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# On integrals with respect to Lévy processes\*

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## Abstract

Assume  $L$  is a non-deterministic real valued Lévy process and  $f$  is a smooth function on  $[0, t]$ . If for some Borel function  $H$   $P$ -almost sure the equality

$$H\left(\int_{[0,t]} f(s)dL_s\right) = L_t$$

holds, then  $f$  is constant on  $[0, t]$ .

**AMS-Classification:** 60G51, 60E07, 60H20

**Key words.** Lévy process, compound Poisson process, integration with respect to Lévy process.

## 1 Introduction

In modeling term structures of interest rates driven by Lévy processes  $L = (L_s, s \geq 0)$  arises the question, if for a fixed positive  $t$  and for a given deterministic continuously differentiable function  $f$  on  $[0, t]$  there may exist a (deterministic) Borel function  $H$  with

$$H\left(\int_{[0,t]} f(s)dL_s\right) = L_t \quad P - a.s. \quad (1)$$

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See Küchler, Naumann [3]. Obviously (1) holds with  $H(x) = c^{-1}x$  if  $f$  equals a nonzero constant  $c$ . In this note we will show that for any non-deterministic Lévy process  $L$  the existence of a measurable function  $H$  with (1) implies  $f$  to be constant.

Note that under assumption (1) the support of the distribution of the random vector  $(I_t, L_t)$  with  $I_t = \int_{[0,t]} f(s)dL_s$  is a zero set with respect to the (two-dimensional) Lebesgue measure  $\lambda_2$ . Therefore the assertion immediately follows if one can ensure that  $(I_t, L_t)$  has a common density with respect to  $\lambda_2$ . This is the case if  $L$  is a Wiener process with positive diffusion coefficient and arbitrary drift. Indeed, if  $f$  is not a constant then  $(F_t, L_t)$  is two-dimensional Gaussian distributed with a regular covariance matrix. For other Lévy processes  $L$  Eberlein and Raible [1] proved that the following condition on the characteristic function of  $L_t$  is sufficient for  $(I_t, L_t)$  to have a common density: There exist real constants  $C, \gamma, \eta > 0$  such that

$$|E[\exp(i u L_1)]| \leq C \cdot \exp(-\gamma|u|^\eta) \quad \forall u \in R \quad (2)$$

But not for every Lévy process  $L$  the vector  $(I_t, L_t)$  has a common density, consider for example Poisson processes. Analyzing the arguments above it turns out to be enough for our purpose to show that the common distribution of  $(I_t, L_t)$  has a nonzero absolutely continuous part with respect to  $\lambda_2$  if  $f$  is not equal to a constant. The key point of the proof of the theorem below is to show that this property holds for every non-deterministic Lévy process  $L$ .

## 2 Lévy processes

Assume  $L = (L_s, s \geq 0)$  to be a real valued Lévy process defined on some probability space  $(\Omega, \mathfrak{A}, P)$ . This means

- (i)  $P(L_0 = 0) = 1$ .
- (ii)  $L$  has independent and stationary increments.
- (iii) All trajectories  $(L_s(\omega), s \geq 0), \omega \in \Omega$ , are cadlag, i.e. continuous from the right and having limits from the left.

Examples of Lévy process are

- the Wiener process  $W$  with drift  $\mu$  and diffusion  $\sigma^2 > 0$ , in this case  $W_t - W_s$  is  $N(\mu(t - s), \sigma^2(t - s))$ -distributed, and all trajectories are continuous,

- the Poisson process  $N$  with jump intensity  $\lambda > 0$ , in this case  $N_t - N_s$  is Poisson-distributed with parameter  $\lambda(t - s)$  and the trajectories are piecewise constant, non-decreasing, jumping at times  $\tau_k, k \geq 1$ , with jump size one where  $(\tau_k - \tau_{k-1}), k \geq 1$ , with  $\tau_0 = 0$  are mutual independent exponential distributed with parameter  $\lambda$  random variables.
- the compound Poisson process  $Y_t = \sum_{k=1}^{N_t} Z_k$ , where  $N = (N_s, s \geq 0)$  is a Poisson process with jump intensity  $\lambda > 0$  and the  $(Z_k, k \geq 1)$  are independent identically distributed random variables the distribution function  $F$  of which satisfies  $F(0+) = F(0-)$ . Moreover, the  $(Z_k, k \geq 1)$  and  $N$  are independent. The  $Z_k$  form the jump sizes of  $(Y_t, t \geq 0)$ .

