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Rent-Seeking Games and the All-Pay Auction

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Rent-Seeking Games and the All-Pay Auction*

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Abstract. This paper considers rent-seeking games in which a small percentage change in a player's bid has a large percentage impact on her odds of winning, i.e., on the ratio of her respective probabilities of winning and losing. An example is the Tullock contest with a high R. The analysis provides a fairly complete characterization of the equilibrium set. In particular, for "sufficiently generic" valuations, any equilibrium of the rent-seeking game is shown to be both payoff- and revenue-equivalent to the first-price all-pay auction. For general valuations, the analysis establishes a robustness property of the all-pay auction.

Keywords. Rent-seeking games, mixed-strategy Nash equilibrium, robustness of the all-pay auction, Tullock contest.

JEL-Codes. C72, D45, D72, L12.

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1. Introduction

The concept of rent-seeking (Tullock, 1967; Krueger, 1974; Bhagwati, 1982) has found wide-spread application in economics and political theory.[1] Two main classes of game-theoretic models have been distinguished in the literature.[2] In one class of models, the prize is allocated according to a stochastic success function, where a higher bid tends to increase a player's probability of winning—but typically not too strongly (Tullock, 1980; Pérez-Castrillo and Verdier, 1992; Nti, 1999; Cornes and Hartley, 2005). In another class of models, represented by the (first-price) all-pay auction, the prize is always awarded to the highest bidder (Hillman and Samet, 1987; Hillman and Riley, 1989; Baye et al., 1996).[3] These two classes of models have traditionally been analyzed separately and sometimes with different conclusions.[4]

The first paper that ventured into the no man's land between the two classes of models was Baye et al. (1994), who showed that the symmetric two-player Tullock contest with finite $R > 2$ allows a mixed-strategy Nash equilibrium in which both players have an expected payoff of zero. Che and Gale (2000) addressed similar issues in a somewhat different framework. Considering a success function of the difference form, they characterized two main classes of equilibria, and showed that, as the noise diminishes, equilibrium payoffs in the contest converge to the corresponding payoffs of the

[1]See, e.g., the collective volume by Congleton et al. (2008).

[2]These are sometimes referred to as imperfectly and perfectly discriminating contests (Hillman and Riley, 1989; Nitzan, 1994).

[3]Needless to say, the basic model of the all-pay auction has been extended in numerous ways. See, for instance, Clark and Rijs (1998), Che and Gale (1998), Konrad (2002), and Siegel (2009, 2010).

[4]For example, while in an all-pay auction, it may be optimal for a revenue-maximizing politician to exclude the lobbyist with the highest valuation (Baye et al., 1993), this is never the case for the lottery contest (Fang, 2002).

all-pay auction. For success functions in the tradition of the Tullock contest, however, Che and Gale (2000, p. 23) noted that it was not known if the qualitative properties of the mixed equilibrium in the all-pay auction (with heterogeneous valuations) would be similarly preserved if the success function is slightly perturbed.

Substantial progress in this regard has been made by Alcade and Dahm (2010). Specifically, they identified conditions under which a given rent-seeking game allows an *all-pay auction equilibrium* that, as the term indicates, shares important characteristics with an equilibrium of the corresponding all-pay auction.[5] The construction starts from a symmetric equilibrium with complete rent-dissipation in a two-player contest with homogeneous valuations, and exploits the fact that, with two players, introducing a mass point at the zero bid of one player is equivalent to a proportional reduction of the valuation of the other player. Since additional, low-valuation players have little incentive to enter the active contest, this indeed allows to construct an all-pay auction equilibrium in the rent-seeking game. As Alcade and Dahm (2010, p. 5) conclude, however, their results are partial without an improved understanding of the entire equilibrium set. As a matter of fact, one can show that additional, payoff-inequivalent equilibria do exist.[6]

In this paper, we consider rent-seeking games in which a small percentage change in a player's bid has a large percentage impact on her odds of winning, i.e., on the ratio between her respective probabilities of winning

[5] Provided that valuations are "sufficiently generic," i.e., provided that the all-pay auction has a unique equilibrium, an all-pay auction equilibrium is equivalent in terms of participation probabilities, average bid levels, winning probabilities, and expected payoffs. In general, the equivalence is required for a selected equilibrium of the all-pay auction.

[6] See Appendix B for an example.

and losing. Under this condition, it is shown that, for "sufficiently generic" valuations, actually any equilibrium of the rent-seeking game is an all-pay auction equilibrium and, thus, both payoff- and revenue-equivalent to the outcome of the corresponding all-pay auction. Moreover, for general valuations, we establish a robustness property of the all-pay auction that addresses the above-mentioned issues, in particular by making a statement about the entire equilibrium set.

The analysis proceeds as follows. We first show, by extending existing arguments, that a mixed-strategy Nash equilibrium always exists in a rent-seeking game with heterogeneous valuations. Then, the equilibrium set is examined with a focus on the minimum of the support of the players' bid distributions, as previously done by Baye at al. (1994) and Alcade and Dahm (2010) in discrete settings. In that part of the analysis, to accomplish the step from homogeneous to heterogeneous valuations, we essentially reverse the construction of Alcade and Dahm (2010). However, we also make use of a couple of new inequalities for winning probabilities when bids are close to each other. Finally, we re-use the tools developed mainly for the characterization of the equilibrium set to derive the robustness result.

The example of the n-player Tullock contest with heterogeneous valuations illustrates our findings.[7] First, for $n = 2$ and $R > 2$, any equilibrium is

[7] It might be useful to briefly recall what is known about the equilibrium set of the Tullock contest. For low values of the usual parameter R (see Section 6), pure-strategy Nash equilibria have been characterized through the players' first-order conditions (Pérez-Castrillo and Verdier, 1992; Nti, 1999; Cornes and Hartley, 2005). For high values of R, the equilibrium set is less completely explored. Baye et al. (1994) proved the existence of a symmetric equilibrium with complete rent dissipation in two-player contests with $R > 2$. Alcade and Dahm (2010) found a symmetric equilibrium in n-player contests with $R > 2$, as well as an all-pay auction equilibrium in n-player contests with heterogeneous valuations and $R > 2$. Finally, in a companion paper (2015), it is shown that any symmetric

shown to be an all-pay auction equilibrium. Second, for "sufficiently generic" valuations, and R sufficiently large, all but the two strongest players remain passive in any equilibrium, so that the equilibrium may be characterized just as in the case $n = 2$. Further, if R not sufficiently large, and if $n \geq 3$, then there may be multiple, payoff-inequivalent equilibria in which the preemptive rent need not go to the strongest player. Finally, for general valuations, any sequence of equilibria associated with a sequence of Tullock contests with $R \to \infty$ induces a sequence of payoff profiles that converges to the unique payoff profile of the corresponding all-pay auction.

