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The Krusell-Smith Algorithm: Are Self-fulfilling Equilibria Likely?

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Abstract

I investigate whether the popular Krusell and Smith algorithm used to solve heterogeneous-agent economies with aggregate uncertainty and incomplete markets is likely to be subject to multiple self-fulfilling equilibria. In a benchmark economy, the parameters representing the equilibrium aggregate law of motion are randomly perturbed 500 times, and are used as the new initial guess to compute the equilibrium with this algorithm. In a sequence of cases, differing only in the magnitude of the perturbations, I do not find evidence of multiple self-fulfilling equilibria. The economic reason behind the result lies in a self-correcting mechanism present in the algorithm: compared to the equilibrium law of motion, a candidate one implying a higher (lower) expected future capital reduces (increases) the equilibrium interest rates, increasing (reducing) the savings of the wealth-rich agents only. These, on the other hand, account for a small fraction of the population and cannot compensate for the opposite change triggered by the wealth-poor agents, who enjoy higher (lower) future wages and increase (reduce) their current consumption. Quantitatively, the change in behavior of the wealth-rich agents has a negligible impact on the determination of the change in the aggregate savings, inducing stability in the algorithm as a by-product.

JEL Classification Codes: C63, C68, E21, E32.

Keywords: Unemployment Risk, Business Cycles, Incomplete Markets, Heterogeneous Agents, Numerical Methods, Self-fulfilling Equilibria.

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1 Introduction

This paper investigates whether the popular Krusell and Smith (KS) algorithm, used to solve heterogeneous-agent economies with aggregate uncertainty and incomplete markets, is likely to be subject to multiple Self-Fulfilling Equilibria (SFE). This possibility arises because the equilibrium Aggregate Law of Motion (ALM) is unknown, and it needs to be computed through a guess-and-verify iterative procedure. Crucially, the agents' optimal decision rules have to be calculated at each step of this fixed-point problem, but they in turn depend on the ALM being tried. In principle, this process can lead to a complementarity between the guess related to the agents' perception of the evolution of future prices and their implied choices.

This method was first proposed by Krusell and Smith (1998), and it has been successfully applied to a wide variety of problems. Notable examples include the magnitude of welfare costs due to business cycles (Castaneda, Diaz-Gimenez, and Rios-Rull (1998) and Storesletten, Telmer, and Yaron (2001)), fluctuations in frictional labor markets (Gomes, Greenwood and Rebelo (2001) and Nakajima (2012)), the determinants of fiscal policy (Heathcote (2005)), the pricing and allocation of risky and safe assets (Krusell and Smith (1997) and Pijoan-Mas (2007)), and the analysis of rising wage inequality in a political economy framework (Corbae, D'Erasmo, and Kuruscu (2009)).

Where does the possibility of multiple SFE stem from? In the simplest environment, where households' only choice concerns their savings, if postulating an ALM for capital that is above (below) the equilibrium one leads to more (less) resources being saved by the households in the aggregate, we would be in a situation displaying complementarity between the guessed aggregate dynamics of capital and the resulting saving behavior, which could lead to multiple SFE. This instance has been acknowledged by Krusell and Smith (2006), who argued for the absence of multiple SFE by analyzing a simple two-period model. This paper undertakes a systematic study on the subject, considering the full-blown version of their model, with both infinitely-lived agents and preference heterogeneity.

My investigation is complementary to the analysis performed in Young (2005), who assesses the robustness of the KS algorithm along several dimensions. In particular, he argues for the absence of SFE by working with a version of the model with heterogeneous beliefs. Unlike him, I rely on a Monte Carlo analysis of the KS economy. Through appropriately designed perturbation experiments, I do not find evidence supporting the existence of SFE such that the parameters representing the ALM converge to different values, depending on the initial guess. Although some replications do converge to ALM parameters that differ from the equilibrium one, the discrepancy is always quantitatively negligible, making it difficult to disentangle this gap from the numerical error induced by the discretization of the state space, the sampling variability arising from the simulations and the convergence criteria. In terms of substance, even if one is willing to consider these alternative ALM as different equilibria, the

implied differences are quantitatively minimal.

The rest of the paper is organized as follows. Section 2 briefly presents the model and the calibration. Section 3 discusses the role of the ALM in the KS model. Section 4 outlines the perturbation experiments. Section 5 describes the main results, while Section 6 concludes. Two appendices discuss in more detail the numerical methods used and report the complete calibration. Another appendix presents some additional results and a set of robustness exercises.

2 Model

The model is similar to the set-up in Krusell and Smith (1998) with preference heterogeneity, except for the availability of unemployment benefits for workers without a job. Following den Haan, Judd and Juillard (2010), I assume a budget-balanced Unemployment Insurance (UI) scheme.

I consider a production economy with aggregate shocks in which agents face different employment histories and self-insure by accumulating a single risky asset. A borrowing constraint (b) potentially prevents agents from borrowing the desired amount of resources in periods where they obtain a low income.

Technology: The production side of the economy is modeled as a constant returns to scale technology of the Cobb-Douglas form, which relies on aggregate capital K_t and labor L_t to produce final output $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$. The aggregate shock takes only two values: $z_t = \{z_G, z_B\}$, with $z_G = 1.01 > z_B = 0.99$. The aggregate shock follows a symmetric first-order Markov chain, whose transition matrix is reported in Appendix B, and it is such that booms and recessions last the same number of periods. Capital depreciates at the exogenous rate δ and firms hire capital and labor every period from competitive markets. Total labor services are $L_t = lN_t$, namely they are the product of the employment rate N_t and l, the share of the time endowment devoted to market activities.[1] The first order conditions give the expressions for the net real return to capital r_t and the wage rate w_t:

$$r_t = \alpha z_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta, \tag{1}$$

$$w_t = (1-\alpha)\, z_t \left(\frac{K_t}{L_t}\right)^{\alpha}. \tag{2}$$

Government: The government taxes the employed agents' labor income at rate τ_t to finance a budget-balanced UI scheme. Unemployed agents receive UI benefits equal to a fixed replacement rate

[1] The total time endowment is normalized to 1.

