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Awareness of Unawareness: A Theory of Decision Making in the Face of Ignorance

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Awareness of Unawareness: A Theory of Decision Making in the Face of Ignorance

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Abstract

In the wake of growing awareness, decision makers anticipate that they might become aware of material possibilities and ideas that, in their current state of ignorance, are unimaginable. This anticipation manifests itself in their choice behavior. This paper models this awareness of unawareness and axiomatizes a probabilistic sophisticated representation of beliefs about ignorance and subjective expected utility representation, in an enriched framework, that assigns utility to the unknown while maintaining, in both instances, the flavor of reverse Bayesianism of Karni and Vierø (2013, 2014).

Keywords: Awareness, unawareness, ignorance, reverse Bayesianism

JEL classification: D8, D81, D83

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1 Introduction

Habituation to technologies and ideas that, prior to their discovery, were unimaginable or, in some instances, for lack of appropriate language, indescribable, is an important aspect of human experience. The anticipation of additional such discoveries shapes our future outlook and manifests itself in our choice behavior.

On the more mundane level, decision makers might have experienced situations in which, because of lack of imagination or lack of attention, their decisions resulted in outcomes that they failed to foresee and take into account. Therefore, when making decisions, one might suspect that one might fail to take into account all the potential consequences. Awareness of one's limited awareness of the consequences of one's actions affects individual choice behavior in a way similar to the anticipation of discoveries of new consequences.

In this paper, which builds upon Karni and Vierø (2013, 2014), we propose a dynamic, choice-based, theory designed to capture decision makers anticipation of becoming aware of consequences which they are currently unaware of, because they are unimaginable, are not explicitly specifiable, or for lack of attention have been neglected, and analyze its behavioral implications.

The main thrust of Karni and Vierø (2013, 2014) is the evolution of decision makers' beliefs as they become aware of new acts, consequences, and the links among them. In these models, however, decision makers are myopic, believing at every point that they are fully aware of the scope of their universe. Formally, decision makers consider the state space that resolves the uncertainty associated with the alternative feasible courses of action and consequences that they can describe, to be a sure event. Put differently, even though it happened before, decision makers do not anticipate the possibility of discoveries that would change their conception of the universe and require an expansion of the state space. In this paper, we extend the analytical framework to incorporate decision makers' awareness of their potential ignorance and the anticipation that new discoveries may reveal consequences that were unspecified in the formulation of the decision problem either because they are unimaginable or were neglected for lack of attention. Within the extended analytical framework we develop axiomatic models of choice under uncertainty and analyze the behavioral implications of a decision maker's awareness of his unawareness, including the evolution of his beliefs about his ignorance in the wake of discovery of new consequences.

Depending on the nature of the discoveries, the sense of ignorance, or the ‘residual’ unawareness, may shrink, grow, or remain constant. For instance, as unknown and unsuspected regions of the Earth or the solar system were discovered (or rediscovered), fewer regions remained to be discovered, and the sense of ignorance diminished. By contrast, some scientific discoveries, such as relativity, atoms, or the structure of the DNA, resolved certain outstanding issues in physics and biology and, at the same time, opened up new, unsuspected, vistas. These discoveries enhanced the sense that our ignorance is, in fact, greater than what was perceived before these discoveries were made. Our model is designed to accommodate both types of growing awareness.

The sense that there might be consequences, lurking in the background, of which one is unaware may inspire fear or excitement and affect individual choice behavior. In this work we present a model that assigns utility to the unknown that captures this aspect of the decision problem. The “utility of unknown consequences” represents the attitudes of the decision maker towards discovering unknown consequences and the emotions it evokes.

If fear is the predominant emotion evoked by the unknown, then confidence that one is unlikely to encounter unknown consequences would beget boldness of action while the lack of it would induce more prudent behavior. If acting in an environment, be it physical, cultural, legal or political, that one is accustomed to is accompanied by greater degree of such confidence, lack of familiarity with foreign cultural, legal, or political landscapes may contribute to the reported domestic bias and ‘insufficient’ diversification in domestic and foreign financial investments.

In the next section we present the analytical framework and the basic preference structure. In section 3 we introduce axioms that provide links among distinct levels of unawareness and present the representation theorem assigning probabilistic beliefs to making new discoveries and providing rules for updating beliefs. We also present the axiomatic structure depicting shrinking and growing sense of ignorance. In section 4 we enrich the framework and introduce a subjective expected utility model with utility of the unknown. Concluding remarks and a brief review of the related literature appear in section 5. The proofs are provided in Section 6.

2 The Analytical Framework

In our earlier work (see Karni and Vierø [2013, 2014]), dubbed ‘reverse Bayesianism,’ we modeled and analyzed the evolution of a decision maker’s beliefs when his universe, formalized as a state space, expands. In these works, the state space expands as a result of discoveries of new acts and/or consequences. There is, however, a fundamental difference between discoveries of acts and consequences. Discovery of new acts, such as the introduction of derivatives in financial markets, the introduction of new means of transportation by constructing jet-propelled airplanes, or the generation of electricity using nuclear energy, are the result of innovative designs. By contrast, the discovery of new consequences, such as new diseases, (e.g., the discovery of syphilis by the Europeans), the beneficial effects of Penicillium fungi (penicillin) in fighting certain bacterial infections, or the depletion of the ozone layer by photodissociation of man-made halocarbon refrigerants, is arrived at coincidentally, through observation and/or scientific experimentation. Insofar as this paper is concerned, the crucial difference between the two types of discoveries is that the discovery of new acts refine the existing state space while the discovery of new consequences expands the state space. Put differently, when a new feasible act is designed, each element of the prior state space (the state space that existed prior to the introduction of the new act) becomes a non-degenerate event in the posterior state space (the state space following the introduction of the new act). By contrast, when a new consequence is discovered the prior state space is augmented as additional states come into being. Thus, unlike the discovery or invention of new feasible acts whose effect on the structure of the state space is fully anticipated, the discovery of new consequences uncovers elements of the state space that were unimaginable, or indescribable, in the formulation of the decision problem.

In this work we investigate behavioral and cognitive aspects of awareness of unawareness. This investigation focuses on the effects of the anticipation and discovery of new consequences on decision makers’ awareness of their ignorance.

2.1 Conceivable states and acts

Let F be a finite, nonempty set of *feasible acts*, and C_0 be a finite, nonempty set of *feasible consequences*. We define $x_0 = \neg C_0$ to be the abstract “consequence” that has the

interpretation “none of the above”.¹ Let $\hat{C}_0 = C_0 \cup \{x_0\}$. Together these sets determine the *augmented conceivable state space*, \hat{C}_0^F , which is, by definition, exhaustive.² They also determine the subset of *fully describable states*, C_0^F . As an illustration, let there be two feasible acts, $F = \{f_1, f_2\}$, and two feasible consequences, $C_0 = \{c_1, c_2\}$. The resulting augmented conceivable state space consists of nine states as depicted in the following matrix:

$$\begin{array}{rcccccccccc}
 F \setminus \hat{C}_0^{F_0} & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\
 f_1 & c_1 & c_2 & c_1 & c_2 & x_0 & x_0 & c_1 & c_2 & x_0 \\
 f_2 & c_1 & c_1 & c_2 & c_2 & c_1 & c_2 & x_0 & x_0 & x_0
 \end{array} \tag{1}$$

The subset of fully describable states in this example is $C_0^F = \{s_1, \dots, s_4\}$.

The set of *conceivable acts*, consists of all the mappings from the augmented conceivable state space to lotteries on the set of feasible consequences.³ Formally,

$$\hat{F}_0 := \{f : \hat{C}_0^F \rightarrow \Delta(C_0)\}, \tag{2}$$

where $\Delta(C_0)$ is the set of all lotteries with consequences in C_0 as prizes.⁴ Conceivable acts can be interpreted as bets on the outcomes of the feasible acts whose payoffs are lotteries over feasible consequences. The reason we restrict the payoffs to lotteries over feasible consequences is that, under the level of awareness described by F and C_0 these are the only payoffs that can be meaningfully specified in every state. Consequently, the set of consequences that defines the payoffs of the conceivable acts is a strict subset of the set of “consequences” that defines the augmented conceivable state space.⁵

¹Since there is no universal set of consequences in the background, the addition of the abstract consequence x_0 to the set C_0 generates a set of consequences that is, by definition, universal. The element x_0 is defined “negatively” using the set of feasible consequences. If x_0 is the empty set, then C_0 is the universal set of consequences.