The all-pay auction is known to be robust with respect to the introduction of private information. Indeed, as Amann and Leininger (1996) have shown, a symmetric two-player all-pay auction with independent types has a unique Bayesian equilibrium that converges to the complete information outcome as the distributions of valuations degenerate. A microfoundation of effort choice in the all-pay auction has been provided by Lang et al. (2014). Specifically, if individual production is a Poisson process for which the respective player chooses a stopping time, then the distribution of breakthroughs corresponds to the distribution of equilibrium efforts in the all-pay auction.

The remainder of this paper is structured as follows. Section 2 introduces the set-up. Section 3 establishes existence. Section 4 presents the equilibrium characterization. Section 5 discusses the robustness of the all-pay auction. An example is presented in Section 6. Section 7 concludes. Most proofs have been collected in Appendix A. Appendix B contains an extended example.

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equilibrium in an n-player contest with $R > n/(n-1)$ involves arbitrarily small positive bids and exhibits complete rent dissipation.

2. Set-up and notation

There are $n \geq 2$ players. Player $i = 1, ..., n$ values the prize of the contest at $V_i > 0$, and chooses a bid $b_i \geq 0$. Decisions are made simultaneously and independently. Without loss of generality, players are renamed such that $V_1 \geq V_2 \geq ... \geq V_n > 0$.

By a *contest success function* Ψ, or CSF, we mean a vector of functions $\Psi_i : \mathbb{R}^n_+ \to [0, 1]$, one for each player $i = 1, ..., n$, such that $\sum_{i=1}^n \Psi_i(b) = 1$ for any $b = (b_1, ..., b_n) \in \mathbb{R}^n_+$.[8] For convenience, we will use the notation $\Psi_i(b) = \Psi_i(b_i, b_{-i}) = \Psi_{i,j}(b_i, b_j, b_{-i,j})$, where $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$ and $b_{-i,j} = b_{-j,i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_{j-1}, b_{j+1}, ..., b_n)$ for $j \neq i$. The following assumptions will be imposed on the CSF:

Assumption 1. (Monotonicity) *$\Psi_i(b_i, b_{-i})$ is weakly increasing in b_i, for any $b_{-i} \in \mathbb{R}^{n-1}_+$, and any $i \in \{1, ..., n\}$; moreover, $\Psi_{i,j}(b_i, b_j, b_{-i,j})$ is weakly declining in b_j, for any $(b_i, b_{-i,j}) \in \mathbb{R}^{n-1}_+$, and any $i, j \in \{1, ..., n\}$ such that $i \neq j$.*

Assumption 2. (Zero bids) *$\Psi_i(0, b_{-i}) = 0$ for any $b_{-i} \neq 0$, and any $i \in \{1, ..., n\}$; moreover, $\Psi_i(b_i, b_{-i}) > 0$ for any $b_i > 0$, and any $i \in \{1, ..., n\}$.*

Assumption 3. (Anonymity) *$\Psi_i(b) = \Psi_{\varphi(i)}(b_{\varphi(1)}, ..., b_{\varphi(n)})$ for any permutation $\varphi : \{1, ..., n\} \to \{1, ..., n\}$, any $b \in \mathbb{R}^n_+$, and any $i \in \{1, ..., n\}$.*

Assumption 4. (Smoothness) *Ψ_i is continuous on $\mathbb{R}^n_+ \backslash \{0\}$, for any $i \in \{1, ..., n\}$; moreover, the partial derivative $\partial \Psi_i(b_i, b_{-i})/\partial b_i$ exists and is continuous in b_{-i}, for any $b_i > 0$, and any $i \in \{1, ..., n\}$.*

[8] The axiomatic approach to CSF has been pioneered by Skaperdas (1996).

Smoothness is imposed for convenience, and could probably be relaxed. For example, while the limit case of the all-pay auction does not satisfy Assumption 4, our equilibrium characterizations hold also in the limit (Baye et al., 1996). Assumptions 1-4 above will be used in the sequel without explicit mentioning.

Player i's payoff is given by $\Pi_i(b_i, b_{-i}) = \Psi_i(b_i, b_{-i})V_i - b_i$. Note that bids exceeding V_i are strictly dominated by a zero bid. Thus, without loss of generality, it may be assumed that each player $i = 1, ..., n$ chooses her bid from the compact and non-empty interval $B_i = [0, V_i]$. Given valuations $(V_1, ..., V_n)$, we will refer to the resulting non-cooperative game as the *n-player rent-seeking game* with CSF Ψ.

Following Dasgupta and Maskin (1986), a *mixed strategy* for player i is a probability measure μ_i on the Borel sets of B_i. We write $D(B_i)$ for the set of player i's mixed strategies, where pure strategies are contained as degenerate measures. The support of a mixed strategy $\mu_i \in D(B_i)$ will be denoted by $S(\mu_i)$, with $\underline{b}_i = \min S(\mu_i)$ and $\bar{b}_i = \max S(\mu_i)$. We refer to $\pi_i = 1 - \mu_i(\{0\})$ as player i's *probability of participation.* Player i will be called *passive* if $\pi_i = 0$, *active* if $\pi_i > 0$, and *always active* if $\pi_i = 1$. If player i is active, then $\underline{b}_i^+ = \inf S(\mu_i)\backslash\{0\}$ will be referred to as player i's *lowest positive bid.* If $\underline{b}_i^+ = 0$, then player i will be said to use *arbitrarily small positive bids.*

