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# THE COST OF SEGREGATION IN SOCIAL NETWORKS

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ABSTRACT. This paper investigates the private provision of public goods in segregated societies. While most research agrees that segregation undermines public provision, the findings are mixed for private provision: social interactions, being strong within groups and limited across groups, may either increase or impede voluntary contributions. Moreover, although efficiency concerns generally provide a rationale for government intervention, surprisingly, little light is shed in the literature on the potential effectiveness of such intervention in a segregated society. This paper first develops an index based on social interactions, which, roughly speaking, measures the welfare impact of income redistribution in an arbitrary society. It then shows that the proposed index vanishes when applied to large segregated societies, which suggests an “asymptotic neutrality” of redistributive policies.

*JEL classification:* C72, D31, H41.

*Keywords:* public goods, segregated society, private provision, networks, Bonacich transfer index.

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## 1. INTRODUCTION

Diversity is becoming a pervasive feature of most societies. Yet, in spite of the numerous gains from cultural differences within society, diversity often breeds segregation, which is detrimental to public goods provision. Segregation may occur along one or a few lines such as ethnicity, religion, language, and income, and its main aspect of limited social interactions across different groups is perceived to undermine the quality of public amenities and hamper public projects. There is robust empirical evidence in the literature; in fact, amongst others, to quote Banerjee, Iyer, and Somanathan (2005), it is “*One of the most powerful hypotheses in political economy...*”.

The literature is furnished with a variety of mechanisms to explore the channels through which segregation operates on public provision of public goods. The divergence in preferences across groups for public goods - languages of instruction at school or the location of the highway - sharply dilutes the support for their provision in Alesina, Baqir, and Easterly (1999) and restricts the choice of optimal funding policies in Fernández and Levy (2008). Ethnic fragmentation also results in less spending on education in Poterba (1997) and Goldin and Katz (1999) and reduces growth in Easterly and Levine (1997) and Alesina and La Ferrara (2005). Besley, Pande, Rahman, and Rao (2004) show that leaders provide public goods essentially to their ethnic groups, largely excluding others, while Vigdor (2004) observes a low demand for public goods due to minimal altruistic preferences.

This paper investigates the private provision of public goods in segregated societies. In general, public goods are provided by both government and individuals. Private contributions account for the provision of many important public goods ranging from charitable education and health care to essential infrastructure. The access to private contributions, however, may often be constrained by geographical location or social interactions, benefiting neighbors and acquaintances, while effectively excluding others. A recent literature, by Bramoullé and Kranton (2007), Bramoullé, Kranton, and D’Amours (2012), and Allouch (2012), has investigated public goods games, where consumers may benefit only from neighbors’ provision,

generalizing the standard model of private provision of pure public goods. In addition, Bloch and Zenginobuz (2007) examine a standard local public good model with spillovers between jurisdictions, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) incorporate private information, Galeotti and Goyal (2010) investigate issues of network formation, Rébillé and Richefort (2012) provide a welfare analysis, and Elliott and Golub (2013) explore decentralized mechanisms for efficient provision. However, unlike the case of public provision, the implication of segregation for private provision is unclear. On one hand, segregation may raise private provision due to the strong feeling of solidarity within groups as found in Fong and Luttmer (2009), but, on the other hand, it may decrease private provision due to the weak social attachments across groups as found in Miguel and Gugerty (2005). In addition, while efficiency concerns generally provide a rationale for government intervention in private markets, it remains to be seen whether such intervention is effective in segregated societies.

We begin our analysis by investigating the impact of government intervention in a society with an arbitrary but fixed network of social interactions. Government intervention in private provision aims to achieve socially optimal outcomes, which is very much in the spirit of the second welfare theorem, although, unlike competitive equilibrium, the Nash equilibrium outcomes will typically be inefficient. The channel of government intervention is lump-sum income redistribution, which plays a central role in economics for achieving various redistributive goals, and is often employed as a benchmark for other channels of intervention. The scale of income redistribution is crucial to our analysis since, similar to the private provision literature, we focus on budget-balanced transfers of relatively small magnitude so that the set of contributors remains unchanged. It is well known from Warr (1983) and Bergstrom, Blume, and Varian (1986) that the private provision model of pure public goods is subject to a strong neutrality result, whereby income redistribution has no effect on either the aggregate provision of public goods or the consumption of private goods.<sup>1</sup>

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<sup>1</sup>The neutrality result, further analyzed in Bernheim (1986) and Andreoni (1989), is equivalent to complete crowding-out, “dollar for dollar” for tax-financed government provision, which has traditionally been the focus of much attention.

