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# Analytical and Numerical Solution of a Poisson RBC model\*

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## Abstract:

This paper analyses a RBC model in continuous time featuring deterministic incremental development of technology and stochastic fundamental inventions arriving according to a Poisson process. Other than in standard RBC models, shocks are uncorrelated, irregular and rather seldom. In two special cases analytical solutions are presented. In the general case a delay differential equation (DDE) has to be solved. Standard numerical solution methods fail, because the steady state is path dependent. A new solution based on a modified method of steps for DDEs provides not only approximations but also upper and lower bounds for optimal consumption path and steady state.

JEL-Classification: C61,C65,C68, E32, O41

Keywords: Business cycle models with poisson shocks, RBC models in continuous time, Delay differential equations.

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# 1 Introduction

RBC models are an often used tool in studying aggregate economic fluctuations. One reason for their popularity is that simulation results are often very good in terms of matching observed Business Cycle facts. Another reason is that solution methods are well established. However, it is well known that the basic RBC framework on which most models are based has some significant shortcomings (see e.g. King and Rebello [10] or Stadler [16]). To be of relevance new models should offer qualitative advantages as well as implementable solution methods. The model presented in this paper offers some advantages over standard RBC models: Technology shocks are seldom and have completely irregular timing, a single shock is able to generate a full business cycle, no auto-correlation of subsequent shocks is necessary to generate persistence in output fluctuations, long-term growth is already included in the model mechanism. The model develops in continuous time and uses a Poisson process as the source for shocks which are fundamental technology inventions. Analytical solutions can be obtained in two special cases. The main contribution, however, is a new numerical solution method which is based on the method of steps for delay differential equations.

Many RBC authors focus on special parameter cases where analytical solutions for the optimal dynamics of models are available. In discrete time the best-known is the case with logarithmic utility and Cobb-Douglas production function assuming full depreciation, first presented by Long and Plosser [13]. As Benhabib and Rustichini [2] showed, this can be extended to the case with utility and production function both of CES-form with the same elasticity parameter. In a continuous time framework the most widely mentioned is the case where the production elasticity of capital in the Cobb-Douglas function  $\alpha$  equals the inverse of intertemporal elasticity of substitution  $\sigma$  in a standard deterministic Ramsey model. Xie [19] showed that in deterministic models this case can be extended to more general functional forms of utility and production function, where both are connected in a certain way. This paper extends the  $\alpha = \sigma$  case to a model with Poisson shocks on technology. It is shown that there exists a unique optimal policy function which is linear in the capital stock. A second special parameter case which results in optimal consumption being a fixed share of output is also presented. By proving a general invariance property of optimal policy functions, it is further shown that all possible analytical solutions have to be of a special form.

General forms of RBC models always require numerical solution methods. Even the basic deterministic neoclassical growth model on which all RBC models are based, does not allow an analytical solution for general parameterizations of utility and production function. This paper presents a numerical solution method for the Poisson RBC model, which is completely different from the usual perturbation methods used for the neoclassical growth model and standard RBC models<sup>1</sup>. There it is in general possible to calculate the steady state or at least an approximation of it a priori from the optimality equations. The transitional dynamics are then obtained by a linearization (or higher degree perturbation) around the steady state. This approach fails with the Poisson RBC model, because the steady state is path-dependent. It can not be calculated or approximated before the optimal consumption path is known. The solution method applied here works as follows: Reducing the model to a stationary system and eliminating time dependence leads to a delay differential equation (DDE) with a single multiplicative delay argument. The solution of this DDE is the optimal consumption

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<sup>1</sup>See King, Plosser and Rebello [9] or Judd [7], ch. 13.

policy. The delay argument is the level of optimal consumption after an invention, i.e. a jump in the Poisson process. The DDE is solved using a modified method of steps. It is based on the method of steps for functional differential equations by Hale [6] modified to the case of a DDE with multiplicative delay. It works by splitting the DDE into a series of ordinary differential equations which can be solved applying the Runge-Kutta or other finite difference methods (see e.g. Judd [7], ch. 10). A similar solution method generating approximations was described by Boucekkine, Licandro and Paul [3]. The present paper extends their work, since not only approximations are provided, but also enveloping paths for the optimal path and upper and lower bounds for the steady state, allowing to measure accuracy of the numerical results.

Real Business Cycle models in continuous time have not received much attention in the literature although they inherit some natural advantages over discrete time models: The timing of shocks is not bound to distinct time points but completely irregular and the development of variables is more smooth. One RBC model in continuous time has been studied by Boucekkine, Del Rio and Licandro [4], who combine it with vintage capital and Leontief production technology. Closer to the here presented work is the model of Wälde [18], who also considers Poisson shocks in a continuous time framework. His work is extended here by two important aspects. It is shown in general that the reduced stationary system of the model stays on its optimal path after a jump in the Poisson process. Further the model is solved in two special cases providing analytical expressions for the optimal consumption policy.

The paper is organized as follows: Section 2 presents the model and introduces the Poisson SDE for the level of technology which causes growth and cycles. The optimal control problem is formulated and necessary and sufficient conditions for a solution are given. Section 3 shows that for all possible analytical solutions the optimal consumption policy has to be of a special form. Then the two already mentioned special parameter cases are considered. Analytical expressions for optimal policy function and value function are presented and their existence and uniqueness proven. Section 4 starts with deducing the reduced stationary system of the model and the basic DDE. Then the modified method of steps is described. Starting with an initial function a sequence of ODE initial value problems is solved. The solutions of these are partial solutions of the basic DDE on subsequent subintervals. Using a standard existence and uniqueness result for DDEs together with a transformation by Kato and McLeod [8] it is then shown that for every initial function the solution of the basic DDE and hence the optimal path is unique. Then a theorem is proved which provides upper and lower bounds for the true steady state values, if enveloping paths for the true optimal path are available. Finally it is described how these enveloping path can be defined formally and how they can be obtained in a practical implementation. The section ends with presenting a numerical example. Section 5 works out differences between the special cases and the general case and discusses in how far the analytical results can give insights for the general case. Properties of the model and the nature of cycles are deduced theoretically and illustrated with a qualitatively motivated simulation.

## 2 The Model

There is a single good in the economy which is used for investment as well as consumption. Its price is set to unity. Output  $Y_t$  is produced employing capital  $K_t$  and labor  $L$  according to a Cobb-Douglas production function.

$$Y_t = K_t^\alpha (A_t L)^{1-\alpha} \quad , \quad \alpha \in (0, 1) \quad (1)$$

There are  $L$  households in the economy, each supplying one unit of labor. Output  $Y_t$  is used for gross investment  $I_t$  and consumption  $C_t$ . Capital  $K_t$  depreciates with rate  $\delta$  and is accumulated according to

$$dK_t = (Y_t - \delta K_t - C_t)dt \quad (2)$$

The level of aggregate technology  $A_t$  grows deterministically over time with an exogenous rate  $g > 0$  and is subject to positive<sup>2</sup> stochastic shocks of size  $\gamma \in (0, 1)$  which arrive according to a Poisson process  $(q_t)$  with arrival rate  $\lambda$ . Following the approach of Redding [14] there are two types of technological progress: Incremental development of existing technologies and fundamental inventions. Incremental development is a permanent and deterministic process taking place all over the economy at every time. Small technological advances aggregate up to a continuously growing aggregate production technology. The fundamental inventions are of very different nature. They are very seldom, occurring only every 4-6 years. Their effect is a leap forward in aggregate production technology. The level of  $A_t$  is increased by the factor  $\gamma$ . The occurrences of fundamental inventions are the jump times of the Poisson process  $(q_t)$ .