ρ of the going labor income. Since labor supply is fixed, and the aggregate unemployment rate can only take two values ($u_t = 0.04$ when $z_t = z_G$ and $u_t = 0.10$ when $z_t = z_B$), the equilibrium tax rate is $\tau_t = \rho(1 - N_t)/N_t$, with $N_t = 1 - u_t$. The transition matrix for the employment opportunities is reported in Appendix B.

Households: Agents' preferences are assumed to be represented by a time-separable utility function $U(.)$. Every household $h \in [0, 1]$ chooses consumption ($c_{h,t}$) and future asset holdings ($a_{h,t+1}$), and the households objective function is:

$$\max_{\{c_{h,t}, a_{h,t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_h^t \frac{c_{h,t}^{1-\gamma} - 1}{1-\gamma}$$

where E is the expectation operator. $\beta_h \in (0, 1)$ is the agents' discount factor, which varies over time according to the transition matrix reported in Appendix B, and can take three different values. Agents can be employed, $s = e$, or unemployed, $s = u$. The employment probabilities follow a first-order Markov process, depend on both the idiosyncratic employment status s and on the aggregate state of the economy z, and the related transition matrix is reported in Appendix B. I use recursive methods to solve the model, and the value function associated with this problem is denoted with $V(a, s, \beta, z, K)$. This represents the expected lifetime utility of an agent whose current asset holdings are equal to a, whose current employment status is s, whose current discount factor is β, facing the aggregate shock z and in an economy with K units of aggregate capital. The Bellman equation is:

$$V(a, s, \beta, z, K) = \max_{c, a'} \left\{ \frac{c^{1-\gamma} - 1}{1-\gamma} + \beta E_{\beta', s', z' | \beta, s, z} V\left(a', s', \beta', z', K'\right) \right\}$$

$$s.t.$$

$$c + a' = (1 + r)\, a + (1 - \tau)\, wl, \text{ if } s = e$$

$$c + a' = (1 + r)\, a + \rho wl, \text{ if } s = u$$

$$c \geq 0, \quad a' \geq b$$

$$\ln K' = \theta_{0,G} + \theta_{1,G} \ln K, \text{ if } z = z_G \tag{3}$$

$$\ln K' = \theta_{0,B} + \theta_{1,B} \ln K, \text{ if } z = z_B \tag{4}$$

Agents have to optimally set their consumption/savings plans. They enjoy utility from consumption, and face some uncertain events in the future. Notice that, according to the KS algorithm, the relevant state variable in the agents' problem is just aggregate capital K, rather than the whole current endogenous distribution over idiosyncratic states. Hence, the agents forecast future prices relying on the (equilibrium) evolution of the aggregate capital stock, the ALM being specified as the pair of equations (3) and (4), which are commented upon in more detail in the following section.

The calibration of the model's parameters is standard and they are presented in Table 1.

[Table 1 about here]

The calibration follows for the most part Krusell and Smith (1998) and den Haan, Judd and Juillard (2010). The only differences pertain the CRRA parameter γ and the borrowing constraint b. While Krusell and Smith (1998) and den Haan, Judd and Juillard (2010) set $\gamma = 1$, I work with $\gamma = 2$. Although my choice for this parameter is still well inside the range of available estimates, compared to the log case it implies a relatively stronger income effect. As it will be discussed below, a higher γ increases the likelihood of SFE. Krusell and Smith (1998) and den Haan, Judd and Juillard (2010) consider $b = 0$, namely the extreme case of a no-borrowing constraint. I set $b = -2$ instead, for the time series average of the share of households in debt to be approximately 10%, which is a more empirically relevant value.[2]

3 The ALM in the Krusell and Smith (1998) model

As for the ALM, the system (3) and (4) specifies its functional form. Following Krusell and Smith (1998), and most of the papers thereafter, I use a log-linear specification.

Let Θ denote the vector of four parameters $\theta_{i,z}$ representing the ALM, with Θ^* referring to their equilibrium values. Θ^* is obtained by guessing a Θ^g, solving and simulating the model under this guess, computing an update $\Theta^{g\prime}$ as the parameter estimates of OLS regressions on the simulated data, and repeating these steps until the four parameters in Θ converge. For the economy under study, the values of Θ^* are reported in the system (5).[3] Notice that z_t stands for the time-t aggregate shock, while K_t stands for aggregate capital, and that there are two parameters (an intercept θ_{0,z_t} and a slope θ_{1,z_t}) per aggregate shock.

[2] The results with the alternative value $b = 0$ are reported in an appendix.