Neutrality of income redistribution can be a serious problem for public goods that rely mostly on private provision as it may limit the effectiveness of government intervention. In the case of local public goods, where not all consumers are necessarily linked to each other, it is unclear how much of the income redistribution affects consumers' welfare or, equivalently, how much of the government intervention is negated by consumers' actions. To this effect, we show that, under a standard utilitarian approach, the impact of income redistribution on social welfare is determined by the Bonacich centrality. Bonacich centrality, due to Bonacich (1987), is a vector that measures power and prestige in social networks and is shown to be related to the Nash equilibrium outcomes of a game by the key contribution of Ballester, Calvó-Armengol, and Zenou (2006).<sup>2</sup> Quite different from the Nash–Bonacich linkage, Allouch (2012) shows that the impact of income redistribution on the aggregate provision of public goods is also determined by the Bonacich centrality.

In order to compare the welfare impact of income redistribution across societies of different sizes and different networks of social interactions, we introduce a new index, called the Bonacich transfer index, which measures the potential per capita welfare gain after income redistribution. Understandably, the proposed index is closely related to the Bonacich centrality vector since it is the norm of its projection on the hyperplane of budget-balanced transfers normalized by the size of the society. Intuitively, the higher is the Bonacich transfer index, the more per capita welfare gains may be achieved from income redistribution and, actually, in this regard, we show that the index may take a wide range of values. For instance, for a society with a regular network of social interactions the Bonacich transfer index is zero, whereas for a society with a star network of social interactions the Bonacich transfer index may be unbounded. Therefore, as developed, the Bonacich transfer index may be thought of as a summary statistic of the efficacy of government intervention based on the complex network of social interactions.

Finally, we further conduct our analysis of the welfare impact of government intervention in segregated societies. Social interactions in segregated societies are

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<sup>2</sup>Related results include Candogan, Bimpikis, and Ozdaglar (2010) for monopoly pricing and İlkiliç (2011) for the tragedy of commons.

represented by network structures, whereby the density of inward social ties for each group is greater than the density of outward social ties. Segregation can emerge from very different social processes and network formation dynamics. Schelling (1969) provides a simple, yet powerful, model showing that very mild individual preferences for having neighbors of the same type may lead to full segregation, even though no individual prefers the final outcome. The case of strong individual preferences is referred to as the homophily principle in sociology: the tendency of individuals to disproportionately form social ties with others similar to themselves. Homophily is a well-documented pattern of social networks and often called upon to understand various social interactions such as friendship and marriage, job market outcomes, speed of information diffusion, and even social mobility. There is an emerging literature in the economics of social networks, by Currarini, Jackson, and Pin (2009), Bramoullé, Currarini, Jackson, Pin, and Rodgers (2012), and Golub and Jackson (2012), that models a random process of network formation strongly influenced by homophily.

Our approach, although it is quite different, takes advantage of the insights of the above-mentioned literatures, since we investigate societies with fixed network structures of social interactions that already display segregation and not the matching processes nor the network formation dynamics leading to them. As such, the Bonacich transfer index, developed in this paper, enables us to investigate the impact of income redistribution on welfare in segregated societies with particular network structures of social interactions. More specifically, our spectral analysis shows that the Bonacich transfer index vanishes in large segregated societies, which implies an “asymptotic neutrality” of income redistribution. Although this result mirrors the widely-known neutrality result for pure public goods, it is quite different in interpretation. More specifically, the asymptotic neutrality, unlike neutrality, allows for the possibility of income redistribution raising social welfare, but rules out the possibility of it raising per capita welfare.

The paper is organized as follows. Section 2 introduces the private provision of public goods in networks. Section 3 relates the impact of income redistribution on welfare to the Bonacich centrality vector. Section 4 introduces the Bonacich transfer

index for a society with an arbitrary but fixed network of social interactions. Section 5 applies the Bonacich transfer index to segregated societies. Section 6 provides an example of integrated versus segregated society meant to give an intuitive feel for the proposed index. We conclude the paper in Section 7 and prove some of our results in Section 8.