$$dA_t = gA_t dt + \gamma A_t dq_t \quad (3)$$

All households have the same preferences represented by an instantaneous utility function  $u$  and a time preference rate  $\rho > 0$ . The economy is governed by a central planner choosing the future consumption path  $\{C_s : s \geq t\}$  in an optimal way. As usual, this future consumption path is described by a consumption policy  $\hat{c}$  which is a function of the state variables  $K_t$  and  $A_t$  and returns the control variable by  $C_t = \hat{c}(K_t, A_t)$ . The instantaneous utility function  $u$  shall be of standard CES form  $u(C) = \frac{C^{1-\sigma}-1}{1-\sigma}$  for  $C \in [1, \infty)$  with  $\sigma > 0$ <sup>3</sup>. The optimal control problem of the model is to maximize expected discounted total utility

$$V(K_t, A_t) = \max_{\hat{c}} E_t^{\hat{c}} \int_t^\infty e^{-\rho(s-t)} u(C_s) ds \quad (4)$$

subject to (2) which is an ordinary differential equation, and (3) which is a Poisson SDE<sup>4</sup>. A solution to the control problem is a pair  $(V, \hat{c})$ , where  $V$  is the Value function and  $\hat{c}$  an optimal policy function. A necessary condition for a solution  $(V, \hat{c})$  is the following Keynes-Ramsey rule<sup>5</sup>

$$dC_t = \left\{ \frac{\alpha}{\sigma} \left( \frac{A_t L}{K_t} \right)^{1-\alpha} - \frac{1}{\sigma} (\rho + \delta - \lambda [(\frac{\tilde{C}_t}{C_t})^{-\sigma} - 1]) \right\} C_t dt + [\tilde{C}_t - C_t] dq_t \quad (5)$$

<sup>2</sup>The analytical results also apply to the case of negative shocks on technology, i.e.  $\gamma \in (-1, 0)$ .

<sup>3</sup>If  $\sigma < 1$  the instant utility function  $u(C)$  is potentially unbounded. To solve the control problem, consumption then has to be assumed bounded by some very large constant.

<sup>4</sup>Stochastic differential equation with respect to a Poisson process

<sup>5</sup>The Keynes-Ramsey rule can be deduced from Bellmann equation of the control problem applying a change of variable formula for Poisson SDE. The derivation follows the lines of Wälde [18].

where  $\tilde{C}_t = \hat{c}(K_t, (1 + \gamma)A_t)$  is the level of consumption that will be attained, if the Poisson process jumps at time  $t$ . The Keynes-Ramsey rule describes the development of optimal consumption  $C_t$  by a Poisson SDE. A sufficient condition for a solution  $(V, \hat{c})$  are the following three equations:

(i) Maximized Bellmann equation

$$\begin{aligned} \rho V(K_t, A_t) = & u(\hat{c}(K_t, A_t)) + [K_t^\alpha (A_t L)^{1-\alpha} - \delta K_t - \hat{c}(K_t, A_t)] V_K(K_t, A_t) \\ & + g A_t V_A(K_t, A_t) + \lambda [V(K_t, (1 + \gamma)A_t) - V(K_t, A_t)] \end{aligned}$$

(ii) First order condition

$$V_K(K_t, A_t) = u'(C_t)$$

(iii) Limiting condition

$$\lim_{t \rightarrow \infty} E e^{-\rho t} V(K_t, A_t) = 0$$

Every pair of functions  $(V, \hat{c})$  satisfying the three equations is a solution to the optimal control problem (4)<sup>6</sup>.

### 3 Specific Analytical Solutions

This section presents a general result for analytical solutions and two examples which arise under certain parameter conditions. The general result states that all possible analytical solutions are of a special form.

**Theorem 1** *Suppose a solution  $(V, \hat{c})$  to the control problem (4) exists. Then the optimal consumption policy  $\hat{c}$  is of the form*

$$\hat{c}(K_t, A_t) = \sum_i \varphi_i K_t^{\nu_i} A_t^{1-\nu_i} \quad \text{with} \quad \varphi_i, \nu_i \in \mathbb{R}, \quad i = 1, 2, \dots \quad (6)$$

The theorem is an immediate consequence of the following invariance property, which is proved in the appendix. It states that every optimal consumption policy  $\hat{c}$  must be homogenous of degree one<sup>7</sup>. This is called invariance property, because it makes  $\hat{c}$  invariant to multiplicative changes in the state variables. Clearly no other algebraic closed-form expression for  $\hat{c}$  than (6) satisfies this property.

**Invariance Property:** *Suppose a solution  $(V, \hat{c})$  to the control problem (4) exists and let  $(K_t, A_t)$  be an arbitrary vector of state variables. Then*

$$\hat{c}(K_t, A_t) = \theta \hat{c}\left(\frac{K_t}{\theta}, \frac{A_t}{\theta}\right) \quad \text{for any} \quad \theta > 0 \quad (7)$$

For optimal control problems like the one considered here it is not in general possible to obtain an analytical solution even in deterministic models. Special cases where this is possible include linear-quadratic models, models with log-utility and models, where optimal

<sup>6</sup>See Schlegel [15] or Kushner and Dupuis [12].

<sup>7</sup>This is a consequence of the constant elasticity of scale of the Cobb-Douglas production function and the linear dynamics of  $(A_t)$ .

consumption is a fixed share of output. Another special case arises, if the production elasticity of capital in the Cobb-Douglas function  $\alpha$  equals the inverse of intertemporal elasticity of substitution  $\sigma$  in a standard deterministic Ramsey model. The following subsections generalize the last two cases to the here considered stochastic Poisson model. It will be shown that there exist unique optimal policy functions and that they indeed are of the general form predicted by Theorem 1.

### 3.1 Special Case $\alpha = \sigma$

If the model parameters satisfy  $\alpha = \sigma$  the control problem (4) has an unique analytical solution  $(V, \hat{c})$ . The optimal consumption policy  $\hat{c}$  is a linear function of capital. The following theorem establishes the closed-form expressions for Value function  $V$  and optimal policy function  $\hat{c}$  and shows the uniqueness:

**Theorem 2** Consider the control problem (4).

- (a) If an optimal consumption policy  $\hat{c}$  with  $\hat{c}(K_t, A_t) = \mu K_t$  exists, then  $\alpha = \sigma$  and the linear factor  $\mu$  is given by  $\mu = \sigma^{-1}\{\rho + (1 - \sigma)\delta\}$ .
- (b) If  $\alpha = \sigma$  and the parameters are such that either  $\sigma < 1$  and  $\gamma < 0$  or  $\sigma < 1 + \frac{\rho}{\delta}$  and  $\rho > (1 - \sigma)g + \lambda[(1 + \gamma)^{1-\sigma} - 1]$  then the linear function  $\hat{c}(K_t, A_t) = \mu K_t$  with  $\mu$  as in (a) is an optimal consumption policy and the value function  $V$  is given by

$$V(K_t, A_t) = \frac{\mu^{-\sigma} K_t^{1-\sigma}}{1 - \sigma} - \frac{1}{\rho(1 - \sigma)} - \frac{\mu^{-\sigma} (A_t L)^{1-\sigma}}{(1 - \sigma)g + \lambda[(1 + \gamma)^{1-\sigma} - 1] - \rho} \quad (8)$$

Part (a) provides necessary conditions for the optimal policy. It is a consequence of the Keynes-Ramsey rule. Part (b) gives sufficient conditions. This part can be proved verifying equations (i) - (iii) of section 2. Whereas part (b) establishes the existence of an optimal policy function, part (a) establishes its uniqueness. The proof is in the appendix. A remarkable fact is that the linear factor  $\mu$  is completely independent of the dynamics of technological progress  $A_t$ . The optimal consumption policy is determined only by the deterministic parameters of the model  $\sigma, \rho$  and  $\delta$ . The frequency  $\lambda$  and the size  $\gamma$  of fundamental technology inventions do not change it. The parameter case of practical interest is that with positive jumps in technology,  $\gamma > 0$ . Then part (b) of the theorem demands two parameter conditions. The first one,  $\sigma < 1 + \frac{\rho}{\delta}$ , is not very restrictive, since in observed economies  $\sigma$  is always around one. The second condition  $\rho > (1 - \sigma)g + \lambda[(1 + \gamma)^{1-\sigma} - 1]$  is slightly more restrictive. It is not satisfied, if fundamental inventions are very large and frequent (high  $\gamma$  and high  $\lambda$ ) and  $\sigma < 1$ <sup>8</sup>. However, no problems arise, if technology inventions are small and infrequent, which is appropriate in practical application.