[3] It goes without saying that the computed Θ^* could be only one of the potentially many equilibria.

$$\begin{cases} \ln K_{t+1} = \theta^*_{0,G} + \theta^*_{1,G} \ln K_t = 0.072692 + 0.971528 \times \ln K_t, & \text{for } z_t = z_G = 1.01 \\ \\ \ln K_{t+1} = \theta^*_{0,B} + \theta^*_{1,B} \ln K_t = 0.064257 + 0.972821 \times \ln K_t, & \text{for } z_t = z_B = 0.99 \end{cases} \tag{5}$$

Can this algorithm display SFE? Figure 1 shows the differential response of savings for selected household types, under two different specifications of the ALM. The solid line plots the saving function for the equilibrium ALM, while the dashed line for an alternative ALM, obtained by increasing $\theta^*_{0,G}$ and $\theta^*_{0,B}$ until the implied forecasted aggregate capital is approximately 10% higher than its equilibrium counterpart.[4]

[Figure 1 about here]

It is clear from the figure that asset-rich and asset-poor households react differently to the perturbation of the equilibrium ALM. In particular, rich agents *increase* their savings, while poor ones *reduce* them. The onset of a complementarity between the ALM and individual savings is indeed a possibility, and it crucially depends on both the strength of the individual responses and the shape of the wealth distribution. This is why I focus on the KS economy with preference heterogeneity. Only in this case the economy is able to match a wealth Gini index of 0.8. Not only this is a desirable feature in a model of endogenous wealth accumulation, but also I am making sure that there is the right mass of wealth-rich individuals, affecting the likelihood of the above mentioned complementarity.

To further investigate this issue, I conduct a sequence of experiments where the four parameters in Θ^* are perturbed randomly. Each $\theta^*_{i,z}$ is multiplied by the realization of a random variate, drawn from independent uniform distributions, whose supports are set to different ranges in the sequence of experiments. The range of the perturbations is progressively increased, from $\pm 1\%$ to $\pm 25\%$, to allow for a deteriorating quality of the initial guess. Finally, the ALM is updated until it converges again and the procedure is repeated several times, with different initial perturbations.

Loosely speaking, this Monte Carlo procedure mimics the actual steps that researchers follow in their search for the equilibrium ALM.[5] As some authors point out, a good initial guess for the ALM is

[4] All the saving functions are for employed agents ($s = e$), during a boom ($z = z_G$), and with an aggregate capital close to its time average computed in the simulations ($K = 12$). The saving functions for other combinations of state variables are qualitatively similar to the ones reported.

[5] Several other aspects of the computational framework are usually fine-tuned with a trial and error approach: the choice of the grids, Horvath (2012), the specification of the number of moments needed to accurately forecast the future endogenous state variables, Young (2005), and the functional forms of the forecasting rules. As these additional features are less controversial, I focus on the parameters of the ALM.

often crucial to the success of the procedure. A sensible choice is represented by the ALM computed for the corresponding economy with complete markets. Here I take a different route, to subject the KS algorithm to a thorough stress test.

4 The Perturbation Experiment

The procedure used to check for the existence of SFE is:

1. For a given calibration, solve the benchmark economy and store the vector of four parameters $\Theta^* = \left[\theta^*_{0,G}, \theta^*_{1,G}; \theta^*_{0,B}, \theta^*_{1,B}\right]$ representing the equilibrium ALM.
2. Choose a grid $X = \{x_1, x_2, ..., x_n\}$ for the perturbation factor x.
3. Set $x = x_1$.
4. Perturb the four parameters by drawing four random variates from a uniform distribution with support $[-x\%, +x\%]$.
5. Check that the new candidate ALM satisfies a requirement of non-explosive dynamics. If not, discard the perturbation.
6. Given the guesses $\Theta^g = \left[\theta^g_{0,G}, \theta^g_{1,G}; \theta^g_{0,B}, \theta^g_{1,B}\right]$, solve the households problem, simulate the economy and update the ALM parameters with a weighted average between the current guess and the parameters resulting from state-dependent OLS regressions on the simulated data.
7. Iterate until convergence of each of the four parameters in Θ and store them.
8. Repeat the procedure 500 times.
9. Move to the next case for x and redo the whole sequence of steps.

In a first set of experiments, I set $n = 4$ and $X = \{1, 2, 3, 4\}$. In particular, in these experiments all the parameters in Θ^* share the common perturbation factor x.

In another set of experiments, I still set $n = 4$, but I change the perturbation scheme. For three out of the four parameters the perturbation factor is set to a very small value $x = 0.1$ so that these parameters are always extremely close to their equilibrium ALM values. Differently, the support of the perturbation factor for the remaining parameter is set to a very large value. This corresponds to $[-10\%, +10\%]$ for $\theta^*_{1,G}$ and $\theta^*_{1,B}$, and to $[-25\%, +25\%]$ for $\theta^*_{0,G}$ and $\theta^*_{0,B}$.[6]

[6]Some experimentation showed that wider supports resulted in several cases either being discarded for not meeting the non-explosive dynamics restriction, or leading to a collapse in the ALM after some iterations. These instances do

The outcome of these procedures are distributions of the converged parameters, obtained by perturbing in a systematic way the equilibrium ones.

5 Results: SFE are not likely

Tables 2 and 3 report a set of statistics of the distributions of the converged ALM, after having randomly perturbed Θ^* 500 times. Table 2 considers the four sets of experiments where all four parameters are perturbed together, while Table 3 considers another four cases where only one parameter is perturbed by a sizeable amount, while the others are de-facto kept at their equilibrium values.

[Table 2 about here]

The statistics reported are the minimum, maximum, mean, median and standard deviation for each converged ALM parameter in the sequence of 500 perturbations. By inspecting their values, it is apparent that there is no evidence of multiple SFE. The range of the converged parameters is always tiny, and each parameter differs from its counterpart in Θ^* in the fourth decimal place, at worse. In particular, the equilibrium value for the first parameter is $\theta^*_{0,G} = 0.07269$, while its widest range combining the four experiments is $[0.07259, 0.07294]$. Similarly, $\theta^*_{1,G} = 0.97153$, while its widest range is $[0.97142, 0.97157]$, $\theta^*_{0,B} = 0.06426$, while its widest range is $[0.06412, 0.06459]$, and $\theta^*_{1,B} = 0.97282$, while its widest range is $[0.97269, 0.97287]$.[7]

Also when considering even poorer guesses, with one of the parameters being perturbed wildly, the values in Table 3 do not alter the picture: all the parameters converge in a neighborhood of the equilibrium ones.