A *mixed equilibrium* is an n-tuple $\mu^* = (\mu_1^*, ..., \mu_n^*)$, with $\mu_i^* \in D(B_i)$, such that each player maximizes her ex-ante expected payoff, i.e., such that

$$E[\Pi_i(b_i, b_{-i})|\mu_i^*, \mu_{-i}^*] = \max_{\mu_i \in D(B_i)} E[\Pi_i(b_i, b_{-i})|\mu_i, \mu_{-i}^*], \tag{1}$$

for any $i = 1, ..., n$, where $\mu_{-i}^* = \prod_{j \neq i} \mu_j^*$ is the product measure. For a given equilibrium μ^*, denote by $p_i^* = E[\Psi_i(b_i, b_{-i})|\mu_i^*, \mu_{-i}^*]$ and $b_i^* = E[b_i|\mu_i^*]$, respectively, player i's ex-ante probability of winning, and player i's average bid level. Player i's equilibrium payoff, or *rent*, may then be written as $\Pi_i^* = p_i^* V_i - b_i^*$. Note that $\Pi_i^* = E[\Pi_i(b_i, b_{-i})|\mu_{-i}^*]$, for any $b_i \in S(\mu_i^*)\backslash\{0\}$.

3. Existence

Since any CSF satisfying Assumption 2 is discontinuous, Glicksberg's theorem does not apply. Existence of a mixed equilibrium is, therefore, shown below. Related results of existence not covering the present situation can be found in Baye et al. (1994), Yang (1994), and Alcade and Dahm (2010).

Lemma 1. *A mixed equilibrium μ^* exists in any n-player rent-seeking game with heterogeneous valuations.*

Proof. Assume that, instead of maximizing her payoff $\Pi_i(b_i, b_{-i})$, each player $i = 1, ..., n$ maximizes $U_i(b_i, b_{-i}) = \Pi_i(b_i, b_{-i})/V_i$. Clearly, this does not affect the equilibrium set. We check the conditions of Dasgupta and Maskin (1986, Th. 5 & fn. 14). Note first that the set of discontinuities of U_i is $A^{**}(i) = \{0\} \subseteq A^*(i)$, where $A^*(i) = \{(b_1, ..., b_n) \in \prod_{j=1}^n B_j : \exists j \neq i \text{ s.t. } b_j = b_i\}$. Moreover,

$$\sum_{i=1}^{n} U_i(b_i, b_{-i}) = 1 - \sum_{i=1}^{n} \frac{b_i}{V_i} \tag{2}$$

is continuous in $b = (b_1, ..., b_n)$. Next, $U_i(b_i, b_{-i})$ is bounded and continuous in b_i, for any $b_{-i} \neq 0$, and any $i = 1, ..., n$. Finally,

$$U_i(b_i, 0) = \begin{cases} 1 - b_i/V_i & \text{if } b_i > 0 \\ \\ 1/n & \text{if } b_i = 0 \end{cases} \tag{3}$$

is bounded and lower semi-continuous in b_i, for any $i = 1, ..., n$. The claim follows. $\square$

4. Equilibrium characterization

We start by introducing, on the set of CSFs, a measure of proximity to the technology of the all-pay auction. Following Che and Gale (1997), we think here of a proxy for the political culture or of a choice variable for a politician.

For a given CSF Ψ, the ratio $Q_i = \Psi_i/(1-\Psi_i)$ will be referred to as player i's *odds of winning*. At a bid vector $(b_i, b_{-i}) \in \mathbb{R}^n_+$, the own-bid elasticity of player i's odds of winning is $\rho_i = \partial \ln Q_i / \partial \ln b_i = b_i(\partial \Psi_i/\partial b_i)/\Psi_i(1 - \Psi_i)$. Note that ρ_i is well-defined if $b_i > 0$ and $b_{-i} \neq 0$. Define now the *decisiveness* $\rho = \rho(\Psi) = \inf\{\rho_i(b) : i \in \{1, ..., N\}, b \in \mathbb{R}^n_+ \text{ s.t. } b_i > 0, b_{-i} \neq 0\}$, i.e., as the joint infimum of the functions ρ_i over the relevant domains.

As an elasticity, the parameter ρ allows the usual graphical interpretation. For instance, in the limit case of the all-pay auction, the odds of winning would jump from zero to infinity at the highest competing bid, which corresponds to a perfectly elastic odds of winning. In the other extreme, a pure lottery, the odds of winning are constant, and hence, perfectly inelastic. We are interested here in the case where ρ is large, which means that a small percentage change in the bid has a large percentage impact on the odds of winning. In other words, there is little "noise" in the contest technology.[9]

Conditions on a rent-seeking game such that all of its equilibria share key characteristics with the corresponding equilibrium of the all-pay auction are

[9] Indeed, many commonly used CSF allow a microfoundation in terms of either uncertainty (Jia, 2008; Fu and Lu, 2012) or incomplete information (Eccles and Wegner, 2014).

provided below. We start with the case of two contestants, for which the equivalence is the strongest.

Proposition 1. *Consider a two-player rent-seeking game with CSF* Ψ *and valuations* $V_1 \geq V_2 > 0$*. Then, provided that* $\rho(\Psi) > 2$*, any mixed equilibrium* μ^* *satisfies the following two properties:*

(A1) Player 1 participates with probability $\pi_1 = 1$*, uses arbitrarily small positive bids, bids an average amount of* $b_1^* = V_2/2$*, wins with probability* $p_1^* = 1 - V_2/2V_1$*, and receives a rent of* $V_1 - V_2$*.*

(A2) Player 2 participates with probability $\pi_2 = V_2/V_1$*, uses arbitrarily small positive bids, bids an average amount of* $b_2^* = (V_2)^2/2V_1$*, wins with probability* $p_2^* = V_2/2V_1$*, and receives no rent.*

Proof. See the Appendix. □

The conclusions of Proposition 1 match those for the all-pay auction (Hillman and Riley, 1989, Prop. 2). In particular, if $n = 2$ and $\rho(\Psi) > 2$, then the rent-seeking game with CSF Ψ and the all-pay auction are equivalent both in terms of expected payoffs and in terms of expected revenue.