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<sup>8</sup>For example the parameter combination  $\rho = 0.04$ ,  $g = 0.02$ ,  $\sigma = 0.7$ ,  $\gamma = 0.3$ ,  $\lambda = 0.5$  does not satisfy the condition.

### 3.2 Special Case $\sigma = \frac{\rho + \delta - \lambda[(1 + \gamma)^{1 - \alpha} - 1]}{\alpha\delta - (1 - \alpha)g}$

Another special case where the control problem (4) has an unique analytical solution  $(V, \hat{c})$  is the one which results in a constant savings rate. Under the above parameter condition the optimal consumption policy  $\hat{c}$  is a linear function of output. The following theorem<sup>9</sup> establishes the closed-form expressions for Value function  $V$  and optimal policy function  $\hat{c}$  and shows the uniqueness. It can be proved in a completely analogous way to the previous.

**Theorem 3** *Consider the control problem (4). If and only if the parameters are such that  $\sigma = \frac{\rho + \delta - \lambda[(1 + \gamma)^{1 - \alpha} - 1]}{\alpha\delta - (1 - \alpha)g} > 1$ , then the function  $\hat{c}$  given by*

$$\hat{c}(K_t, A_t) = \nu Y_t = \nu K_t^\alpha (A_t L)^{1 - \alpha} \quad \text{with} \quad \nu = 1 - \frac{1}{\sigma}$$

is an optimal consumption policy and the value function  $V$  is given by

$$V(K_t, A_t) = \frac{\nu^{-\sigma}}{1 - \alpha\sigma} K_t^{1 - \alpha\sigma} (A_t L)^{-\sigma(1 - \alpha)} - \frac{1}{\rho(1 - \sigma)}$$

In both special cases it can be seen from the expressions for  $\hat{c}$  that the optimal policy functions indeed possess the invariance property and are of the general form (6).

## 4 General Numerical Solution

A prerequisite for numerical solution of stochastic optimal control problems is the guarantee of abstract existence of a solution. Hence the following assumption is made:

**Assumption 1:** *There exists a solution  $(V, \hat{c})$  to the control problem (4).*

### 4.1 A Delay Differential Equation

By adjusting the aggregate variables output, capital and consumption for labor and the level of technology a reduced system of the model can be obtained<sup>10</sup>.

$$c_t = \frac{C_t}{A_t L} \quad , \quad k_t = \frac{K_t}{A_t L} \quad , \quad y_t = \frac{Y_t}{A_t L}$$

From the resource constraint (2) and the Keynes-Ramsey rule (5) it is possible to derive two

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<sup>9</sup>In an equivalent formulation the theorem can also be split into two parts like the previous one: A part (a) with necessary conditions establishing uniqueness proved by the Keynes-Ramsey rule and a part (b) with sufficient conditions establishing existence proved by equations (i) - (iii) of section 2.

<sup>10</sup>The new adjusted variables may be called effective per capita variables, variables per efficiency labor or just stationary variables. If no confusion with the original growing variables can arise, they are from time to time just referred to as output, capital or consumption.

equations in  $k_t$  and  $c_t$  which completely describe the development of the economy:

$$dk_t = (y_t - \delta k_t - gk_t - c_t)dt - \frac{\gamma}{1 + \gamma} k_t dq_t \quad (9)$$

$$dc_t = \left\{ \frac{\alpha}{\sigma} k_t^{\alpha-1} - \frac{1}{\sigma} (\rho + \delta + \sigma g - \lambda [(1 + \gamma)^{-\sigma} (\frac{\tilde{c}_t}{c_t})^{-\sigma} - 1]) \right\} c_t dt + (\tilde{c}_t - c_t) dq_t \quad (10)$$

This system (9) and (10) defines the reduced system of the economy. The system is stationary, the increasing stochastic processes ( $q_t$ ) and ( $A_t$ ) no longer appear in the equations. For any given state ( $K_t, A_t$ ) there exists an optimal value of  $C_t$  given by  $\hat{c}(K_t, A_t)$ , where  $\hat{c}$  is the optimal policy function that was assumed to exist in Assumption 1. By the invariance property the optimal value  $c_t$  in the reduced system is only dependent on  $k_t = \frac{K_t}{A_t L}$  and not on the absolute values of  $K_t$  and  $A_t$ , since

$$c_t = \frac{C_t}{A_t L} = \frac{1}{A_t L} \hat{c}(K_t, A_t) = \hat{c}\left(\frac{K_t}{A_t L}, \frac{1}{L}\right) = \hat{c}\left(k_t, \frac{1}{L}\right)$$

Hence a new function  $c$  can be defined by  $c(k) := \hat{c}(k, \frac{1}{L})$  giving the optimal stationary consumption value by  $c_t = c(k_t)$ . The function  $c$  describes the optimal path of the reduced system in the 2-dimensional  $(k, c)$ -plane. As can be seen from (9) a jump in the Poisson process shifts the level of  $k_t$  backwards. The resulting optimal value  $c_t$  is given by  $c(k_t)$  and hence the system is on the same path described by the function  $c$  just further to the left. The system never leaves the optimal path.

By eliminating time dependence the two equations (9) and (10) can be combined to just one equation. Replacing  $c_t$  by  $c(k_t)$  in (10) and applying  $c'(k) = \frac{dc}{dk} (\frac{dk}{dt})^{-1}$  the following single equation for the function  $c$  is obtained:

$$c'(k) = \frac{\frac{\alpha}{\sigma} k^{\alpha-1} c(k) - \frac{1}{\sigma} (\delta + g + \rho + \lambda) c(k) + \frac{\lambda}{\sigma} (1 + \gamma)^{-\sigma} [c(k)]^{1+\sigma} [c(\frac{k}{1+\gamma})]^{-\sigma}}{k^\alpha - c(k) - (\delta + g)k} \quad (11)$$

This is a non-linear delay differential equation (DDE) with the single multiplicative delay argument  $c(\hat{\gamma}k)$ , where  $\hat{\gamma} = \frac{1}{1+\gamma}$ . It is of the general form  $c'(k) = f[k, c(k), c(\hat{\gamma}k)]$  with  $\hat{\gamma} \in (0, 1)$ . It shall be called the basic DDE. The unknown optimal function  $c$  is the solution to it. The task is to solve the basic DDE numerically, or rather find a good approximation for the solution. The optimal consumption policy  $\hat{c}$  or an approximation of it is then given by

$$\hat{c}(K_t, A_t) = A_t L * c\left(\frac{K_t}{A_t L}\right) \quad (12)$$

## 4.2 Solution by a Method of Steps

The solution method applied is a modified method of steps. The method of steps was described by Hale [6] for the case of a DDE with single additive delay argument  $c(k - \tau)$ . In the economic literature Boucekkine, Licandro and Paul [3] have described - also for the additive case - how this method may be applied in continuous-time modelling of a vintage capital growth model. Kato and McLeod [8] study a linear DDE with a single multiplicative

delay argument. The present DDE is non-linear, but a modification of the method described by Hale adapted to the multiplicative case is possible.