We summarize some well known facts on Lévy processes that will be used below. For proofs and further details see for example Sato [4]. In general, to every Lévy process  $L$  is assigned a uniquely determined *characteristic triple*  $\mathfrak{L} := (\mu, \sigma^2, \nu)$  where  $\mu \in R, \sigma^2 \geq 0$  and  $\nu$  is a measure on the real axis  $R$  satisfying

$$\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \nu(dx) < \infty \quad (3)$$

such that it holds

$$E \exp(iyL_t) = \exp[t \cdot \psi(y)], \quad t \geq 0, y \in R$$

with

$$\psi(y) = i\mu y - \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} (e^{iyx} - 1 - \frac{iyx}{1+x^2}) \nu(dx), y \in R.$$

Conversely, to any triple  $\mathfrak{L} = (\mu, \sigma^2, \nu)$  with (3) there is a uniquely in law determined Lévy process  $L$  having  $\mathfrak{L}$  as its characteristic triple. If  $\sigma^2 + \nu(R) = 0$  it holds  $L_s = \mu s, s \geq 0$ . We call this case the deterministic one.

If  $\nu \equiv 0$  then we obtain a Wiener process with  $\mu$  as drift and with  $\sigma^2$  as diffusion coefficient.

Let  $L$  be a Lévy process. For any  $\varepsilon > 0$  one can decompose  $L$  into two mutual independent Lévy processes  $L^{\leq \varepsilon}$  and  $L^{> \varepsilon}$  defined by

$$L_s^{> \varepsilon} := \sum_{u \leq s} \mathbb{I}_{\{|\Delta L_u| > \varepsilon\}} \Delta L_u, s \geq 0, \text{ where}$$

$$\Delta L_u := L_u - L_{u-0},$$

and

$$L_s^{\leq \varepsilon} := L_s - L_s^{> \varepsilon}, \quad s \geq 0.$$

The process  $L^{>\varepsilon}$  is a compound Poisson process with jump intensity  $\lambda_\varepsilon := \nu(R \setminus [-\varepsilon, \varepsilon])$  and jump size distribution  $F_\varepsilon(dz) := \lambda_\varepsilon^{-1} \nu(dz) \cdot \mathbb{1}_{R \setminus [-\varepsilon, \varepsilon]}(z)$ .

If  $f$  is a continuously differentiable function on  $[0, t]$  we define the integral

$\int_{[0,t]} f(s) dL_s$  by

$$\int_{[0,t]} f(s) dL_s := f(t)L_t - \int_{[0,t]} L_s \frac{d}{ds} f(s) ds$$

and denote it shortly by  $I_t$ .

Note that  $L$  is a (possibly degenerated to the deterministic case if  $\sigma^2 = 0$ ) Wiener process if and only if  $L_t^{>\varepsilon} \equiv 0$  for all  $\varepsilon > 0$ .

### 3 Results

The result of this note is the following theorem.

**Theorem 3.1.** *Fix a positive number  $t$ , let  $f$  be a real valued continuously differentiable function on  $[0, t]$  and assume  $L = (L_s, s \geq 0)$  to be a non-deterministic Lévy process. If for some Borel function  $H$  it holds*

$$H\left(\int_{[0,t]} f(s) dL_s\right) = L_t \quad P - a.s. \quad (4)$$

then  $f$  is necessarily a constant.

*Proof.* Let us firstly consider the case that  $L$  is a Wiener process with parameters  $\mu \in R$  and  $\sigma^2 > 0$ . Then  $(I_t, L_t) = \left(\int_0^t f(s) dL_s, L_t\right)$  is a Gaussian vector. Its distribution is degenerated by assumption (4). Thus its covariance matrix is singular, because of  $\sigma^2 > 0$  this implies  $\left(\int_{[0,t]} f(s) d s\right)^2 = t \cdot \int_{[0,t]} f^2(s) d s$ . By the Cauchy-Schwarz inequality this is only possible if  $f$  is a constant.