[Table 3 about here]

The same results can be appreciated visually from the kernel density estimates of the distributions of the converged parameters, reported in Figure 2. Interestingly, the densities are found to be bimodal. The plots show that each converged parameter tends to cluster around two distinct values. There are

not represent an issue for the perturbation experiment, as a researcher would throw them out and try the model's solution with a different guess.

[7] As for the parameters converging to the boundaries of their support, for a subset of them I restart the procedure, using their values as the new initial guesses. Since they converge to values closer to Θ^*, these cases are likely due to the numerical approximation.

two possible interpretations for this outcome: a) these are genuinely two separate equilibria, or b) the clustering is induced by the numerical error. Given the size of the gap between the two values, it is safe to speculate that these differences are induced by the convergence criteria. Nevertheless, even if these were two distinct SFE, quantitatively the discrepancy is so minimal that it doesn't have any discernible effect on the outcomes of interest.[8] The business cycles statistics, correlations among endogenous variables, time series behavior of prices, Gini coefficients of wealth, percentages of households in debt, and ergodic distributions of aggregate capital are all virtually identical to the ones obtained with Θ^*.

[Figure 2 about here]

It is worthwhile to consider the economic reason behind the absence of multiple SFE. Fundamentally, this result lies in a self-correcting mechanism present in the algorithm. Compared to the equilibrium ALM, a candidate one implying a higher (lower) expected future capital reduces (increases) the equilibrium interest rates, increasing (reducing) the savings of the wealthy agents only. These, on the other hand, account for a small fraction of the population and cannot compensate for the opposite change triggered by the poor agents, who enjoy higher (lower) future wages and increase (reduce) their current consumption. Quantitatively, the change in behavior of the wealth-rich agents has a negligible impact on the determination of the change in the aggregate savings, inducing stability in the algorithm as a by-product.[9]

Given the simple market structure that is typically assumed in this class of models, with competitive markets there is a known relationship between the value of aggregate capital and the equilibrium prices, namely equations (1) and (2). This is one of the reasons underlying the effectiveness of the algorithm: for the agents to accurately predict the future prices, instead of using the whole endogenous distribution over the state variables (an infinite dimensional object), they use only a finite number of its moments (typically just the mean), contributing to deliver the celebrated approximate aggregation result.

[8] Some experimentation with different calibrations (e.g., log preferences) led to similar results. It is worth mentioning that a systematic study wasn't always possible in those alternative cases, because of the tendency for the ALM to diverge or collapse with some perturbations.

[9] This result is related to the findings in Giusto (2014). However, his stability result is local, while the current analysis is not, to capture the fact that a researcher might start the quest for the equilibrium ALM with a poor guess. As well, the benchmark KS model without preference heterogeneity he focuses on is less prone to SFE, because wealth concentration is very low, leading to a negligible mass of agents having a response in their savings going in the same direction as the forecasted change in K_{t+1}.

Mechanically, postulating parameters for the ALM that are above the equilibrium ones leads the agents to believe that more capital will be available in all future periods. Consequently, a higher future capital reduces the future interest rates and increases the future wages. As usual, these changes trigger three different effects affecting the intertemporal motive of savings: a human wealth effect, through increased wages from equation (2), and income and substitution effects, through decreased interest rates from equation (1). Furthermore, there is a complex response of precautionary savings: higher wages and UI benefits make the borrowing constraint less likely to be binding, while lower interest rates make the already accumulated wealth a less effective instrument to smooth consumption in the bad states of the world. The change in precautionary savings can go either way, but it is typically found to be quantitatively small.[10] Whether the individuals increase or decrease their savings (compared to their behavior under the equilibrium ALM) depends on their accumulated wealth, the fraction of income they obtain from capital and the relative change in prices.

All agents experience the three effects mentioned above, and the overall response of their savings depends on which ones dominate (assuming that the net effect on precautionary savings is always unimportant). As argued already, in principle there is indeed the possibility of multiple SFE. Because of consumption smoothing, wealthy individuals increase their savings, as lower interest rates decrease their future income, leading them to save more in the current period. Only for this class of agents the negative income effect more than compensates the sum of the human wealth and substitution effects. In contrast, poor individuals will enjoy higher wages and unemployment benefits in the future, a positive human wealth effect, which together with the substitution effect drives their savings down. For this class of agents, the sum of the human wealth and substitution effects more than offsets the negative income effect. Theoretically, it is hard to state whether in the aggregate the increased savings of the first group will more than compensate the decreased savings of the second one, namely to sign unambiguously the relative strengths of the human wealth, income and substitution effects. What makes this hard is finding the threshold value for accumulated wealth such that the negative income effects starts to dominate. Furthermore, there are several such thresholds, one for each possible combination of state variables (with the exception of individual wealth). However, these considerations should suggest that in the KS economy there is a self-correcting mechanism with respect to wrong guesses in the ALM, because the human wealth effect, the income effect for agents in debt, and the substitution effect, move the aggregate savings in the opposite direction. Quantitatively it turns out that, for a plausible calibration of the model, these thresholds are more than 100 times higher than the average income. Since the share of agents holding such high wealth levels is well below 1%, multiple

[10] Just like in Krusell and Smith (1998), I find that the aggregate capital (averaged over time) is higher in the incomplete markets economy compared to its complete markets counterpart by approximately 2%. This represents a measure of the importance of precautionary savings in the aggregate wealth.

SFE are not likely to arise in this model.

As a final remark, it goes without saying that both lower values of the CRRA parameter and/or preference homogeneity decrease the magnitude of the negative income effect, making the occurence of SFE even less likely.