## 2. THE MODEL

We consider a society comprising  $n$  consumers embedded on a fixed network  $\mathbf{g}$  of social interactions. We denote by  $\mathcal{N}_i$  consumer  $i = 1, \dots, n$ 's neighbors in the network  $\mathbf{g}$ . The preferences of each consumer  $i$  are represented by a twice continuously differentiable, strictly increasing, and strictly quasi-concave utility function  $u_i(x_i, q_i + Q_{-i})$ , where  $x_i$  is consumer  $i$ 's private good consumption,  $q_i$  is consumer  $i$ 's public good provision, and  $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$  is the sum of public good provisions of consumer  $i$ 's neighbors in the society. Furthermore, the public good can be produced from the private good via a unit-linear production technology. Therefore, the prices of the private good and the public good can be normalized to  $\mathbf{p} = (p_x, p_Q) = (1, 1)$ . Each consumer  $i$  faces the utility maximization problem

$$\begin{aligned} \max_{x_i, q_i} & u_i(x_i, q_i + Q_{-i}) \\ \text{s.t.} & \quad x_i + q_i = w_i \text{ and } q_i \geq 0, \end{aligned}$$

where  $w_i$  is his income (exogenously fixed). The utility maximization problem can be represented equivalently as

$$\begin{aligned} \max_{x_i, Q_i} & u_i(x_i, Q_i) \\ \text{s.t.} & \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i}, \end{aligned} \tag{1}$$

where consumer  $i$  chooses his (local) public good consumption,  $Q_i = q_i + Q_{-i}$ . Let  $\gamma_i$  be the Engel curve of consumer  $i$ . Then consumer  $i$ 's local public good demand is

$$Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\},$$

or, equivalently,

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\}. \quad (2)$$

Let  $\mathbf{G} = [g_{ij}]$  denote the adjacency matrix of the network  $\mathbf{g}$ , where  $g_{ij} = 1$  indicates that consumer  $i \neq j$  are neighbors and  $g_{ij} = 0$  otherwise. The adjacency matrix of the network,  $\mathbf{G}$ , is symmetric with nonnegative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by  $\lambda_{\max}(\mathbf{G}) = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = \lambda_{\min}(\mathbf{G})$ , where  $\lambda_{\max}(\mathbf{G})$  is the largest eigenvalue and  $\lambda_{\min}(\mathbf{G})$  is the lowest eigenvalue of  $\mathbf{G}$ . By the Perron–Frobenius Theorem, it holds that  $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G}) > 0$ .

We consider the following network-specific normality assumption:

**Network normality.** For each consumer  $i = 1, \dots, n$ , the Engel curve  $\gamma_i$  is differentiable and it holds that  $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma'_i(\cdot) < 1$ .

**Proposition 1.** *Assume network normality. Then there exists a unique Nash equilibrium for the private provision.*

**Proof.** See Allouch (2012).□

The network normality assumption amounts to both the normality of the private good and a strong normality of the public good. The seminal contribution of Bergstrom, Blume, and Varian (1986) shows that the assumption that both the private good and the public good are normal is sufficient to guarantee the existence and uniqueness of a Nash equilibrium in the standard model of private provision. Bramoullé, Kranton, and D’Amours (2012) investigate the existence and uniqueness of a Nash equilibrium in games of strategic substitutes on networks with linear best-reply functions. More generally, their contribution shows that the lowest eigenvalue,  $\lambda_{\min}(\mathbf{G})$ , is key to equilibrium analysis. Building on the above important contributions, Allouch (2012) introduces the assumption of network normality and establishes the existence and uniqueness of a Nash equilibrium in the private provision of public goods on networks, which simultaneously extends Bergstrom, Blume, and Varian (1986) to networks and Bramoullé, Kranton, and D’Amours (2012) to nonlinear best-reply functions.