²This method of constructing the state space from the primitive sets of feasible acts and consequences appears in Schmeidler and Wakker (1987) and Karni and Schmeidler (1991). It was used in Karni and Vierø (2013, 2014). The augmentation due to “none of the above” is specific to the present paper.

³Here we invoke the analytical framework of Anscombe and Aumann (1963).

⁴Formally, $p \in \Delta(C_0)$ is a function $p : C_0 \rightarrow [0, 1]$ satisfying $\sum_{c \in C_0} p(c) = 1$. Notice that with this definition of $\Delta(C_0)$ we have that, for any $C_0 \subset C_1$, any $p \in \Delta(C_0)$ is also an element of $\Delta(C_1)$ with $p(c) = 0$ for all $c \in C_1 - C_0$. Likewise, $q \in \Delta(C_1)$ is an element of $\Delta(C_0)$ if $q(c) = 0$ for all $c \in C_1 - C_0$.

⁵We revisit this assertion in Section 4.

Suppose that a new consequence, $c' \notin C_0$, is discovered. This discovery expands the set of feasible consequences to $C_1 = C_0 \cup \{c'\}$. At the same time the abstract “consequence” that has the interpretation “none of the above” becomes $x_1 = \neg C_1$ and the augmented set of consequences becomes $\hat{C}_1 = C_1 \cup \{x_1\}$. The posterior augmented conceivable state space is \hat{C}_1^F . In our illustrating example, if a new consequence c_3 is discovered, the augmented conceivable state space becomes

$$\begin{array}{l}
F \setminus \hat{C}_1^F \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s'_5 \quad s_5 \quad s'_6 \quad s_6 \quad s'_7 \quad s_7 \quad s'_8 \quad s_8 \quad s'_9 \quad s''_9 \quad s'''_9 \quad s_9 \\
f_1 \quad c_1 \quad c_2 \quad c_1 \quad c_2 \quad c_3 \quad x_1 \quad c_3 \quad x_1 \quad c_1 \quad c_1 \quad c_2 \quad c_2 \quad c_3 \quad c_3 \quad x_1 \quad x_1 \quad (3) \\
f_2 \quad c_1 \quad c_1 \quad c_2 \quad c_2 \quad c_1 \quad c_1 \quad c_2 \quad c_2 \quad c_3 \quad x_1 \quad c_3 \quad x_1 \quad c_3 \quad x_1 \quad c_3 \quad x_1
\end{array}$$

The set of fully describable states also expands and is now $C_1^F = C_0^F \cup \{s'_5, s'_6, s'_7, s'_8, s'_9\}$. Thus, when a new feasible consequence is discovered, each of the prior fully describable states remains as before, while each of the prior imperfectly describable states is split into a fully describable state and one or more posterior imperfectly describable states. Hence, points are added to the subset of fully describable states, and simultaneously the number of imperfectly describable states also increases.

As the decision maker’s augmented conceivable state space expands, so does the set of conceivable acts, $\hat{F}_1 := \{f : \hat{C}_1^F \rightarrow \Delta(C_1)\}$. Because the set of conceivable acts is a variable in our model, a decision maker is characterized by a collection of preference relations, one for each level of awareness over the corresponding set of conceivable acts. We denote the preference relation on \hat{F}_i by \succsim_i , $i = 0, 1$. In particular, the prior preference relation is denoted by \succsim_0 on \hat{F}_0 and the posterior preference relation is denoted by \succsim_1 on \hat{F}_1 . Denote by \succ_i and \sim_i the asymmetric and symmetric parts of \succsim_i , respectively. These derived relations are given the usual interpretation of strict preference and indifference, respectively. With the usual abuse of notation, we denote by p the constant act that assigns p to each $s \in \hat{C}_i^F$ and by c or δ^c the degenerate lottery that assigns the unit probability mass to the consequence c .

For all $f, g \in \hat{F}_i$ and $\alpha \in [0, 1]$ define the convex combination $\alpha f + (1 - \alpha)g \in \hat{F}_i$ by: $(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s)$, for all $s \in \hat{C}_i^F$. Then, \hat{F}_i is a convex subset in a linear space.

Consider a decision maker whose choices are characterized by a preference relation \succsim_i on \hat{F}_i . For any $f, g \in \hat{F}_i$, and $E \subset \hat{C}_i^F$, let $g_E f$ be the act in \hat{F}_i obtained from f by for each $s \in E$ replacing its s -th coordinate with $g(s)$. Following Savage (1954), a state $s \in \hat{C}_i^F$ is

said to be *null* if $p_{\{s\}}f \sim_i q_{\{s\}}f$, for all $p, q \in \Delta(C_i)$, for all $f \in \hat{F}_i$. A state is said to be *nonnull* if it is not null.

2.2 Basic preference structure

We assume throughout that the set of consequences of which the decision maker is aware prior to the discovery has a most preferred and a least preferred element. Formally, there exist $c^*, c_* \in C_0$ such that the constant act that assigns c^* to every state is strictly preferred over any other constant act in \hat{F}_0 and the constant act that assigns c_* to every state is strictly less preferred than any other constant act in \hat{F}_0 .

For all $p, q \in \Delta(C_i)$, p first-order stochastically dominates q according to \succsim_i if $\sum_{\{j|c \succsim_i c_j\}} p(c_j) \leq \sum_{\{j|c \succsim_i c_j\}} q(c_j)$ for all $c \in C_i$, and p strictly first-order stochastically dominates q according to \succsim_i if p first-order stochastically dominates q according to \succsim_i and, in addition, $\sum_{\{j|c \succ_i c_j\}} p(c_j) < \sum_{\{j|c \succ_i c_j\}} q(c_j)$ for some $c \in C_i$. We denote these domination relations by $p \geq^1 q$ and $p >^1 q$, respectively.

As described above, when the state space changes in the wake of discoveries of new consequences, the set of conceivable acts must be expanded and the preference relation must be redefined on the extended domain. Following Machina and Schmeidler (1995), we assume that, for each \hat{F}_i , \succsim_i adheres to the following axioms, which ensure probabilistic sophistication.

(A.1) (Weak order) For every \hat{F}_i , the preference relation \succsim_i is transitive and complete.

(A.2) (Mixture continuity) For each \hat{F}_i and all $f, g, h \in \hat{F}_i$, if $f \succ_i g$ and $g \succ_i h$ then there exists $\alpha \in (0, 1)$ such that $\alpha f + (1 - \alpha) h \sim_i g$.

(A.3) (Monotonicity) For every \hat{F}_i and $p, q \in \Delta(C_i)$, if $p \geq^1 q$ then $p_{E_k} h \succsim_i q_{E_k} h$, for all partitions $\{E_1, \dots, E_n\}$ of \hat{C}_i^F and all $h \in \hat{F}_i$, with $p_{E_k} h \succ_i q_{E_k} h$ if $p >^1 q$ and E_k is nonnull.

(A.4) (Replacement) For every \hat{F}_i and any partition $\{E_1, \dots, E_n\}$ of \hat{C}_i^F , if

$$\delta_{E_k}^{c^*} \left(\delta_{E_j}^{c^*} \delta^{c_*} \right) \sim_i \left(\alpha \delta^{c^*} + (1 - \alpha) \delta^{c_*} \right)_{E_k \cup E_j} \delta^{c_*}$$

for some $\alpha \in [0, 1]$ and pair of events E_i, E_j , then

$$p_{E_k}(q_{E_j} h) \sim_i (\alpha p + (1 - \alpha) q)_{E_k \cup E_j} h$$

for all $p, q \in \Delta(C_i)$ and $h \in \hat{F}_i$.

(A.5) (Nontriviality) For every $\hat{F}_i, \succ_i \neq \emptyset$.

To link the preference relations across expanding sets of conceivable acts, we invoke the relevant part of the invariant risk preferences axiom introduced in Karni and Vierø (2013), asserting the commonality of risk attitudes across levels of awareness.

(A.6) (Invariant risk preferences) For all $C_0 \subset C_1$ and the corresponding \succ_0 on \hat{F}_0 and \succ_1 on \hat{F}_1 , $p \succ_0 q$ if and only if $p \succ_1 q$, for all $p, q \in \Delta(C_0)$.