As a second step we assume that  $L$  is a compound Poisson process with jump times  $\tau_1, \tau_2, \dots$  and jumps sizes  $Z_k$  at time  $\tau_k, k \geq 1$ . Denote the distribution function of  $Z_k$  by  $F$ . Let  $z$  be a point of increase of  $F$ . This means  $P(z - \varepsilon < Z_1 < z + \varepsilon) > 0$  for all  $\varepsilon > 0$ .

We may suppose  $z > 0$ , otherwise consider  $-L$  instead of  $L$ . If  $f$  is not a constant function, based on the assumptions on  $f$  we find a subinterval  $U := (u_0, u_1)$  of  $(0, t)$  with  $u_0 < u_1$  and  $f'(x) \neq 0$  at all points  $x$  from  $U$ . In particular  $f'$  is either strictly positive or strictly negative on  $U$ . Thus  $f$  is strictly monotone on  $U$  and consequently maps  $U$  one-to-one on an interval  $V = (v_0, v_1)$  with  $v_0 < v_1$ .

Without restriction we can assume  $v_0 > 0$ , otherwise restrict  $U$  to a smaller open non-void interval and/or consider, if it is needed,  $-f$  instead of  $f$ .

We continue the proof by choosing a positive  $\varepsilon$  such that

$$\varepsilon < z \cdot \frac{c-1}{c+1} \quad (5)$$

with  $c := (v_1/v_0)^{\frac{1}{2}}$ . For every  $n \geq 1$  put

$$C_{\varepsilon,n} := \{\omega \in \Omega \mid Z_k(\omega) \in (z - \varepsilon, z + \varepsilon), k = 1, \dots, n\}$$

and

$$D_n := \{\omega \in \Omega : u_0 < \tau_1(\omega) < \dots < \tau_n(\omega) < u_1, \tau_{n+1}(\omega) < t\}.$$

Because of the special choice of the point  $z$  and the independence of the  $Z_k, k \geq 1$  we have  $P(C_{\varepsilon,n}) > 0$ . The jump times  $\tau_n, n \geq 1$  have the property that the differences  $\tau_{k+1} - \tau_k, k \geq 1$ , are independent and exponentially distributed. This implies in particular  $P(D_n) > 0$ . Now using the independence of the  $(Z_k, k \geq 1)$  from the  $(\tau_k, k \geq 1)$  we conclude that

$$P(D_n \cap C_{\varepsilon,n}) = P(D_n)P(C_{\varepsilon,n}) > 0. \quad (6)$$

We have  $P$ -almost surely for all  $\omega \in D_n \cap C_{\varepsilon,n}$

$$I_t = \int_{[0,t]} f(s) dL_s = \sum_{k=1}^n Z_k f(\tau_k) \in (n(z - \varepsilon)v_0, n(z + \varepsilon)v_1) \quad (7)$$

Consequently, the distribution of  $I_t$  given  $D_n \cap C_{\varepsilon,n}$  can be expressed as

$$P(I_t \in B \mid D_n \cap C_{\varepsilon,n}) = [P(C_{\varepsilon,n})]^{-1} \int_{(z-\varepsilon, z+\varepsilon)^n} P\left(\sum_{k=1}^n z_k f(\tau_k) \in B \mid D_n\right) F(dz_1) \dots F(dz_n) \quad (8)$$

where  $B$  is any Borel set. Here we have used once again the independence of the  $(Z_k, k \geq 1)$  from the  $(\tau_k, k \geq 1)$ .

At this point we formulate and prove an auxiliary result on the distribution of  $\sum_{k=1}^n z_k f(\tau_k)$  as a lemma.