### 3. GOVERNMENT INTERVENTION IN PRIVATE PROVISION

This section investigates the impact of government intervention on private provision of public goods. The government aims to achieve socially optimal outcomes by drawing on income redistribution as a policy instrument. Income redistribution takes the form of lump-sum transfers, which are traditionally viewed as a reference point for other policy instruments. We denote by a budget-balanced transfer, a  $\mathbf{t} = (t_1, t_2, \dots, t_n)^T \in \mathbb{R}^n$  such that  $\sum_{i=1}^n t_i = 0$ . Let  $\mathbf{q}^* = (q_1^*, \dots, q_n^*)^T$  denote the Nash equilibrium corresponding to the income distribution  $\mathbf{w} = (w_1, \dots, w_n)^T$  and  $\mathbf{q}^{\mathbf{t}} = (q_1^{\mathbf{t}}, \dots, q_n^{\mathbf{t}})^T$  denote the Nash equilibrium corresponding to the income distribution  $\mathbf{w} + \mathbf{t} = (w_1 + t_1, \dots, w_n + t_n)^T$ . Similar to Warr (1983) and Bergstrom, Blume, and Varian (1986), we will focus our analysis on income redistributions that leave the set of contributors unchanged, and we will refer to them as “relatively small”.

For simplicity, from now on we will focus our analysis on particular preferences:

**Gorman polar form preferences.** There exists a real number  $a$  such that, for each consumer  $i = 1, \dots, n$ , it holds that  $\gamma'_i(\cdot) = 1 - a$ .

Although the assumption of Gorman polar form preferences is quite restrictive, it includes some interesting and important classes of preferences; for instance, both Cobb–Douglas preferences and quasi-linear preferences with respect to a common numeraire satisfy this assumption.

In general, there are compelling reasons for presuming that not all consumers will be contributing to public goods. For simplicity also, passing to subnetworks if necessary, we assume that all consumers are contributors.<sup>3</sup> Finally, we will assume throughout the paper network normality, which is equivalent to  $a \in ]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$ .

**3.1. The Bonacich centrality measure.** In the social networks literature, a variety of network measures have been proposed to explore the potential importance, power, and influence of individuals (or institutions) in social interactions. The most intuitive network measure is degree centrality, defined as the number of immediate

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<sup>3</sup>Notice that an income distribution almost proportional to the eigenvector centrality, the unique unit eigenvector associated with  $\lambda_{\max}(\mathbf{G})$ , will always lead to an interior Nash equilibrium.

neighbors in the network, which gives importance to individuals with more connections. Obviously, the fact that degree centrality overlooks indirect influences from distant neighbors gives rise to the use of rather global network measures. The key contribution of Ballester, Calvó-Armengol, and Zenou (2006) relate the Nash equilibrium outcomes of a game to the Bonacich centrality, due to Bonacich (1987), defined by

$$\mathbf{b}(\mathbf{G}, \delta) = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1},$$

where  $\mathbf{I}$  is the identity matrix,  $\mathbf{1}$  is the  $n$ -dimensional vector with all components equal to one, and  $\delta$  is the attenuation parameter. Since for  $|\delta| < \frac{1}{\lambda_{\max}(\mathbf{G})}$  it holds that

$$\mathbf{b}(\mathbf{G}, \delta) = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1} = \sum_{k=0}^{+\infty} \delta^k \mathbf{G}^k \mathbf{1}$$

the Bonacich centrality of a consumer  $i$  can be expressed as follows:

$$\mathbf{b}_i(\mathbf{G}, \delta) = \sum_{k=0}^{+\infty} \delta^k \sum_{j=1}^n (\mathbf{G}^k)_{ij}.$$

Given that  $(\mathbf{G}^k)_{ij}$  counts the total number of walks of length  $k$  emanating from  $i$  and terminating at  $j$ , it follows that the Bonacich centrality of consumer  $i$  counts the number of walks emanating from  $i$  discounted by  $\delta$  to the power of their length. Notice that the attenuation parameter  $\delta$  captures the decay of influence of distant consumers.

Quite different from the Nash–Bonacich linkage, Allouch (2012) shows that the impact of income redistribution on the aggregate provision of public goods,  $Q = \sum_{i=1}^n q_i$ , is related to a generalization of the Bonacich centrality. The following proposition reproduces the result for the special case of preferences of the Gorman polar form investigated in this paper.