3 Representation and Evolution of Beliefs

3.1 The general result

The following two axioms link the preference relations across different levels of awareness. The first axiom, dubbed Refinement Consistency I, asserts that the decision maker's ranking of objective versus subjective uncertainty, conditional on the initial set of fully describable states, remains unchanged when the set of fully describable states expands in the wake of discovery of new consequences. The intuition is that, while the discovery of new consequences may change the decision maker's sense of ignorance, such discoveries do not affect the part of his preferences that only concerns the fully describable and well-understood part of his universe.

(A.7) (Refinement Consistency I) For all $C_0 \subset C_1$ and the corresponding sets of conceivable acts \hat{F}_0 and \hat{F}_1 , all $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$, if $g = g' = \eta \delta^{c^*} + (1 - \eta) \delta^{c^*}$ for some $\eta \in [0, 1]$ and $f = f' = \delta_{\{s\}}^{c^*} \left(\delta_{C_0^F}^{c^*} g \right)$, for some $s \in C_0^F$, then it holds that $f \succ_0 g$ if and only if $f' \succ_1 g'$.

Note that g and g' are roulette lotteries interpreted as constant acts on the relevant state space. What happens when a new consequence is discovered is that the event on which the acts f and g as well as f' and g' agree is partitioned more finely. The axiom states that such refinement does not alter preferences conditional on the event that is unaffected by the change. In a nutshell, Refinement Consistency I ensures robustness of the decision maker's preferences, conditional on the a priori fully describable event, with respect to discovery of new consequences.

The second axiom, dubbed Refinement Consistency II, asserts that, in the wake of discovery of new consequences, and conditional on the set of a priori imperfectly describable states, a decision maker's ranking of objective uncertainty versus subjective uncertainty regarding a state is the same as that of objective uncertainty versus subjective uncertainty regarding the corresponding event in the a posteriori state space. To state this idea formally we introduce the following additional notations: If $C_0 \subset C_1$ then for each $s \in \hat{C}_0^F - C_0^F$ there corresponds an event $E(s) \subset \hat{C}_1^F - C_0^F$ defined by $E(s) = \{\hat{s} \in \hat{C}_1^F - C_0^F \mid \forall f \in F$ if $f(s) \in C_0$, then $f(\hat{s}) = f(s)$ and if $f(s) = x_0$ then $f(\hat{s}) \in \{x_1\} \cup (C_1 - C_0)\}$.⁶

(A.8) (Refinement Consistency II) For all $C_0 \subset C_1$ and the corresponding sets of conceivable acts \hat{F}_0 and \hat{F}_1 , all $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$, if $g = g' = \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}$ for some $\eta \in [0, 1]$, $f = \delta_{\{s\}}^{c^*}(\delta_{(\hat{C}_0^F - C_0^F)}^{c^*}g)$, and $f' = \delta_{E(s)}^{c^*}(\delta_{(\hat{C}_1^F - C_0^F)}^{c^*}g')$, for some $s \in \hat{C}_0^F - C_0^F$, then it holds that $f \succcurlyeq_0 g$ if and only if $f' \succcurlyeq_1 g'$.

Note that the acts f, g, f' , and g' all agree on the, initially fully describable, event C_0^F . The intuition behind the axiom is that, conditional on the event that is not fully describable a priori, the decision maker views the relative likelihoods of a priori measurable sub-events as being independent of the extent to which he can describe the events.⁷

Theorem 1 below concerns the existence and uniqueness properties of a probabilistically sophisticated representation of preference relations satisfying the aforementioned axioms. To state the theorem we invoke the following definitions: A function V is said to be *strictly monotonic* if $V(p) \geq V(q)$ whenever p dominates q according to first-order stochastic dominance, with strict inequality in the case of strict dominance. A function V is said to be *mixture continuous* if $V(\alpha p + (1 - \alpha)q)$ is continuous in α for all p and q .

Theorem 1 *For each $C_0 \subset C_1$ and the corresponding preference relations \succcurlyeq_0 on \hat{F}_0 and \succcurlyeq_1 on \hat{F}_1 , the following two conditions are equivalent:*

- (i) \succcurlyeq_0 and \succcurlyeq_1 each satisfy (A.1) - (A.5) and jointly, \succcurlyeq_0 and \succcurlyeq_1 satisfy (A.6) - (A.8).
- (ii) *There exist real-valued, mixture continuous, strictly monotonic functions, V_0 on $\Delta(C_0)$ and V_1 on $\Delta(C_1)$, and probability measures, π_0 on \hat{C}_0^F and π_1 on \hat{C}_1^F , such that for*

⁶It may be helpful to look at the matrices (1) and (3) to see what this notation captures.

⁷An event E is measurable with respect to the prior state space if there is an act, $f \in \hat{F}_0$ and consequence $c \in \hat{C}_0$ such that $f^{-1}(c) = E$.

all $f, g \in \hat{F}_0$,

$$f \succcurlyeq_0 g \Leftrightarrow V_0 \left(\sum_{s \in \hat{C}_0^F} \pi_0(s) f(s) \right) \geq V_0 \left(\sum_{s \in \hat{C}_0^F} \pi_0(s) g(s) \right). \quad (4)$$

and, for all $f', g' \in \hat{F}_1$,

$$f' \succcurlyeq_1 g' \Leftrightarrow V_1 \left(\sum_{s \in \hat{C}_1^F} \pi_1(s) f'(s) \right) \geq V_1 \left(\sum_{s \in \hat{C}_1^F} \pi_1(s) g'(s) \right). \quad (5)$$

The functions V_0 and V_1 are unique up to positive transformations and $V_0(p) = V_1(p)$ for all $p \in \Delta(C_0)$, the probability measures π_0 and π_1 are unique and, for all $s, s' \in C_0^F$,

$$\frac{\pi_0(s)}{\pi_0(s')} = \frac{\pi_1(s)}{\pi_1(s')} \quad (6)$$

and, for all $s, s' \in \hat{C}_0^F - C_0^F$,

$$\frac{\pi_0(s)}{\pi_0(s')} = \frac{\pi_1(E(s))}{\pi_1(E(s'))}. \quad (7)$$

Property (6) in Theorem 1 states that the decision maker's subjective beliefs about the relative likelihoods of fully describable states, conditional of the initial set of fully describable states, remain unchanged in the wake of discoveries of new consequences. Property (7) states that the decision maker's subjective beliefs about the relative likelihood of a priori measurable sub-events, conditional of the set of states that he cannot fully describe a priori, remains unchanged in the wake of discoveries of new consequences. Property (6) is reverse Bayesian updating following the discovery of a new consequence as it occurs in Karni and Vierø (2013, 2014). Thus, insofar as the discovery of new consequences is concerned, the model and preference structures of Karni and Vierø (2013, 2014) are nested within the present one and correspond to the special case when $\pi_i(C_i^F) = 1$ for all i . That is, in Karni and Vierø (2013, 2014), for any level of awareness, the decision maker assigns probability zero to future expansions of his awareness.

3.2 Decreasing and increasing sense of ignorance

Discoveries of new consequences expand the decision maker's universe and, depending on their nature, may be accompanied by diminishing, growing or unchanged sense of ignorance. These reflect that intuitively there are three different possible reactions to making a discovery of a new consequence: One could think that it leaves fewer consequences to

discover or that new discoveries will be harder to make. Alternatively, one could become more focused on the possibility of making new discoveries, perhaps because the discovery of new consequences poses new questions. Finally, one could consider the current discovery as having no effect on the likelihood of future discoveries. As we now show, each of these reactions is axiomatically founded.

The next axiom captures the preferential expression of a decreasing sense of ignorance. The case of an increasing sense of unawareness is symmetric and can be treated formally in the same way. For both decreasing and increasing sense of ignorance, the axioms describe the decision maker's willingness to bet on or against making discoveries of new consequences.

(A.9) (Decreasing Sense of Ignorance) For all $C_0 \subset C_1$ and the corresponding sets of conceivable acts \hat{F}_0 and \hat{F}_1 , for all $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$, such that $g = \eta \delta^{c^*} + (1 - \eta) \delta^{c^*}$ for some $\eta \in [0, 1]$, $f = \delta_{C_0}^{c^*} \delta^{c^*}$ and $f' = \delta_{C_1}^{c^*} \delta^{c^*}$, we have that $f \sim_0 g$ implies $f' \succcurlyeq_1 g'$.