Proposition 1 strengthens the conclusions of Alcade and Dahm (2010) by showing that properties (A1-A2) are definitely not the result of a purposeful construction, but hold for any mixed equilibrium of the rent-seeking game. The result is obtained, in particular, by relying on a somewhat different set of assumptions.[10]

[10] Alcade and Dahm (2010) use two main assumptions, (E1) and (E2). Roughly speaking, (E1) imposes an inverse U-shape on the best-response mapping, whereas (E2) captures that success probabilities are sensitive to bids when bids are close. The present analysis drops (E1), yet strengthens (E2).

The idea of the proof is simple. If a CFS is very decisive, then any range of bids used with positive probability may be rationalized only if the opponent makes use of strictly smaller positive bids. Therefore, there is a sense in which the equilibrium unravels, and both players must use arbitrarily small positive bids. However, arbitrarily small positive bids can win with substantial probability only against a zero bid. Thus, either both players are always active and dissipate all rents, or there is one player that is not always active. Either way, there is at most one player with a positive rent. By reverting the construction in Alcade and Dahm (2010), it can then be shown that any rent accrues to player 1 and equals $V_1 - V_2$.

The analysis of the rent-seeking game complicates for $n \geq 3$ players. One reason for this is the well-known fact that the all-pay auction allows a continuum of equilibria if $V_2 = V_3$ (Baye et al., 1996). While those equilibria are all payoff-equivalent, they need not be equivalent in terms of revenue. In fact, in this case, equilibria may also differ in terms of lowest positive bids, participation probabilities, average bid levels, and expected win probabilities. The following result covers at least all those "sufficiently generic" valuation profiles for which the all-pay auction has a unique equilibrium.

Proposition 2. *Consider an n-player rent-seeking game with CSF Ψ and valuations $V_1 \geq V_2 > V_3 \geq ... \geq V_n > 0$. Then, provided that $\rho(\Psi)$ is sufficiently large, any mixed equilibrium μ^* satisfies properties (A1-A2). Moreover, players $3, ..., n$ remain passive.*

Proof. See the Appendix. □

Proposition 2 may be understood as a variation of the corresponding uniqueness result for the all-pay auction (Hillman and Riley, 1989, Prop. 4). The main point to prove is that there is no equilibrium in which any of the players $3, ..., n$ is active. This is accomplished by showing that if, say, player 3 was active, then players 1 and 2 could individually ensure a positive expected payoff by deviating to $\bar{b}_3$. But this would imply that more than one player earns a positive rent, which is impossible, as we show, even in n-player rent-seeking games with $n \geq 3$. Consistent with that intuition, the conclusions of Proposition 2 rely crucially on the assumption that ρ is sufficiently large, and therefore, as mentioned before, need not hold under more flexible conditions of Alcade and Dahm (2010).[11]

5. Robustness of the all-pay auction

We arrive at the promised robustness result. Considered is a setting with $n \geq 2$ players and arbitrary valuations.

Proposition 3. *Fix valuations $V_1 \geq V_2 \geq V_3 \geq ... \geq V_n > 0$. Consider a sequence of CSFs $\{\Psi^m\}_{m=1}^{\infty}$ with $\rho(\Psi^m) \to \infty$ as $m \to \infty$. Then, as $m \to \infty$, the profile of rents $\{(\Pi_1^m, ..., \Pi_n^m)\}_{m=1}^{\infty}$ resulting from any corresponding sequence of mixed equilibria $\{\mu^m\}_{m=1}^{\infty}$ converges to the equilibrium payoff profile $(V_1 - V_2, 0, ..., 0)$ of the all-pay auction.*

Proof. See the Appendix. $\square$

Thus, a small change in the technology of the all-pay auction does not affect the equilibrium set very much. For generic valuations, the limit is reached

[11] The present analysis does not require a characterization of the equilibrium set for the case $V_2 = V_3$. We conjecture, however, that payoff-equivalence breaks down for asymmetric equilibria with more than two active players.

already for a finite m. For non-generic payoffs, however, taking the limit seems necessary.

6. Example

Following Tullock (1980), suppose that

$$\Psi_i(b_i, b_{-i}) = \begin{cases} b_i^R / \sum_{j=1}^n b_j^R & \text{if } \sum_{j=1}^n b_j > 0 \\ \\ 1/n & \text{if } \sum_{j=1}^n b_j = 0, \end{cases} \tag{4}$$

where $R > 0$. One can easily check that $\rho(\Psi) = R$.[12] Indeed,

$$\begin{aligned} \rho_i(b_i, b_{-i}) &= \frac{b_i}{\Psi_i(b_i, b_{-i})(1 - \Psi_i(b_i, b_{-i}))} \frac{\partial \Psi_i(b_i, b_{-i})}{\partial b_i} && (5) \\ &= \frac{b_i(\sum_{j=1}^n b_j^R)^2}{b_i^R(\sum_{j \neq i} b_j^R)} \frac{R b_i^{R-1}(\sum_{j \neq i} b_j^R)}{(\sum_{j=1}^n b_j^R)^2} && (6) \\ &= R, && (7) \end{aligned}$$

which implies the claim. Thus, as outlined in the Introduction, immediate variants of the results above hold, in particular, for the Tullock contest.[13]

7. Conclusion

This paper has continued the study of the relationship between rent-seeking games and the all-pay auction. The own-bid elasticity of a player's odds of winning has been employed as an intuitive measure of the proximity between a rent-seeking game and the all-pay auction. Conditions have been provided under which any mixed-strategy Nash equilibrium of an n-player rent-seeking game is both payoff- and revenue-equivalent to the outcome of the corresponding all-pay auction. Moreover, it was shown that, as the rent-seeking

[12] Cf. also Wang (2010, fn. 4).

[13] Mixed equilibria of the Tullock contest are characterized more explicitly in a companion paper (2015).

game approaches the all-pay auction, the payoff profile of any equilibrium converges to the payoff profile of the corresponding all-pay auction.

The analysis has also provided a general existence result for mixed equilibria in rent-seeking games with heterogeneous valuations, and has revealed some potentially important properties of the equilibrium set of the asymmetric n-player Tullock contest with large R.[14]

Our findings have immediate implications for policy work such as Ellingsen (1991). For example, even if the political contest is not perfectly discriminating, costly lobbying by the demand side will unambiguously increase welfare. Furthermore, excluding the possibility of alternative, say collusive, equilibria is obviously important for the reliability of empirical estimates of rent-seeking expenditures.[15] Last but not least, the analysis has shown that the exclusion principle of Baye et al. (1993) extends quite robustly to a large class of rent-seeking games.