**Proposition 2.** *For any relatively small transfer  $\mathbf{t}$ , it holds that*

$$\mathbf{q}^{\mathbf{t}} - \mathbf{q}^* = (1 - a)(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{t} \quad \text{and hence} \quad Q^{\mathbf{t}} - Q^* = (1 - a) \mathbf{b}(\mathbf{G}, -a) \cdot \mathbf{t}.$$

**Proof.** See the Appendix.  $\square$

**Remark 1.** The assumption of network normality, needed for the uniqueness of a Nash equilibrium, may be relaxed and Proposition 2 still holds partially. Indeed, for almost any  $a \in ]0, 1[$ , the matrix  $\mathbf{I} + a\mathbf{G}$  is invertible. Moreover, similar to Theorem 1 in Bergstrom, Blume, and Varian (1986), for a Nash equilibrium  $\mathbf{q}^*$  corresponding to the income distribution  $\mathbf{w} = (w_1, \dots, w_n)^T$  and any transfer of relatively small magnitude  $\mathbf{t}$ , it may be easily checked that  $\mathbf{q}^\dagger = \mathbf{q}^* + (1 - a)(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{t}$  is a Nash equilibrium corresponding to the income distribution  $\mathbf{w} + \mathbf{t} = (w_1 + t_1, \dots, w_n + t_n)^T$ .

**3.2. Welfare analysis.** In order to investigate the welfare impact of income redistribution, we take a standard utilitarian approach. More specifically, we consider the (indirect) social welfare function

$$\mathcal{SW}(\mathbf{w}) \stackrel{\text{def}}{=} \sum_{i=1}^n u_i(x_i^*, Q_i^*),$$

which is the sum of utilities achieved by consumers at the unique Nash equilibrium with income distribution  $\mathbf{w} = (w_1, \dots, w_n)$ .

**Proposition 3.** *Assume network normality. Then there exists a positive real number  $\kappa$  such that for any relatively small transfer  $\mathbf{t}$  it holds that*

$$\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w}) = -\kappa \mathbf{b}(\mathbf{G}, -a) \cdot \mathbf{t}.$$

**Proof.** See the Appendix.  $\square$

Proposition 3 shows that the impact of income redistribution on welfare is determined by the Bonacich centrality vector.

One of the most deeply ingrained ideas when thinking about public goods is that they are always underprovided by a system of private provision. Surprisingly, Propositions 2 and 3 show that the impacts of income redistribution on aggregate provision and social welfare are determined by the Bonacich centrality vector, although by pulling the income redistribution in opposite directions. More precisely, an income transfer carried out from a high Bonacich centrality consumer to a low Bonacich centrality consumer always increases social welfare and decreases aggregate provision. As a consequence, one may conclude that when public goods are

provided solely by voluntary contribution, raising social welfare and raising aggregate provision are sharply conflicting policy objectives.

The underlying economic intuition for the above observation may be explained as follows. First, each consumer cares only about the sum of own and neighbors' public goods provision, which may well be different from the aggregate provision. Moreover, a second-best welfare maximization argument stipulates transferring income to consumers that, due to their network position, face a low social cost to produce public goods. Understandably, such transfers of income simultaneously increase social welfare and reduce aggregate provision.

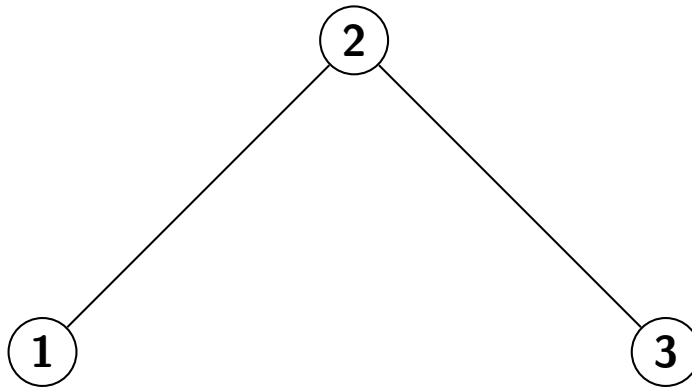


Figure 1: The star network with three consumers

**Example 1.** Consider a society with three consumers and the star network of social interactions described in Figure 1. The Bonacich centrality vector of the network of social interactions for  $a \in ]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$  is

$$\mathbf{b}(\mathbf{G}, -a) = (\mathbf{b}_1(\mathbf{G}, -a), \mathbf{b}_2(\mathbf{G}, -a), \mathbf{b}_3(\mathbf{G}, -a))^T = \frac{1}{1 - 2a^2}(1 - a, 1 - 2a, 1 - a)^T.$$

We first observe that the ranking of consumers produced by the Bonacich centrality vector is insensitive to the value of  $a$ . Indeed, consumers 1 and 3 are equally ranked and always have a higher Bonacich centrality than consumer 2.