6 Concluding remarks

Several algorithms tackling economies with heterogeneous agents and aggregate shocks have been recently developed, and their relative performance is discussed in den Haan (2010). The simplicity of the KS procedure, together with its successful implementation in many (and diverse) applications, make it often the method of choice. This note has showed that a potential threat to this methodology, multiple SFE, does not appear to be a relevant problem for a canonical version of the incomplete markets model with aggregate shocks and preference heterogeneity. However, it is not straightforward that this result will hold in substantially more complicated models, with several endogenous variables appearing in the specification of the ALM. Researchers applying this method should provide further evidence on the absence of multiple SFE: considering large systematic perturbations to the equilibrium ALM, as done here, is now feasible for a large set of models solved with the KS algorithm.

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Parameter	*Value*	*Target*
β - *Rate of time preference*	$\{0.9858, 0.9894, 0.9930\}$	*Average wealth Gini index* ≈ 0.80, *average interest rate* $\approx 1\%$
γ - *CRRA*	2.0	*Micro estimates on the Elasticity of Intertemporal Substitution*
δ - *Capital depreciation rate*	0.025	*Average investment share of output* $\approx 26\%$
α - *Capital share*	0.36	*Capital share of output* $= 36\%$
b - *Borrowing limit*	-2.0	*Average share of households in debt* $\approx 10\%$
l - *Labor supply*	0.3271	*Share of market time, with total time endowment* $= 1$
ρ - *UI Replacement Rate*	0.40	*Average UI replacement rate*

Table 1: Calibration.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×100*
Perturbation Range: ±1%					
$\theta^*_{0,G}$	0.07259	0.07284	0.07269	0.07269	0.00221
$\theta^*_{1,G}$	0.97147	0.97157	0.97153	0.97153	0.00091
$\theta^*_{0,B}$	0.06412	0.06446	0.06426	0.06425	0.00335
$\theta^*_{1,B}$	0.97274	0.97287	0.97282	0.97282	0.00134
Perturbation Range: ±2%					
$\theta^*_{0,G}$	0.07260	0.07290	0.07270	0.07269	0.00215
$\theta^*_{1,G}$	0.97144	0.97157	0.97153	0.97153	0.00089
$\theta^*_{0,B}$	0.06413	0.06453	0.06427	0.06425	0.00323
$\theta^*_{1,B}$	0.97271	0.97287	0.97282	0.97282	0.00129
Perturbation Range: ±3%					
$\theta^*_{0,G}$	0.07263	0.07294	0.07270	0.07269	0.00226
$\theta^*_{1,G}$	0.97142	0.97156	0.97153	0.97153	0.00095
$\theta^*_{0,B}$	0.06416	0.06459	0.06427	0.06425	0.00336
$\theta^*_{1,B}$	0.97269	0.97286	0.97282	0.97282	0.00135
Perturbation Range: ±4%					
$\theta^*_{0,G}$	0.07263	0.07281	0.07270	0.07269	0.00190
$\theta^*_{1,G}$	0.97148	0.97155	0.97153	0.97153	0.00079
$\theta^*_{0,B}$	0.06417	0.06441	0.06426	0.06425	0.00292
$\theta^*_{1,B}$	0.97276	0.97286	0.97282	0.97282	0.00117

Table 2: Results - 500 perturbations, per perturbation range; all ALM parameters are perturbed by independent random draws from a uniform distribution. The Equilibrium ALM is: $\theta^*_{0,G} = 0.07269$, $\theta^*_{1,G} = 0.97153$, $\theta^*_{0,G} = 0.06426$, $\theta^*_{0,G} = 0.97282$.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×100*
Perturbation Range for $\theta_{0,G}$*:* $\pm 25\%$					
$\theta^*_{0,G}$	0.07260	0.07288	0.07270	0.07269	0.00189
$\theta^*_{1,G}$	0.97145	0.97157	0.97153	0.97153	0.00078
$\theta^*_{0,B}$	0.06413	0.06450	0.06426	0.06425	0.00273
$\theta^*_{1,B}$	0.97273	0.97287	0.97282	0.97282	0.00109
Perturbation Range for $\theta_{1,G}$*:* $\pm 10\%$					
$\theta^*_{0,G}$	0.07258	0.07280	0.07269	0.07269	0.00098
$\theta^*_{1,G}$	0.97148	0.97157	0.97153	0.97153	0.00041
$\theta^*_{0,B}$	0.06411	0.06440	0.06425	0.06425	0.00152
$\theta^*_{1,B}$	0.97276	0.97288	0.97282	0.97282	0.00061
Perturbation Range for $\theta_{0,B}$*:* $\pm 25\%$					
$\theta^*_{0,G}$	0.07260	0.07282	0.07269	0.07269	0.00253
$\theta^*_{1,G}$	0.97148	0.97156	0.97153	0.97153	0.00104
$\theta^*_{0,B}$	0.06413	0.06443	0.06426	0.06425	0.00391
$\theta^*_{1,B}$	0.97275	0.97287	0.97282	0.97282	0.00157
Perturbation Range for $\theta_{1,B}$*:* $\pm 10\%$					
$\theta^*_{0,G}$	0.07260	0.07292	0.07270	0.07271	0.00205
$\theta^*_{1,G}$	0.97143	0.97157	0.97152	0.97152	0.00085
$\theta^*_{0,B}$	0.06413	0.06456	0.06427	0.06429	0.00310
$\theta^*_{1,B}$	0.97270	0.97287	0.97281	0.97281	0.00124

Table 3: Results - 500 perturbations, per perturbation range; only one ALM parameter is perturbed by a random draw from a uniform distribution. The Equilibrium ALM is: $\theta^*_{0,G} = 0.07269$, $\theta^*_{1,G} = 0.97153$, $\theta^*_{0,G} = 0.06426$, $\theta^*_{0,G} = 0.97282$.