Starting at the initial level of capital  $k_0$  the  $k$ -axis is divided into intervals

$$I_0 = \left[\frac{k_0}{1+\gamma}, k_0\right] \quad \text{and} \quad I_i = [x(i-1), x(i)] \quad , \quad i = 1, 2, \dots$$

with the interval boundaries given by  $x(i) = k_0(1+\gamma)^i$  for  $i = 0, 1, 2, \dots$ . The initial interval  $I_0$  lies to the left of  $k_0$ , all other intervals to the right. The intervals have the property that for any  $k \in I_i$  the value  $\frac{k}{1+\gamma}$  lies in the adjoining interval to the left  $I_{i-1}$ . The basic idea of the method of steps is to assume, a solution  $c_{i-1}(\cdot)$  to (11) on the interval  $I_{i-1}$  is known. Replacing the delay term  $c(\frac{k}{1+\gamma})$  in (11) by this known function  $c_{i-1}(\frac{k}{1+\gamma})$  for  $k \in I_i$ , equation (11) reduces to an ordinary differential equation on the interval  $I_i$  of the form

$$c'(k) = f\left[k, c(k), c_{i-1}\left(\frac{k}{1+\gamma}\right)\right] \quad , \quad k \in I_i$$

with a known function  $c_{i-1}$  and an initial condition at the left interval boundary

$$c(x(i-1)) = c_{i-1}(x(i-1))$$

This initial value problem may be solved numerically using standard methods (e.g. Runge-Kutta) yielding a solution  $c_i(\cdot)$  on the interval  $I_i$ . The general solution  $c(\cdot)$  may then be defined piecewise by

$$c(k) = c_i(k) \quad , \quad k \in I_i$$

Repeating the method extends the solution  $c(\cdot)$  to the right step by step. The procedure is stopped at the interval  $I_n$  which includes the steady state value  $k^*$  of the reduced system, i.e. the value such that the point  $(k^*, c(k^*))$  generates no more deterministic movement in the system (9) and (10). In general, the smaller the jump size  $\gamma$  of the Poisson process, the more intervals and hence steps are needed until the solution  $c(k)$  is sufficiently determined. To start the method an initial function  $\phi(k)$  on the initial interval  $I_0 = [\frac{k_0}{1+\gamma}, k_0]$  is necessary. The method generates very smooth solutions. Since at the interval boundaries  $x(i)$  the partial solutions  $c_i$  and  $c_{i+1}$  equal each other, the overall solution  $c$  is continuous. Kolmanovskii and Myshkis [11] show, that the partial solutions  $c_i$  are  $i$  times differentiable. The authors further show that the partial solutions  $c_i$  depend less and less on the initial function  $\phi$  proceeding further to the right.

The following theorem shows that given an initial function  $\phi$  on the initial interval, the basic DDE (11) has a unique solution. The theorem is an application of the standard result on existence and uniqueness of solutions to an initial value problem for a DDE with a single additive delay argument (see e.g. Hale [6], Theorem 5.2) transformed to the case with a single multiplicative delay argument. The proof uses a transformation by Kato and McLeod [8].

**Theorem 4** Consider the open set  $D := \{(k, u, v) \in (0, \infty) \times \mathbb{R}^2 : k^\alpha - u - (\delta + g)k > 0\}$ . Let  $\phi$  be a continuous function on the interval  $I_0 = [\frac{k_0}{1+\gamma}, k_0]$  such that  $(k_0, \phi(k_0), \phi(\frac{k_0}{1+\gamma})) \in D$ . Then the initial value problem with the DDE (11) and initial condition  $c(k) = \phi(k)$  ,  $k \in I_0$  has a unique solution.

The set  $D$  corresponds to the half of the  $(k, c)$ -plane below the zero-motion line for capital. The existence of a unique solution is guaranteed only in the case of an initial function below this zero motion line. The solution is the increasing branch of the optimal path. With an initial function above the zero motion line and the set  $D' = \{(k, u, v) \in (0, \infty) \times \mathbb{R}^2 : k^\alpha - u - (\delta + g)k < 0\}$  the theorem implies the existence of another unique solution on this set  $D'$ . This solution is the decreasing branch of the optimal path. Other than in deterministic models the two branches do not intersect in a unique steady state. There are two different steady states.

The zero motion lines  $H$  for capital and  $G$  for consumption are those points in the  $(k, c)$ -plane at which there is no deterministic movement in (9) or (10) respectively, i.e. where  $dk = 0$  or  $dc = 0$ . Both can be expressed as functions of  $k$ :

$$\begin{aligned} H(k) &= k^\alpha - (\delta + g)k \\ G(k) &= \left[ \frac{1}{\lambda}(\rho + \delta + \sigma g + \lambda) - \frac{\alpha}{\lambda}k^{\alpha-1} \right]^{1/\sigma} (1 + \gamma) c\left(\frac{k}{1 + \gamma}\right) \end{aligned}$$

Whereas the zero motion line for capital  $H(k)$  is completely determined by the above functional expression, the zero motion line for consumption  $G(k)$  is unknown, unless the optimal policy function  $c$  is specified, since it depends on the delay term  $c(\frac{k}{1+\gamma})$ . The zero motion line for capital is independent from the optimal path, but the zero motion line for consumption is path dependent. Since the steady state lies at the intersection of both zero motion lines, it is path dependent as well. Let  $c(\cdot, \phi)$  be an approximation for the solution of the basic DDE obtained from the method of steps with initial function  $\phi$ . Then an approximation  $G(\cdot, \phi)$  for  $G$  is given by

$$G(k, \phi) = \left[ \frac{1}{\lambda}(\rho + \delta + \sigma g + \lambda) - \frac{\alpha}{\lambda}k^{\alpha-1} \right]^{1/\sigma} (1 + \gamma) c\left(\frac{k}{1 + \gamma}, \phi\right) \quad (13)$$

Numerically generated approximations  $c(\cdot, \phi)$  always do one of two things: Hit the zero motion line for capital  $H(k)$  and turn left upwards or attain a maximum and turn right downwards before hitting  $H(k)$ . According to which behavior is observed the parameters of the initial function have to be adjusted in one or the other direction. Repeating this procedure many times provides a good first approximation for the optimal path. An algorithm providing such a reference path is given in the appendix. The following illustration shows the two zero motion lines, the unknown true optimal path  $c(k)$  and the two possible behaviors of approximating paths.

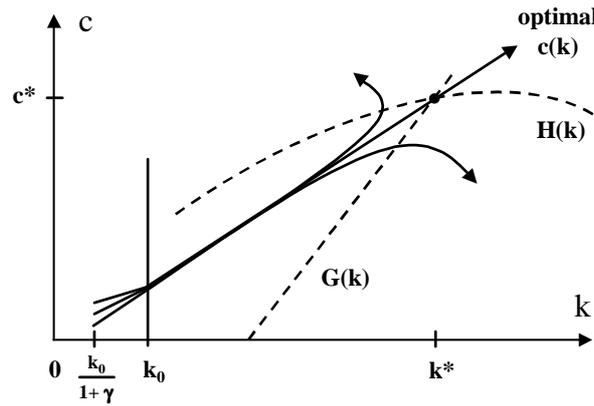


Illustration of possible path behavior

### 4.3 Boundaries for the Solution

For a rigorous numerical solution it is necessary to obtain upper and lower bounds for the optimal path and the steady state. The next theorem describes how bounds for the steady state can be found, if bounds for the optimal path are already known. In the following then a method for generating these bounds for the optimal path is presented. The theorem uses the fact that, if the optimal path  $c(k)$  is known, the steady state  $(k^*, c^*)$  can be determined by the intersection of the zero motion lines  $H(k)$  and  $G(k)$ . It states that bounds for the steady state are determined by the intersection of  $H(k)$  with different  $G(k, \phi)$  generated according to (13) by different approximations  $c(k, \phi)$  resulting from different initial functions  $\phi$ . The initial functions  $\phi$  have to be chosen in such a way, that the approximations  $c(k, \phi)$  envelope the true optimal path  $c(k)$ . The proof is in the appendix.