Applications are not restricted to static settings. By backwards induction, decisions in any stage depend solely on expected payoffs in later stages. Thus, revenue equivalence applies to pairwise elimination tournaments (Rosen, 1986; Groh et al., 2012) and implies, e.g., that optimal seedings do not change when some noise enters the respective success functions for semifinals and finals. Analogous conclusions may be drawn for sequential contests (e.g., Konrad, 2004; Klumpp and Polborn, 2006; Konrad and Kovenock; 2009; Sela, 2012).

[14] E.g., our results might prove useful for verifying or rejecting the conjecture of Franke et al. (2014, p. 125), which says that optimally biased n-player Tullock contests with sufficiently large R extract a revenue of $(V_1 + V_2)/2$.

[15] Cf. the survey by Del Rosal (2011, Sec. 2.1).

Appendix A: Proofs

This appendix contains proofs of Propositions 1 through 3. We prepare those proofs with altogether six lemmas.

Lemma A.1. *Consider an n-player rent-seeking game. Then, in any mixed equilibrium μ^*, there are players $i \neq j$ such that $\pi_i = 1$ and $\pi_j > 0$.*

Proof. Suppose that $\pi_i < 1$ for $i = 1, ..., n$. Then player 1, say, could raise her zero bids to some small $\varepsilon > 0$, thereby increasing the probability to win against coincident zero bids of players $2, ..., n$ from $\Psi_1(0, ..., 0) = 1/n$ to $\Psi_1(\varepsilon, 0, ..., 0) = 1$. Further, if $\pi_j = 0$ for any $j \neq i$, then player i could profitably shade her positive bids. $\square$

Lemma A.2. *Consider an n-player rent-seeking game with $\rho(\Psi) > 2$. Then, in any mixed equilibrium μ^* with $\pi_i = 1$, it holds that, (i) $\underline{b}_j = 0$ for any $j \neq i$, and (ii) $\Pi_j^* = 0$ for any $j \neq i$.*

Proof. (i) Suppose that $\underline{b}_j > 0$ for some $j \neq i$. The main case to consider is $\underline{b}_j > \underline{b}_i > 0$. In this case, $b_j \in S(\mu_j^*)$ implies $\Psi_{i,j}(\underline{b}_i, b_j, b_{-i,j}) \leq 1/2$ for any $b_{-i,j} \in \mathbb{R}_+^{n-2}$, so that with $b_{-i} = (b_j, b_{-i,j})$ and $\rho_i(\underline{b}_i, b_{-i}) > 2$, one obtains $(1 - \Psi_i(\underline{b}_i, b_{-i}))\rho_i(\underline{b}_i, b_{-i}) > 1$, or equivalently, $\partial\Psi_i(\underline{b}_i, b_{-i})/\partial b_i > \Psi_i(\underline{b}_i, b_{-i})/\underline{b}_i$. Hence, using that $\partial\Pi_i(\underline{b}_i, b_{-i})/\partial b_i$ is continuous in b_{-i},

$$\left.\frac{\partial E\left[\Pi_i(b_i, b_{-i})|\, \mu_{-i}^*\right]}{\partial b_i}\right|_{b_i = \underline{b}_i} = E\left[\frac{\partial\Psi_i(\underline{b}_i, b_{-i})}{\partial b_i}V_i - 1 \middle|\, \mu_{-i}^*\right] \tag{8}$$

$$> \frac{1}{\underline{b}_i}E\left[\Psi_i(\underline{b}_i, b_{-i})V_i - \underline{b}_i|\, \mu_{-i}^*\right]. \tag{9}$$

However, since $\underline{b}_i > 0$ is an optimum, the left-hand side of (8) vanishes, whereas the expected payoff in (9) is nonnegative, delivering a contradiction.

Thus, $\underline{b}_j = 0$, as claimed. The case where $\underline{b}_i > \underline{b}_j > 0$ follows by symmetry. Finally, if $\underline{b}_i = 0$ then, because $\pi_i = 1$, we have $\underline{b}_i^+ = 0$. Hence, we find some $b_i^+ \in S(\mu_i^*)$ with $\underline{b}_j > b_i^+ > 0$. The proof now proceeds as in the main case, with $\underline{b}_i$ replaced by b_i^+.

(ii) Clearly, $\Pi_j^* = 0$ if $\pi_j < 1$. Assume, therefore, that $\pi_j = 1$. Since $\underline{b}_j = 0$ by part (i), this implies $\underline{b}_j^+ = 0$. Hence, there is a sequence $\{b_j^{(\nu)}\}_{\nu=1}^{\infty}$ in $S(\mu_j^*)\backslash\{0\}$ such that $\lim_{\nu\to\infty} b_j^{(\nu)} = 0$. Clearly, for any ν, we have $\Pi_j^* = E[\Pi_j(b_j^{(\nu)}, b_{-j})\,|\mu_{-j}^*]$. Using Lebesgue's theorem, this implies

$$\Pi_j^* = \lim_{\nu\to\infty} E\left[\Pi_j(b_j^{(\nu)}, b_{-j})\middle|\, \mu_{-j}^*\right] = E\left[\lim_{\nu\to\infty} \Pi_j(b_j^{(\nu)}, b_{-j})\middle|\, \mu_{-j}^*\right]. \tag{10}$$

If $b_{-j} \neq 0$, then $\lim_{\nu\to\infty} \Pi_j(b_j^{(\nu)}, b_{-j}) = \Pi_j(\lim_{\nu\to\infty} b_j^{(\nu)}, b_{-j}) = \Pi_j(0, b_{-j}) = 0$. If, however, $b_{-j} = 0$, then $\lim_{\nu\to\infty} \Pi_j(b_j^{(\nu)}, 0) = \lim_{\nu\to\infty} V_j - b_j^{(\nu)} = V_j$. It follows that $\Pi_j^* = V_j \cdot \prod_{k\neq j}(1 - \pi_k) = 0$. Thus, from $\pi_i = 1$ and $i \neq j$, one obtains $\Pi_j^* = 0$, as claimed. $\square$