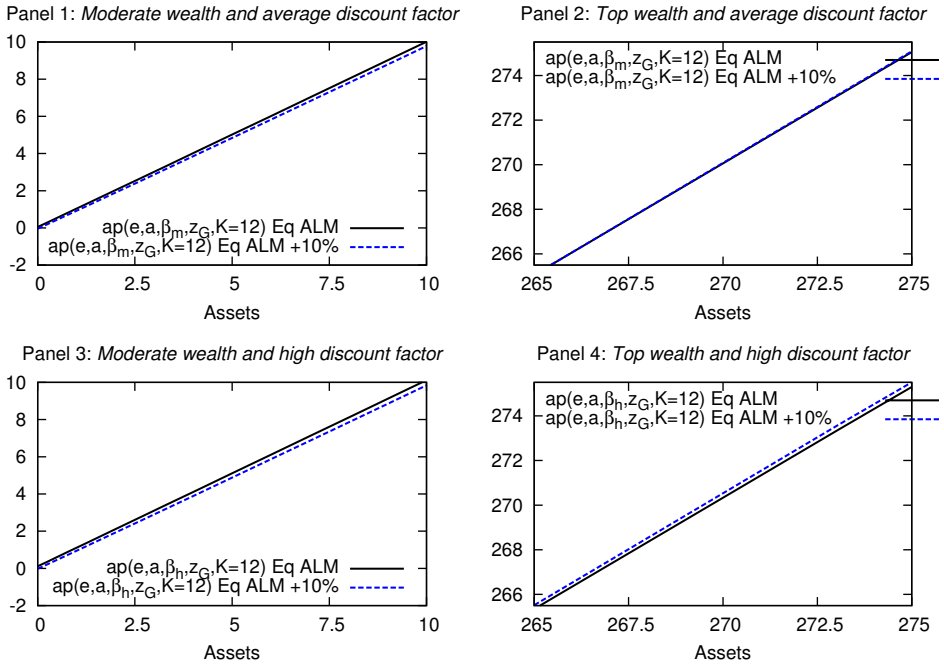

Figure 1: Differential response of savings for selected household types, Equilibrium ALM (solid line) Vs. Perturbation = 10% (dashed line).

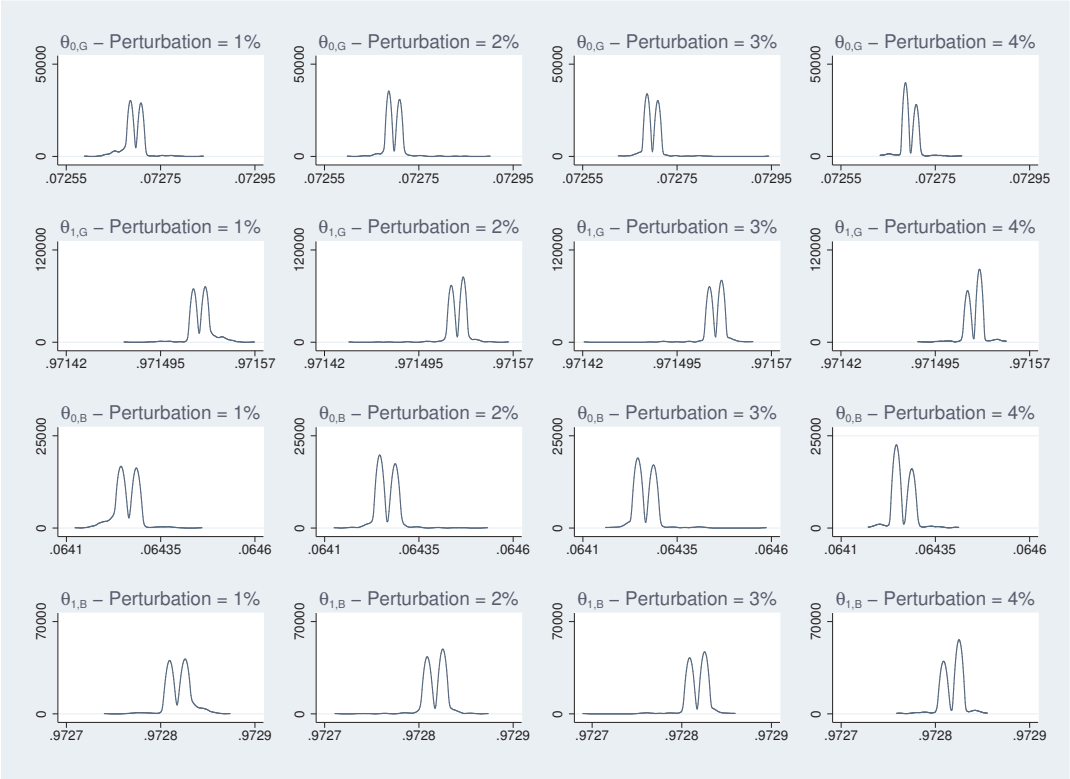

Figure 2: Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.

Appendix A - Computation

- All codes were written in the FORTRAN 95 language, relying on the Intel Fortran Compiler, build 11.1.048 (with the IMSL library). They were compiled selecting the O3 option (maximize speed), and without automatic parallelization. They were executed on a 64-bit PC platform, running Windows 7 Professional Edition, with an Intel $i7-2600k$ Quad Core processor clocked at 4.6 Ghz.

- The replications take more than 8 days to complete. Notice that 500 equilibria have to be computed, and typically 7 to 16 iterations on the ALM are needed to find each equilibrium.