Moreover, given the network structure of social interactions, one may naturally expect consumer 2 having the highest centrality in most network measures; however, it is clear that the Bonacich centrality was not intended to capture the importance of

consumers but rather their social cost to produce public goods. For instance, starting from a private provision equilibrium where the three consumers are contributors, a relatively small income transfer  $\mathbf{t} = (t_1, t_2, t_3)$  with  $t_2 > 0$  always raises social welfare and decreases aggregate provision.

#### 4. AN ECONOMIC INDEX FOR INCOME REDISTRIBUTION

**4.1. The Bonacich transfer index.** Our investigation shows that the Bonacich centrality vector is key to understanding the impact of income redistribution on welfare. In the following, we would like to compare the welfare impact of income redistribution across societies of different sizes and different networks of social interactions. To be able to do so, we introduce a new network measure, called the Bonacich transfer index, defined by

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}, -a) \stackrel{\text{def}}{=} \max_{\mathbf{t} \in \mathcal{B}_{\mathcal{T}}} \frac{\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w})}{n},$$

where  $\mathcal{B}_{\mathcal{T}} = \{t \in \mathbf{1}^{\perp} \mid \|t\| \leq 1\}$  denote the unit ball in the hyperplane of budget-balanced transfers  $\mathbf{1}^{\perp}$ . The Bonacich transfer index measures the potential per capita welfare gain per unit of redistribution for a society based on the network of social interactions. This corresponds to an average utilitarian approach to welfare, which is adequate to deal with a change in the size of the society.<sup>4</sup>

The following result shows that the Bonacich transfer index corresponds to the norm of the projection of the Bonacich centrality vector on the hyperplane of budget-balanced transfers  $\mathbf{1}^{\perp}$  normalized by the size of the society.

**Proposition 4.**

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}, -a) = \frac{\kappa}{n} \|\text{proj}_{\mathbf{1}^{\perp}} \mathbf{b}(\mathbf{G}, -a)\|.$$

**Proof.** From Proposition 3 it follows that

$$\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w}) = -\kappa \mathbf{b}(\mathbf{G}, -a) \cdot \mathbf{t} = -\kappa (\text{proj}_{\mathbf{1}^{\perp}} \mathbf{b}(\mathbf{G}, -a)) \cdot \mathbf{t}.$$

If  $\text{proj}_{\mathbf{1}^{\perp}} \mathbf{b}(\mathbf{G}, -a) = 0$  then the equality  $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}, -a) = \frac{\kappa}{n} \|\text{proj}_{\mathbf{1}^{\perp}} \mathbf{b}(\mathbf{G}, -a)\| = 0$  holds trivially. If  $\text{proj}_{\mathbf{1}^{\perp}} \mathbf{b}(\mathbf{G}, -a) \neq 0$ , the maximum of  $\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w})$ , for

<sup>4</sup>It is worth noting that when the size of the society is fixed, the average utilitarian approach is identical in its policy recommendations to the standard utilitarian approach.

$\mathbf{t} \in \mathcal{B}_\tau$ , occurs at  $-\frac{\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)}{\|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|}$ . Hence

$$\begin{aligned} \mathbf{b}^{\mathcal{T}I}(\mathbf{G}, -a) &= \max_{\mathbf{t} \in \mathcal{B}_\tau} \frac{\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w})}{n} = -\frac{\kappa}{n} \mathbf{b}(\mathbf{G}, -a) \cdot \frac{\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)}{\|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|} \\ &= \frac{\kappa}{n} \frac{\|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|^2}{\|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|} = \frac{\kappa}{n} \|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|. \square \end{aligned}$$

**4.2. Spectral analysis of the Bonacich transfer index.** In this section, we provide an alternative formulation for the Bonacich transfer index based on the spectral analysis of the network of social interactions. More specifically, similar to a recent result established by Allouch (2012) for the Bonacich centrality vector, we show that the Bonacich transfer index may be expressed from a selection of the spectrum<sup>5</sup> of the network of social interactions. The intuition is as follows: the Bonacich centrality vector is closely related to the number of walks in the network of social interactions, which in turn is determined only by a subset of the eigenvalues of the network of social interactions.