**Theorem 5** *If there are positive, increasing approximation functions  $c(\cdot, \phi_1)$  and  $c(\cdot, \phi_2)$  enveloping the true optimal path  $c(\cdot)$  on the second last interval*

$$c(k, \phi_1) \leq c(k) \leq c(k, \phi_2) \quad \text{for all } k \in I_{n-1}$$

*then there is a lower bound  $k_2$  and an upper bound  $k_1$  for the  $k$ -value of the steady state*

$$k_2 \leq k^* \leq k_1$$

*The bounds  $k_1$  and  $k_2$  are the intersection points of the zero motion lines for capital and consumption and can be determined as the unique solutions to the following equations*

$$G(k_i, \phi_i) = H(k_i) \quad , \quad i = 1, 2$$

*Upper and lower bounds for the  $c$ -value of the steady state are among  $H(k_1)$ ,  $H(k_2)$  and  $H(k_{\max})$ , where  $k_{\max} = [\frac{1}{\alpha}(\delta + g)]^{1/(\alpha-1)}$  is the value at which the zero motion line  $H(k)$  for capital attains its maximum.*

Now a formal way of defining the enveloping approximation functions is presented. Later it is described, how this rather formal definition may be implemented. Let  $\overline{M}$  be the set of all

continuous initial functions on the interval  $I_0$  and  $M$  the subset of all linear initial functions.

$$M = \{\phi \in \overline{M} : \exists(\mu, c_0) : \phi(k) = \mu(k - k_0) + c_0\}$$

Restricting initial functions to the set  $M$  does not result in loss of generalization. For suppose there is a function  $\phi \in \overline{M}$ . Since  $\phi$  is continuous it attains maximum and minimum on the interval  $I_0$ . Let  $c_0 = \phi(k_0)$ . Then one can define linear functions  $\phi_u, \phi_o \in M$  through the point  $(k_0, c_0)$  with the property that  $\phi_u$  and  $\phi_o$  envelope  $\phi$  on  $I_0$ , i.e. that  $\phi_u(k) \leq \phi(k) \leq \phi_o(k)$  for  $k \in I_0$ . From (11) it can be deduced that the resulting approximation functions on the first interval  $I_1$  are such that  $c(k, \phi_u) \geq c(k, \phi) \geq c(k, \phi_o)$  for  $k \in I_1$ . Hence it is sufficient to consider only the set  $M$  of linear initial functions. By definition of  $M$  a linear initial function is completely described by a pair  $(\mu, c_0)$ . All members of  $M$  may hence be denoted by  $\phi(\mu, c_0)$ . With a small  $\varepsilon > 0$  now define the following subset

$$M_\varepsilon = \{\phi(\mu, c_0) \in M : \nu_u \leq \mu \leq \nu_o, c(\cdot, \phi(\mu, c_0 + \varepsilon)) \text{ turns left upwards} \\ \text{and } c(\cdot, \phi(\mu, c_0 - \varepsilon)) \text{ attains maximum and turns right downwards}\}$$

The members of  $M_\varepsilon$  are such that a small deviation of size  $\varepsilon$  in the value of  $c_0$  in either direction leads to opposite path behavior. This is motivated by the fact, that a small deviation from the true optimal path also leads to opposite path behavior. In the 2-dimensional  $(\mu, c_0)$ -plane in which every point corresponds to an initial function  $\phi(\mu, c_0)$ , the set  $M_\varepsilon$  can be illustrated as a strip of thickness  $\varepsilon$ . Only initial functions taken from this strip  $M_\varepsilon$  may lead to good approximation functions. Good here is understood in the sense, that the approximation functions are so close to the optimal path, that a small deviation of size  $\varepsilon$  to either side changes the path behavior. Initial functions from outside the strip  $M_\varepsilon$  lead to bad approximation functions in this sense. Now consider the following error bounds:

$$\delta(i) := \max_{\substack{\phi \in M_\varepsilon \\ k \in I_i}} c(k, \phi) - \min_{\substack{\phi \in M_\varepsilon \\ k \in I_i}} c(k, \phi) \quad , \quad i = 1, \dots, n$$

The value  $\delta(i)$  is the maximum difference that any two approximation functions resulting from initial functions out of  $M_\varepsilon$  may attain on the interval  $I_i$ . Let  $\phi_r$  be any element of  $M_\varepsilon$  and let  $c_r = c(\cdot, \phi_r)$  be the resulting approximation function called the reference function or reference path. This reference path should be the best guess for the true optimal path. It can be generated from the algorithm presented in the appendix. Then define

$$c_{\min}(k) = \widehat{c}(k) - \delta(i) \quad , \quad k \in I_i \\ c_{\max}(k) = \widehat{c}(k) + \delta(i) \quad , \quad k \in I_i$$

The two functions  $c_{\min}$  and  $c_{\max}$  envelope all paths  $c(\cdot, \phi)$  generated by initial functions  $\phi \in M_\varepsilon$ . Hence the true optimal path is enveloped by  $c_{\min}$  and  $c_{\max}$  as well and the two functions may serve as enveloping approximation functions in Theorem 5<sup>11</sup>. Summing up, the following results were achieved: An approximation function  $c_r$  for the true optimal path

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<sup>11</sup>By way of their definition both are not continuous, but have jumps at every interval border. This can be avoided, when using a global  $\delta = \max_{i=1, \dots, n} \delta(i)$ . But it turns out that this is not useful. Within the first few intervals, the values  $\delta(i)$  are rather large, but diminish quickly moving away from the initial interval. Hence the above definition allows smaller error bounds for the reference function  $c_r(k)$  for most values of  $k$  than can be achieved with a global error bound  $\delta$ .

$c$  was determined. Error bounds  $\delta(i)$  for every interval  $I_i$  were defined such that

$$|c_r(k) - c(k)| \leq \delta(i) \quad , \quad k \in I_i$$

Finally, lower  $(k_l^*, c_l^*)$  and upper bounds  $(k_u^*, c_u^*)$  for the true steady state  $(k^*, c^*)$  were obtained according to Theorem 5.

In a practical implementation, clearly not all linear initial functions from the set  $M_\varepsilon$  can be considered to define the error bounds  $\delta(i)$  and thus the enveloping paths  $c_{\min}$  and  $c_{\max}$ . What can be done, is taking a small collection of initial functions  $\{\phi_1, \dots, \phi_d, \phi_r\}$  out of  $M_\varepsilon$  in such a way, that all corner points of  $M_\varepsilon$  are covered and the reference function  $c_r$  is the best guess for the true optimal path. This collection of initial functions then yields a collection of approximation functions  $\{c(\cdot, \phi_1), \dots, c(\cdot, \phi_d), c_r\}$ . From these the following error bounds<sup>12</sup> can be defined

$$\delta(i) = \max_{k \in I_i} \{c(k, \phi_1), \dots, c(k, \phi_d), c_r(k)\} - \min_{k \in I_i} \{c(k, \phi_1), \dots, c(k, \phi_d), c_r(k)\} \quad , \quad i = 1, \dots, n$$

Now the results are illustrated with a numerical example. Let the parameters be

$$\begin{aligned} \alpha &= 0.3 \quad , \quad \delta = 0.02 \quad , \quad \rho = 0.04 \quad , \quad \sigma = 0.8 \\ g &= 0.02 \quad , \quad \lambda = 0.1 \quad , \quad \gamma = 0.1 \quad , \quad k_0 = 1 \end{aligned}$$

and consider the following initial functions:  $\phi_r(k) = \mu k$  and

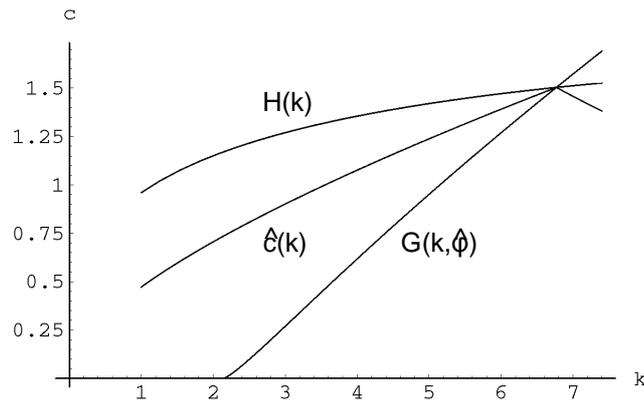
$$\begin{aligned} \phi_1(k) &= 0.2(k - k_0) + \eta_1 \quad , \quad \phi_2(k) = 0.2(k - k_0) + \eta_1 + \varepsilon \\ \phi_3(k) &= 0.4(k - k_0) + \eta_2 \quad , \quad \phi_4(k) = 0.4(k - k_0) + \eta_2 + \varepsilon \\ \phi_5(k) &= 0.6(k - k_0) + \eta_3 \quad , \quad \phi_6(k) = 0.6(k - k_0) + \eta_3 + \varepsilon \end{aligned}$$

$$\mu = 0.472377785302 \quad , \quad \eta_1 = 0.472637231498 \quad , \quad \eta_2 = 0.472448508603 \quad , \quad \eta_3 = 0.472249769050$$

The first one  $\phi_r$  generates the reference path  $c_r$  and the other six cover the corner points of  $M_\varepsilon$  with  $\varepsilon = 10^{-12}$ . The values  $\eta_1, \eta_2, \eta_3$  and  $\mu$  were determined by the recursive approach of adjusting the initial functions according to the observed path behavior (see algorithm in the appendix). They are such that the paths  $c(\cdot, \phi_i)$  generated by  $\phi_1, \phi_3$  and  $\phi_5$  attain a maximum and turn right downwards and the paths generated by  $\phi_2, \phi_4$  and  $\phi_6$  turn left upwards. The following graphic shows the reference path  $c_r(k)$  together with the zero motion line for capital  $H(k)$  and the approximated zero motion line for consumption  $G(k, \phi_r)$ . All three intersect in the approximation of the steady state.

