0)$, $S_1(\omega_{1m}) = (1/m, -1, 0)$ ($m \geq 1$), $S_1(\omega_2) = (0, 1, 0)$, $S_1(\omega_3) = (0, 0, 1)$, $S_1(\omega_4) = (0, 0, -1)$. A strategy H belongs to \mathcal{P}_\geq iff $H_1 \equiv (a, 0, 0)$ for some $a \geq 0$. Hence $\{H^e \bullet S_1 > 0\} = \Omega_1$, and $0 < \alpha(S) < 1$. On $\Omega_2 \cup \Omega_3 \cup \Omega_4$ NA does not hold (consider $H \in \mathcal{P}_\geq(\Omega_2 \cup \Omega_3 \cup \Omega_4)$, given by $H_1 \equiv (0, 1, 0)$). On the other hand, NA holds on $\Omega_3 \cup \Omega_4 = \{dQ/dP > 0\}$, $Q \in \mathcal{M}_S$ given by $Q\{\omega_3\} = Q\{\omega_4\} = 1/2$. Hence $0 < \alpha(S) < \beta(S) < 1$. Modifying this example in an obvious way, gives an example for which $0 < \alpha(S) = \beta(S) < 1$.

2.16 Example. ($T = 1, d = 2; 0 < \alpha(S) < \beta(S) = 1$). Let $\Omega = \Omega_1 \cup \Omega_2$ (a disjoint union) where $\Omega_1 = \{\omega_{1m} : m \geq 1\}$ and $\Omega_2 = \{\omega_2\}$. The filtration is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = \{0, 1\}^\Omega$. Let $P\{\omega\} > 0, \omega \in \Omega$. (S_t) is given by $S_0 \equiv (0, 0)$, $S_1(\omega_{1m}) = (1/m, -1)$ ($m \geq 1$) and $S_1(\omega_2) = (0, 1)$. It is easy to see that $H \in \mathcal{P}_\geq$ implies $H \bullet S_1 = (0, 0)$ on Ω_2 . Furthermore, $\{H^e \bullet S_1 > 0\} = \Omega_1$. On the other hand, let $H^{(n)} \in \mathcal{P}$ and $\phi_n \in L_+^0$ be given by $H_1^{(n)} \equiv (n, 1)$, $\phi_n(\omega_{1m}) = (n/m - 2)^+$ ($m \geq 1$), and $\phi_n(\omega_2) = 0$. Since $H^{(n)} \bullet S_1(\omega) - \phi_n(\omega) \rightarrow 1$ ($\omega \in \Omega$), we have $P(\zeta^e > 0) = \beta(S) = 1$. Hence, by Corollary 2.13, $\mathcal{M}_S = \emptyset$.

We conclude this section with outlining an alternative proof of Theorem 2.12(a). We shall use the following result which generalizes a certain version of Yan's [8] theorem (compare Theorem 3.1 in [6]). Using similar arguments as in the proof of Theorem 3.1 in [6], we obtain

2.17 Theorem. *Let $0 \leq \epsilon < 1$ be fixed. Let $K \subset L^1$ be a convex cone which is closed with respect to the norm topology on L^1 . Suppose that*

$$(2.17.1) \quad K \supset -L_+^1 := \{\xi : -\xi \in L^1 \cap L_+^0\}$$

and

$$(2.17.2) \quad K \cap L_+^1 \subset L_+^{(\epsilon)}.$$

Then there exists a random variable \widehat{Z} such that

$$(2.17.3) \quad 0 \leq \widehat{Z} \leq 1 \quad a.s.,$$

$$(2.17.4) \quad P(\widehat{Z} = 0) \leq \epsilon,$$

and

$$(2.17.5) \quad \mathbb{E}[\xi \widehat{Z}] \leq 0, \quad \xi \in K.$$

If we have additionally that

$$(2.17.6) \quad K \cap L_+^1 \not\subset L_+^{(\delta)} \quad \text{for all } 0 \leq \delta < \epsilon,$$

then the above \widehat{Z} can be chosen in such a way that

$$(2.17.7) \quad P(\widehat{Z} = 0) = \epsilon.$$

Proof of Theorem 2.12(a): We proceed similarly as in the proof of Theorem 1.1 in [4]. Fix any extreme random variable ζ^e . Choose any probability measure $\widehat{P} \sim P$ such that ζ^e and the random variables S_t^i are \widehat{P} -integrable, and

$$(2.17.8) \quad d\widehat{P}/dP \in L^\infty.$$

Note that $\bar{A}_T \cap L_+^0[\widehat{P}]$ equals $\bar{A}_T \cap L_+^0$, and ζ^e is also extreme with respect to \widehat{P} . Let $\widehat{\beta}(S) := \widehat{P}(\zeta^e > 0)$ and $K := \bar{A}_T \cap L^1(\widehat{P})$. Then K is a closed convex cone in $L^1(\widehat{P})$, and satisfies (2.17.1) as well as (2.17.2) and (2.17.6) for $\epsilon = \widehat{\beta}(S)$. According to Theorem 2.17 there exists a random variable \widehat{Z} satisfying (2.17.3) and (2.17.5) with respect to \widehat{P} , and we have

$$(2.17.9) \quad \widehat{P}(\widehat{Z} = 0) = \widehat{\beta}(S).$$

Let $Q^* \ll \widehat{P}$ be given by $dQ^*/d\widehat{P} = \widehat{Z}$. Then, by (2.17.8), $dQ^*/dP \in L^\infty$. By (2.17.5), (S_t) is a Q^* -martingale, and $Q^* \in \mathcal{M}_S$. It follows from (2.17.9) and Corollary 2.11 that

$$\left\{ \frac{dQ^*}{dP} > 0 \right\} = \left\{ \frac{dQ^*}{d\widehat{P}} > 0 \right\} = \{\zeta^e = 0\} \quad \widehat{P} - \text{a.s.}$$

Since $\widehat{P} \sim P$, this shows that Q^* satisfies (2.12.1). □

3 Conclusion

The main objective of this paper has been to characterize those subsets of $F^e = \{H^e \bullet S_T = 0\}$, on which NA holds. In particular we showed that $\{\zeta^e = 0\}$ is the largest set with this property. Theorem 2.14 gives conditions which are necessary and sufficient for (S_t) to satisfy the no-arbitrage condition on F^e .

The intuitive reason for the fact that, in general, NA does not hold on F^e is the following: Knowing in advance that F^e occurs may provide “too much information” on the evolution of (S_t) . Hence it would be interesting to find conditions equivalent to those in Theorem 2.14, which are formulated in terms of certain notions of information theory (e.g. entropy).

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