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# substitutability and the social cost of carbon in a solvable growth model with irreversible climate change

by Martin F. Quaas and Johannes Bröcker

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# Substitutability and the Social Cost of Carbon in a Solvable Growth Model with Irreversible Climate Change\*

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## Abstract

We develop an overlapping generations endogenous growth model with stocks of produced capital, human capital, a non-renewable resource, and irreversibly accumulated greenhouse gases in deterministic and stochastic versions. The model allows for analyzing different elasticities of substitution. We present a full analytical solution and characterization of the transition dynamics. We show that, as a rule of thumb, the social cost of carbon grow at a rate equal to the economy's growth rate divided by the elasticity of substitution. We analytically study sensitivity of the social cost of carbon with respect to key parameters: the intergenerational discount rate, the elasticity of substitution, and climate uncertainty. We show that the social cost of carbon explode at a finite level of log-normally distributed climate uncertainty. We illustrate results in a calibrated version of the model.

Keywords: overlapping generations; substitutes vs. complements; stochastic resource dynamics; optimum growth; climate policy

JEL: Q54; O44; Q32

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# 1 Introduction

A growing literature studies the economics of climate change and exhaustible resource use in quantitative integrated assessment models (Nordhaus 1993, Stern 2006, and many others). These models typically do not allow for analytical solutions, which makes it difficult to assess the general properties of the (numerically computed) model solutions, and in particular it is almost impossible to conduct a global sensitivity analysis. This may be problematic, as several important parameter values are largely unknown, including the time preference rate, the elasticity of intertemporal substitution, or the parameters of the climate damage function.

Thus, it is useful to construct models that are simple enough to allow for closed-form solutions, and yet rich enough to encompass the most relevant processes to study the issues of climate change, exhaustible resource use, economic growth and sustainability. Analytical models have been recently developed by Bretschger and Karydas (2013), Golosov et al. (2014), Gerlagh and Liski (2012), and Traeger (2015). To enable a closed-form solution, these models follow the tradition of the Brock and Mirman (1972) model and assume that the production function is Cobb-Douglas and that the utility function is logarithmic. A standard result from these models is that, “as a rule of thumb, the optimal carbon tax grows at approximately the same rate as GDP” (Smulders et al. 2014, 435).

The assumptions of a Cobb-Douglas production function and logarithmic utility imply that both the elasticity of substitution between production factors and the elasticity of intertemporal substitution are both assumed to be equal to one. This may be a too restrictive assumption, as it is well-known that the elasticity of substitution may play a crucial role for the possibilities of sustainable growth (Dasgupta and Heal 1979). Also the more recent endogenous growth literature has shown that substitution possibilities play an important role for the long-run development. In this vein, Acemoglu et al. (2012) show that, when accumulable and non-accumulable production factors are complements, policy cannot avoid an environmental disaster, at least if a deterioration in environmental quality is irreversible. On the other hand, complementarity between renewable and non-renewable resource input can also create incentives for structural change towards renewables (Bretschger and Smulders 2012).

Our model contributes to the literature on analytically solvable integrated assessment models by proposing and analyzing a model that augments existing approaches in three respects: First, our model allows to deviate from the standard assumption that the production function is Cobb-Douglas and utility is logarithmic. While we maintain the

assumption that the elasticities of substitution between production factors is the same as the elasticity of intertemporal substitution, in order to obtain a closed-form solution of the model, our model enables us to analyze the effects of an elasticity of substitution smaller or larger than one. As discussed above, this is particularly important for studying the long-term effects of climate policy and sustainability. Indeed, assuming that the elasticity of substitution between production factors is equal to one does not seem to be the most realistic specification. Krusell et al. (2000), for example, find that the elasticity of substitution between skilled labor and capital equipment in production is  $\sigma = 2/3$ . Also with respect to the elasticity of intertemporal substitution, a value below one is much more common in numerical integrated assessment models. Nordhaus and Sztorc (2013), Pindyck and Wang (2013), and Cai et al. (2015), for example, specify the elasticity of intertemporal substitution to be  $\sigma = 2/3$ .

Second, we study an overlapping generations general equilibrium model with endogenous growth driven by human capital accumulation. Deviating from the standard approach of considering an infinitely-lived agent allows to disentangle the effects of time preferences within and across generations (Schneider et al. 2012).

Third, our model captures the essence of irreversible climate change in a stylized fashion. The stock variables of the model comprise two accumulating production factors, a man-made, consumable capital stock and human capital, and two exhaustible resources/natural sinks, a non-renewable resource and accumulated atmospheric carbon. Consistent with the most recent IPCC report (Collins et al. 2013), and with earlier contributions by Stollery (1998) and Bretschger and Karydas (2013), climate change is considered to be irreversible. This makes the remaining budget for carbon emissions a non-renewable resource as well, which plays a crucial role in the analysis.

We present a full analytical characterization of the optimal dynamics in the model, including the transition dynamics. The new aspects of the model allow to qualify and augment existing results. Exploiting the overlapping generations structure of the model, we show that only short-run transition dynamics are affected by the intergenerational time preference rate, which applies to the intertemporal distribution of utility within a generation. The optimal carbon tax and the long-run dynamics of the model are affected mainly by the intergenerational discount rate. Confirming results from the numerical integrated assessment models, we show that the optimal carbon tax is highly sensitive to the intergenerational discount rate. We attribute this to the fact that carbon emissions lead to largely irreversible effects (Collins et al. 2013). Moreover, we find that uncertainty about climate dynamics has a strong impact on the social cost of carbon if the elasticity of substitution is below one and uncertainty is relatively large.

We show that allowing an elasticity of substitution different from one qualifies analytical ‘rules of thumb’ on the optimal carbon tax that have been derived using the standard Cobb-Douglas/logarithmic utility assumption. Specifically, the optimal carbon tax does not grow at a rate equal to the economy’s growth rate, but at a rate approximately equal to the economy’s growth rate divided by the elasticity of substitution. We further show that the optimal carbon tax is very sensitive to the elasticity of substitution. Thus, a misspecification of this parameter can lead to grossly wrong results.

The paper is structured as follows. In the next section, we describe the model. The main results of the paper are contained in Section 3 where we present the closed-form solution of the deterministic model; characterize the growth dynamics and the efficient carbon tax; extend the model to take stochastic capital dynamics with Epstein-Zin preferences into account. Section 4 presents a numerical illustration with a calibrated version of the model. The final section discusses our analysis and concludes.

## 2 OLG Endogenous Growth Model with Irreversible Climate Change

In the following we set up a discrete-time overlapping generations (OLG) model where a representative household lives for two periods in time. In the first period, say  $t$ , they receive a bequest  $B_t = Y_t - R_t$  of consumable capital from their parent, which they divide into consumption  $C_t$  and productive capital  $K_t$ . In the second period of their life, they divide their earned income into retirement consumption  $R_{t+1}$  and the bequest they leave to their children.

In addition to produced capital ( $K_t$ ) there is human capital ( $H_t$ ) as another accumulable production factor, and a non-renewable resource ( $S_t$ ). When this resource, i.e. fossil fuels such as coal or oil, is used in the production of goods, it generates emissions  $E_t$  that accumulate to a stock causing future climate damages.

Production output  $Y_{t+1}$  becomes available at the beginning of the next period. Using  $E_t \in [0, S_t]$  to denote resource input, and  $L_t \in [0, H_t]$  to denote skilled labor used in production, the production function is assumed to be given by

$$Y_{t+1} = F(K_t, E_t, L_t) = \left( \alpha K_t^{\frac{\sigma-1}{\sigma}} + \beta E_t^{\frac{\sigma-1}{\sigma}} + \gamma L_t^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (1)$$

where  $F(K_t, E_t, L_t)$  is a constant-returns-to-scale ( $\alpha, \beta, \gamma > 0, \alpha + \beta + \gamma = 1$ ) production

function, with  $\sigma > 0$  specifying the constant elasticity of substitution (CES). For the special case  $\sigma = 1$ , the production function (1) becomes the Cobb-Douglas function with exponents  $\alpha$  for capital  $K_t$ ,  $\beta$  for resource input  $E_t$ , and  $\gamma$  for skilled labor  $L_t$ .

In a discrete-time framework, some assumption about the timing of consumption and production within a period has to be made. The assumption here is that consumption takes place before production of goods, and the remaining capital stock  $K_t$  is employed in production. The output produced in period  $t$  is divided into consumption and productive capital at the beginning of the next period  $t + 1$ . The accounting equation is  $Y_{t+1} = K_{t+1} + C_{t+1} + R_{t+1}$ .

Human capital is a growing production factor which can be produced independently of the resource. With  $L_t$  being skilled labor employed in production, the remainder  $H_t - L_t$  is used to build up the human capital stock according to

$$H_{t+1} = (1 - \varepsilon) H_t + \kappa (H_t - L_t) \quad (2)$$

with  $\kappa > 0$  being the productivity of human capital accumulation and  $0 \leq \varepsilon \leq 1$  being the rate of human capital depreciation. As opposed to the input of nonrenewable resources, human capital accumulation tends to increase future consumption possibilities. We assume  $\kappa > \varepsilon$ , such that human capital will accumulate for sure if all time is spent in human capital accumulation and none in the production of goods, i.e. if  $L_t = 0$ .

The stock of the non-renewable resource follows the usual equation of motion

$$S_{t+1} = S_t - E_t, \quad (3)$$

i.e. the remaining resource stock  $S_t$  is simply reduced by the quantity extracted. The overall size of the resource available is limited by the initial stock  $S_0$ , thus resource input in production must eventually decrease. This tends to impose a limit on future consumption possibilities. In a sense, the model thus is a discrete-time version of the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal 1974, Solow 1974, Stiglitz 1974a,b), extended to take into account human capital formation and climate damage.

We adopt a very simple description of climate dynamics, using a single state variable ( $X_t$ ). This is a strong simplification, as it neglects time lags between emissions and temperature increase (the effect of such a lag on optimal climate policies is studied by Bretschger and Karydas 2013). However, our focus is on the long-term sustainability effects of climate change. In that regard, the most important aspect is that carbon emissions have consequences for consumption possibilities in the long run. Specifically,

we assume that climate damage is irreversible. This is in line with previous studies, including Stollery (1998) and Bretschger and Karydas (2013). More importantly, there is clear evidence from climate science that the consequences of emitting carbon are by and large irreversible (Collins et al. 2013). Before formally describing the dynamics of the atmospheric carbon stock, we briefly turn to assumptions on the utility function.

Utility depends on the consumption of private consumption ( $C_t$  and  $R_{t+1}$ ) and on climate damage, which, for simplicity, is assumed to be a function of the stock of atmospheric carbon. This way of capturing climate damage is common in the literature, and used, among others, by Tahvonen and Kuuluvainen (1991), Hoel and Sterner (2007), Sterner and Persson (2008), Llavador et al. (2011), and van der Ploeg and Withagen (2014). Following this literature, we assume that there is an upper threshold level for the stock of atmospheric carbon beyond which it causes intolerable damages. Given this assumptions on utility and the irreversibility of carbon emissions, there is a remaining “budget”  $X_t$  of carbon emissions, which is the difference between the upper threshold level and the current stock of atmospheric carbon.

The remaining carbon budget develops according to

$$X_{t+1} = X_t - E_t, \quad (4)$$

with initial carbon budget  $X_0$ , and  $E_t \leq \min\{S_t, X_t\}$ . Thus, the intertemporal use of carbon is limited by two constraints: One is the stock  $S_0$  of the carbon resource initially available in ground; the other one is the initial budget  $X_0$  of carbon emissions into the atmosphere. Only the tighter of these two constraints will be effective along an optimal path. According to Collins et al. (2013) and McGlade and Ekins (2015), the effective constraint is the remaining carbon budget. We thus assume  $X_0 < S_0$ . This means that the remaining budget of carbon emissions is the scarce resource and the efficient emission tax will capture the corresponding scarcity rent (Asheim 2013).