Note that this is a decreasing sense of ignorance in the weak sense. It includes the cases of strictly decreasing sense of ignorance ($f' \succ_1 g'$) and constant sense of ignorance ($f' \sim_1 g'$) as special instances. A decision maker has a constant sense of ignorance if he is equally inclined to bet against something unforeseen before and after the discovery of a new consequence. He has a strictly decreasing sense of ignorance if he is more inclined to bet against something unforeseen after the discovery.

Theorem 2 below quantifies the decreasing sense of unawareness by subjective probabilities. Specifically, if growing awareness is accompanied by decreasing sense of ignorance, the subjective probability assigned to the 'residual' unawareness diminishes.

Theorem 2 For each pair $C_0 \subset C_1$ and the corresponding preference relations \succcurlyeq_0 on \hat{F}_0 and \succcurlyeq_1 on \hat{F}_1 , the following statements are equivalent:

- (i) \succcurlyeq_0 and \succcurlyeq_1 each satisfy (A.1) - (A.5), and jointly \succcurlyeq_0 and \succcurlyeq_1 satisfy (A.6) - (A.9).
- (ii) There exists a representation as in Theorem 1 and, in addition,

$$\pi_0(\hat{C}_0^F - C_0^F) \geq \pi_1(\hat{C}_1^F - C_1^F). \quad (8)$$

Inequality (8) includes the case of strictly shrinking ignorance, $\pi_0(\hat{C}_0^F - C_0^F) > \pi_1(\hat{C}_1^F - C_1^F)$, and the case of constant ignorance, $\pi_0(\hat{C}_0^F - C_0^F) = \pi_1(\hat{C}_1^F - C_1^F)$, as special instances.

Clearly, it is possible to formulate the notion of a strictly *increasing sense of ignorance* by changing the conclusion of Axiom (A.9) as follows:

(A.9') (**Increasing Sense of Ignorance**) For all $C_0 \subset C_1$ and the corresponding sets of conceivable acts \hat{F}_0 and \hat{F}_1 , all $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$, such that $g = g' = \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}$ for some $\eta \in [0, 1]$, $f = \delta^{c^*}_{C_0^F}$ and $f' = \delta^{c^*}_{C_1^F}$, $f \sim_0 g$ implies $g' \succ_1 f'$.

A decision maker has an increasing sense of ignorance if he is less inclined to bet against a future increase in awareness after a new consequence is discovered. Correspondingly, we have the following:

Corollary 1 For all $C_0 \subset C_1$ and the corresponding preference relations \succsim_0 on \hat{F}_0 and \succsim_1 on \hat{F}_1 , the following statements are equivalent:

- (i) \succsim_0 and \succsim_1 each satisfy (A.1) - (A.5) and jointly, \succsim_0 and \succsim_1 satisfy (A.6) - (A.8) and (A.9').
- (ii) There exists a representation as in Theorem 1 and, in addition,

$$\pi_0(\hat{C}_0^F - C_0^F) < \pi_1(\hat{C}_1^F - C_1^F). \quad (9)$$

Constant or strictly increasing sense of ignorance necessitates that the decision maker views the world as infinite. There will, in his view, always be more consequences to discover. On the other hand, with a decreasing sense of ignorance, both finite and infinite views of the universe are possible.

As the above analysis shows, the model of Karni and Vierø (2013, 2014) is the special case of growing awareness in which the decision maker exhibits a constant sense of unawareness assigning zero probability to discovery of new consequences. In Karni and Vierø (2013, 2014), new discoveries were outside of the decision maker's conception. However, the same situation arises if the decision maker can in fact conceive of new discoveries, but considers them impossible.

4 Utility of Unknown Consequences

4.1 Extended conceivable acts

Conceivable acts are mappings from the set of states to the set of lotteries on feasible consequences. This specification is the most general possible if the consequences (the lotteries) are to be meaningfully described in every state. In other words, including the abstract consequence “none of the above,” or x_0 , in the supports of the lotteries would create a conceptual problem in states that are fully characterized by feasible consequences (e.g., the states s_1, \dots, s_4 in the first example in Section 2.1). In these states, x_0 remains abstract, so a lottery with x_0 in its support cannot be specified and, therefore, is meaningless. By contrast, in states whose partial or complete descriptions include x_0 , this abstract consequence takes a concrete meaning. Consequently, only in those states lotteries whose supports include x_0 can be specified. In the first example in Section 2.1, with the state space depicted in (1), a lottery that assigns 10% chance to winning c_1 , 75% to winning c_2 , and 15% chance to winning a prize, which is not yet known, but is neither c_1 nor c_2 and will be discovered once the event $\{s_5, \dots, s_9\}$ obtains, is well defined in the states s_5, \dots, s_9 .

As we have shown, the specification of the conceivable acts that restricts the support of the lotteries to feasible consequences is *sufficient to obtain subjective probabilities and utility representation on conceivable acts* (see Theorem 1). These subjective probabilities also apply in the special case in which the representation takes the form of subjective expected utility. In case of subjective expected utility, however, if only the aforementioned conceivable acts are considered, it is possible to specify the utilities of the feasible consequences but not that of the unspecified consequence x_0 . Put differently, the framework of Sections 2 and 3 makes it possible to assign probabilistic beliefs to discovery of new consequences, but is not sufficiently rich to give us a measuring rod for “the utility of unknown consequences” that may not even exist. Assigning utility to the unspecified consequence, x_0 , would allow an explicit and formal expression of the decision maker’s sentiments (e.g., fear or excitement) associated with the prospect of discovering consequences of which he is currently unaware. Presumably, the sentiments associated with discovering unknown aspects of the universe affect individual (and social) choice behavior.

To explore the possibility of assigning utility to unknown consequences, x_0 , we extend the range of the set of conceivable acts. Formally, let $\Delta(\hat{C}_0)$ be the set of extended lotteries that include x_0 in their supports. Let \hat{F}_0 be as defined in (2) and $\tilde{F} := \{\tilde{f} : (\hat{C}_0^F - C_0^F) \rightarrow$

$\Delta(\hat{C}_0)\}$. That is, \tilde{F} is the set of all functions from the set of imperfectly describable states to the set of extended lotteries. Define a set of *extended conceivable acts* F^* as follows:

$$F^* = \{\tilde{f}_{(\hat{C}_0^F - C_0^F)} f \mid f \in \hat{F}_0, \tilde{f} \in \tilde{F}\}.$$

Note that $F^* \supset \hat{F}_0$ and, in particular, that F^* does not include the constant acts whose image is in $\Delta(\hat{C}_0) - \Delta(C_0)$. The set of extended conceivable acts F^* is therefore not an Anscombe and Aumann (1963) set of acts. Rather, the set is the limit of what can be meaningfully expressed as bets that can in fact be resolved, given the decision maker's awareness. For all $f^* = \tilde{f}_{(\hat{C}_0^F - C_0^F)} f$ and $g^* = \tilde{g}_{(\hat{C}_0^F - C_0^F)} g$ in F^* and $\alpha \in (0, 1)$, define $(\alpha f^* + (1 - \alpha) g^*)(s) = \alpha f^*(s) + (1 - \alpha) g^*(s)$, for all $s \in \hat{C}_0^F$ (that is, $\alpha f^* + (1 - \alpha) g^* = (\alpha \tilde{f} + (1 - \alpha) \tilde{g})_{(\hat{C}_0^F - C_0^F)} (\alpha f + (1 - \alpha) g)$). Then F^* is a convex set in a linear space.

4.2 Extended preferences and their representation

Let \succ^* be a preference relation on F^* and assume that the restriction of \succ^* to \hat{F}_0 agrees with \succ (that is, $\succ^* = \succ$ on \hat{F}_0). Note that the definition of null and nonnull events from Section 2.1 still applies.

We state the following well-known subjective expected utility axioms for a weak order \succ^* on a generic set of acts \mathbb{F} and a generic set of consequences C .⁸

(A.10) (Archimedean) For all $f, g, h \in \mathbb{F}$, if $f \succ^* g$ and $g \succ^* h$ then $\alpha f + (1 - \alpha) h \succ^* g$ and $g \succ^* \beta f + (1 - \beta) h$, for some $\alpha, \beta \in (0, 1)$.

(A.11) (Independence) For all $f, g, h \in \mathbb{F}$, and $\alpha \in (0, 1]$, $f \succ^* g$ if and only if $\alpha f + (1 - \alpha) h \succ^* \alpha g + (1 - \alpha) h$.