**Lemma 3.1.** *Assume  $n \geq 1$ , as well as  $z_k > 0, k = 1, \dots, n$ , to be fixed. Then the distribution of  $S_n := \sum_{k=1}^n z_k f(\tau_k)$  given  $D_n$  has a density being strictly positive on*

$$\mathcal{J} := \left( v_0 \sum_{k=1}^n z_k, v_1 \sum_{k=1}^n z_k \right).$$

*Proof.* Given  $D_n^0 := \{\tau_n \leq t < \tau_{n+1}\}$  the random vector  $(\tau_1, \tau_2, \dots, \tau_n)$  has a strictly positive density on  $\Delta_n := \{(s_1, \dots, s_n) : 0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq t\}$ , indeed it is uniformly distributed thereon (see for example Sato [4], Chapter 1.3). This implies immediately that given  $D_n$  the vector  $(\tau_1, \dots, \tau_n)$  is uniformly distributed on  $\Delta_n \cap (u_0, u_1)^n$ .

Put  $S_n(s_1, \dots, s_n) := \sum_{k=1}^n z_k f(s_k)$ . The function  $G$  defined by  $G(s_1, \dots, s_n) = (s_1, s_2, \dots, s_{n-1}, S_n(s_1, \dots, s_n))$  maps  $\Delta_n \cap (u_0, u_1)^n$  continuously differentiable one-to-one on  $(\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times \mathcal{J}$ . This is a consequence of the supposed smoothness and the strict monotonicity of the mapping  $f$  from  $(u_0, u_1)$  onto  $(v_0, v_1)$ . Thus, given  $D_n$  the vector  $G(\tau_1, \dots, \tau_n) = (\tau_1, \dots, \tau_{n-1}, \sum_{k=1}^n z_k f(\tau_k))$  has a density being strictly positive on  $(\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times \mathcal{J}$  which can be expressed as

$$\gamma \cdot \mathbb{I}_{\Delta_n \cap (u_0, u_1)^n}(G^{-1}(v_1, v_2, \dots, v_n))(z_n |f'(v_n)|)^{-1}$$

where  $\gamma$  is the normalizing constant. Consequently given  $D_n$  the random variable  $S_n = S_n(\tau_1, \dots, \tau_n)$  has a density  $\psi_{S_n}(v)$  equal to

$$\begin{aligned} \gamma z_n^{-1} \cdot \int_{\Delta_{n-1} \cap (u_0, u_1)^{n-1}} \mathbb{I}_{\Delta_n \cap (u_0, u_1)^n}(G^{-1}(v_1, \dots, v_{n-1}, v)) dv_1, \dots, dv_{n-1} \\ \cdot |f'(v)|^{-1} \cdot \mathbb{I}_{\mathcal{J}}(v) \quad , \quad v \in R, \end{aligned}$$

which is strictly positive on  $\mathcal{J}$ . Thus the lemma is proved.  $\square$

Corollary 3.1: *Given  $D_n \cap C_{\varepsilon, n}$  the integral  $I_t$  has a density being strictly positive at least on the non-void interval  $(nv_0(z + \varepsilon), nv_1(z - \varepsilon))$ .*

*Proof.* We use the Lemma 3.1 to express the integrand in formula (8)

$$P\left(\sum_{k=1}^n z_k f(\tau_k) \in B \mid D_n\right) = \int_B \psi_{S_n}(v; z_1, \dots, z_n) dv \quad (9)$$

Inserting (9) into (8) and changing the order of integration we get

$$\begin{aligned} P(I_t \in B \mid D_n \cap C_{\varepsilon, n}) \cdot P(C_{\varepsilon, n}) = \\ \int_B \left( \int_{(z-\varepsilon, z+\varepsilon)^n} \psi_{S_n}(v; z_1, \dots, z_n) F(dz_1) \dots F(dz_n) \right) dv \end{aligned}$$

Thus given  $D_n \cap C_{\varepsilon, n}$  the integral  $I_t$  has a density which is strictly positive on  $(nv_0(z + \varepsilon), nv_1(z - \varepsilon))$ .