Using  $U(C_t, X_t)$  to denote the instantaneous utility function, a representative household’s intertemporal utility is given by

$$V(B_t, X_t, H_t) = U(C_t, X_t) + \psi U(R_{t+1}, X_{t+1}) + \varphi V(B_{t+1}, X_{t+1}, H_{t+1}). \quad (5)$$

Here,  $\psi \in (0, 1)$  is the individual utility discount factor, and  $\varphi \in (0, 1)$  is the degree of altruism towards the child, where  $V(B_{t+1}, X_{t+1}, H_{t+1})$  denotes intertemporal utility of the representative dynasty that is endowed with the bequest package consisting of the bequest  $B_{t+1} = Y_{t+1} - R_{t+1}$  of production output, resource stock  $X_{t+1}$ , and human capital  $H_{t+1}$ .



We will refer to  $\psi$  as the intragenerational discount factor and to  $\varphi$  as the intergenerational discount factor. For intergenerational discounting, we shall also analyze how the endogenous variables, in particular the optimal tax on carbon emissions, depend on the intergenerational discount rate  $\rho > 0$ , which is related to the intergenerational discount factor as  $\varphi = 1/(1 + \rho)$ .

We assume that the instantaneous utility function is

$$U(C_t, X_t) = \begin{cases} \frac{\sigma}{\sigma-1} \left( C_t^{\frac{\sigma-1}{\sigma}} + \delta X_t^{\frac{\sigma-1}{\sigma}} \right) & \text{for } \sigma \neq 1 \\ \ln(C_t) + \delta \ln(X_t) & \text{for } \sigma = 1. \end{cases} \quad (6)$$

Specification (6) of the instantaneous utility function assumes that the intertemporal elasticity of substitution is constant and equal to the elasticity of substitution between inputs in production. Using this assumption the model becomes solvable in closed form. There is no obvious theoretical economic argument in support of this assumption. However, as discussed above, a specification  $\sigma = 2/3$  would be in line both with empirical findings for the elasticity of substitution between production factors (Krusell et al. 2000), and with an elasticity of intertemporal substitution common in numerical integrated assessment models (Nordhaus and Sztorc 2013, Pindyck and Wang 2013, Cai et al. 2015).

Given the specification (6), where  $\delta > 0$  is a preference parameter that measures the importance of climate damage relative to utility from consumption of private goods, the stock of atmospheric carbon will never exceed the upper limit along an optimal path, i.e. the remaining carbon budget will remain positive,  $X_t > 0$ .

### 3 Growth Dynamics and Optimal Carbon Tax

The optimal intertemporal allocation is obtained by solving the optimization problem

$$V(B_t, X_t, H_t) = \max_{C_t, R_{t+1}, L_t, E_t} \left\{ U(C_t, X_t) + \psi U(R_{t+1}, X_{t+1}) + \varphi V(B_{t+1}, X_{t+1}, H_{t+1}) \right\} \quad (7)$$

subject to (1), (2), (4), and non-negativity constraints. The following proposition characterizes the closed-form solution for the optimal feedback policies.

**Proposition 1.** *The optimal policy functions are*

$$C_t = c^* Y_t; \quad R_t = (1 - b^*) Y_t; \quad E_t = e^* X_t; \quad L_t = l^* H_t \quad (8)$$

with constants

$$c^* = \frac{1 - (\varphi \alpha)^\sigma}{1 + \left(\frac{\psi}{\varphi}\right)^\sigma} \quad (9a)$$

$$b^* = \frac{1 + (\psi \alpha)^\sigma}{1 + \left(\frac{\psi}{\varphi}\right)^\sigma} \quad (9b)$$

$$\beta e^{*\frac{-1}{\sigma}} = \delta \frac{\psi + \varphi}{1 - \varphi (1 - e^*)^{\frac{-1}{\sigma}}} c^{*\frac{1}{\sigma}} (1 - e^*)^{\frac{-1}{\sigma}} \quad (9c)$$

$$l^* = 1 - \frac{1}{\kappa} (\varphi^\sigma (1 - \varepsilon + \kappa)^\sigma - (1 - \varepsilon)) \quad (9d)$$

and the value function has the form

$$V(B, X, H) = \frac{\sigma}{\sigma - 1} \left( \Xi B^{\frac{\sigma-1}{\sigma}} + \Delta X^{\frac{\sigma-1}{\sigma}} + \Gamma H^{\frac{\sigma-1}{\sigma}} \right) \quad (10)$$

with constants  $\Xi$ ,  $\Delta$ , and  $\Gamma$  given in Appendix A.

*Proof.* See Appendix A. □

We are particularly interested in the fraction  $e^*$  of the remaining carbon budget that is used up in each period, which is determined by equation (9c). This equation is derived from the familiar optimality condition that marginal abatement costs should equal marginal damage, i.e.

$$F_{E_t} = \frac{(\psi \delta + \varphi \Delta) X_{t+1}^{-\frac{1}{\sigma}}}{\psi R_{t+1}^{-\frac{1}{\sigma}}} \quad (11)$$

The marginal product of carbon ( $F_{E_t}$ ) on the left hand side of this condition is a measure for marginal abatement cost in terms of consumption possibilities at  $t+1$ . The right-hand side of (11) equals marginal climate damage, given by the marginal rate of substitution between climate damage and consumption in  $t+1$ . Total marginal climate damage, given in utility terms by the numerator on the right-hand side of (11), consists of the marginal damage from emissions at  $t$  that the emitters will suffer themselves in the second period of their life ( $\psi \delta X_{t+1}^{-1/\sigma}$ ) and the marginal damage that will accrue to future generations ( $\varphi \Delta X_{t+1}^{-1/\sigma}$ ). Given the assumptions of the model, all variables cancel out and this condition simplifies to (9c), which determines  $e^*$ .

The comparative statics of  $e^*$  leads to intuitive results. The rate  $e^*$  at which the remaining carbon budget is reduced decreases with the parameter  $\delta$  capturing marginal damage from greenhouse gas emissions. It increases with the parameter  $\beta$  that cap-

tures the marginal abatement costs of atmospheric carbon. In line with intuition,  $e^*$  approaches zero as the discount factor  $\varphi$  approaches one, or, equivalently, as the intergenerational discount rate  $\rho$  approaches zero. To study this relationship further, we are interested in how  $e^*$  changes with  $\rho$  at small values of  $\rho$ , and, hence, at small values of  $e^*$ . This elasticity depends on  $\sigma$  (see Appendix A):

$$\lim_{e^* \rightarrow 0} \frac{\rho}{e^*} \frac{de^*}{d\rho} = \begin{cases} 1 & \text{for } \sigma \leq 0 \\ \sigma & \text{for } \sigma \geq 0. \end{cases} \quad (12)$$

For limited substitution possibilities,  $\sigma \leq 1$ , and a small intergenerational discount rate  $\rho$ , a one-percent increase in the discount rate leads to a one-percent increase in the rate at which the remaining carbon budget is used. For favorable substitution possibilities,  $\sigma > 1$ , the relationship is even stronger, with a one-percent increase in the discount rate leading to an increase of  $e^*$  by  $\sigma$  percent. We will come back to this issue when we discuss the optimal carbon tax in Section 3.2.

In a similar fashion, we derive the semi-elasticity of  $e^*$  with respect to the elasticity of substitution in Appendix A. For a small value of  $e^*$ , and keeping the savings rate  $1 - c^*$  fixed, this semi-elasticity is

$$\lim_{e^* \rightarrow 0} \frac{1}{e^*} \frac{de^*}{d\sigma} = \ln \left( \frac{\beta}{\delta} \frac{1 - \varphi}{1 + \frac{\psi}{\varphi}} \right). \quad (13)$$

Unless  $\beta$  is much larger than  $\delta$ , the result is that  $e^*$  is decreasing with  $\sigma$ . The rate  $e^*$  at which the remaining carbon budget is reduced is particularly sensitive to the exact specification of  $\sigma$  if  $\beta \ll \delta$ ,  $\varphi$  is close to one, and if the intragenerational discount rate is large compared to the intergenerational discount rate,  $\psi \gg \varphi$ .

### 3.1 Growth dynamics

To further study growth dynamics, we use the result from Proposition 1 in the equations of motion (1), (2), and (4). The state variables, each taken to the power  $\frac{\sigma-1}{\sigma}$ , develop according to the following linear system

$$\begin{pmatrix} Y_{t+1}^{\frac{\sigma-1}{\sigma}} \\ H_{t+1}^{\frac{\sigma-1}{\sigma}} \\ X_{t+1}^{\frac{\sigma-1}{\sigma}} \end{pmatrix} = \begin{pmatrix} \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} & \gamma l^{\frac{\sigma-1}{\sigma}} & \beta e^{\frac{\sigma-1}{\sigma}} \\ 0 & (1 - \varepsilon + \kappa(1 - l^*))^{\frac{\sigma-1}{\sigma}} & 0 \\ 0 & 0 & (1 - e^*)^{\frac{\sigma-1}{\sigma}} \end{pmatrix} \begin{pmatrix} Y_t^{\frac{\sigma-1}{\sigma}} \\ H_t^{\frac{\sigma-1}{\sigma}} \\ X_t^{\frac{\sigma-1}{\sigma}} \end{pmatrix} \quad (14)$$

As stated in Proposition 1 the feedback policies for all three capital and resource stocks are to utilize a constant fraction in production. For the carbon budget and human capital this implies constant growth rates.

The optimal consumption growth rate, by contrast, changes over time and depends on the initial state of the economy. The following proposition characterizes the growth patterns of the economy depending on the value of  $\sigma$ .

**Proposition 2** (Optimal growth dynamics). *There exists a value  $\underline{Y}_0(H_0, X_0)$  such that growth dynamics are characterized as follows.*

1. *If  $\sigma > 1$ , growth monotonically accelerates if  $Y_0 > \underline{Y}_0(H_0, X_0)$ , while growth first slows down and later accelerates if  $Y_0 < \underline{Y}_0(H_0, X_0)$ .*
2. *If  $\sigma < 1$ , growth monotonically slows down if  $Y_0 < \underline{Y}_0(H_0, X_0)$ , while growth first accelerates and later slows down if  $Y_0 > \underline{Y}_0(H_0, X_0)$ . If  $\sigma < 1$  and the initial carbon budget is large enough, there will be a temporary phase with an intermediate steady state during which the economy grows at a rate  $\hat{H}$ .*
3. *If  $\sigma = 1$   $\bar{g} \equiv (1 - e^*)^{\frac{\beta}{1-\alpha}}(1 + \hat{H})^{\frac{\gamma}{1-\alpha}} > 0$ , growth monotonically accelerates if  $Y_0 < \underline{Y}_0(H_0, X_0)$ , while it first slows down and later accelerates if  $Y_0 > \underline{Y}_0(H_0, X_0)$ . If  $\sigma = 1$  and  $\bar{g} < 0$ , growth will first accelerate and later slow down if  $Y_0 < \underline{Y}_0(H_0, X_0)$ , while it will monotonically slow down if  $Y_0 > \underline{Y}_0(H_0, X_0)$ .*

*Proof.* See Appendix B. □

The intuition for this result is that there is first a phase of catching up if the economy initially is poor in produced capital relative to human capital and the carbon budget. Thus, the initial growth rate is high in this case.

In the longer run, the economy's growth rate is determined by the most abundant factor (human capital) if the factors are substitutes,  $\sigma > 1$ . In this case, growth ultimately accelerates approaching the growth rate of human capital. If the factors are complements,  $\sigma < 1$ , the long-run growth rate is limited by the growth of the scarcest factor, and the economy's growth rate ultimately falls to this level. For the case  $\sigma = 1$  the relative productivities  $\alpha$ ,  $\beta$ , and  $\gamma$  matter for the long-run growth rate as well.

## 3.2 Social cost of carbon

Now we shall analyze how the social cost of carbon, which correspond to the optimal carbon emission tax in a market setting, depend on the state of the economy and important parameters. The optimal emission tax is equal to both marginal abatement cost

(the marginal product of carbon emissions) and the marginal damage from atmospheric carbon. According to condition (11) these two expressions are equal along an optimal path. The following result is proven in Appendix C.