Lemma A.3. *Consider a mixed equilibrium (μ_1^*, μ_2^*) in a two-player rent-seeking game such that $\Pi_1^* = \Pi_2^* = 0$. Then, $V_1 = V_2$ and $p_1^* = p_2^* = \frac{1}{2}$.*

Proof. Since $\Pi_2^* = 0$, a deviation by player 2 to player 1's equilibrium strategy μ_1^* would necessarily yield weakly negative expected payoffs for player 2, i.e., $V_2/2 - b_1^* \leq 0$. Combining this with $\Pi_1^* = p_1^* V_1 - b_1^* = 0$ yields

$$p_1^* V_1 \geq \frac{V_2}{2}. \tag{11}$$

By symmetry,

$$p_2^* V_2 \geq \frac{V_1}{2}. \tag{12}$$

Summing up yields

$$p_1^* V_1 + p_2^* V_2 \geq \frac{V_1 + V_2}{2}, \tag{13}$$

with $p_1^* + p_2^* = 1$. If $V_1 > V_2$, then (13) implies $p_2^* \leq 1/2$, so that, using (11), $V_1 \leq V_2$, a contradiction. Similarly, $V_1 < V_2$ leads to a contradiction. Hence, $V_1 = V_2$, and (13) is an equality. But then, (11) and (12) must be equalities as well. Thus, also $p_1^* = p_2^* = \frac{1}{2}$. $\square$

Lemma A.4. $\Psi_{i,j}(b_i, 0, b_{-i,j}) \leq 2\Psi_{i,j}(b_i, b_i, b_{-i,j})$, *for any* $(b_i, b_{-i,j}) \in \mathbb{R}_+^{n-2}$.

Proof. Since winning probabilities across players sum up to one,

$$\Psi_{i,j}(b_i, b_j, b_{-i,j}) + \Psi_{j,i}(b_j, b_i, b_{-j,i}) + \sum_{k \neq i,j} \Psi_{k,j}(b_k, b_j, b_{-k,j}) = 1, \tag{14}$$

for any $b \in \mathbb{R}_+^n$. Evaluating at $b_j = b_i$, one obtains

$$2\Psi_{i,j}(b_i, b_i, b_{-i,j}) = 1 - \sum_{k \neq i,j} \Psi_{k,j}(b_k, b_i, b_{-k,j}) \tag{15}$$

$$\geq 1 - \sum_{k \neq i,j} \Psi_{k,j}(b_k, 0, b_{-k,j}) \tag{16}$$

$$\geq 1 - \Psi_{j,i}(0, b_i, b_{-j,i}) - \sum_{k \neq i,j} \Psi_{k,j}(b_k, 0, b_{-k,j}) \tag{17}$$

Using (14) with $b_j = 0$, the lemma follows. $\square$

Lemma A.5. *Fix* $\kappa > 1$. *Then, provided that* $\rho(\Psi)$ *is sufficiently large,* $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) > \Psi_{i,j}(b_j, 0, b_{-i,j})/\kappa$ *for any* $b_j > 0$ *and any* $b_{-i,j} \in \mathbb{R}_+^{n-2}$.

Proof. Suppose that $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) \leq \Psi_{i,j}(b_j, 0, b_{-i,j})/\kappa$ for some $b_j > 0$ and some $b_{-i,j} \in \mathbb{R}_+^{n-2}$. Then, obviously, $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) \leq 1/\kappa$, and hence, $\Psi_{i,j}(\tilde{\kappa} b_j, b_j, b_{-i,j}) \leq 1/\tilde{\kappa}$ for any $\tilde{\kappa} \in [1, \kappa]$. Writing $b_{-i} = (b_j, b_{-i,j})$, this implies

$$\frac{\partial \ln \Psi_i(\tilde{\kappa} b_j, b_{-i})}{\partial \ln \tilde{\kappa}} = \frac{\tilde{\kappa}}{\Psi_i(\tilde{\kappa} b_j, b_{-i})} \cdot \frac{\partial \Psi_i(\tilde{\kappa} b_j, b_{-i})}{\partial \tilde{\kappa}} \tag{18}$$

$$= (1 - \Psi_i(\tilde{\kappa} b_j, b_{-i}))\rho_i(\tilde{\kappa} b_j, b_{-i}) \tag{19}$$

$$\geq (1 - \frac{1}{\tilde{\kappa}})\,\rho \tag{20}$$

for any $\tilde{\kappa} \in [1, \kappa]$. Consequently, using the fundamental theorem of calculus,

$$\ln\left(\frac{\Psi_i(\kappa b_j, b_{-i})}{\Psi_i(b_j, b_{-i})}\right) = \int_1^{\kappa} \frac{\partial \ln \Psi_i(\tilde{\kappa} b_j, b_{-i})}{\tilde{\kappa}\,\partial \ln \tilde{\kappa}}\, d\tilde{\kappa} \geq (\ln \kappa + \frac{1}{\kappa} - 1)\,\rho. \tag{21}$$

But, $\ln \kappa + \frac{1}{\kappa} - 1 > 0$. Hence, for ρ sufficiently large, $(\ln \kappa + \frac{1}{\kappa} - 1)\rho > \ln(2/\kappa)$, and thus, $\Psi_i(\kappa b_j, b_{-i}) > (2/\kappa)\Psi_{i,j}(b_j, b_j, b_{-i,j})$. Applying Lemma A.4 leads to a contradiction. The lemma follows. □

Lemma A.6. *Fix $V_i > V_j$, for some $i \neq j$. Then, in any n-player rent-seeking game with $\rho(\Psi)$ sufficiently large, $\pi_j > 0$ implies $\Pi_i^* > 0$.*

Proof. Note that $\pi_j > 0$ ensures $\bar{b}_j > 0$. Since a zero bid guarantees a nonnegative expected payoff, this implies $E\left[\Psi_j(\bar{b}_j, b_{-j}) \middle| \mu_{-j}^*\right] V_j - \bar{b}_j \geq 0$. Multiplying through with $\kappa \equiv \sqrt{V_i/V_j} > 1$ delivers

$$E\left[\frac{\Psi_j(\bar{b}_j, b_{-j})}{\kappa} \middle| \mu_{-j}^*\right] V_i - \kappa \bar{b}_j \geq 0. \tag{22}$$