(A.12) (Monotonicity) For all $p, q \in \Delta(C)$, $f \in \mathbb{F}$, and nonnull event $E \subseteq \hat{C}_0^F$, $pEf \succ^* qEf$ if and only if $p \succ^* q$.

(A.13) (Nontriviality) There are $f, g \in \mathbb{F}$ such that $f \succ^* g$.

By the Theorem of Anscombe and Aumann (1963), the restriction of \succ^* to \hat{F}_0 is a weak order satisfying the Archimedean, Independence, Monotonicity and Nontriviality

⁸We believe these axioms are sufficiently famous that they do not require further discussion. For readers not familiar with these axioms, Fishburn (1970) and Kreps (1988) give excellent discussions.

axioms with $\mathbb{F} = \hat{F}_0$ and $C = C_0$ if and only if there exists a non-constant, real-valued, affine function, U on $\Delta(C_0)$, unique up to positive linear transformation, and a unique probability measure π on \hat{C}_0^F such that for all $f, g \in \hat{F}_0$,

$$f \succ^* g \Leftrightarrow \sum_{s \in \hat{C}_0^F} U(f(s)) \pi(s) \geq \sum_{s \in \hat{C}_0^F} U(g(s)) \pi(s). \quad (10)$$

Representation (10) gives us the utility of feasible consequences as well as the probability measure over the augmented conceivable state space. To extend the representation to also provide the utility of the abstract consequence x_0 , we define sets of *conditional extended conceivable acts* as follows: For every $f \in \hat{F}_0$, let

$$F_{(\hat{C}_0^F - C_0^F)}(f) := \{\tilde{f}_{(\hat{C}_0^F - C_0^F)} f \in F^* \mid \tilde{f} \in \tilde{F}\}.$$

That is, $F_{(\hat{C}_0^F - C_0^F)}(f)$ is the set of all acts in F^* that are extensions of a particular $f \in F_0$. With this we can obtain a subjective utility representation on each of the sets of conditional extended conceivable acts.

Proposition 1 *For every given $f \in \hat{F}_0$ the restriction of \succ^* to $F_{(\hat{C}_0^F - C_0^F)}(f)$ is a weak order satisfying (A.10) - (A.13) with $\mathbb{F} = F_{(\hat{C}_0^F - C_0^F)}(f)$ and $C = \hat{C}_0$ and if and only if there exist a real-valued, non-constant, affine function U_f^* on $\Delta(\hat{C}_0)$ and a conditional probability measure μ on $(\hat{C}_0^F - C_0^F)$ such that, for all $\tilde{f}_{(\hat{C}_0^F - C_0^F)} f$ and $\tilde{g}_{(\hat{C}_0^F - C_0^F)} f$ in $F_{(\hat{C}_0^F - C_0^F)}(f)$,*

$$\tilde{f}_{(\hat{C}_0^F - C_0^F)} f \succ^* \tilde{g}_{(\hat{C}_0^F - C_0^F)} f \Leftrightarrow \sum_{s \in \hat{C}_0^F - C_0^F} U_f^*(\tilde{f}(s)) \mu(s) \geq \sum_{s \in \hat{C}_0^F - C_0^F} U_f^*(\tilde{g}(s)) \mu(s), \quad (11)$$

where U_f^* is unique up to positive affine transformation, μ is unique and $\mu(s) = 0$ if and only if s is null.

The proof is an immediate implication of Anscombe and Aumann (1963).

Since \succ^* agrees with \succ on \hat{F}_0 , the representations in (10) and (11) together imply that for all $f \in \hat{F}_0$ and $p \in \Delta(C_0)$, $U_f^*(p) = U(p)$, i.e. independent of f , and that $\mu(s) = \pi(s)/\pi(\hat{C}_0^F - C_0^F)$ for all $s \in \hat{C}_0^F - C_0^F$. However, the utility of the abstract consequence x_0 , $U_f^*(x_0)$, may depend on the act f . The axiom below, which we call Separability, connects

the different conditional representations in Proposition 1. The axiom requires that the ranking of acts that agree on the set of fully describable states, C_0^F and are constant on the set of partially describable states, $\hat{C}_0^F - C_0^F$, be independent of the part on which they agree. This separability is not implied by the independence axiom because lotteries in $\Delta(\hat{C}_0^F)$ are not defined on the subset of fully describable states, which lends the choice set a “non-rectangular shape.” Formally,

(A.14) (Separability) For all $f, g \in \hat{F}_0$ and $\hat{p}, \hat{q} \in \Delta(\hat{C}_0)$, $\hat{q}_{(\hat{C}_0^F - C_0^F)} f \succ^* \hat{p}_{(\hat{C}_0^F - C_0^F)} f$ if and only if $\hat{q}_{(\hat{C}_0^F - C_0^F)} g \succ^* \hat{p}_{(\hat{C}_0^F - C_0^F)} g$.

The next theorem formally combines the representations (10) and (11) to obtain a general subjective expected utility representation with a utility of the unknown.

Theorem 3 *The following conditions are equivalent:*

- (i) *For every given $f \in F^*$, the preference relation \succ^* on F^* is a weak order satisfying axioms (A.10) - (A.13) with $C = C_0$ and $\mathbb{F} = \hat{F}_0$ and also with $C = \hat{C}_0$ and $\mathbb{F} = F_{(\hat{C}_0^F - C_0^F)}(f)$ for all $f \in \hat{F}_0$, as well as axiom (A.14).*
- (ii) *There exist real-valued, non-constant, affine functions, U on $\Delta(C_0)$ and U^* on $\Delta(\hat{C}_0)$, and a probability measure π on \hat{C}_0^F such that, for all $f^*, g^* \in F^*$,*

$$f^* \succ^* g^* \Leftrightarrow \sum_{s \in C_0^F} U(f^*(s))\pi(s) + \sum_{s \in \hat{C}_0^F - C_0^F} U^*(f^*(s))\pi(s) \geq \sum_{s \in C_0^F} U(g^*(s))\pi(s) + \sum_{s \in \hat{C}_0^F - C_0^F} U^*(g^*(s))\pi(s). \quad (12)$$

The functions U and U^ are unique up to positive linear transformation and they agree on $\Delta(C_0)$. The probability measure is unique, with $\pi(s) = 0$ if and only if s is null and, for all $s \in \hat{C}_0^F - C_0^F$, $\pi(s) / \pi(\hat{C}_0^F - C_0^F) = \mu(s)$.*

The framework of sections 2 and 3 allowed us to obtain the decision maker’s beliefs, including those assigned to the less than fully describable event and its measurable sub-events. As we have just shown, enriching the framework to include extended conceivable acts further allows us to obtain the utility of the unknown. This utility will reflect whether the decision maker faces the unknown with fear or excitement.

4.3 Applications

A strength of our framework is that it distinguishes among states in which different feasible acts result in new consequences, as illustrated in the matrix (1). It therefore allows for the decision maker viewing different acts as being more or less likely to increase awareness. If familiarity begets boldness while lack of it begets prudence, acts that are perceived as less likely to result in unforeseeable consequences are expected to be preferred over similar acts that are more likely to result in unforeseeable consequences. As an illustration, consider again the example in matrix (1) in Section 2.1. Suppose that the decision maker is confident that the act f_1 is unlikely to lead to unforeseen consequences. Specifically, f_1 is taking a familiar route from Spain to India around the Cape of Good Hope. For simplicity of exposition, suppose that the decision maker believes that if he chooses f_1 either the consequence c_1 “getting to India safely” or c_2 “sinking in the ocean” will obtain. In other words, on the basis of past experience, under f_1 the possibility that “neither c_1 nor c_2 ” (that is, x_0) will obtain is believed to be impossible. Formally, the event $\{s_5, s_6, s_9\}$ is considered null. By contrast, x_0 is considered a real possibility if f_2 , a route that was not tried before, such as going to India by sailing westward, is chosen. Thus, the event $\{s_7, s_8\}$ is assigned positive probability. By the representation (12),

$$f_1 \mapsto U(\delta_{c_1})[\pi_0(s_1) + \pi_0(s_3) + \pi_0(s_7)] + U(\delta_{c_2})[\pi_0(s_2) + \pi_0(s_4) + \pi_0(s_8)].$$

and

$$f_2 \mapsto U(\delta_{c_1})[\pi_0(s_1) + \pi_0(s_2)] + U(\delta_{c_2})[\pi_0(s_3) + \pi_0(s_4)] + U^*(\delta_{x_0})[\pi_0(s_7) + \pi_0(s_8)].$$

Therefore, a choice of f_2 over f_1 yields higher probability of encountering an “unknown” consequence, x_0 . If $U(\delta_{c_1}) > U^*(\delta_{x_0})$ and $\pi_0(s_3) \geq \pi_0(s_2) + \pi_0(s_8)$, then $f_1 \succ f_2$.