That this interval is non void follows from assumption (5). Indeed, (5) implies  $z + \varepsilon < c(z - \varepsilon)$  as well as  $z > \varepsilon$  and thus we have

$$\frac{v_1(z - \varepsilon)}{v_0(z + \varepsilon)} = c^2 \frac{(z - \varepsilon)^2}{(z + \varepsilon)^2} \cdot \frac{z + \varepsilon}{z - \varepsilon} > \frac{z + \varepsilon}{z - \varepsilon} > 1. \quad (10)$$

This completes the proof of the corollary.  $\square$

We continue the proof of the theorem. By assumption we have

$$H(I_t) = L_t \quad P - a.s.$$

For any  $n \geq 1$  this implies

$$H(I_t) = \sum_{k=1}^n Z_k \in (n(z - \varepsilon), n(z + \varepsilon)) \quad P(\cdot | D_n \cap C_{\varepsilon, n}) - a.s. \quad (11)$$

Now choose two positive integers  $l$  and  $m$  with

$$\frac{z + \varepsilon}{z - \varepsilon} < \frac{l}{m} < c^2 \frac{z - \varepsilon}{z + \varepsilon} \quad (12)$$

which is possible because of (10).

Introduce for any  $n \geq 1$  the two intervals

$$V_n := (n(z - \varepsilon), n(z + \varepsilon)) \text{ and } W_n := (nv_0(z + \varepsilon), nv_1(z - \varepsilon)).$$

By construction of  $l$  and  $m$  we get from (12)  $l > m$  and

$$\underline{w} := lv_0(z + \varepsilon) < mv_1(z - \varepsilon) =: \bar{w}. \quad (13)$$

Thus it holds

$$W_l \cap W_m = (\underline{w}, \bar{w}) \neq \emptyset. \quad (14)$$

From Corollary 3.1. and (11) it follows for  $n = l$

$$H(v) \in V_l \quad \text{Lebesgue - a.e. on } W_l$$

and for  $n = m$

$$H(v) \in V_m \quad \text{Lebesgue - a.e. on } W_m.$$



Thus (14) implies

$$H(v) \in V_l \cap V_m \quad \text{for all } v \text{ from } (\underline{w}, \overline{w}). \quad (15)$$

But from the first inequality of (12) we have  $l > m$  and  $l(z - \varepsilon) > m(z + \varepsilon)$ , this means  $V_l \cap V_m = \emptyset$ . This is a contradiction to (15). Consequently, the assumption, that  $f$  is not constant cannot be valid.

The proof of the theorem is finished for  $L$  being a compound Poisson process or a Wiener process, both non degenerated.

In the third part of the proof we assume that  $L$  is a general non-deterministic Lévy process. We choose an  $\varepsilon > 0$  such that  $L^{>\varepsilon}$  is non-trivial. (If no such  $\varepsilon$  exists then  $L$  necessarily is a Wiener process, and for this case the proof was given at the beginning.)

By assumption we have

$$H(I_t^{>\varepsilon} + I_t^{\leq\varepsilon}) = L_t^{>\varepsilon} + L_t^{\leq\varepsilon} \quad \text{P-a.s.} \quad (16)$$

with

$$I_t^{>\varepsilon} := \int_{[0,t]} f(s) dL_s^{>\varepsilon}, \quad I_t^{\leq\varepsilon} := \int_{[0,t]} f(s) dL_s^{\leq\varepsilon}.$$

We know that  $(I_t^{>\varepsilon}, L_t^{>\varepsilon})$  is independent of  $(I_t^{\leq\varepsilon}, L_t^{\leq\varepsilon})$ .

Therefore from (16) we get

$$H(I_t^{>\varepsilon} + y) = L_t^{>\varepsilon} + z \quad \text{P-a.s.} \quad (17)$$

for  $P^{(I_t^{\leq\varepsilon}, L_t^{\leq\varepsilon})}$  - almost all  $(y, z)$ . Thus there exists at least one pair  $(y, z)$  such that (17) holds. As a consequence from the second part of the proof we obtain

$$\tilde{H}(I_t^{>\varepsilon}) = L_t^{>\varepsilon} \quad \text{P-a.s.} \quad (18)$$

for the Borel function  $\tilde{H}(x) := H(x+y) - z$ . Because  $L_t^{>\varepsilon}$  is a nontrivial compound Poisson process, the function  $f$  has to be constant. Now the proof of the theorem is complete.  $\square$

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