Moreover, choosing ρ sufficiently large, Lemma A.5 implies that for any $b_{-j} = (b_i, b_{-j,i}) \in \mathbb{R}_+^{n-2}$ and any $b_j \in S(\mu_j^*)$,

$$\frac{\Psi_j(\bar{b}_j, b_{-j})}{\kappa} \leq \frac{\Psi_{j,i}(\bar{b}_j, 0, b_{-j,i})}{\kappa} \tag{23}$$

$$= \frac{\Psi_{i,j}(\bar{b}_j, 0, b_{-i,j})}{\kappa} \tag{24}$$

$$< \Psi_{i,j}(\kappa \bar{b}_j, \bar{b}_j, b_{-i,j}) \tag{25}$$

$$\leq \Psi_{i,j}(\kappa \bar{b}_j, b_j, b_{-i,j}). \tag{26}$$

Combining this with (22) yields $E\left[\Psi_{i,j}(\kappa\bar{b}_j, b_j, b_{-i,j}) \middle| \mu^*_{-j}\right] V_i - \kappa\bar{b}_j > 0$. Taking expectations with respect to μ^*_j, and noting that $\Psi_{i,j}(\kappa\bar{b}_j, b_j, b_{-i,j})$ does not depend on b_i, one arrives at $E\left[\Psi_i(\kappa\bar{b}_j, b_{-i}) \middle| \mu^*_{-i}\right] V_i - \kappa\bar{b}_j > 0$. I.e., player i's expected payoff from choosing $b_i = \kappa\bar{b}_j$ is positive. Thus, $\Pi^*_i > 0$. $\square$

Proof of Proposition 1. By Lemma A.1, there are players $i, j \in \{1, 2\}$ with $i \neq j$ such that $\pi_i = 1$ and $\pi_j > 0$. Consider the *modified contest* in which player i's valuation is scaled down to $V'_i \equiv V_i \cdot \pi_j > 0$. Define a mixed strategy $\widehat{\mu}_j$ in the modified contest via $\widehat{\mu}_j(Y_j) = \mu^*_j(Y_j \backslash \{0\})/\pi_j$, for any Borel set $Y_j \subseteq B_j$. It is straightforward to check that $(\mu^*_i, \widehat{\mu}_j)$ is a mixed equilibrium in the modified contest such that both players are always active. By Lemma A.2, there is complete rent dissipation in the modified contest for both players. Lemma A.3 implies $V'_i = V_j$. Hence, $\pi_j = V_j/V_i$, showing that $i = 1$ and $j = 2$. Moreover, since each player wins with probability 1/2 in the modified contest, player 1 realizes a rent of $(1 - \pi_2)V_1 = V_1 - V_2$ in the original contest. The remaining claims are now immediate. $\square$

Proof of Proposition 2. Suppose that there is a mixed equilibrium μ^* in which some player $j \in \{3, ..., n\}$ is active. Since $V_1 \geq V_2 > V_3 \geq V_j$, Lemma A.6 implies that $\Pi^*_1 > 0$ and $\Pi^*_2 > 0$. But then, $\pi_1 = \pi_2 = 1$, which is impossible in view of Lemma A.2. $\square$

Proof of Proposition 3. The proof has three parts.

Part I: $\limsup_{m\to\infty} \Pi^m_1 \geq V_1 - V_2$. The claim is obvious for $V_1 = V_2$. Assume, therefore, that $V_1 > V_2$. Suppose that player 1 chooses $\hat{b}_1 = V_2$ in the rent-seeking game with CSF Ψ^m. Then, since players $2, ..., n$ never bid above V_2, it follows that $\Psi_1^m(\hat{b}_1, b_{-1}) \geq 1/n$ or, equivalently, $\Psi_1^m(\hat{b}_1, b_{-1})/(1 - \Psi_1^m(\hat{b}_1, b_{-1})) \geq 1/(n-1)$, for any $b_{-1} \in \prod_{j \neq 1} S(\mu_j^m)$. Moreover, if player 1 raises her bid to $\tilde{b}_1 = (1+\varepsilon)\hat{b}_1$, for some small $\varepsilon > 0$, then

$$\frac{\Psi_1^m(\tilde{b}_1, b_{-1})}{1 - \Psi_1^m(\tilde{b}_1, b_{-1})} \geq (1+\varepsilon)^{\rho(\Psi^m)} \cdot \frac{1}{n-1}, \tag{27}$$

where the right-hand side grows above all finite bounds as $m \to \infty$. Therefore, for m sufficiently large, $\Psi_1^m(\tilde{b}_1, b_{-1}) \geq 1 - \varepsilon$ for any $b_{-1} \in \prod_{j \neq 1} S(\mu_j^m)$. It follows that, for any fixed $\varepsilon > 0$, a sufficiently large choice of m ensures that $\Pi_1^m \geq (1-\varepsilon)V_1 - (1+\varepsilon)V_2 = V_1 - V_2 - \varepsilon(V_1 + V_2)$. Thus, $\limsup_{m\to\infty} \Pi_1^m \geq V_1 - V_2$, as claimed.

Part II: $\lim_{m\to\infty} \Pi_1^m = V_1 - V_2$. Suppose that $\Pi_1^m \geq V_1 - V_2 + \delta$ for some $\delta > 0$ and some m. Then, $\bar{b}_1^m \in S(\mu_1^m)\backslash\{0\}$ and, hence, $\Pi_1^m = E[\Psi_1^m(\bar{b}_1^m, b_{-1})|\mu_{-1}^m]V_1 - \bar{b}_1^m$. It follows that

$$E[\Psi_1^m(\bar{b}_1^m, b_{-1})|\mu_{-1}^m]V_2 - \bar{b}_1^m \geq \delta. \tag{28}$$