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<sup>12</sup>Recalling Theorem 5 it is clear that to get a small confidence interval for the steady state, it is crucial to have a small error bound in the second last interval  $\delta(n-1)$ . To achieve this one may take in the maximum definition of  $\delta(n-1)$  not the whole interval  $I_{n-1}$ , but instead only the part from the left border up to an  $\bar{k}$  which is chosen in such a way that  $\bar{k}(1+\gamma)$  is a clear upper bound for the steady state.



Reference Path with Zero Motion Lines

The following table provides values of the reference path  $c_r$  for the mid point of each interval  $I_i$  and the error bounds  $\delta(i)$  for this interval. It can be observed that the error bounds quickly become very small and only close to the steady state in the last few intervals become a bit larger again<sup>13</sup>.

Interval	$k$	$\hat{c}(k)$	Error $\delta(i)$	Interval	$k$	$\hat{c}(k)$	Error $\delta(i)$
1	1.050	0.486	$3.87 * 10^{-4}$	11	2.723	0.850	$6.71 * 10^{-10}$
2	1.155	0.513	$3.11 * 10^{-6}$	12	2.996	0.901	$7.56 * 10^{-10}$
3	1.271	0.542	$2.34 * 10^{-8}$	13	3.295	0.955	$8.23 * 10^{-10}$
4	1.398	0.572	$2.11 * 10^{-10}$	14	3.635	1.013	$1.46 * 10^{-9}$
5	1.537	0.605	$2.75 * 10^{-10}$	15	3.987	1.074	$1.97 * 10^{-9}$
6	1.691	0.640	$3.00 * 10^{-10}$	16	4.386	1.140	$2.79 * 10^{-9}$
7	1.860	0.676	$3.42 * 10^{-10}$	17	4.825	1.211	$4.31 * 10^{-9}$
8	2.046	0.716	$4.49 * 10^{-10}$	18	5.307	1.286	$7.79 * 10^{-9}$
9	2.251	0.758	$5.50 * 10^{-10}$	19	5.838	1.367	$1.99 * 10^{-8}$
10	2.476	0.802	$5.64 * 10^{-10}$	20	6.422	1.453	$1.84 * 10^{-7}$

The upper and lower bounds for the steady state determined as the intersection points of the zero motion line for capital  $H(k)$  with the two approximated zero motion lines for consumption  $G(k, c_{\min})$  and  $G(k, c_{\max})$  show a high degree of accuracy:

Steady State Value	Lower Bound	Upper Bound	Difference
$k^*$	6.7688046	6.7688053	$7.39 * 10^{-7}$
$c^*$	1.5040648	1.5040649	$2.85 * 10^{-8}$

## 5 Robustness of Analytical Results, Simulation and Properties of the Model

In the special case  $\alpha = \sigma$  the optimal path is linear with  $\frac{c(k)}{k} = \mu$  constant. In the presented numerical example with  $\alpha = 0.3$  and  $\sigma = 0.8$  the values  $\frac{c(k)}{k}$  range from 0.46 in the first

<sup>13</sup>The quality of the approximation can also be measured by the global Euler test (see Judd [7], ch. 13).

interval down to 0.23 in the last interval. Hence the optimal path is by far not linear, it is concave. It turns out that the degree of concavity is high, if  $\alpha$  is a lot smaller than  $\sigma$  and that the optimal path is close to linear, if the two parameters are close together. For example with  $\alpha = 0.75$  and  $\sigma = 0.8$  the values  $\frac{c(k)}{k}$  range from 0.064 to 0.061, which is quite close to the linear factor  $\mu = 0.06^{14}$ . What are the differences between two economies with a concave and an (almost) linear optimal consumption path? In the first one, the speed of convergence towards the steady state is slower. The speed of convergence is largely determined by the deterministic growth of capital per efficiency labor:

$$dk = k^\alpha - (\delta + g)k - c(k)dt$$

It is easy to see that a more concave optimal path  $c(k)$  leads c.p. to slower capital growth, since for small  $k$ -values the corresponding  $c$ -values are relatively high diminishing  $dk$ . The underlying mechanism is the following: A high value of  $\sigma$  corresponds to a low intertemporal elasticity of substitution. There is a strong preference for today's consumption in the economy and thus investment is low. A low  $\alpha$  moreover implies a relatively low level of output given a stock of capital. Hence in an economy with  $\alpha \ll \sigma$  and correspondingly concave optimal consumption path, output is low and from this low output the most is consumed. Therefore capital accumulation and the speed of convergence is low. If  $\alpha$  and  $\sigma$  are close to each other, implying relatively high  $\alpha$  and/or low  $\sigma$ , the opposite is true. High preference for savings and high levels of output from a given capital stock imply high speed of capital accumulation and convergence towards the steady state. If only physical capital is taken into account, the case  $\alpha = \sigma$  is not realistic. Estimated values for  $\alpha$  range usually between 0.3 and 0.4, whereas estimations of  $\sigma$  yield values around one. If capital is viewed more broadly to include human and physical components, however, Barro, Mankiw and Sala-i-Martin [1] note that a value of  $\alpha = 0.8$  is reasonable<sup>15</sup>.

In the special cases as well as in the general case, the Poisson RBC model is able to generate cycles, although in the presented form only in a rather simple and regular manner. A boom is caused by a new fundamental technology invention. Following this invention output immediately jumps upwards. The new technology is more productive and hence with the given resources more output can be produced. The higher output is initially mainly used for new investment. Consumption does not jump with output, but develops very gradually. It is optimal for the economy first to accumulate more capital to make use of the more productive technology and to postpone a strong rise in consumption. After the initial jump output rises further with growth rates above trend as more capital is accumulated. Then due to diminishing returns investment declines and the speed of capital accumulation slows down. In this phase consumption grows more strongly, as it is optimal now to allocate more resources to producing consumption goods and less to investment in new capital goods. Eventually the growth in both output and consumption levels off and falls below long-term trend as the effect from the productivity rise caused by the new fundamental technology invention fades out and only incremental progress is made.

The following graphic shows a simulation of the model over 100 quarters<sup>16</sup> with very large

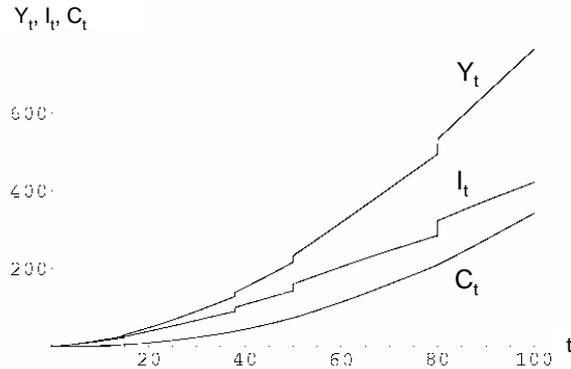
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<sup>14</sup>Evaluated for  $\alpha = \sigma = 0.75$

<sup>15</sup>The authors point out that the neoclassical growth model is able to explain observed rates of convergence towards the steady state, if economies are closed and  $\alpha = 0.8$ .

<sup>16</sup>The following parameters were used:  $\alpha = \sigma = 2/3$ ,  $\delta = 2\%$ ,  $\rho = 1.5\%$ , and  $g = 0.5\%$ . Time horizon

shock intensity (size of inventions)  $\gamma = 0.25$ . Given a realization of the Poisson process  $(q_t)$  the time paths for all aggregate variables were calculated.