- In the actual solution of the model I need to discretize the continuous state variables a and K (the employment status s, the preference heterogeneity β, and the aggregate productivity shock z are already discrete). For the household assets a I rely on an unevenly spaced grid, with the distance between two consecutive points increasing geometrically. This is done to allow for a high precision of the policy rules at low values of a, where the change in curvature is more pronounced. In order to keep the computational burden manageable, I use 51 grid points on the household assets space, the lowest value being the borrowing constraint b and the highest one being a value $a_{\max}$ high enough for the saving functions to cut the 45 degree line ($a_{\max} = 300$). Following Horvath (2012), I use a grid in the aggregate capital dimension K that has more points around its average value (computed in the simulations). Over the $[3, 27]$ interval I use 25 points, which are far more than the typical 4-6. However, in the iterative process on the ALM parameters, the simulations do visit regions of the state space that are very far from the support of the ergodic equilibrium distribution, causing convergence issues when using a coarse grid.

- As for the solution method for the household problem, I use a time iteration procedure on the set of Euler equations, guessing the future saving functions, and with linear interpolation in the (a, K) dimensions. This method proved to be more stable than the relatively common value function iteration scheme with cubic spline interpolation.

- A collocation method is implemented, that is I look for the policy functions such that the residuals of the Euler equations are (close to) zero at the collocation points (which correspond to the asset grid). It follows that for all possible combinations of state variables I need to solve a non-linear equation. To get the optimal policy functions, I compute the first order conditions with respect to a' and through the envelope condition I obtain a set of Euler equations, whose unknowns are the policy functions, $a'(a, s, \beta, z, K)$. I start from a set of guesses, $a'(a, s, \beta, z, K)_0$, and keep on iterating until a fixed point is reached, i.e. until two successive iterations satisfy:

$\underset{a}{Sup}\,|a'(a,s,\beta,z,K)_{n+1} - a'(a,s,\beta,z,K)_n| < 10^{-6}$, $\forall s, \forall \beta, \forall z, \forall K$.

- The aggregate dynamics are computed by simulating a large sample of $30,000$ individuals for $5,000$ periods, with the first $1,000$ periods being discarded as a burn-in. As for the approximation method, I rely on a linear approximation scheme for the saving functions, for values of a and K falling outside the grid.

Appendix B - The Complete Calibration

The transition matrix of the individual shocks, conditional on the aggregate ones, is (each entry refers to the probability $\pi_{z,z';s,s'}$):

$$\pi\left(z,z',s,s'\right) = \begin{bmatrix} \pi_{G,G;u,u} & \pi_{G,G;u,e} & \pi_{G,B;u,u} & \pi_{G,B;u,e} \\ \pi_{G,G;e,u} & \pi_{G,G;e,e} & \pi_{G,B;e,u} & \pi_{G,B;e,e} \\ \pi_{B,G;u,u} & \pi_{B,G;u,e} & \pi_{B,B;u,u} & \pi_{B,B;u,e} \\ \pi_{B,G;e,u} & \pi_{B,G;e,e} & \pi_{B,B;e,u} & \pi_{B,B;e,e} \end{bmatrix} = \begin{bmatrix} 0.292 & 0.583 & 0.094 & 0.031 \\ 0.024 & 0.851 & 0.009 & 0.116 \\ 0.031 & 0.094 & 0.525 & 0.350 \\ 0.002 & 0.123 & 0.039 & 0.836 \end{bmatrix}$$

The transition matrix for the preference heterogeneity is:

$$\pi\left(\beta,\beta'\right) = \begin{bmatrix} 0.995 & 0.005 & 0.000 \\ 0.000625 & 0.99875 & 0.000625 \\ 0.000 & 0.005 & 0.995 \end{bmatrix}$$

The transition matrix of the aggregate shocks is:

$$\pi\left(z,z'\right) = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$$

Appendix C - Additional Results

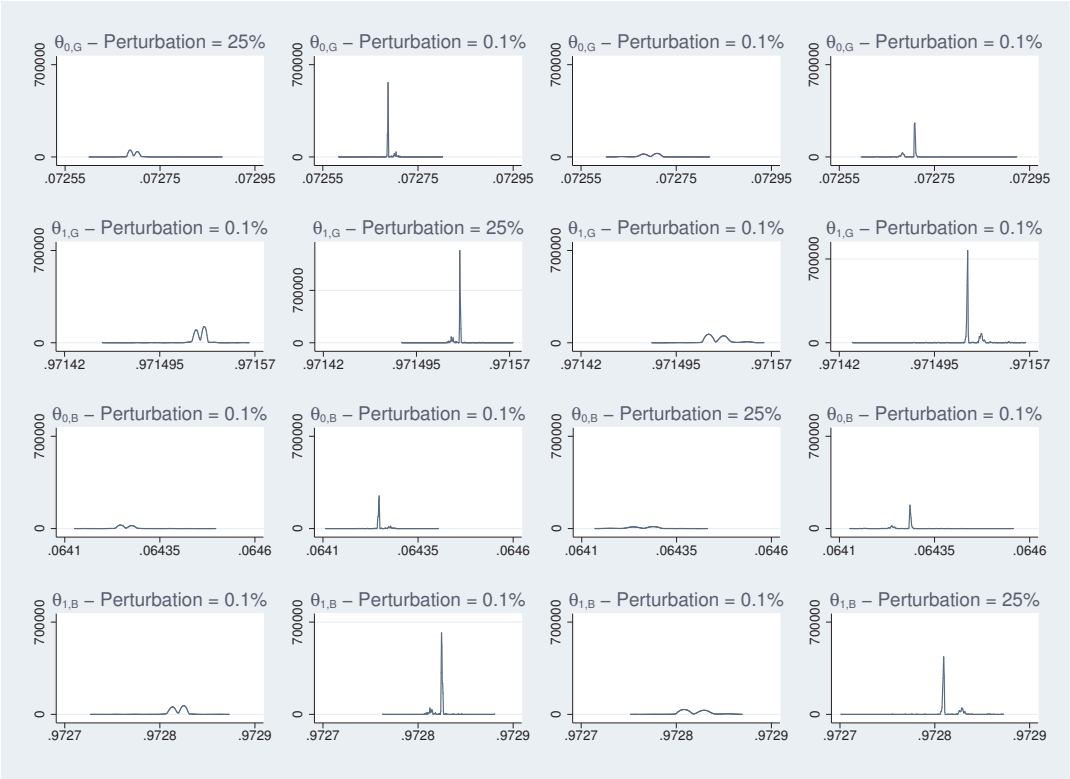

Figure 3: Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×100*
Perturbation Range: ±1%					
$\theta^*_{0,G}$	0.07289	0.07327	0.07302	0.07302	0.00324
$\theta^*_{1,G}$	0.97133	0.97149	0.97143	0.97144	0.00137
$\theta^*_{0,B}$	0.06456	0.06503	0.06473	0.06472	0.00432
$\theta^*_{1,B}$	0.97256	0.97274	0.97268	0.97268	0.00173
Perturbation Range: ±2%					
$\theta^*_{0,G}$	0.07297	0.07325	0.07303	0.07302	0.00291
$\theta^*_{1,G}$	0.97134	0.97145	0.97143	0.97144	0.00124
$\theta^*_{0,B}$	0.06466	0.06501	0.06474	0.06472	0.00386
$\theta^*_{1,B}$	0.97257	0.97271	0.97267	0.97268	0.00155
Perturbation Range: ±3%					
$\theta^*_{0,G}$	0.07299	0.07328	0.07302	0.07302	0.00266
$\theta^*_{1,G}$	0.97132	0.97144	0.97143	0.97144	0.00113
$\theta^*_{0,B}$	0.06469	0.06504	0.06473	0.06472	0.00348
$\theta^*_{1,B}$	0.97256	0.97270	0.97268	0.97268	0.00139
Perturbation Range: ±4%					
$\theta^*_{0,G}$	0.07298	0.07325	0.07303	0.07302	0.00238
$\theta^*_{1,G}$	0.97134	0.97145	0.97143	0.97144	0.00101
$\theta^*_{0,B}$	0.06466	0.06501	0.06474	0.06472	0.00328
$\theta^*_{1,B}$	0.97257	0.97270	0.97268	0.97268	0.00131

Table 4: Robustness ($b = 0$) - 500 perturbations, per perturbation range; all ALM parameters are perturbed by independent random draws from a uniform distribution. The Equilibrium ALM is: $\theta^*_{0,G} = 0.07304$, $\theta^*_{1,G} = 0.97143$, $\theta^*_{0,G} = 0.06475$, $\theta^*_{0,G} = 0.97267$.

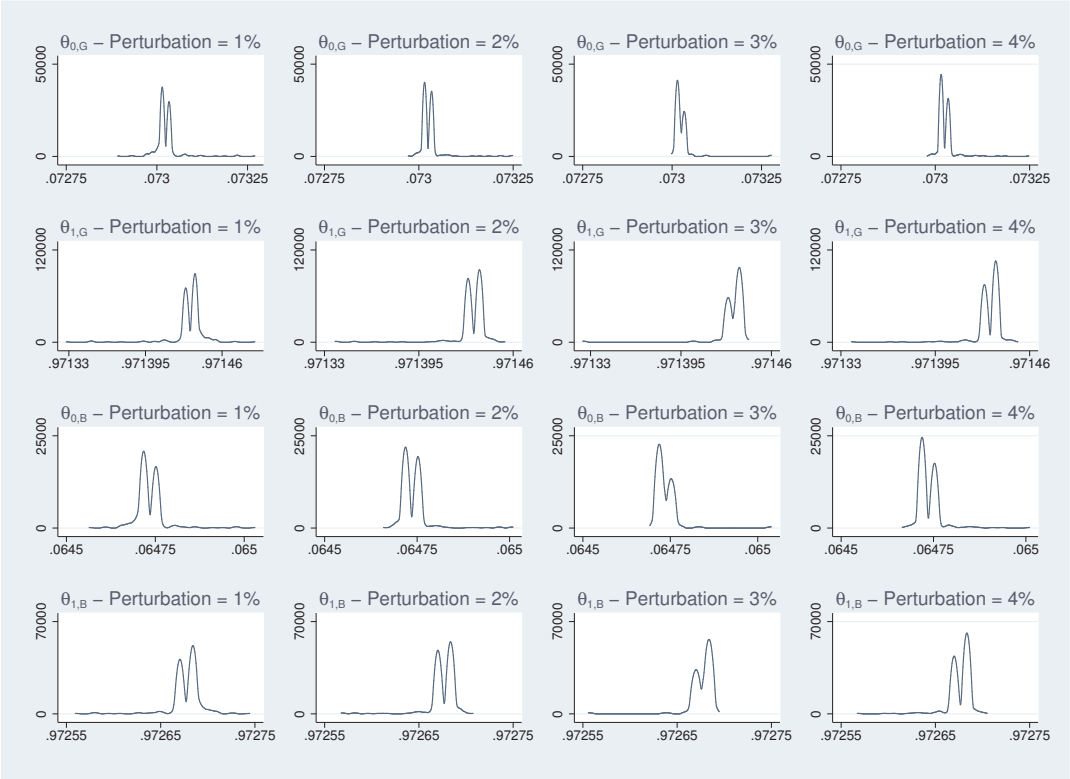

Figure 4: Robustness ($b = 0$) - Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.