This type of reasoning might explain the reluctance to invest in foreign markets governed by legal rules and customs that are less familiar. Investment in such environments may involve consequences of which the investors are unaware. Awareness of such unawareness may result in more prudent behavior and produce the well-known domestic bias in financial investments.

5 Concluding Remarks

5.1 The evolution of beliefs about describable events

Theorem 1 concerns the evolution of the relative likelihoods of fully describable (or not fully describable) events in the wake of discovery of new consequences, but is silent on the likelihood themselves. By contrast, Theorem 2 concerns the evolution of the likelihoods of the not fully describable events. Therefore, combining the results of the two theorems makes it possible to discuss the magnitudes of the change in the beliefs about the likelihoods of fully describable events. For instance, suppose that a new discovery is accompanied by a sense of constant unawareness. By Theorem 2, $\pi_0(\hat{C}_0^F - C_0^F) - \pi_1(\hat{C}_1^F - C_1^F) = 0$. But

$$\sum_{s \in C_0^F} \pi_0(s) + \pi_0(\hat{C}_0^F - C_0^F) = 1$$

and

$$\sum_{s \in C_0^F} \pi_1(s) + \sum_{s \in (C_1^F - C_0^F)} \pi_1(s) + \pi_1(\hat{C}_1^F - C_1^F) = 1.$$

Hence, probability mass must be shifted from C_0^F to $C_1^F - C_0^F$, proportionally (that is, the probabilities of all the states in C_0^F must be reduced equiproportionally). Similarly, increasing sense of unawareness requires that probability mass must be shifted from C_0^F to $\hat{C}_1^F - C_0^F$, proportionally and that some of this probability must be shifted to $\hat{C}_1^F - C_1^F$. Finally, decreasing sense of unawareness implies that some probability mass of the event $\hat{C}_0^F - C_0^F$ is shifted towards the newly describable event $C_1^F - C_0^F$. In the latter instance, the effect of growing awareness on the subjective probability assigned to the set of originally describable states, C_0^F , is unpredictable.

The present paper shows that the model of Karni and Vierø (2013) is, in fact, the special case of growing awareness in which the decision maker exhibits not only a constant sense of ignorance, but a constant sense of ignorance assigning zero probability to discovering new consequences. Such a decision maker can be thought of as being myopic with respect to growing awareness, believing at every point that he is fully aware of the scope of his universe. The present paper thus gives an explicit and formal meaning to this type of myopia.

5.2 Related literature

The exploration of the issue of (un)awareness in the literature has invoked at least three different approaches; the epistemic approach, the game-theoretic or interactive decision making approach, and the choice-theoretic approach.

The epistemic approach is taken in Fagin and Halpern (1988), Dekel, Lipman, and Rustichini (1998) Modica and Rustichini (1999), Halpern (2001), Heifetz, Meier, and Schipper (2006, 2008), Li (2009), Hill (2010), Board and Chung (2011), Walker (2011), and Halpern and Rego (2009, 2013a). Of these, Board and Chung (2011), Walker (2011), and Halpern and Rego (2009, 2013a) consider awareness of unawareness. Schipper (2013a) provides an excellent overview of the epistemic literature as well as of the literature on awareness and unawareness more generally.

The game-theoretic, or interactive decision making, approach is taken in Halpern and Rego (2008, 2013b), Heifetz, Meier, and Schipper (2013a, 2013b), Heinsalu (2014), and Grant and Quiggin (2013). The latter develops a model of games with awareness in which inductive reasoning may cause an individual to entertain the possibility that her awareness is limited. Individuals thus have inductive support for propositions expressing their own unawareness. In this paper, we implicitly assume inductive reasoning to motivate considering awareness of unawareness.

The choice-theoretic approach to unawareness or related issues is taken in Li (2008), Ahn and Ergin (2010), Schipper (2013b), Lehrer and Teper (2014), Kochov (2010), Walker and Dietz (2011), and Alon (2014). The former four are discussed in detail in Karni and Vierø (2013). Walter and Dietz (2011) and Kochov (2010) consider decision makers who are aware of their potential unawareness, and are thus the papers closest related to the present paper.

Walker and Dietz (2011) take a choice theoretic approach to static choice under “conscious unawareness.” In their model, unawareness materializes in the form of coarse contingencies (that is, their state space does not resolve all uncertainty). Their representation is similar to Klibanoff, Marinacci, and Mukerji’s (2005) smooth ambiguity model. The model of Walker and Dietz (2011) differs from ours in several respects: theirs is a static model and thus does not consider the issue of updating when awareness increases, their approach to modeling the state space differs from ours, and in their model a decision maker’s beliefs are not represented by a single probability measure.

Kochov (2010) develops an axiomatic model of dynamic choice in which the decision maker knows that her perception of the environment may be incomplete. This causes the decision maker's beliefs to be represented by a non-singleton set of priors, with prior by prior Bayesian updating as the decision maker's perception of the universe becomes more precise. Kochov's work differs from ours in the way the state space and its evolution are modeled, and the representation of decision makers' beliefs.

Alon (2014) considers a decision maker in a Savage framework. The axioms she imposes imply that the decision maker acts as if he completes the state space with an extra state to which he assigns the worst consequence obtainable from every act. The decision maker is a subjective expected utility maximizer over the set of extended acts. An interpretation of the model is that the decision maker acts as if she faces some unforeseen event. Unlike the model of this paper, Alon's model is static and thus begs the issue of updating. Moreover, since the range of acts is simply the standard set of consequences, Alon's model does not extend the utility to unknown consequences.

The separation of probabilistic sophistication from the expected utility hypothesis was first done in a Savage framework in Machina and Schmeidler (1992). Machina and Schmeidler (1995) followed up with the result for an Anscombe and Aumann framework. Grant and Polak (2006) proposed an alternative axiomatization of probabilistically sophisticated choice behavior.

In the framework of preferences over menus, Dekel, Lipman and Rustichini (2001) propose "... a model that allows for unforeseen contingencies in the sense that the agent does not have an exogenously given list of all possible states of the world." (p. 893). The agent in their model knows that there may be considerations that she cannot specify. While this sounds similar, the content is completely different from the model of this paper. Specifically, the states in Dekel, Lipman and Rustichini are alternative preferences that the decision maker might entertain at the time he has to choose from the menu. These "mental states" resolve the uncertainty concerning the decision maker's own preferences rather than the payoffs of the feasible acts.

By definition, when a decision maker is unaware of a consequence, he cannot pay attention to that consequence. This aspect of our model is shared by the recent literature on revealed attention (see Masatlioglu, Nakajima, and Ozbay (2012)). In the case where the decision maker assigns probability zero to future discoveries of new consequences, our model is related to Ortoleva's (2012) model of non-Bayesian reactions to unexpected news.

6 Proofs

6.1 Proof of Theorem 1

(i) \Rightarrow (ii). Since \succcurlyeq_0 and \succcurlyeq_1 satisfy (A.1) - (A.5), the Theorem of Machina and Schmeidler (1995) implies the existence of mixture continuous, monotonic real-valued functions, V_0 and V_1 satisfying (4) and (5) as well as the uniqueness of V_0 and V_1 and of π_0 and π_1 . By (4) and (5), the restriction of \succcurlyeq_0 and \succcurlyeq_1 to the constant acts $p, q \in \Delta(C_0)$ imply that $V_0(p) \geq V_0(q)$ if and only if $p \succcurlyeq_0 q$ and that $V_1(p) \geq V_1(q)$ if and only if $p \succcurlyeq_1 q$. By (A.6), $p \succcurlyeq_0 q$ if and only if $p \succcurlyeq_1 q$. Thus, by the uniqueness of the representations, V_0 and V_1 can be chosen so that $V_0 = V_1$ on $\Delta(C_0)$.