**Proposition 3.** *The optimal tax rate on emissions is given by*

$$\tau_t = F_{E_t} = \frac{Y_{t+1}^{\frac{1}{\sigma}}}{X_t^{\frac{1}{\sigma}}} \beta e^{*\frac{-1}{\sigma}}. \quad (15)$$

The optimal tax rate increases at a rate  $\hat{\tau}_t$  given by

$$1 + \hat{\tau} = \left( \frac{1 + \hat{Y}_t}{1 - e^*} \right)^{\frac{1}{\sigma}},$$

where  $\hat{\tau}$  has the following properties.

1. If  $\sigma \leq 1$ , the optimal tax rate increases at a rate larger than the economy's growth rate,  $\hat{\tau} > \hat{Y}_t$  for all  $t$ .
2. If  $\sigma > 1$ , the optimal tax rate increases at a rate larger, equal, or smaller than the economy's growth rate according to

$$\hat{\tau} \gtrless \hat{Y}_t \quad \Leftrightarrow \quad 1 + \hat{Y}_t \lesseqgtr (1 - e^*)^{-\frac{1}{\sigma-1}}. \quad (16)$$

3. As a 'rule of thumb',

$$\hat{\tau} \approx (1 - e^*)^{-\frac{1}{\sigma}} \frac{1}{\sigma} \hat{Y}_t > \frac{\hat{Y}_t}{\sigma}. \quad (17)$$

For an initial capital stock which is not unreasonably large, it will always be the case that  $\hat{Y}_t > -e^*$  (cf. Proposition 1). Thus, the optimal tax rate will grow at a positive rate.

For the case of logarithmic utility, the optimal tax rate is proportional to consumption, as under the optimal policy,  $Y_{t+1} = C_{t+1}/c$ . This result is in line with previous findings (Golosov et al. 2014, Bretschger and Karydas 2013). It does not generalize for the case  $\sigma \neq 1$ , however. For  $\sigma > 1$ , the optimal tax rate is concave in consumption; for  $\sigma < 1$ , the optimal tax rate is convex in consumption. Furthermore, the optimal carbon tax rate does not increase at the same rate as GDP. Thus the 'rule of thumb' stated in Smulders et al. (2014, 435) does not hold in general. Rather, the tax rate grows at a rate approximately equal to the the economy's growth rate divided by the elasticity of

substitution. This may make a big difference. For a specification  $\sigma = 2/3$ , for example, the growth rate of the optimal carbon tax is 50% larger than the economy's growth rate.

Furthermore, in our modeling framework, the optimal carbon tax rate grows faster than GDP even for the case of logarithmic utility. This difference to previous analytical integrated assessment models (Golosov et al. 2014, Gerlagh and Liski 2012, Traeger 2015) arises, as we assume that climate damage is irreversible. Thus, the optimal tax rate increases over time not only because of economic growth, but also because the remaining carbon budget is decreasing over time.

**Proposition 4** (Sensitivity of the Social Cost of Carbon to intergenerational discounting). *For a small value of the intergenerational discount rate  $\rho$ , and given output  $Y_{t+1}$  and carbon budget  $X_t$ , the elasticity of the carbon tax rate to the social discount rate is given by*

$$\lim_{\rho \rightarrow 0} \frac{\rho}{\tau_t} \frac{d\tau_t}{d\rho} = \begin{cases} -1 & \text{for } \sigma \geq 1 \\ -\frac{1}{\sigma} & \text{for } \sigma \leq 1 \end{cases}. \quad (18)$$

*Proof.* The result follows from (12) with (15).  $\square$

For the case of logarithmic utility,  $\sigma = 1$ , this result is in line with Traeger (2015, Proposition 2), who shows that the optimal carbon tax rate is approximately proportional to  $1/\rho$ . This result generalizes to the case  $\sigma \geq 1$ , if the intergenerational discount rate is small. For  $\sigma < 1$ , the optimal carbon tax rate is even more sensitive to the intergenerational discount rate: The elasticity is equal to  $-1/\sigma < -1$ .

Proposition 4 takes the output in period  $t$  as given. If one alternatively considers the sensitivity of the tax rate to the intergenerational discount rate for given stock sizes at  $t$ , one has to take into account that also  $Y_{t+1}$  increases as  $\rho$  decreases. The optimal tax rate becomes even more sensitive to the intergenerational tax rate.

To study the sensitivity of the optimal carbon tax with respect to the elasticity of substitution, we will consider changes in  $\sigma$  that leave the observables unchanged. In particular, we assume that the income share of carbon energy,  $\theta_{E_t} = E_t F_{E_t}/Y_{t+1}$ , output  $Y_{t+1}$ , and the initial carbon budget are fixed.

**Proposition 5** (Sensitivity of the Social Cost of Carbon to the elasticity of substitution). *For a small value of  $e^*$ , and given the income share of carbon energy, output  $Y_{t+1}$ , and carbon budget  $X_{t+1}$ , the semi-elasticity of the carbon tax rate to the elasticity of substitution is given by*

$$\lim_{e^* \rightarrow 0} \frac{1}{\tau_t} \frac{d\tau_t}{d\sigma} = -\ln \left( \frac{\beta}{\delta} \frac{1-\varphi}{1+\frac{\psi}{\varphi}} \right). \quad (19)$$

*Proof.* Differentiating  $F_{E_t} = (\theta_{E_t} Y_{t+1}/X_{t+1}) \frac{1}{e^*}$  with respect to  $\sigma$ , we find  $\lim_{e^* \rightarrow 0} \frac{1}{\tau_t} \frac{d\tau_t}{d\sigma} = - \lim_{e^* \rightarrow 0} \frac{1}{e^*} \frac{de^*}{d\sigma}$ . Using (13) gives the result.  $\square$

The optimal carbon tax rate is rather sensitive to the elasticity of substitution if the damage parameter  $\delta$  is much larger than the productivity parameter  $\beta$ , if the intergenerational discount rate  $\rho$  is small (i.e., if  $\varphi$  is large), and if the intragenerational discount factor is much larger than the intergenerational discount factor,  $\psi \gg \varphi$ . In a sense, the more severe the climate problem is, the more sensitive is the optimal carbon tax to the elasticity of substitution.

### 3.3 Stochastic Climate Damage

The model can be extended to take stochastic capital dynamics into account. Here we focus on the case where climate damage is uncertain. Specifically, we replace (4) by the stochastic equation of motion

$$X_{t+1} = \zeta_t (X_t - E_t), \quad (4')$$

where  $\zeta_t$  is series of iid random shocks to the carbon budget. This captures in a stylized way uncertainty about the remaining carbon budget. For each generation,  $\zeta_t$  may turn out to be larger or smaller than the expected value  $\mathcal{E}[\zeta_t]$ . The effect on the remaining carbon budget is permanent. For example, if  $\zeta_t$  turns out to be larger than expected, the increased carbon budget will benefit the current and all future generations. To focus on the effect of uncertainty, we assume  $\mathcal{E}[\zeta_t] = 1$ , although the model would also be solvable if there was a constant expected decay,  $\mathcal{E}[\zeta_t] < 1$ .

Assuming  $\sigma \neq 1$ , the optimal stochastic feedback policies are found by solving the stochastic optimization problem

$$V(B_t, X_t, H_t) = \frac{\sigma}{\sigma - 1} \max_{C_t, R_{t+1}, L_t, E_t} \left\{ C_t^{\frac{\sigma-1}{\sigma}} + \delta X_t^{\frac{\sigma-1}{\sigma}} + \mathcal{E}_{\zeta_t} \left[ \left( \psi \left( R_{t+1}^{\frac{\sigma-1}{\sigma}} + \delta X_{t+1}^{\frac{\sigma-1}{\sigma}} \right) + \varphi V(B_{t+1}, X_{t+1}, H_{t+1}) \right) \right] \right\} \quad (20)$$

In Appendix D we prove the following result, which is a generalization of Proposition 1.

**Proposition 6.** *If  $\sigma \geq 1$ , or if  $\sigma < 1$  and uncertainty is small such that  $\varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] < 1$ ,*

the optimal policy functions are

$$\tilde{C}_t = c^* Y_t; \quad \tilde{R}_t = (1 - b^*) Y_t; \quad \tilde{E}_t = \tilde{e} X_t; \quad \tilde{L}_t = l^* H_t \quad (21)$$

with constants  $c^*$ ,  $b^*$ , and  $l^*$  given by (9a), (9b), and (9d), and  $\tilde{e}$  determined by

$$(1 - e^*)^{\frac{1}{\sigma}} = \varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] + \frac{\delta}{\beta} \frac{\psi + \varphi}{\varphi} c^{*\frac{1}{\sigma}} \tilde{e}^{\frac{1}{\sigma}} \quad (22)$$

Uncertainty about the remaining carbon budget has no direct effect on the policy functions corresponding to the stocks other than the climate system. Indirectly and in the longer run, however, the change in carbon emissions will affect the growth rate of the economy as well. For a somewhat simpler interpretation of the effect of uncertainty, we have organized terms in (22) slightly different than in (9c). It is easy to see that the two conditions are identical in the case of no uncertainty  $\mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] = 1$ .

The rate  $\tilde{e}$  depends on the product of the intergenerational discount factors  $\varphi$  and  $\psi$  and the effect of uncertainty,  $\mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right]$ . Obviously,  $\mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \leq 1$  for the case  $\sigma \geq 1$ . If, however,  $\sigma < 1$ , we have  $\mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \geq 1$ , and the product of this expression and the intergenerational discount factor in the first term on the right-hand side of (22) may even exceed one if uncertainty is large enough. We thus have the following result, where we assume that  $\zeta_t$  is log-normally distributed.

**Proposition 7.** *If  $\sigma < 1$  and  $\zeta_t$  is log-normally distributed with mean one and variance  $v_\zeta$ , the stochastic optimization problem has a solution only if*

$$v_\zeta < \bar{v}_\zeta \equiv (1 + \rho)^{\frac{2\sigma^2}{1-\sigma}} - 1 \quad (23)$$

*Proof.* The stochastic optimization problem has a solution only if  $\varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] < 1$ . For a log-normal distribution with mean one and variance  $v_\zeta$ , we have  $\mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] = (1 + v_\zeta)^{\frac{1-\sigma}{2\sigma^2}}$ . Thus, the stochastic optimization problem has a solution only if (23) holds.  $\square$

This result can be seen as a variant of Weitzman's (2009) 'dismal theorem'. If uncertainty becomes too large, the standard approach to determining the social cost of carbon has no solution. Note, however, that here the result is obtained with a very standard assumption about the distribution of uncertainty. It is a pure consequence of the irreversibility of climate change and lasting uncertainty about the remaining carbon budget.

Climate uncertainty affects the optimal carbon tax rate. In a similar way as in



Proposition 3 it is straightforward to show that the optimal carbon tax rate under climate uncertainty is given by

$$\tilde{\tau}_t = \frac{Y_{t+1}^{\frac{1}{\sigma}}}{X_t^{\frac{1}{\sigma}}} \beta \tilde{e}^{-\frac{1}{\sigma}}. \quad (15')$$

The optimal carbon tax is rather sensitive to climate uncertainty, if  $\sigma < 1$ , and uncertainty is relatively large. The following result is proven in Appendix D.

**Proposition 8** (Sensitivity of the Social Cost of Carbon to climate risk). *Under the assumptions of Proposition 7, if the variance of climate uncertainty ( $v_\zeta$ ) approaches  $\bar{v}_\zeta$  (defined in 23) from below, and given output  $Y_{t+1}$  and carbon budget  $X_t$ , the elasticity of the optimal carbon tax rate to climate uncertainty is given by*

$$\lim_{v_\zeta \rightarrow \bar{v}_\zeta} \frac{v_\zeta}{\tau_t} \frac{d\tau_t}{dv_\zeta} = \frac{1}{\sigma}. \quad (24)$$

Thus, if uncertainty is relatively large, and if the elasticity of substitution is below one, the social cost of carbon becomes very sensitive to further increases in uncertainty.