An eigenvalue  $\mu$  of  $\mathbf{G}$ , which has an associated eigenvector not orthogonal to the vector  $\mathbf{1}$ , is said to be a main eigenvalue (Cvetković (1970)). By the Perron–Frobenius Theorem, the maximum eigenvalue of  $\mathbf{G}$  has an associated eigenvector with all its entries positive and, therefore, is a main eigenvalue. The distinct main eigenvalues,  $\mu_1, \mu_2, \dots, \mu_s$  ( $\mu_1 > \mu_2 > \dots > \mu_s$ ), of  $\mathbf{G}$  form the main part of the spectrum, denoted by  $\mathcal{M}$  (Harary and Schwenk (1979)). The cosine of the angle between the eigenspace of  $\mu_i$ ,  $\mathcal{E}_{\mathbf{G}}(\mu_i)$ , and the vector  $\mathbf{1}$ , denoted by  $\beta_i$ , is called a main angle of  $\mathbf{G}$ . Obviously,  $\mu_i$  is a main eigenvalue if and only if  $\beta_i \neq 0$ . Moreover, it holds that  $\sum_{i=1}^s \beta_i^2 = 1$ .

The following result shows that the Bonacich transfer index may be expressed from the main part of the spectrum  $\mathcal{M}$ .

**Theorem 1.**

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}, -a) = \kappa \sqrt{\frac{1}{n} \left( \left( \sum_{i=1}^s \frac{\beta_i^2}{(1 + a\mu_i)^2} \right) - \left( \sum_{i=1}^s \frac{\beta_i^2}{1 + a\mu_i} \right)^2 \right)}$$

<sup>5</sup>Not always a proper selection.

**Proof.** See the Appendix.  $\square$

The Bonacich transfer index has, in view of Theorem 1, a natural geometric interpretation. Specifically, the Bonacich transfer index is related to the gap in Jensen's inequality<sup>6</sup> for the convex function  $f(x) = x^2$ , applied to the convex combination of  $\frac{1}{1+a\mu_1}, \frac{1}{1+a\mu_2}, \dots, \frac{1}{1+a\mu_s}$  with weights  $\beta_1^2, \beta_2^2, \dots, \beta_s^2$ . Indeed, recall that  $\sum_{i=1}^s \beta_i^2 = 1$  and it may be easily checked that

$$\mathbf{b}^{TI}(\mathbf{G}, -a) = \kappa \sqrt{\frac{1}{n} \left( \sum_{i=1}^s \beta_i^2 f\left(\frac{1}{1+a\mu_i}\right) \right) - f\left(\sum_{i=1}^s \frac{\beta_i^2}{1+a\mu_i}\right)}.$$

**Corollary 1.**  $\mathbf{b}^{TI}(\mathbf{G}, -a) = 0$  if and only if the network is regular.

**Proof.** See the Appendix.  $\square$

Corollary 1 shows that a zero Bonacich transfer index characterizes regular networks. Hence, a relatively small budget-balanced transfer will have no impact on welfare in regular networks.

Notice that, since, in view of Proposition 2, the Bonacich transfer index may also be related to the impact of income redistribution on aggregate provision, it follows that regular networks are the only instance where the two policy objectives of raising welfare and raising aggregate provision coincide as they are both redundant.

**Corollary 2.** If  $\lambda_{\min}(\mathbf{G}) = \mu_s$ , then

$$\lim_{a \uparrow \frac{-1}{\lambda_{\min}(\mathbf{G})}} \mathbf{b}^{TI}(\mathbf{G}, -a) = +\infty.$$

**Proof.** See the Appendix.  $\square$

Corollary 2 shows that the Bonacich transfer index may also be unbounded from above, which, together with Corollary 1, implies that the Bonacich transfer index may take a wide range of values.