Let $g = g' = \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}$, and for some $s \in C_0^F$, and let f, f' be as in Axiom (A.7). Suppose that $f \sim_0 g$. But $f \sim_0 g$ if and only if

$$\delta_{\{s\}}^{c^*} \left(\delta_{C_0^F}^{c^*} (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \sim_0 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (13)$$

By the representation in (4) the last indifference holds if and only if

$$\begin{aligned} V_0 \left(\pi_0(s)\delta^{c^*} + (\pi_0(C_0^F) - \pi_0(s))\delta^{c^*} + (1 - \pi_0(C_0^F))(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \\ = V_0 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \end{aligned} \quad (14)$$

But, by first-order stochastic dominance, (14) holds if and only if $\pi_0(s) + (1 - \pi_0(C_0^F))\eta = \eta$. Hence,

$$\eta = \frac{\pi_0(s)}{\pi_0(C_0^F)}. \quad (15)$$

By Axiom (A.7), $f \sim_0 g$ if and only if $f' \sim_1 g'$, the latter of which is equivalent to

$$\delta_{\{s\}}^{c^*} \left(\delta_{C_0^F}^{c^*} (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \sim_1 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (16)$$

By the representation in (5), (16) holds if and only if

$$\begin{aligned} V_1 \left(\pi_1(s)\delta^{c^*} + (\pi_1(C_0^F) - \pi_1(s))\delta^{c^*} + (1 - \pi_1(C_0^F))(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \\ = V_1 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \end{aligned} \quad (17)$$

But (17) holds if and only if $\pi_1(s) + (1 - \pi_1(C_0^F))\eta = \eta$. Thus, $f' \sim_1 g'$ if and only if

$$\eta = \frac{\pi_1(s)}{\pi_1(C_0^F)}. \quad (18)$$

By (15) and (18) we have that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} = \frac{\pi_1(s)}{\pi_1(C_0^F)}. \quad (19)$$

An analogous argument applies for any $s' \in C_0^F$. We therefore also have that, for any $s' \in C_0^F$,

$$\frac{\pi_0(s')}{\pi_0(C_0^F)} = \frac{\pi_1(s')}{\pi_1(C_0^F)}. \quad (20)$$

Together, (19) and (20) imply that

$$\frac{\pi_1(s)}{\pi_1(s')} = \frac{\pi_0(s)}{\pi_0(s')}. \quad (21)$$

Now, let again $g = g' = \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}$, and for some $s \in \hat{C}_0^F - C_0^F$, let f, f' be as in Axiom (A.8). Suppose that $f \sim_0 g$. But $f \sim_0 g$ if and only if

$$\delta_{\{s\}}^{c^*} \left(\delta_{\hat{C}_0^F - C_0^F}^{c^*} (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \sim_0 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (22)$$

By the representation in (4) the last indifference holds if and only if

$$\begin{aligned} & V_0 \left(\pi_0(s)\delta^{c^*} + (1 - \pi_0(C_0^F) - \pi_0(s))\delta^{c^*} + \pi_0(C_0^F) \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \right) \\ &= V_0 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \end{aligned} \quad (23)$$

using that $\pi_0(\hat{C}_0^F - C_0^F) = 1 - \pi_0(C_0^F)$. But (23) holds if and only if $\pi_0(s) + \pi_0(C_0^F)\eta = \eta$. Hence,

$$\eta = \frac{\pi_0(s)}{1 - \pi_0(C_0^F)}. \quad (24)$$

By Axiom (A.8), $f \sim_0 g$ if and only if $f' \sim_1 g'$, the latter of which is equivalent to

$$\delta_{E(s)}^{c^*} \left(\delta_{\hat{C}_1^F - C_0^F}^{c^*} (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \sim_1 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (25)$$

By the representation in (5), (25) holds if and only if

$$\begin{aligned} & V_1 \left(\pi_1(E(s))\delta^{c^*} + \left(\pi_1(\hat{C}_1^F - C_0^F) - \pi_1(E(s)) \right) \delta^{c^*} + \pi_1(C_0^F) (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \right) \\ &= V_1 (\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}) \end{aligned} \quad (26)$$

But (26) holds if and only if $\pi_1(E(s)) + \pi_1(C_0^F)\eta = \eta$. Thus, $f' \sim_1 g'$ if and only if

$$\eta = \frac{\pi_1(E(s))}{1 - \pi_1(C_0^F)}. \quad (27)$$

By (24) and (27) we have that

$$\frac{\pi_0(s)}{1 - \pi_0(C_0^F)} = \frac{\pi_1(E(s))}{1 - \pi_1(C_0^F)}. \quad (28)$$

An analogous argument applies for any $s' \in \hat{C}_1^F - C_0^F$. We therefore also have that, for any $s' \in \hat{C}_1^F - C_0^F$,

$$\frac{\pi_0(s')}{1 - \pi_0(C_0^F)} = \frac{\pi_1(E(s'))}{1 - \pi_1(C_0^F)}. \quad (29)$$

Together (28) and (29) imply that

$$\frac{\pi_1(E(s))}{\pi_1(E(s'))} = \frac{\pi_0(s)}{\pi_0(s')}. \quad (30)$$

(ii) \Rightarrow (i). That \succcurlyeq_0 and \succcurlyeq_1 satisfy (A.1) - (A.5) is an implication of the Theorem of Machina and Schmeidler (1995). Invariant risk preferences, (A.6), follows from the equality of V_0 and V_1 on $\Delta(C_0)$.

To show that (A.7) holds, let $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$ be as in (A.7). By (4), $f \succcurlyeq_0 g$ if and only if

$$\begin{aligned} & V_0 \left(\pi_0(s)\delta^{c^*} + (\pi_0(C_0^F) - \pi_0(s))\delta^{c^*} + (1 - \pi_0(C_0^F)) \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \right) \\ & \geq V_0 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right). \end{aligned}$$

By first order stochastic dominance, the last inequality holds if and only if

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} \geq \eta. \quad (31)$$

Suppose that $g' \succ_1 f'$. By (5), $g' \succ_1 f'$ if and only if

$$\begin{aligned} & V_1 \left(\pi_1(s)\delta^{c^*} + (\pi_1(C_0^F) - \pi_1(s))\delta^{c^*} + (1 - \pi_1(C_0^F)) \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \right) \\ & < V_1 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right). \end{aligned}$$

By first order stochastic dominance, this holds if and only if $\pi_1(s) + (1 - \pi_1(C_0^F))\eta < \eta$. Hence,

$$\eta > \frac{\pi_1(s)}{\pi_1(C_0^F)}. \quad (32)$$

Now, expressions (31) and (32) imply that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} > \frac{\pi_1(s)}{\pi_1(C_0^F)}. \quad (33)$$

However, by (6),

$$\frac{\pi_0(s')}{\pi_0(s)} = \frac{\pi_1(s')}{\pi_1(s)} \quad (34)$$

for all $s, s' \in C_0^F$. Summing over $s' \in C_0^F$ and rearranging, (34) implies that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} = \frac{\pi_1(s)}{\pi_1(C_0^F)}$$

which contradicts (33).

To show that (A.8) holds, let $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$ be as in (A.8). By (4), $f \succ_0 g$ if and only if

$$\begin{aligned} & V_0 \left(\pi_0(s)\delta^{c^*} + (1 - \pi_0(C_0^F) - \pi_0(s))\delta^{c^*} + \pi_0(C_0^F) \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \right) \\ & \geq V_0 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right). \end{aligned}$$

By first order stochastic dominance, the last inequality holds if and only if

$$\frac{\pi_0(s)}{1 - \pi_0(C_0^F)} \geq \eta. \quad (35)$$

Suppose that $g' \succ_1 f'$. By (5), $g' \succ_1 f'$ if and only if

$$\begin{aligned} & V_1 \left(\pi_1(E(s))\delta^{c^*} + (1 - \pi_1(C_0^F) - \pi_1(E(s)))\delta^{c^*} + \pi_1(C_0^F) \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right) \right) \\ & < V_1 \left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*} \right). \end{aligned}$$

By first order stochastic dominance, this holds if and only if $\pi_1(E(s)) + \pi_1(C_0^F)\eta < \eta$. Hence,

$$\eta > \frac{\pi_1(E(s))}{1 - \pi_1(C_0^F)}. \quad (36)$$

Now, expressions (35) and (36) imply that

$$\frac{\pi_0(s)}{1 - \pi_0(C_0^F)} > \frac{\pi_1(E(s))}{1 - \pi_1(C_0^F)}. \quad (37)$$