## 4 Calibration and Sensitivity of the Social Cost of Carbon to key Parameters

To explore quantitative effects we calibrate the model in the following. We calibrate the deterministic version of the model. Our focus in this section is on the sensitivity of the optimal carbon tax rate  $\tau$  with respect to the key parameters  $\rho$ ,  $\sigma$ , and  $v_\zeta$ .

Symbol	$\sigma$	$\alpha$	$\beta$	$\gamma$	$\delta$	$\varepsilon$	$\kappa$	$\psi$	$\varphi$	$X_0$	$Y_0$	$H_0$
Value	2/3	.06	.53	.41	43.2	.71	4.03	.82	.59	3000	.54	2.28

Table 1: Calibrated parameter values. Rates are per generation of 30 years;  $X_0$  is in gigatons of carbon;  $Y_0$  and  $H_0$  are in units of world output in the first generation.

For the calibration we use a generation time of  $G = 30$  years. In our benchmark calibration we assume an elasticity of substitution between skilled labor and capital equipment of  $\sigma = 2/3$  found by Krusell et al. (2000). Given the assumption that the elasticity of intertemporal substitution is the same as the elasticity of substitution between the production factors, this implies an a value for the preference for consumption smoothing  $1/\sigma = 1.5$ , which is not uncommon in the literature (e.g. Nordhaus and Sztorec 2013, Pindyck and Wang 2013, Cai et al. 2015). Some authors argue for the somewhat larger value  $1/\sigma = 2$  (Weitzman 2007; Sterner and Persson 2008; Gollier 2012, 2015).

An alternative calibration for  $\sigma = 1/2$  is straightforward and leads to more pronounced effects of complementarity in our model framework.

In addition to the assumptions on generation time and  $\sigma$ , we base our calibration on data for the growth rate of the global economy,  $\hat{Y}_t$ , the pure rate of return on capital  $r_t$ , and income shares for capital, labor, and carbon energy. For the growth rate of per capita consumption we use  $\hat{Y}_t = 0.021$  from Piketty (2014, Figure 2.4). We use  $r_t = 0.05$  for the pure rate of return on capital, which is roughly the average of the estimates  $r_t^B = 0.046$  for Britain and  $r_t^F = 0.053$  for France from Piketty (2014, Figures 6.3 and 6.4). For the income share of capital ( $\theta_K = 0.25$ ) and skilled labor ( $\theta_L = 0.70$ ) we approximately use the figures of Piketty (2014, Figures 6.1 and 6.2) for Britain and France in the past 50 years, allowing for an carbon energy share of  $\theta_E = 0.05$  (in line with IEA 2015), which is not included in Piketty's figures.

The main focus of the numerical investigation is the comprehensive sensitivity analysis of the optimal carbon tax. To this end, we set up a benchmark calibration of the model such that the optimal quantity of emissions is similar to values obtained in standard integrated assessment models. According to Golosov et al. (2014, Figure 3) optimal carbon emissions are initially about 60% of the laissez-faire carbon emissions. Carbon dioxide emissions from fossil fuel combustion in the 30 years from 1985 to 2015 were about 690 gigatons, equivalent to 188 gigatons of carbon (IEA 2015, :48). We use these figures for our calibration, assuming  $e^* X_0 = 0.6 \times 188$  gigatons. Based on IPCC we consider a budget of  $X_0 = 3000$  gigatons of carbon (Collins et al. 2013, Friedlingstein et al. 2014). Thus,

$$e^* = \frac{0.6 \times 188}{3000} = 0.038, \quad (25)$$

i.e. every generation uses 3.8 percent of the remaining carbon budget. Units of measurement are chosen such that world output in  $t = 1$  is normalized to one,  $Y_1 = 1$ .

In what follows we focus on the calibration of parameter values needed to derive the optimal carbon tax. Further steps of calibration an numerical analysis can be found in Appendix E. Resulting parameter values are summarized in Table 1.

For the calibration of the intergenerational discount factor, we use that the interest factor is given by  $F_{K_t} = \alpha K_t^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}} = \alpha (b^* - c^*)^{-\frac{1}{\sigma}} \left(1 + \hat{Y}_t\right)^{\frac{1}{\sigma}}$ . Using a result from

Appendix A, we have  $1/\varphi = \alpha (b^* - c^*)^{-\frac{1}{\sigma}}$ . Rearranging yields

$$\varphi = \frac{(1 + \hat{Y}_t)^{\frac{1}{\sigma}}}{F_{K_t}} = \left( \frac{1.021^{1.5}}{1.05} \right)^G = 0.59, \quad (26)$$

The resulting intergenerational discount factor of  $\varphi = 0.59$  per generation corresponds to an intergenerational discount factor of 0.983 per year, or an intergenerational discount rate of 1.7% per year.

For calibrating the parameter values for  $\alpha$ ,  $\beta$ , and  $\gamma$  we use the income shares

$$\theta_{K_t} = \frac{F_{K_0} K_0}{Y_0} = \alpha K_0^{\frac{\sigma-1}{\sigma}} = 0.25 \quad (27a)$$

$$\theta_{E_t} = \frac{F_{E_0} E_0}{Y_0} = \beta E_0^{\frac{\sigma-1}{\sigma}} = 0.70 \quad (27b)$$

$$\theta_{L_t} = \frac{F_{L_0} L_0}{Y_0} = \gamma L_0^{\frac{\sigma-1}{\sigma}} = 0.05 \quad (27c)$$

Using the above calibration  $e^* = 0.0376$  and  $X_0 = 3000$  gigatons of carbon, we obtain  $\beta = 0.53$ . As detailed in Appendix E, we further calibrate  $\alpha = 0.06$ . To calibrate the intragenerational discount factor, we assume that in both periods of life equal fraction of output is consumed,  $1 - b^* = c^*$ . From this condition we obtain  $(\psi/\varphi)^\sigma - (\psi\alpha)^\sigma = 1 + (\varphi\alpha)^\sigma$ , and thus

$$\psi = \varphi \left( \frac{1 + (\varphi\alpha)^\sigma}{1 - (\varphi\alpha)^\sigma} \right)^{\frac{1}{\sigma}} = 0.81. \quad (28)$$

Using these values in (9c) yields

$$\delta = \beta \frac{(1 - e^*)^{\frac{1}{\sigma}} - \varphi}{\frac{\psi + \varphi}{\varphi} (c^* e^*)^{\frac{1}{\sigma}}} = 43.2. \quad (29)$$

We use this calibration of the model to study how the optimal carbon tax rate is affected by important parameters of the model, specifically the elasticity of substitution  $\sigma$  and the intergenerational discount rate  $\rho$ . To this end, we vary the parameters  $\sigma$  and  $\rho$ , keeping the other parameters fixed at their benchmark levels given in Table 1, and compute the optimal share of energy use  $e^*$ . Figure 1 shows the optimal carbon tax rate, relative to the optimal carbon tax in the baseline parameterization, as a function of the elasticity of substitution; with the vertical axis in logarithmic scale. Similarly, Figure 2 shows the social cost of carbon as a function of the intergenerational discount rate.

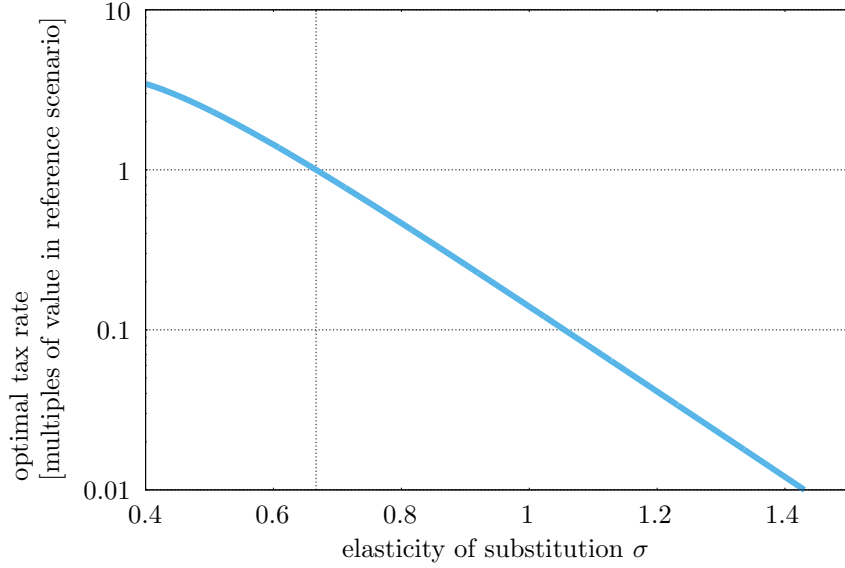


Figure 1: Optimal tax rate as a function of the elasticity of substitution  $\sigma$ . Calibrated parameter values are given in Table 1.

For both parameters, the optimal tax rate is very sensitive to the exact specification. For the elasticity of substitution, values between  $\sigma = 1/2$  and 1 are common in the literature. Yet, the resulting optimal carbon tax rate varies by more than two orders of magnitude for this range of different elasticities of substitution. The optimal carbon tax is similarly sensitive to the intergenerational discount rate. As formally shown in Proposition 4 for the case of very small discount rates, the elasticity at which the social cost of carbon change with the intergenerational discount rate is given by the preference for consumption smoothing,  $1/\sigma$ , if  $\rho$  is small. As also seen in Figure 2 the optimal carbon tax is extremely sensitive against the exact specification of  $\rho$ , in particular for small values of  $\rho$ . This is a direct consequence of the fact that carbon emissions lead to largely irreversible consequences. Thus the marginal abatement costs have to be traded off against damage costs for very many generations to come in the future, which have a high weight in decision-making if  $\rho$  is small.

We finally use the calibrated model to assess the effect of risk on the optimal carbon tax. Figure 3 shows how the tax rate depends on the variance a log-normally distributed climate shock. For small levels of the variance, the effect of climate uncertainty on the social cost of carbon is moderate, but as the variance approaches the critical level given in (23), the social cost of carbon explodes with further increasing uncertainty.

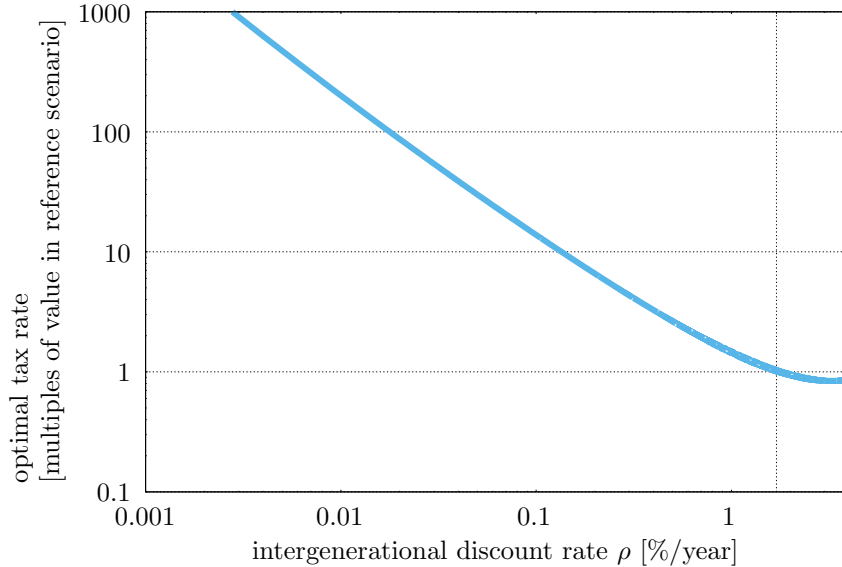


Figure 2: Optimal tax rate as a function of the intergenerational discount rate. Calibrated parameter values are given in Table 1.