Fix some $\kappa > 1$. Then, provided that m is sufficiently large, Lemma A.5 implies that, for any $b_{-1} = (b_2, b_{-1,2}) \in \mathbb{R}_+^{n-1}$ and any $b_1 \in S(\mu_1^m)$,

$$\Psi_1^m(\bar{b}_1^m, b_{-1}) \leq \Psi_{1,2}^m(\bar{b}_1^m, 0, b_{-1,2}) \tag{29}$$

$$= \Psi_{2,1}^m(\bar{b}_1^m, 0, b_{-2,1}) \tag{30}$$

$$< \kappa\Psi_{2,1}^m(\kappa\bar{b}_1^m, \bar{b}_1^m, b_{-2,1}) \tag{31}$$

$$\leq \kappa\Psi_{2,1}^m(\kappa\bar{b}_1^m, b_1, b_{-2,1}). \tag{32}$$

Combining this with (28), taking expectations with respect to μ_1^m, and noting that $\Psi_{2,1}^m(\kappa\bar{b}_1^m, b_1, b_{-2,1})$ does not depend on b_2, one arrives at

$$\kappa E[\Psi_2^m(\kappa\bar{b}_1^m, b_{-2})|\mu_{-2}^m]V_2 - \bar{b}_1^m \geq \delta. \tag{33}$$

Dividing (33) by κ, and choosing κ sufficiently close to one, ensures that

$$E[\Psi_2^m(\kappa\bar{b}_1^m, b_{-2})|\mu_{-2}^m]V_2 - \kappa\bar{b}_1^m \geq \frac{\delta}{\kappa} - (\kappa - \frac{1}{\kappa})\bar{b}_1^m > 0. \tag{34}$$

But then $\Pi_2^m > 0$ and, hence, $\pi_2^m = 1$. Lemma A.2 implies $\Pi_1^m = 0$, a contradiction. Thus, $\liminf_{m\to\infty}\Pi_1^m \leq V_1 - V_2$, and the claim follows from part I.

Part III: $\lim_{m\to\infty}\Pi_j^m = 0$ *for any* $j \neq 1$*.* There are two cases. If $V_j = V_1$ then, after a renaming of the players, part II delivers $\lim_{m\to\infty}\Pi_j^m = V_j - V_1 = 0$. If $V_j < V_1$, however, then $\Pi_j^m > 0$ would imply $\pi_j^m = 1$. But then, $\Pi_1^* = 0$ by Lemma A.2, and $\Pi_1^* > 0$ by Lemma A.6, a contradiction. This proves the final claim, and hence, the proposition. □

Appendix B

In this appendix, it is shown that, under the assumptions of Alcade and Dahm (2010), a rent-seeking game may possess a mixed equilibrium that is not an all-pay auction equilibrium. For this, consider a three-player Tullock contest with $V_1 > V_2 > V_3 > 0$ and $R > 2$. Then, there exists an all-pay auction equilibrium (μ_2^*, μ_3^*) in the two-player contest between players 2 and 3. Suppose that player 1 intends to enter the active contest with a bid $b_1 > 0$. If player 1 were to face only player 2, then player 1 would win with probability $p_{12} = E\left[b_1^R/(b_1^R + b_2^R)\,\middle|\,\mu_2^*\right] > 0$, and have an expected payoff of $\Pi_{12}^* = p_{12}V_1 - b_1$. We claim that $\Pi_{12}^* \leq p_{12}(V_1 - V_3)$. Suppose not. Then, in the two-player contest between 2 and 3, player 3's expected payoff from submitting a bid equal to b_1 (instead of playing μ_3^*) would amount to $p_{12}V_3 - b_1 = p_{12}V_1 - b_1 - p_{12}(V_1 - V_3) > 0$, contradicting the fact that (μ_2^*, μ_3^*)

is an all-pay auction equilibrium. Thus, indeed, $\Pi_{12}^* \leq p_{12}(V_1 - V_3)$. Note next that player 1's probability of winning, once μ_3^* is taken into account, is lowered by

$$
\begin{aligned}
&E\left[\frac{b_1^R}{b_1^R + b_2^R} - \frac{b_1^R}{b_1^R + b_2^R + b_3^R} \,\middle|\, \mu_2^* \times \mu_3^* \right] \\
&= E\left[\frac{b_1^R}{b_1^R + b_2^R}(1 - \frac{b_1^R + b_2^R}{b_1^R + b_2^R + b_3^R}) \,\middle|\, \mu_2^* \times \mu_3^* \right] && (35) \\
&= E\left[\frac{b_1^R}{b_1^R + b_2^R} E\left[\frac{b_3^R}{b_1^R + b_2^R + b_3^R} \,\middle|\, \mu_3^* \right] \,\middle|\, \mu_2^* \right] && (36) \\
&\geq E\left[\frac{b_1^R}{b_1^R + b_2^R} \,\middle|\, \mu_2^* \right] \cdot \frac{E\left[\, b_3^R \,\middle|\, \mu_3^* \right]}{3V_1^R} && (37) \\
&\geq p_{12} \cdot \frac{(E\left[\, b_3 \mid \mu_3^* \right])^R}{3V_1^R} && (38) \\
&= p_{12} \cdot \frac{1}{3V_1^R} \cdot (\frac{V_3^2}{2V_2})^R && (39) \\
&> p_{12} \cdot \frac{1}{2^{R+2}} \cdot (\frac{V_3}{V_1})^{2R}, && (40)
\end{aligned}
$$

where Jensen's inequality has been used. Therefore, player 1's expected payoff from bidding b_1 in the three-player contest against μ_2^* and μ_3^* is

$$
E\left[\frac{b_1^R V_1}{b_1^R + b_2^R + b_3^R} - b_1 \,\middle|\, \mu_2^* \times \mu_3^* \right] < p_{12} \cdot \left\{ V_1 - V_3 - \frac{V_1}{2^{R+2}} \cdot (\frac{V_3}{V_1})^{2R} \right\}. \quad (41)
$$

If V_1 and V_3 are sufficiently close to each other, so that $V_1 - V_3 < V_1/2^{R+3}$ and $(V_3/V_1)^{2R} > 1/2$, then the right-hand side of (41) is easily seen to be negative. Thus, with μ_1^* giving all weight to the zero bid, $(\mu_1^*, \mu_2^*, \mu_3^*)$ is an equilibrium in the three-payer contest with $\Pi_1^* = 0, \Pi_2^* = V_2 - V_3$, and $\Pi_3^* = 0$.

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