However, by (6),

$$\frac{\pi_0(s')}{\pi_0(s)} = \frac{\pi_1(E(s'))}{\pi_1(E(s))} \quad (38)$$

for all $s, s' \in \hat{C}_0^F - C_0^F$. Summing over $s' \in \hat{C}_0^F - C_0^F$ and rearranging, (38) implies that

$$\frac{\pi_0(s)}{1 - \pi_0(C_0^F)} = \frac{\pi_1(E(s))}{1 - \pi_1(C_0^F)}$$

which contradicts (37). ♠

6.2 Proof of Theorem 2

Sufficiency of axioms: That the axioms imply existence of a representation as in Theorem 1 follows from the proof of Theorem 1. Let $g = g' = \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}$, and let f, f' be as in Axiom (A.9). Suppose that $f \sim_0 g$. But $f \sim_0 g$ if and only if

$$\delta_{C_0^F}^{c^*} \delta^{c^*} \sim_0 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (39)$$

By the representation in (4) the last indifference holds if and only if

$$V_0\left(\pi_0(C_0^F)\delta^{c^*} + (1 - \pi_0(C_0^F))\delta^{c^*}\right) = V_0\left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}\right) \quad (40)$$

But, by first-order stochastic dominance, (40) holds if and only if

$$\eta = \pi_0(C_0^F). \quad (41)$$

By Axiom (A.9), $f \sim_0 g$ implies that $f' \succeq_1 g'$, which is equivalent to

$$\delta_{C_1^F}^{c^*} \delta^{c^*} \succeq_1 \eta\delta^{c^*} + (1 - \eta)\delta^{c^*}. \quad (42)$$

By the representation in (5), (42) holds if and only if

$$V_1\left(\pi_1(C_1^F)\delta^{c^*} + (1 - \pi_1(C_1^F))\delta^{c^*}\right) \geq V_1\left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}\right) \quad (43)$$

But, by first-order stochastic dominance, (43) holds if and only if

$$\pi_1(C_1^F) \geq \eta. \quad (44)$$

By (41) and (44) we have that

$$\pi_1(C_1^F) \geq \pi_0(C_0^F), \quad (45)$$

which is equivalent to $\pi_1(\hat{C}_1^F - C_1^F) \leq \pi_0(\hat{C}_0^F - C_0^F)$. The inequality in (45) is strict if and only if $f' \succ_1 g'$ in Axiom (A.9), and holds with equality if and only if $f' \sim_1 g'$ in Axiom (A.9).

Necessity of Axioms: Necessity of axioms (A.1)-(A.8) follows from the proof of Theorem 1. To show that (A.9) holds, let $f, g \in \hat{F}_0$ and $f', g' \in \hat{F}_1$ be as in (A.9). By (4), $f \sim_0 g$ if and only if

$$V_0\left(\pi_0(C_0^F)\delta^{c^*} + (1 - \pi_0(C_0^F))\delta^{c^*}\right) = V_0\left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}\right) \quad (46)$$

By first order stochastic dominance, (46) holds if and only if

$$\eta = \pi_0(C_0^F). \quad (47)$$

Suppose now that $g' \succ_1 f'$. By (5), $g' \succ_1 f'$ if and only if

$$\begin{aligned} & V_1\left(\pi_1(C_1^F)\delta^{c^*} + (1 - \pi_1(C_1^F))\delta^{c^*}\right) \\ & < V_1\left(\eta\delta^{c^*} + (1 - \eta)\delta^{c^*}\right). \end{aligned}$$

By first order stochastic dominance, this holds if and only

$$\pi_1(C_1^F) < \eta. \quad (48)$$

Now, expressions (47) and (48) imply that

$$\pi_0(C_0^F) > \pi_1(C_1^F). \quad (49)$$

However, by (8), $\pi_0(C_0^F) \leq \pi_1(C_1^F)$, which contradicts (49). ♠

6.3 Proof of Theorem 3

Sufficiency of axioms: We here provide the part of the proof that does not follow directly from (10) or Proposition 1. The agreement of \succ^* and \succ on \hat{F}_0 and the representations (10) and (11) imply that, for all $\tilde{f}, \tilde{g} \in \tilde{F}$ such that $\tilde{f}(s), \tilde{g}(s) \in \Delta(C_0)$ for all $s \in \hat{C}_0^F - C_0^F$, and for all $f \in \hat{F}_0$

$$\tilde{f}_{(\hat{C}_0^F - C_0^F)} f \succ^* \tilde{g}_{(\hat{C}_0^F - C_0^F)} f \Leftrightarrow \sum_{s \in \hat{C}_0^F - C_0^F} U(\tilde{f}(s)) \pi(s) \geq \sum_{s \in \hat{C}_0^F - C_0^F} U(\tilde{g}(s)) \pi(s). \quad (50)$$

Hence, with appropriate normalization, for all $p \in \Delta(C_0)$, $U_f^*(p) = U(p)$, for all $f \in \hat{F}_0$. Therefore $U^*(p)$ is independent of f .

Suppose that $c^* \succ^* x_0 \succ c_*$, let $\hat{p} = \alpha c^* + (1 - \alpha) c_*$ be such that $\hat{p}_{(\hat{C}_0^F - C_0^F)} f \sim^* x_{0(\hat{C}_0^F - C_0^F)} f$. Then, by the representation (11), $U_f^*(\hat{p}) = U_f^*(x_0)$. Then, by Axiom (A.14) and the representation (11) we have that $U_g^*(\hat{p}) = U_g^*(x_0)$. But $U_f^*(\hat{p}) = U_f^*(x_0)$ is equivalent to

$$u_f^*(x_0) = \alpha U(c^*) + (1 - \alpha) U(c_*)$$

and $U_g^*(\hat{p}) = U_g^*(x_0)$ is equivalent to

$$u_g^*(x_0) = \alpha U(c^*) + (1 - \alpha) U(c_*).$$

Hence, $u_f^*(x_0) = u_g^*(x_0) \equiv u(x_0)$ for all $f, g \in \hat{F}_0$.

Suppose instead that $x_0 \succ^* c^* \succ c_*$, and let $\hat{p} = \alpha x_0 + (1 - \alpha) c_*$ be such that $\hat{p}_{(\hat{C}_0^F - C_0^F)} f \sim^* c_{(\hat{C}_0^F - C_0^F)}^* f$. Then, by the representation (11), $U_f^*(\hat{p}) = U_f^*(x_0)$. Then, by Axiom (A.14) and representation (11) we have that $U_g^*(\hat{p}) = U_g^*(x_0)$. But $U_f^*(\hat{p}) = U_f^*(x_0)$ is equivalent to

$$\alpha u_f^*(x_0) + (1 - \alpha) U(c_*) = U(c^*),$$

and $U_g^*(\hat{p}) = U_g^*(x_0)$ is equivalent to

$$\alpha u_g^*(x_0) + (1 - \alpha) U(c_*) = U(c^*).$$

Solving for $u_f^*(x_0)$ and $u_g^*(x_0)$ we get,

$$u_f^*(x_0) = u_g^*(x_0) = \frac{U(c^*) - U(c_*)}{\alpha} + U(c_*) \equiv u(x_0)$$

for all $f, g \in \hat{F}_0$.

Finally, if $c^* \succ^* c_* \succ^* x_0$ let $\hat{p} = \alpha c^* + (1 - \alpha) x_0$ such that $\hat{p}_{(\hat{C}_0^F - C_0^F)} f \sim^* c_{*(\hat{C}_0^F - C_0^F)} f$ then, by the same argument,

$$u_f^*(x_0) = u_g^*(x_0) = \frac{U(c_*) - \alpha U(c^*)}{1 - \alpha} \equiv u(x_0)$$

for all $f, g \in \hat{F}_0$.

It follows that $U^*(\hat{q}) = \sum_{c \in C_0} \hat{q}(c) U(\delta_c) + \hat{q}(x_0) u(x_0)$ for any $\hat{q} \in \Delta(\hat{C}_0)$.

The uniqueness of the subjective probabilities is implied by the uniqueness of the subjective probabilities in (10).⁹

Necessity of axioms: Necessity of axioms (A.10)-(A.13) on the respective domains follows from Anscombe and Aumann (1963). The necessity of (A.14) is immediate. ♠

⁹The uniqueness of π in conjunction with Proposition 1 imply that $\mu(s) = \pi(s) / \pi(\hat{C}_0^F - C_0^F)$ for all $s \in \hat{C}_0^F - C_0^F$.

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