## 5 Conclusion

In this paper we have proposed and analyzed an overlapping generations endogenous growth model in discrete time with stocks of produced capital, human capital, and irreversible climate change. We fully characterized optimal dynamics and the social cost of carbon both in a deterministic setting and with long-term shocks on the climate system. Our approach differs from standard analytical integrated assessment models by allowing the elasticity of substitution between production factors and intertemporal consumption to differ from one.

Our analysis qualifies the standard ‘rule of thumb’ that the social cost of carbon are proportional to GDP and that the social cost of carbon grow at the same rate as GDP. These results have been found in models that assume that the elasticity of substitution is equal to one (Bretschger and Karydas 2013, Golosov et al. 2014, Gerlagh and Liski 2012, Traeger 2015). Here, we have shown that the social cost of carbon are proportional to GDP to the power of  $1/\sigma$ , and that the social cost of carbon grow at a rate somewhat larger than the GDP growth rate divided by the elasticity of substitution. Our analysis confirms the result for the case  $\sigma = 1$  considered in standard analytical integrated assessment models. However, a value  $\sigma = 2/3$  seems more in line with the quantitative evidence and numerical integrated assessment modeling (Krusell et al. 2000, Nordhaus and Sztorc 2013, Pindyck and Wang 2013, Cai et al. 2015). Given this elasticity of

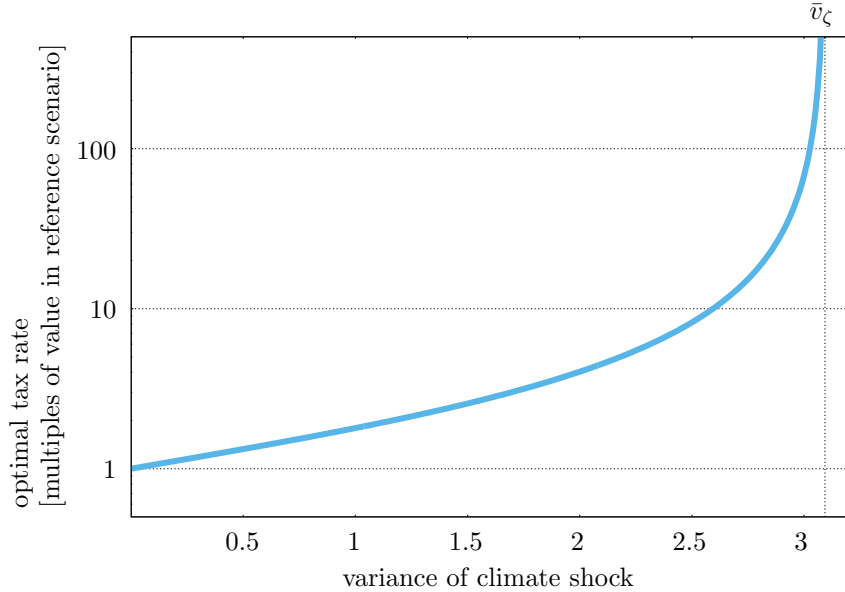


Figure 3: Optimal tax rate as a function of the variance of log-normally distributed climate shocks. Calibrated parameter values are given in Table 1.

substitution, the social cost of carbon grow at a rate much larger than the economy’s GDP growth rate.

Moreover, our analysis allows to assess analytically how sensitive the social cost of carbon is to key parameters. We confirm that varying the intergenerational discount rate in a range of values generally deemed plausible (Drupp et al. 2015), the social cost of carbon varies over several orders of magnitude, making it very difficult to derive a conclusive numerical value for the social cost of carbon (Pindyck 2013).

We have further shown that the social cost of carbon are similarly sensitive to the specification of the elasticity of substitution. This puts into question the common approach in analytical integrated assessment models that assume an elasticity of substitution equal to one. An interesting question for future research will be to analyze whether the social cost of carbon are more sensitive toward the elasticity of substitution between production factors or the elasticity of intertemporal substitution.

Finally, we have studied the case where the remaining carbon budget is unknown and subject to long-lasting shocks. We have found that for the case  $\sigma < 1$  there is no solution for the stochastic optimization problem if uncertainty is too large. This result, which has a similar flavor as Weitzman’s (2009) ‘dismal theorem’, comes about because the motive for precautionary reduction of emissions that cause unknown and irreversible climate damage becomes overly dominant. For smaller levels of uncertainty, the social cost of carbon become increasingly sensitive to climate shocks as uncertainty increases.

# Appendix

## A Proof of Proposition 1

Using the guess (10) in (7),

$$\begin{aligned} \Xi B_t^{\frac{\sigma-1}{\sigma}} + \Delta X_t^{\frac{\sigma-1}{\sigma}} + \Gamma H_t^{\frac{\sigma-1}{\sigma}} &= \max_{C_t, R_{t+1}, L_t, E_t} \left\{ C_t^{\frac{\sigma-1}{\sigma}} + \delta X_t^{\frac{\sigma-1}{\sigma}} + \psi \left( R_{t+1}^{\frac{\sigma-1}{\sigma}} + \delta (X_t - E_t)^{\frac{\sigma-1}{\sigma}} \right) \right. \\ &\quad \left. + \varphi \left( \Xi (F(B_t - C_t, E_t, L_t) - R_{t+1})^{\frac{\sigma-1}{\sigma}} \right) \right. \\ &\quad \left. + \Delta (X_t - E_t)^{\frac{\sigma-1}{\sigma}} + \Gamma ((1 - \varepsilon) H_t + \kappa (H_t - L_t))^{\frac{\sigma-1}{\sigma}} \right\}, \quad (30) \end{aligned}$$

with  $B_{t+1} = Y_{t+1} - R_{t+1}$ ,  $Y_{t+1} = F(K_t, E_t, L_t)$  given by (1),  $X_{t+1}$  given by (4), and  $H_{t+1}$  given by (2), leads to first-order conditions

$$C_t^{-\frac{1}{\sigma}} = \varphi \Xi F_{K_t} B_{t+1}^{-\frac{1}{\sigma}} \quad (31a)$$

$$\psi R_{t+1}^{-\frac{1}{\sigma}} = \varphi \Xi B_{t+1}^{-\frac{1}{\sigma}} \quad (31b)$$

$$\varphi \Xi F_{E_t} B_{t+1}^{-\frac{1}{\sigma}} = \psi \delta X_{t+1}^{-\frac{1}{\sigma}} + \varphi \Delta X_{t+1}^{-\frac{1}{\sigma}} \quad (31c)$$

$$\Xi F_{L_t} B_{t+1}^{-\frac{1}{\sigma}} = \Gamma \kappa H_{t+1}^{-\frac{1}{\sigma}} \quad (31d)$$

Using (31b) in (31c) leads to condition (11).

Using the guesses (8) and the properties of the production function, i.e.  $F_{K_t} = \alpha K_t^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}}$  and so on, these equations simplify to

$$(c^* Y_t)^{-\frac{1}{\sigma}} = \varphi \Xi \alpha ((b^* - c^*) Y_t)^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}} (b^* Y_{t+1})^{-\frac{1}{\sigma}} \quad (32a)$$

$$\psi ((1 - b^*) Y_{t+1})^{-\frac{1}{\sigma}} = \varphi \Xi (b^* Y_{t+1})^{-\frac{1}{\sigma}} \quad (32b)$$

$$\varphi \Xi \beta (e^* X_t)^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}} (b^* Y_{t+1})^{-\frac{1}{\sigma}} = \psi \delta ((1 - e^*) X_t)^{-\frac{1}{\sigma}} + \varphi \Delta ((1 - e^*) X_t)^{-\frac{1}{\sigma}} \quad (32c)$$

$$\Xi \gamma (l^* H_t)^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}} (b^* Y_{t+1})^{-\frac{1}{\sigma}} = \Gamma \kappa ((1 - \varepsilon + \kappa(1 - l^*)) H_t)^{-\frac{1}{\sigma}} \quad (32d)$$

Canceling common factors and rearranging, we obtain

$$(c^*)^{-\frac{1}{\sigma}} = \varphi \Xi \alpha (b^* - c^*)^{-\frac{1}{\sigma}} (b^*)^{-\frac{1}{\sigma}} \quad (33a)$$

$$\psi (1 - b^*)^{-\frac{1}{\sigma}} = \varphi \Xi (b^*)^{-\frac{1}{\sigma}} \quad (33b)$$

$$\varphi \Xi \beta (e^*)^{-\frac{1}{\sigma}} (b^*)^{-\frac{1}{\sigma}} = \psi \delta (1 - e^*)^{-\frac{1}{\sigma}} + \varphi \Delta (1 - e^*)^{-\frac{1}{\sigma}} \quad (33c)$$

$$\Xi \gamma (l^*)^{-\frac{1}{\sigma}} (b^*)^{-\frac{1}{\sigma}} = \Gamma \kappa (1 - \varepsilon + \kappa(1 - l^*))^{-\frac{1}{\sigma}} \quad (33d)$$

Furthermore, equating coefficients for  $B_t^{\frac{\sigma-1}{\sigma}}$ ,  $X_t^{\frac{\sigma-1}{\sigma}}$ , and  $H_t^{\frac{\sigma-1}{\sigma}}$  in (30) leads to the conditions

$$\Xi (b^*)^{\frac{\sigma-1}{\sigma}} = (c^*)^{\frac{\sigma-1}{\sigma}} + \left( \psi (1 - b^*)^{\frac{\sigma-1}{\sigma}} + \varphi \Xi (b^*)^{\frac{\sigma-1}{\sigma}} \right) \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} \quad (33e)$$

$$\Delta = \delta + (\psi \delta + \varphi \Delta) (1 - e^*)^{\frac{\sigma-1}{\sigma}} + \left( \psi (1 - b^*)^{\frac{\sigma-1}{\sigma}} + \varphi \Xi (b^*)^{\frac{\sigma-1}{\sigma}} \right) \beta (e^*)^{\frac{\sigma-1}{\sigma}} \quad (33f)$$

$$\Gamma = \varphi \Gamma (1 - \varepsilon + \kappa (1 - l^*))^{\frac{\sigma-1}{\sigma}} + \left( \psi (1 - b^*)^{\frac{\sigma-1}{\sigma}} + \varphi \Xi (b^*)^{\frac{\sigma-1}{\sigma}} \right) \gamma (l^*)^{\frac{\sigma-1}{\sigma}} \quad (33g)$$

Solving gives the coefficients  $c^*$ ,  $b^*$ ,  $e^*$ , and  $l^*$  stated in the proposition.