Time paths of  $Y_t, C_t$  and  $I_t$  with large shocks ( $\gamma = 0.25$ )

It is easy to see that at the arrival times of the Poisson process output jumps upwards and that this increase is alone due to a jump in aggregate investment. The response of consumption is very gradual. Overall, investment is more volatile and consumption less volatile than output. If a more realistic shock intensity  $\gamma$  is considered, e.g. 1% instead of the 25% in the above graphic, then the jumps in output and investment are less drastic (and no longer visible in the plot). It is possible to apply the HP-filter to the model generated data and then calibrate the model, but these quantitative steps are not the focus of this paper and the present form of the model is too simple for a realistic calibration.

What are the advantages of the Poisson RBC model? Firstly, the timing of shocks is completely irregular. The standard RBC approach always assumes one shock every quarter. Here two shocks can occur shortly after each other, and there can be long periods without a shock at all. Secondly, shocks are rather seldom, they occur only every 4-6 years. Thirdly, long-term growth is already included in the present model through the deterministic growth rate  $g$ . Finally, the Poisson RBC model does not require autocorrelation of shocks to generate sufficient persistence in output fluctuations. A single shock is able to cause a full and persistent output cycle.

## 6 Conclusion

In a Ramsey model in continuous time with infinitely lived households the level of technology grows deterministically by incremental development of existing technologies and stochastically by fundamental inventions which arrive according to a Poisson process. Every fundamental invention causes a full Business Cycle. The model preserves the basic RBC assumptions, but offers some advantages: Shocks are irregular and seldom and a single shock is able to generate a full cycle, no auto-correlation of subsequent shocks is necessary to generate sufficient persistence.

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$T = 100$ , jumps in  $(q_t)$  at  $t = 15, 38, 50, 80$ . All initial values equal to one.

In two special parameter cases analytical solutions for the optimal consumption problem were derived. Existence of the solutions was proved by verifying maximized Bellmann equation, first order condition and limiting condition. Uniqueness followed from a Keynes-Ramsey rule describing the deterministic and stochastic development of optimal consumption in form of a Poisson SDE. Moreover it was shown that due to an invariance property of the optimal consumption policy all possible analytical solutions have to be of a special form.

Like all models based on the neoclassical growth model in continuous time the Poisson RBC model requires numerical solution for general parameter sets. Standard perturbation methods for numerical solution of RBC models fail, because it is not possible to calculate the steady state of the model in advance. The steady state is path dependent and the optimal consumption path is described by a delay differential equation. A new numerical solution method was presented which allows approximation of the optimal consumption path and the steady state. The method is a modified method of steps for solving the DDE describing the optimal consumption path. The method extends previously available results, since in addition to the approximations it also provides enveloping paths for the optimal consumption path and upper and lower bounds for the steady state, thus allowing to measure the accuracy of the numerical results. Application of the new solution method is not restricted to the Poisson RBC model. It is applicable to all models where a Poisson or other jump process leads to optimality equations which are DDEs with a multiplicative delay argument.

## 7 Appendix

**Proof of Invariance Property:** Let  $\theta > 0$  and define  $\bar{K}_t := \frac{K_t}{\theta}$ ,  $\bar{A}_t := \frac{A_t}{\theta}$  and  $\bar{C}_t := \frac{C_t}{\theta}$ . The constraints of the optimal control problem (4) then become

$$d\bar{K}_t = d\frac{K_t}{\theta} = \frac{1}{\theta}dK_t = \left(\frac{Y_t}{\theta} - \delta\frac{K_t}{\theta} - \frac{C_t}{\theta}\right)dt = (\bar{Y}_t - \delta\bar{K}_t - \bar{C}_t)dt \quad (14)$$

and

$$d\bar{A}_t = d\frac{A_t}{\theta} = \frac{1}{\theta}dA_t = g\frac{A_t}{\theta}dt + \gamma\frac{A_t}{\theta}dq_t = g\bar{A}_tdt + \gamma\bar{A}_tdq_t \quad (15)$$

Consider the following optimization problem:

$$\bar{V}(\bar{K}_t, \bar{A}_t) = \max_{\bar{c}} E_t^{\bar{c}} \int_t^{\infty} e^{-\delta(s-t)} u(\bar{C}_s) ds$$

subject to (14) and (15). This optimization problem has the same structure as (4). Existence of a solution  $(V, \hat{c})$  for (4) is equivalent to existence of a solution  $(\bar{V}, \bar{c})$  with the same structure, i.e.  $\bar{c} = \hat{c}$  and  $\bar{V} = V$ . On the one hand the optimal value  $\bar{C}_t$  is then given by  $\bar{c}(\bar{K}_t, \bar{A}_t)$ . On the other hand  $\bar{C}_t = \frac{C_t}{\theta}$ . Hence for an arbitrary vector of state variables  $(K_t, A_t)$  the following holds:

$$\hat{c}(K_t, A_t) = C_t = \theta\bar{C}_t = \theta\bar{c}(\bar{K}_t, \bar{A}_t) = \theta\bar{c}\left(\frac{K_t}{\theta}, \frac{A_t}{\theta}\right) = \theta\hat{c}\left(\frac{K_t}{\theta}, \frac{A_t}{\theta}\right)$$

**Proof of Theorem 2: Part (a):** Suppose optimal consumption is given by  $C_t = \hat{c}(K_t, A_t) = \mu K_t$  with an unknown constant  $\mu$ . Since  $\hat{c}$  is then independent of  $A_t$  and the devel-

oment of  $K_t$  is completely deterministic, the level of  $C_t$  is independent from a jump in the Poisson process ( $q_t$ ). Hence  $\tilde{C}_t = C_t$  and the Keynes-Ramsey rule (5) becomes  $dC_t = [\frac{\alpha}{\sigma}(\frac{A_t L}{K_t})^{1-\alpha} - \frac{1}{\sigma}(\rho + \delta)]C_t dt$ . This is a linear homogeneous differential equation. Its solution is

$$C_t = C_0 \exp \left\{ \int_0^t \frac{\alpha}{\sigma} \left( \frac{A_s L}{K_s} \right)^{1-\alpha} - \frac{1}{\sigma} (\rho + \delta) ds \right\}$$

Inserting  $C_t = \mu K_t$  into the resource constraint of the economy (2) gives  $dK_t = [(\frac{A_t L}{K_t})^{1-\alpha} - \delta - \mu]K_t dt$ . Although this differential equation is not linear, one may formally apply the same solution formula to get an integral expression for  $K_t$ . Then

$$K_t = K_0 \exp \left\{ \int_0^t \left( \left( \frac{A_s L}{K_s} \right)^{1-\alpha} - \delta - \mu \right) ds \right\}$$

Since  $C_0 = \mu K_0$  and  $C_t = \mu K_t$  for all  $t > 0$  the following identity must hold:

$$\int_0^t \left( \frac{\alpha}{\sigma} - 1 \right) \left( \frac{A_s L}{K_s} \right)^{1-\alpha} + \mu + \delta - \frac{1}{\sigma} (\rho + \delta) ds = 0 \quad , \quad \forall t > 0$$

This condition must also hold for  $t \searrow 0$ . Hence it follows that the integrand must be equal to zero for all  $s \in (0, t)$ . The only way this can be achieved is, if  $\alpha = \sigma$  and  $\mu = \sigma^{-1} \{ \rho + (1 - \sigma) \delta \}$ .