Dividing (33a) by (33b),

$$\frac{c^{*\frac{-1}{\sigma}}}{\psi (1 - b^*)^{\frac{-1}{\sigma}}} = \alpha (b^* - c^*)^{\frac{-1}{\sigma}} \quad (34a)$$

Using (33b) in (33e),

$$\Xi b^{*\frac{\sigma-1}{\sigma}} = c^{*\frac{\sigma-1}{\sigma}} + \psi (1 - b^*)^{\frac{-1}{\sigma}} \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} \stackrel{(34a)}{=} b^* c^{*\frac{-1}{\sigma}} \quad (34b)$$

Using this in (33a) and solving yields

$$b^* - c^* = (\varphi \alpha)^\sigma \quad (34c)$$

Few obvious further steps lead to (9a) and (9b). Furthermore,

$$\Xi = \left( \frac{1 + (\psi \alpha)^\sigma}{1 - (\varphi \alpha)^\sigma} \right)^{\frac{1}{\sigma}} \quad (34d)$$

Using (33b) in (33f),

$$\Delta = \delta + (\psi \delta + \varphi \Delta) (1 - e^*)^{\frac{\sigma-1}{\sigma}} + \varphi \Xi b^{*\frac{-1}{\sigma}} \beta e^{*\frac{-1}{\sigma}} e^* \quad (34e)$$

$$\stackrel{(33c)}{=} \delta + (\psi \delta + \varphi \Delta) (1 - e^*)^{\frac{\sigma-1}{\sigma}} + \psi \delta (1 - e^*)^{\frac{-1}{\sigma}} e^* + \varphi \Delta (1 - e^*)^{\frac{-1}{\sigma}} e^* \\ = \delta \left( 1 + \psi (1 - e^*)^{\frac{-1}{\sigma}} \right) + \varphi \Delta (1 - e^*)^{\frac{-1}{\sigma}} \quad (34f)$$

Solving yields

$$\Delta = \delta \frac{1 + \psi (1 - e^*)^{\frac{-1}{\sigma}}}{1 - \varphi (1 - e^*)^{\frac{-1}{\sigma}}} \quad (34g)$$



Plugging this and (34b) in (33c) we obtain

$$\varphi \beta e^{\star-\frac{1}{\sigma}} c^{\star-\frac{1}{\sigma}} = (\psi \delta + \varphi \Delta) (1 - e^{\star})^{-\frac{1}{\sigma}} \quad (34h)$$

$$= \delta (1 - e^{\star})^{-\frac{1}{\sigma}} \left( \psi + \varphi \frac{1 + \psi (1 - e^{\star})^{-\frac{1}{\sigma}}}{1 - \varphi (1 - e^{\star})^{-\frac{1}{\sigma}}} \right) \quad (34i)$$

$$= \delta (1 - e^{\star})^{-\frac{1}{\sigma}} \frac{\psi + \varphi}{1 - \varphi (1 - e^{\star})^{-\frac{1}{\sigma}}} \quad (34j)$$

rearranging leads to (9c). Using (33b) in (33g),

$$\Gamma = \varphi \Gamma (1 - \varepsilon + \kappa(1 - l^{\star}))^{\frac{\sigma-1}{\sigma}} + \varphi \Xi (b^{\star})^{-\frac{1}{\sigma}} \gamma (l^{\star})^{-\frac{1}{\sigma}} l^{\star} \quad (34k)$$

$$\stackrel{(33d)}{=} \varphi \Gamma (1 - \varepsilon + \kappa(1 - l^{\star}))^{\frac{\sigma-1}{\sigma}} + \varphi \Gamma \kappa (1 - \varepsilon + \kappa(1 - l^{\star}))^{-\frac{1}{\sigma}} l^{\star} \quad (34l)$$

$$= \varphi \Gamma (1 - \varepsilon + \kappa(1 - l^{\star}))^{-\frac{1}{\sigma}} (1 - \varepsilon + \kappa(1 - l^{\star}) + \kappa l^{\star}). \quad (34m)$$

Rearranging leads to (9d). Plugging  $l^{\star}$ ,  $b^{\star}$ , and  $\Xi$  in (33g) yields  $\Gamma$ .

For the comparative statics with respect to  $\rho$ , write (9c) as

$$(1 - e^{\star})^{\frac{1}{\sigma}} = \frac{1}{1 + \rho} + f(\rho) e^{\star \frac{1}{\sigma}} \quad (35)$$

with finite  $f$  and  $f_{\rho}$ . The total differential at  $e^{\star} = \rho = 0$  is

$$\frac{1}{\sigma} de^{\star} = d\rho - f_{\rho} e^{\star \frac{1}{\sigma}} - \frac{1}{\sigma} f e^{\star \frac{1}{\sigma}-1} de^{\star} \quad (36)$$

For  $\sigma < 1$  the second and third term vanish at  $e^{\star} = 0$  implying  $e_{\rho}^{\star} = \sigma$  and thus  $\rho/e^{\star} e_{\rho}^{\star} = 1$ . For  $\sigma = 1$  the second terms still vanishes at  $e^{\star} = 0$  and the third term goes to  $f(0)/\sigma$  implying  $e_{\rho}^{\star} = \sigma/(1 + f(0))$  and thus  $\rho/e^{\star} e_{\rho}^{\star} = 1$ . For  $\sigma > 1$  the second term still vanishes, but the third term goes to infinity implying  $e_{\rho}^{\star} = \sigma/(1 + f e^{\star \frac{1}{\sigma}-1})$  to vanish. For the elasticity we obtain

$$\lim_{\rho \rightarrow 0} \frac{\rho}{e^{\star}} e_{\rho}^{\star} = \sigma \lim_{\rho \rightarrow 0} \frac{\rho}{e^{\star} + f e^{\star \frac{1}{\sigma}}} = \sigma \lim_{\rho \rightarrow 0} \frac{1}{e_{\rho}^{\star} + f_{\rho} e^{\star \frac{1}{\sigma}} + \frac{1}{\sigma} f e^{\star \frac{1}{\sigma}-1} e_{\rho}^{\star}} = \sigma. \quad (37)$$

As for the comparative statics with respect to  $\sigma$ , write (9c) as

$$e^{\star} = \frac{1}{c^{\star}} \left( \frac{(1 - e^{\star})^{\frac{1}{\sigma}} - \varphi}{\frac{\delta}{\beta} \frac{\psi + \varphi}{\varphi}} \right)^{\sigma} \approx \frac{1}{c^{\star}} \left( \frac{\beta}{\delta} \frac{\varphi(1 - \varphi)}{\psi + \varphi} \right)^{\sigma} \quad (38)$$

Thus, the semi-elasticity of  $e^{\star}$  with respect to  $\sigma$  is as given in equation (13).

## B Proof of Proposition 2

Before turning to the proof of the proposition, we first derive some preliminary results on the transition dynamics. We focus on the development of the economy's growth rate  $\hat{Y}_t$ . As  $Y_{t+1} = (1 + \hat{Y}_t) Y_t$ , equation (14) implies

$$Y_{t+1}^{\frac{\sigma-1}{\sigma}} = \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} Y_t^{\frac{\sigma-1}{\sigma}} + \beta e^{\frac{\sigma-1}{\sigma}} X_t^{\frac{\sigma-1}{\sigma}} + \gamma l^{\frac{\sigma-1}{\sigma}} H_t^{\frac{\sigma-1}{\sigma}} \quad (39)$$

$$(1 + g_{t+1})^{\frac{\sigma-1}{\sigma}} Y_{t+1}^{\frac{\sigma-1}{\sigma}} = \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} Y_{t+1}^{\frac{\sigma-1}{\sigma}} + \beta e^{\frac{\sigma-1}{\sigma}} X_{t+1}^{\frac{\sigma-1}{\sigma}} + \gamma l^{\frac{\sigma-1}{\sigma}} H_{t+1}^{\frac{\sigma-1}{\sigma}} \quad (40)$$

As thus also

$$(1 + g_t)^{\frac{\sigma-1}{\sigma}} = \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} + \beta e^{\frac{\sigma-1}{\sigma}} \frac{X_t^{\frac{\sigma-1}{\sigma}}}{Y_t^{\frac{\sigma-1}{\sigma}}} + \gamma l^{\frac{\sigma-1}{\sigma}} \frac{H_t^{\frac{\sigma-1}{\sigma}}}{Y_t^{\frac{\sigma-1}{\sigma}}} \quad (41)$$

Dividing equation (41) for  $t + 1$  by (41) for  $t$ , and solving for  $g_{t+1}$  yields the following result.

**Lemma 1.** *The growth rate of the economy develops over time according to*

$$1 + g_{t+1} = \left( \frac{\alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}}{(1 + g_t)^{\frac{\sigma-1}{\sigma}}} (1 + g_t)^{\frac{\sigma-1}{\sigma}} + \left( 1 - \frac{\alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}}{(1 + g_t)^{\frac{\sigma-1}{\sigma}}} \right) (1 + \hat{Y}_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (42a)$$

*i.e. the growth factor at  $t + 1$  is a generalized mean, with state-dependent weighting factors, of the growth factor at  $t$  and a term  $1 + \hat{Y}_t$  with*

$$1 + \hat{Y}_t \equiv \left( \frac{\beta e^{\frac{\sigma-1}{\sigma}} (X_0 (1 - e^*)^t)^{\frac{\sigma-1}{\sigma}}}{\beta (e^* X_0 (1 - e^*)^t)^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_0 (1 + g_H)^t)^{\frac{\sigma-1}{\sigma}}} (1 - e^*)^{\frac{\sigma-1}{\sigma}} + \frac{\gamma (l^* H_0 (1 + g_H)^t)^{\frac{\sigma-1}{\sigma}}}{\beta (e^* X_0 (1 - e^*)^t)^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_0 (1 + g_H)^t)^{\frac{\sigma-1}{\sigma}}} (1 + g_H)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (42b)$$

*being again a generalized mean, with time-dependent weighting factors, of the factor  $1 - e^*$  at which the remaining carbon budget is decreased and the growth factor  $1 + g_H$  of human capital.*

*Proof.* Consider equation (41) for  $t + 1$

$$\begin{aligned} (1 + g_{t+1})^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} &= \beta e^{\frac{\sigma-1}{\sigma}} \left( \frac{X_{t+1}}{Y_{t+1}} \right)^{\frac{\sigma-1}{\sigma}} + \gamma \left( \frac{l^* H_{t+1}}{Y_{t+1}} \right)^{\frac{\sigma-1}{\sigma}} \\ &= \beta e^{\frac{\sigma-1}{\sigma}} \left( \frac{(1 - e^*) X_t}{(1 + \hat{Y}_t) Y_t} \right)^{\frac{\sigma-1}{\sigma}} + \gamma \left( \frac{l^* (1 + g_H) H_t}{(1 + \hat{Y}_t) Y_t} \right)^{\frac{\sigma-1}{\sigma}} \end{aligned} \quad (43)$$

and divide it by (41) for  $t$ ,

$$(1 + g_t)^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} = \beta e^{\frac{\sigma-1}{\sigma}} \left( \frac{X_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}} + \gamma \left( \frac{l^* H_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}} \quad (44)$$

This leads to

$$\frac{(1 + g_{t+1})^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}}{(1 + g_t)^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}} = \frac{1}{(1 + \hat{Y}_t)^{\frac{\sigma-1}{\sigma}}} \underbrace{\frac{\beta (e^* (1 - e^*) X_t)^{\frac{\sigma-1}{\sigma}} + \gamma (l^* (1 + g_H) H_t)^{\frac{\sigma-1}{\sigma}}}{\beta (e^* X_t)^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_t)^{\frac{\sigma-1}{\sigma}}}}_{\equiv (1 + \hat{Y}_t)^{\frac{\sigma-1}{\sigma}}}$$

In (42b) we have used  $X_t = (1 - e^*)^t X_0$  and  $H_t = (1 + g_H)^t H_0$ . Solving

$$\frac{(1 + g_{t+1})^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}}{(1 + g_t)^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}} = \frac{(1 + \hat{Y}_t)^{\frac{\sigma-1}{\sigma}}}{(1 + \hat{Y}_t)^{\frac{\sigma-1}{\sigma}}}$$

for  $1 + g_{t+1}$  yields the proposed result (42a).  $\square$

This result can be used to fully characterize the optimal dynamics of the model. Consider first equation (42b). This equation defines the expression  $\hat{Y}_t$ , which is a weighted generalized average of the growth rates  $1 - e^*$  of the remaining carbon budget, and  $1 + g_H$  of human capital. This implies that  $\hat{Y}_t$  is in between  $1 - e^* < 0$  and  $1 + g_H > 0$  at any point in time. The weighting factors depend on the current stock sizes of the remaining carbon budget and human capital, and both weights are changing over time in a uniform way. The direction of change is determined by the value of  $\sigma$ . If  $\sigma > 1$ ,  $\hat{Y}_t$  monotonically increases over time, as the weight on  $1 + g_H$  increases over time. If  $\sigma < 1$ ,  $\hat{Y}_t$  monotonically decreases over time, as the weight on  $1 - e^*$  increases over time in this case.