**Part (b):** This part is proved verifying equations (i)-(iii) in section 2 for the candidates  $(V, \hat{c})$ . By Inserting the candidates into (i), using the form of  $u$  and the equality  $\alpha = \sigma$  and finally inserting for  $\mu$  it is after some transformation easily verified that equation (i) is satisfied. Differentiating (8) by capital  $K_t$  and using the form of  $u$  gives (ii):

$$V_K(K_t, A_t) = \mu^{-\sigma} K_t^{-\sigma} = \hat{c}(K_t, A_t)^{-\sigma} = C_t^{-\sigma} = u'(C_t)$$

From the structure of the Value function in (8) it follows that for the limiting condition (iii) it is sufficient to show

$$\lim_{t \rightarrow \infty} e^{-\rho t} E(A_t^{1-\alpha}) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-\rho t} E(K_t^{1-\alpha}) = 0$$

This can be done applying a change of variable formula to the differential equations (3) and (2), inserting  $\alpha = \sigma$  and using the martingale property of the integral with respect to a Poisson process. It follows that  $\lim_{t \rightarrow \infty} e^{-\rho t} E(A_t^{1-\alpha}) = 0$  if and only if

$$\rho > (1 - \sigma)g + \lambda[(1 + \gamma)^{1-\sigma} - 1] \quad (16)$$

Since  $\rho > 0$  this condition is certainly satisfied if  $\sigma < 1$  and  $\gamma < 0$ . Further it can be deduced that  $\lim_{t \rightarrow \infty} e^{-\rho t} E(K_t^{1-\alpha}) = 0$  if condition (16) holds and  $\mu > 0$ . According to the expression for  $\mu$  this is certainly positive if  $\sigma < 1$ . If that is not the case then  $\sigma < \frac{\rho}{\delta} + 1$  is required.

### Algorithm for first numerical approximation (reference path)

**Step 1: Specify parameters and initial values:** Set parameter values for  $\alpha, \sigma, \rho, \delta, g, \lambda$

and  $\gamma$ , initial value  $k_0$ , and number  $n$  of intervals to be considered<sup>17</sup>. Then calculate intervals

$$I_i = [x(i-1), x(i)] \quad \text{for } i = 1, 2, \dots, n \quad \text{with } x(i) = k_0(1 + \gamma)^i$$

Define initial function  $\phi_\mu$  depending on parameters in the parameter vector  $\mu = (\mu_1, \mu_2, \dots, \mu_r)$  and set accuracy level  $\zeta$  for parameters  $\mu$ .

**Step 2: Parameterize initial function  $\phi_\mu$ :** Set values for the parameters  $\mu$  of the initial function

**Step 3: Generate solution  $c(\cdot, \phi_\mu)$ :** Solve equation (11) with  $c(\frac{k}{1+\gamma}) = \phi_\mu(k)$  on  $I_0$  yielding a solution  $c_1(\cdot, \phi_\mu)$  on  $I_1$ . For  $i = 2$  to  $n$  solve equation (11) with  $c(\frac{k}{1+\gamma}) = c_{i-1}(\frac{k}{1+\gamma}, \phi_\mu)$  on  $I_i$  yielding a solution  $c_i(\cdot, \phi_\mu)$  on  $I_i$ . Define overall solution  $c(\cdot, \phi_\mu)$  by  $c(k, \phi_\mu) := c_i(k, \phi_\mu)$ ,  $k \in I_i$ ,  $i = 1, \dots, n$

**Step 4: Test result and repeat:** If  $\mu$  has reached predefined accuracy level  $\zeta$  STOP,  $c(\cdot, \phi_\mu)$  is a good approximation. If  $c(\cdot, \phi_\mu)$  turns left upwards go back to Step 2 and adjust parameters  $\mu$  of initial function (lower). If  $c(\cdot, \phi_\mu)$  attains maximum go back to Step 2 and adjust parameters  $\mu$  of initial function (higher).

**Proof of Theorem 5:** Let  $c(k, \phi_1) \leq c(k, \phi_2)$  for all  $k \in I_{n-1}$ . This is equivalent to  $c(\frac{k}{1+\gamma}, \phi_1) \leq c(\frac{k}{1+\gamma}, \phi_2)$  for  $k \in I_n$ . By (13) the resulting approximations for the zero motion line for consumption satisfy

$$G(k, \phi_1) \leq G(k, \phi_2) \quad , \quad k \in I_n \quad (17)$$

From (13) it is easy to see that all functions  $G(\cdot, \phi_i)$  have a common zero at  $k_G^0 = [\frac{1}{\alpha}(\rho + \delta + \sigma g + \lambda)]^{1/(\alpha-1)}$ . Further it can be shown that the functions  $G(\cdot, \phi_i)$  are increasing for  $k \geq k_G^0$ . To see this let  $k_G^0 \leq x \leq y$ . Then  $-x^{\alpha-1} \leq -y^{\alpha-1}$  and since all parameters are positive and the  $c(\cdot, \phi_i)$  are increasing functions, it follows that

$$\begin{aligned} G(x, \phi_i) &= \left[ \frac{1}{\lambda}(\rho + \delta + \sigma g + \lambda) - \frac{\alpha}{\lambda}x^{\alpha-1} \right]^{1/\sigma} (1 + \gamma) c\left(\frac{x}{1 + \gamma}, \phi_i\right) \\ &\leq \left[ \frac{1}{\lambda}(\rho + \delta + \sigma g + \lambda) - \frac{\alpha}{\lambda}y^{\alpha-1} \right]^{1/\sigma} (1 + \gamma) c\left(\frac{y}{1 + \gamma}, \phi_i\right) = G(y, \phi_i) \end{aligned}$$

It is easily verified that the function  $H(k)$  has a maximum at  $k_{\max} = [\frac{1}{\alpha}(\delta + g)]^{1/(\alpha-1)}$  and is strictly increasing for  $k < k_{\max}$  and strictly decreasing for  $k > k_{\max}$ . Further the function  $H(k)$  has zeros at 0 and  $k_H^0 = (\delta + g)^{1/(\alpha-1)}$ . Since  $0 < k_G^0$  the function  $H$  lies above the two functions  $G(\cdot, \phi_i)$  for small  $k$ . But since the  $G(\cdot, \phi_i)$  increase and  $H$  eventually decreases after  $k_{\max}$  the function  $H$  lies below the  $G(\cdot, \phi_i)$  for large  $k$ . Thus for  $i = 1, 2$  there exist unique values  $k_i$ , where  $H$  and  $G(\cdot, \phi_i)$  intersect. These  $k_i$  are the unique solutions to the equation

$$\begin{aligned} H(k) &= G(k, \phi_i) \\ \Leftrightarrow k^\alpha - (\delta + g)k &= \left[ \frac{1}{\lambda}(\rho + \delta + \sigma g + \lambda) - \frac{\alpha}{\lambda}k^{\alpha-1} \right]^{1/\sigma} (1 + \gamma) c\left(\frac{k}{1 + \gamma}, \phi_i\right) \end{aligned}$$

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<sup>17</sup>The appropriate number  $n$  is not known a priori. Rather it has to be found by a sequential approach of increasing  $n$ .

From (17) it then follows that  $k_2 \leq k_1$ . Repeating the whole line of argument with the true optimal path  $c(k)$  in between the approximation functions

$$c(k, \phi_1) \leq c(k) \leq c(k, \phi_2) \quad \text{for all } k \in I_{n-1}$$

it can be deduced that the values  $k_2$  and  $k_1$  are lower and upper bound for the  $k^*$ -value of the steady state:  $k_2 \leq k^* \leq k_1$ . There are four different cases for the bounds of the  $c^*$ -value of the steady state, depending on whether  $G(\cdot, \phi_1)$  and  $G(\cdot, \phi_2)$  intersect with  $H$  in the increasing or decreasing part, i.e. whether  $k_2$  and/or  $k_1$  are greater or less than  $k_{\max}$ . In the four possible cases the boundaries for  $c^* = H(k^*)$  are as follows

$$\begin{aligned} k_2 \leq k^* \leq k_1 \leq k_{\max} &\implies H(k_2) \leq c^* \leq H(k_1) \\ k_2 \leq k^* \leq k_{\max} \leq k_1 &\implies H(k_2) \leq c^* \leq H(k_{\max}) \\ k_2 \leq k_{\max} \leq k^* \leq k_1 &\implies H(k_1) \leq c^* \leq H(k_{\max}) \\ k_{\max} \leq k_2 \leq k^* \leq k_1 &\implies H(k_1) \leq c^* \leq H(k_2) \end{aligned}$$

Hence upper and lower bounds for  $c^*$  can always be found among  $H(k_1)$ ,  $H(k_2)$  and  $H(k_{\max})$  as stated in the theorem.

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