Now consider (42a). Similar as discussed for  $\hat{Y}_t$ , the economy's growth factor  $1 + g_{t+1}$  at  $t + 1$  is a weighted generalized average, now of the growth factor at  $t$ ,  $1 + g_t$ , and of the above discussed moving average  $1 + \hat{Y}_t$  of  $1 - e^*$  and  $1 + g_H$ . It follows immediately from (41) that  $0 < \frac{\alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}}{(1 + g_t)^{\frac{\sigma-1}{\sigma}}} < 1$ , i.e. both weights are between zero and one, and obviously add up to one. The property that  $1 + g_{t+1}$  is a generalized mean of  $1 + \hat{Y}_t$  and  $1 + \hat{Y}_t$  implies that  $1 + g_{t+1}$  must lie in between the two, and in the long run, the dynamics of  $\hat{Y}_t$  are purely driven by the dynamics of  $\hat{Y}_t$ .

Note that the value of  $\hat{Y}_t$  depends on the intergenerational discount factor  $\varphi$ , but not the intragenerational discount factor  $\psi$ : This part of the dynamics of the growth rate is purely driven by intergenerational distribution, not by intragenerational distribution. Intragenerational distribution matters for short-run dynamics, though. The weights on  $\hat{Y}_t$  and  $\hat{Y}_t$  depend on the intragenerational discount factor,  $\psi$ .

The long-run behavior of the economy immediately follows from Lemma 1: If  $\sigma > 1$ , the

economy's growth rate eventually will reach  $g_H$ , while for  $\sigma < 1$ , it will eventually contract at a rate  $-e^*$ . This result resembles the famous findings from the Dasgupta-Heal-Solow-Stiglitz model (Dasgupta and Heal 1974, Solow 1974, Stiglitz 1974a,b). For  $\sigma = 1$ , the equations from Lemma 1 simplify to

$$1 + g_{t+1} = (1 + g_t)^\alpha (1 - e^*)^\beta (1 + g_H)^\gamma \quad (45a)$$

$$\Leftrightarrow 1 + g_t = \left( (1 - e^*)^{\frac{\beta}{1-\alpha}} (1 + g_H)^{\frac{\gamma}{1-\alpha}} \right)^{1-\alpha^t} (1 + g_0)^{\alpha^t} \quad (45b)$$

As  $1 - \alpha = \beta + \gamma$ , the first term in brackets on the right-hand-side of this equation is a weighted geometric mean of  $1 - e^*$  and  $1 + g_H$ , just as in the general case. This weighted geometric mean determines the long-term growth rate of the economy. The economy will grow in the long-run if  $(1 - e^*)^\beta (1 + g_H)^\gamma > 1$  and contract in the long-run if  $(1 - e^*)^\beta (1 + g_H)^\gamma < 1$ . The growth rate for generation  $t$ ,  $\hat{Y}_t$ , again is a weighted geometric mean, namely of the long-term growth rate and the initial growth rate. In the course of time, the weight on the long-run growth rate increases.

In general, the growth rate at  $t+1$  lies in between the growth rate at  $t$  and  $\hat{Y}_t$ . The transition dynamics in particular depend on whether the economy initially is relatively abundant or relatively poor in produced capital, with the threshold level defined by

$$\underline{Y}_0(H_0, X_0) \equiv \left( \frac{\beta (e^* X_0)^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_0)^{\frac{\sigma-1}{\sigma}}}{(1 + \bar{g}_0)^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (46a)$$

where  $\bar{g}_0$  is given by (42b) for  $t = 0$ . For  $\sigma = 1$ , we have

$$\underline{Y}_0(H_0, X_0) \equiv \lim_{\sigma \rightarrow 1} \left( \frac{\beta}{1-\alpha} (e^* X_0)^{\frac{\sigma-1}{\sigma}} + \frac{\gamma}{1-\alpha} (l^* H_0)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = (e^* X_0)^{\frac{\beta}{1-\alpha}} (l^* H_0)^{1-\frac{\beta+\delta}{1-\alpha}} \quad (46b)$$

Now we turn to the proof of the proposition.

Ad 1. For  $\sigma > 1$ , it follows from (42a) that  $g_1 < g_0$  if

$$\begin{aligned} & (1 + g_0)^{\frac{\sigma-1}{\sigma}} < (1 + \bar{g}_0)^{\frac{\sigma-1}{\sigma}} \\ \Leftrightarrow & \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} + \beta e^{\frac{\sigma-1}{\sigma}} \left( \frac{X_0}{Y_0} \right)^{\frac{\sigma-1}{\sigma}} + \gamma \left( \frac{l^* H_0}{Y_0} \right)^{\frac{\sigma-1}{\sigma}} < (1 + \bar{g}_0)^{\frac{\sigma-1}{\sigma}} \end{aligned}$$

Solving for  $Y_0$  leads to the condition  $Y_0 < \underline{Y}_0(H_0, X_0)$ . If  $Y_0 > \underline{Y}_0(H_0, X_0)$ , it follows that  $g_1 > g_0$ . As  $\hat{Y}_t$  is increasing over time, so must be  $g_{t+1}$ , which is a weighted average of  $\hat{Y}_t$  and  $\bar{g}_t$ , and the weight on  $\hat{Y}_t$  is increasing over time (cf. Lemma 1).

Ad 2. For  $\sigma < 1$ , it follows from (42a) that  $g_1 > g_0$  if

$$\begin{aligned} & (1 + g_0)^{\frac{\sigma-1}{\sigma}} < (1 + \bar{g}_0)^{\frac{\sigma-1}{\sigma}} \\ \Leftrightarrow \quad & \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} + \beta e^{*\frac{\sigma-1}{\sigma}} \left( \frac{X_0}{Y_0} \right)^{\frac{\sigma-1}{\sigma}} + \gamma \left( \frac{l^* H_0}{Y_0} \right)^{\frac{\sigma-1}{\sigma}} < (1 + \bar{g}_0)^{\frac{\sigma-1}{\sigma}} \end{aligned}$$

Solving for  $Y_0$  leads to the condition  $Y_0 < \underline{Y}_0(H_0, X_0)$ . If  $Y_0 > \underline{Y}_0(H_0, X_0)$ , it follows that  $g_1 < g_0$ . As  $\hat{Y}_t$  is decreasing over time, so must be  $g_{t+1}$ , which is a weighted average of  $\hat{Y}_t$  and  $\bar{g}_t$ , and the weight on  $\hat{Y}_t$  is decreasing over time (cf. Lemma 1).

Ad 3. For  $\sigma = 1$  it follows from (45a) that  $g_1 > g_0$  if  $Y_0 < \underline{Y}_0(H_0, X_0)$  and  $g_1 < g_0$  if  $Y_0 > \underline{Y}_0(H_0, X_0)$ . In this case,  $\bar{g}$  is constant over time, and thus  $\hat{Y}_t$  approaches the constant  $\bar{g}$  over time.

## C Proof of Proposition 3

The tax rate in the efficient market equilibrium must equal the marginal product of carbon emissions. Using the fact that  $F_{E_t} = \beta E_t^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}}$ , and  $E_t = e^* X_t$  from Proposition 1, we obtain

$$\tau_t = F_{E_t} = \beta (e^* X_t)^{-\frac{1}{\sigma}} Y_{t+1}^{\frac{1}{\sigma}}, \quad (47)$$

as proposed. We thus further have

$$\frac{\tau_{t+1}}{\tau_t} = \frac{Y_{t+2}^{\frac{1}{\sigma}} X_t^{\frac{1}{\sigma}}}{Y_{t+1}^{\frac{1}{\sigma}} X_{t+1}^{\frac{1}{\sigma}}} = \left( \frac{(1 + \hat{Y}_t) Y_{t+1} X_t}{Y_{t+1} (1 - e^*) X_t} \right)^{\frac{1}{\sigma}} = \left( \frac{1 + \hat{Y}_t}{1 - e^*} \right)^{\frac{1}{\sigma}}. \quad (48)$$

For  $\sigma \leq 1$  we have

$$\left( 1 + \hat{Y}_t \right)^{\frac{\sigma-1}{\sigma}} \geq 1 > (1 - e^*)^{\frac{1}{\sigma}} \quad (49)$$

$$\Rightarrow \quad 1 + \hat{\tau} = \left( \frac{1 + \hat{Y}_t}{1 - e^*} \right)^{\frac{1}{\sigma}} > 1 + \hat{Y}_t. \quad (50)$$

## D Proof of Proposition 6

Guessing the value function

$$V(B, X, H) = \frac{1}{1 - \frac{1}{\sigma}} \left( \Xi B^{\frac{\sigma-1}{\sigma}} + \tilde{\Delta} X^{\frac{\sigma-1}{\sigma}} + \Gamma H^{\frac{\sigma-1}{\sigma}} \right) \quad (51)$$

and using the guesses for the feedback policies in (20) yields

$$\begin{aligned} \Xi (Y_t - R_t)^{\frac{\sigma-1}{\sigma}} + \tilde{\Delta} X_t^{\frac{\sigma-1}{\sigma}} + \Gamma H_t^{\frac{\sigma-1}{\sigma}} = \max_{C_t, R_{t+1}, L_t, E_t} \left\{ C_t^{\frac{\sigma-1}{\sigma}} + \delta X_t^{\frac{\sigma-1}{\sigma}} + \psi R_{t+1}^{\frac{\sigma-1}{\sigma}} \right. \\ \left. + \delta \mathcal{E}_{\zeta_t} \left[ X_{t+1}^{\frac{\sigma-1}{\sigma}} \right] + \varphi \left( \Xi (Y_{t+1} - R_{t+1})^{\frac{\sigma-1}{\sigma}} + \tilde{\Delta} \mathcal{E}_{\zeta_t} \left[ X_{t+1}^{\frac{\sigma-1}{\sigma}} \right] + \Gamma H_{t+1}^{\frac{\sigma-1}{\sigma}} \right) \right\} \end{aligned} \quad (52)$$

with  $Y_{t+1}$  determined by (1). It is straightforward to see that the first-order conditions remain the same for  $C_t$ ,  $R_{t+1}$ , and  $L_t$  both in the deterministic case and in the case with stochastic climate dynamics. Thus, the feedback policies are the same as in the deterministic case.

The first-order condition from optimizing with respect to  $E_t$  leads to

$$\varphi \Xi F_{E_t} B_{t+1}^{-\frac{1}{\sigma}} = \psi \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (\tilde{e} X_t)^{-\frac{1}{\sigma}} + \varphi \Delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (\tilde{e} X_t)^{-\frac{1}{\sigma}} \quad (53)$$

Using a similar line of argument as in Appendix A we find

$$\tilde{\Delta} = \delta + \left( \psi \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] + \varphi \Delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \right) (1 - \tilde{e})^{\frac{\sigma-1}{\sigma}} + \varphi \Xi b^{\star - \frac{1}{\sigma}} \beta \tilde{e}^{-\frac{1}{\sigma}} \tilde{e} \quad (54)$$

$$\begin{aligned} & \stackrel{(53)}{=} \delta + \left( \psi \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] + \varphi \Delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \right) (1 - \tilde{e})^{\frac{\sigma-1}{\sigma}} \\ & \quad + \psi \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \tilde{e} + \varphi \Delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \tilde{e} \\ & = \delta \left( 1 + \psi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \right) + \varphi \Delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \end{aligned} \quad (55)$$

Solving yields

$$\Delta = \delta \frac{1 + \psi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}}{1 - \varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}} \quad (56)$$

For the optimal feedback rule we obtain

$$\varphi \beta \tilde{e}^{-\frac{1}{\sigma}} c^{\star - \frac{1}{\sigma}} = \left( \psi \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] + \varphi \tilde{\Delta} \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \right) (1 - \tilde{e})^{-\frac{1}{\sigma}} \quad (57)$$

$$= \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \left( \psi + \varphi \frac{1 + \psi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}}{1 - \varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}} \right) \quad (58)$$

$$= \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}} \frac{\psi + \varphi}{1 - \varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}} \quad (59)$$

From this, we get the equivalent to (9c).

$$\beta \tilde{e}^{-\frac{1}{\sigma}} = \delta \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] \frac{\psi + \varphi}{1 - \varphi \mathcal{E} \left[ \zeta^{\frac{\sigma-1}{\sigma}} \right] (1 - \tilde{e})^{-\frac{1}{\sigma}}} c^{\star \frac{1}{\sigma}} (1 - e^{\star})^{-\frac{1}{\sigma}} \quad (60)$$

Rearranging leads to (22).

For the comparative statics with respect to  $v_\zeta$  in the case where  $\zeta$  is log-normally distributed with mean one and variance  $v_\zeta$ , write (22) as

$$(1 - \tilde{e})^{\frac{1}{\sigma}} = \left( \frac{1 + v_\zeta}{1 + \bar{v}_\zeta} \right)^{\frac{1-\sigma}{2\sigma^2}} + f \tilde{e}^{\frac{1}{\sigma}} \quad (61)$$

where  $(1 + \bar{v}_\zeta)^{\frac{1-\sigma}{2\sigma^2}} = 1 + \rho$  and  $f$  is independent of  $v_\zeta$ . The total differential at  $e^{\star} = 0$  and  $v_\zeta = \bar{v}_\zeta$  is

$$\frac{1}{\sigma} d\tilde{e} = \frac{1 - \sigma}{2\sigma^2} \frac{dv_\zeta}{1 + \bar{v}_\zeta} - \frac{1}{\sigma} f \tilde{e}^{\frac{1}{\sigma}-1} d\tilde{e} \quad (62)$$

For  $\sigma < 1$  the second term vanishes at  $\tilde{e} = 0$  implying that  $\tilde{e}_{v_\zeta}$  is constant and thus  $v_\zeta / \tilde{e}_{v_\zeta} = 1$ .

## E Further details on calibration of the model

In this appendix we provide some further elements of calibration.

To calibrate the parameters of human capital accumulation, we assume that the global economy is in the ‘intermediate’ steady state where consumption grows at the human capital growth rate, as characterized in Proposition 2. From the condition  $\hat{Y}_t = \varphi^\sigma (1 - \varepsilon + \kappa)^\sigma - 1$  we obtain

$$1 - \varepsilon + \kappa = \frac{(1 + \hat{Y}_t)^{\frac{1}{\sigma}}}{\varphi} = F_{K_t} = 1.05^{30} = 4.32. \quad (63)$$

For the rate of human capital depreciation we use the estimate of roughly 0.04 per year (Heckman 1976, table 1), i.e.  $\varepsilon = 1 - 0.96^G = 0.71$ . This yields  $\kappa = 4.32 - 1 + 0.71 = 4.02$ .

From these parameter values, we obtain the labor share used in production, given by (9d), as

$$l^{\star} = 1 - \frac{1}{\kappa} \left( \hat{Y}_t + \varepsilon \right) = \frac{1 + \kappa - \varepsilon - (1 + \hat{Y}_t)}{\kappa} = \frac{4.32 - 1.021^{30}}{4.02} = 0.61. \quad (64)$$

In the next steps we use the income shares (27) From (41) we obtain

$$\begin{aligned} & \left( (1 + g_0)^{\frac{\sigma-1}{\sigma}} - \alpha (b^* - c^*)^{\frac{\sigma-1}{\sigma}} \right) Y_0^{\frac{\sigma-1}{\sigma}} = \beta E_0^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_0)^{\frac{\sigma-1}{\sigma}} \\ \Leftrightarrow & \left( \frac{(1 + g_0)^{\frac{\sigma-1}{\sigma}}}{\varphi^{\sigma-1} \alpha^\sigma} - 1 \right) \alpha ((b^* - c^*) Y_0)^{\frac{\sigma-1}{\sigma}} = \beta E_0^{\frac{\sigma-1}{\sigma}} + \gamma (l^* H_0)^{\frac{\sigma-1}{\sigma}} \end{aligned} \quad (65)$$

where we have used  $(b^* - c^*)^{\frac{1}{\sigma}} = \varphi \alpha$  (cf. Proposition 1). Using (27),  $1 + g_0 = 1.021^{30}$ , and the above calibrated  $\sigma = \frac{2}{3}$  and  $\varphi = 0.59$  we obtain the condition

$$\left( \frac{(1.021^{30})^{-\frac{1}{2}}}{0.59^{-\frac{1}{3}} \alpha^{\frac{2}{3}}} - 1 \right) 0.25 = 0.05 + 0.70 \quad \Leftrightarrow \quad \alpha = 0.060. \quad (66)$$

Using this in (27b), we obtain  $Y_0 = 0.54$ , measured in units of world output in period 1.

Finally, from (27c),

$$H_0 = \frac{1}{l^*} \left( \frac{0.70}{1 - \alpha - \beta} \right)^{\frac{\sigma}{\sigma-1}} = 2.28 \quad (67)$$

This is roughly four times the initial stock of consumable capital  $Y_0$ , which is in line with empirical evidence (Liu 2014, figure 4).

## References

- Acemoglu, D., Aghion, P., Bursztyn, L., Hemous, D., 2012. The environment and directed technical change. *American Economic Review* 102 (1), 131–166.
- Asheim, G. B., 2013. A distributional argument for supply-side climate policies. *Environmental and Resource Economics* 56 (2), 239–254.
- Bretschger, L., Karydas, C., 2013. Optimum growth and carbon policies with lags in the climate system. CER-ETH Center of Economic Research Working Paper 13/184.
- Bretschger, L., Smulders, S., 2012. Sustainability and substitution of exhaustible natural resources – how structural change affects long-term r&d-investments. *Journal of Economic Dynamics & Control* 36, 536 – 549.
- Brock, W. A., Mirman, L. J., 1972. Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory* 4, 479–533.
- Cai, Y., Judd, K. L., Lontzek, T., 2015. The social cost of carbon with economic and climate risks. arXiv:1504.06909 [q-fin.EC].



- Collins, M., Knutti, R., Arblaster, J., Dufresne, J.-L., Fichefet, T., Friedlingstein, P., Gao, X., Gutowski, W., Johns, T., Krinner, G., Shongwe, M., Tebaldi, C., Weaver, A., Wehner, M., 2013. Long-term climate change: Projections, commitments and irreversibility. In: Stocker, T., Qin, D., Plattner, G.-K., Tignor, M., Allen, S., Boschung, J., Nauels, A., Xia, Y., Bex, V., Midgley, P. (Eds.), *Climate Change 2013: The Physical Science Basis. Contribution of Working Group I to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change*. Cambridge University Press, Cambridge, United Kingdom and New York, NY, USA, pp. 1029–1136.
- Dasgupta, P., Heal, G., 1974. The optimal depletion of exhaustible resources. *Review of Economic Studies* 41, 3–28.
- Dasgupta, P. S., Heal, G. M., 1979. *Economic Theory and Exhaustible Resources*. Cambridge University Press, Cambridge.
- Drupp, M. A., Freeman, M. C., Groom, B., Nesje, F., 2015. Discounting disentangled: An expert survey on the determinants of the long-term social discount rate. Centre for Climate Change Economics and Policy Working paper No. 195.
- Friedlingstein, P., Andrew, R. M., Rogelj, J., Peters, G. P., Canadell, J. G., Knutti, R., Luderer, G., Raupach, M. R., Schaeffer, M., van Vuuren, D. P., Le Quere, C., 2014. Persistent growth of CO<sub>2</sub> emissions and implications for reaching climate targets. *Nature Geoscience* 7 (10), 709–715.
- Gerlagh, R., Liski, M., 2012. Carbon prices for the next thousand years. CESifo Working Paper 3855.
- Gollier, C., 2012. *Pricing the Planet’s Future: The Economics of Discounting in an Uncertain World*. Princeton University Press, Princeton (NJ).
- Gollier, C., 2015. Discounting, inequality and economic convergence. *Journal of Environmental Economics and Management* 69, 53 – 61.
- Golosov, M., Hassler, J., Krusell, P., Tsyvinski, A., 2014. Optimal taxes on fossil fuel in general equilibrium. *Econometrica* 82 (1), 41–88.
- Heckman, J. J., 1976. A life-cycle model of earnings, learning, and consumption. *Journal of Political Economy* 84 (4), S11–S44.
- Hoel, M., Sterner, T., 2007. Discounting and relative prices. *Climatic Change* 84 (3-4), 265–280.

- IEA, 2015. CO2 Emissions From Fuel Combustion Highlights 2015. Tech. rep., International Energy Agency.
- Krusell, P., Ohanian, L., Ríos-Rull, J.-V., Violante, G., 2000. Capital-skill complementarity and inequality: A macroeconomic analysis. *Econometrica* 68 (5), 1029–1053.
- Liu, G., 2014. Measuring the stock of human capital for international and inter-temporal comparisons. In: Jorgenson, D. W., Landefeld, J. S., Schreyer, P. (Eds.), *Measuring Economic Sustainability and Progress*. University of Chicago Press.  
URL <http://www.nber.org/books/jorg12-1>
- Llavador, H., Roemer, J. E., Silvestre, J., 2011. A dynamic analysis of human welfare in a warming planet. *Journal of Public Economics* 95 (11–12), 1607 – 1620.
- McGlade, C., Ekins, P., 2015. The geographical distribution of fossil fuels unused when limiting global warming to 2° C. *Nature* 517 (7533), 187–190.
- Nordhaus, W. D., 1993. Optimal greenhouse-gas reductions and tax policy in the ‘DICE’ model. *American Economic Review* 83 (2), 313–317.
- Nordhaus, W. D., Sztorc, P., 2013. Dice 2013r: Introduction and user’s manual.  
URL [http://www.econ.yale.edu/~nordhaus/homepage/documents/DICE\\_Manual\\_103113r2.pdf](http://www.econ.yale.edu/~nordhaus/homepage/documents/DICE_Manual_103113r2.pdf)
- Piketty, T., 2014. *Capital in the Twenty-First Century*. Belknap Press.
- Pindyck, R. S., 2013. Climate change policy: What do the models tell us? *Journal of Economic Literature* 51 (3), 860–72.
- Pindyck, R. S., Wang, N., 2013. The economic and policy consequences of catastrophes. *American Economic Journal: Economic Policy* 5 (4), 306–39.
- Schneider, M. T., Traeger, C. P., Winkler, R., 2012. Trading off generations: Equity, discounting, and climate change. *European Economic Review* 56 (8), 1621–1644.
- Smulders, S., Toman, M., Withagen, C., 2014. Growth theory and ‘green growth’. *Oxford Review of Economic Policy* 30 (3), 423–446.
- Solow, R. M., 1974. Intergenerational equity and exhaustible resources. *Review of Economic Studies* 41, 29–45.

- Stern, N., 2006. Stern Review on the Economics of Climate Change. Government Economic Service, London.
- Sterner, T., Persson, M., 2008. An even sterner review: introducing relative prices into the discounting debate. *Review of Environmental Economics and Policy* 2 (1), 61–76.
- Stiglitz, J., 1974a. Growth with exhaustible natural resources: Efficient and optimal growth paths. *Review of Economic Studies* 41, 123–137.
- Stiglitz, J., 1974b. Growth with exhaustible natural resources: The competitive economy. *Review of Economic Studies* 41, 139–152.
- Stollery, K., 1998. Constant utility paths and irreversible global warming. *Canadian Journal of Economics* 31 (3), 730–742.
- Tahvonen, O., Kuuluvainen, J., 1991. Optimal growth with renewable resources and pollution. *European Economic Review* 35 (2-3), 650 – 661.
- Traeger, C., 2015. Closed-form integrated assessment and uncertainty. CES-IFO Working Paper No. 5464.
- van der Ploeg, F., Withagen, C., 2014. Growth, renewables, and the optimal carbon tax. *International Economic Review* 55 (1), 283–311.
- Weitzman, M. L., 2007. A review of the Stern Review on the economics of climate change. *Journal of Economic Literature* 45 (3), 703–724.
- Weitzman, M. L., 2009. On modeling and interpreting the economics of catastrophic climate change. *Review of Economics and Statistics* 91 (1), 1–19.