<sup>6</sup>Jensen's inequality is a central inequality in the study of convex functions.

**Example 2.** Consider a society with three consumers and the star network of social interactions described in Figure 1. The adjacency matrix of the network of social interactions is

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The spectral decomposition shows that  $\mathbf{G}$  has three distinct eigenvalues

$$\lambda_1 = \sqrt{2}, \lambda_2 = 0, \text{ and } \lambda_3 = -\sqrt{2}$$

with corresponding unit eigenvectors

$$\mathbf{v}_1 = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right), \mathbf{v}_2 = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right), \text{ and } \mathbf{v}_3 = \left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right).$$

Obviously, unlike  $\mathbf{v}_1$  and  $\mathbf{v}_3$ , the sum of  $\mathbf{v}_2$  coordinates is zero. Hence, the main part of the spectrum is  $\mathcal{M} = \{\sqrt{2}, -\sqrt{2}\}$ . Figure 2 provides the geometric interpretation of the Bonacich transfer index for  $a \in ]0, \frac{\sqrt{2}}{2}[$ .

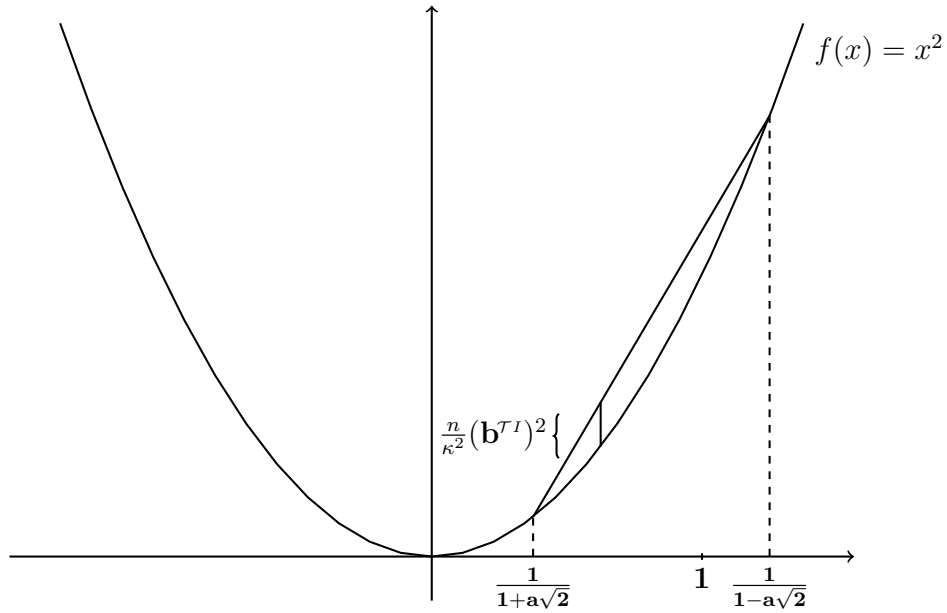


Figure 2: The Bonacich transfer index of a society with three consumers and a star network of social interactions



## 5. SEGREGATED SOCIETY

Now we investigate the impact of income redistribution in a society that is segregated across different groups. More specifically, the society is divided into  $H \geq 2$  non-empty and pairwise disjoint groups,  $C_1, \dots, C_H$ , of consumers of similar attributes, which may involve, amongst other things, ethnicity, religion, language, and income. Consumers may have neighbors from different groups and benefit equally from their neighbors' public goods provision regardless of their group identities.<sup>7</sup>

We introduce the following assumption on the network of social interactions of the society:

**Segregated society:** Let  $C_{h_i}$  denote consumer  $i$ 's own group. Then,

(i) for each consumer  $i = 1, \dots, n$ ,

$$|\mathcal{N}_i \cap C_{h_i}| > \sum_{h \neq h_i} |\mathcal{N}_i \cap C_h|;$$

(ii) if consumers  $i, j$  belong to the same group, that is  $C_{h_i} = C_{h_j}$ , then

$$|\mathcal{N}_i \cap C_h| = |\mathcal{N}_j \cap C_h|, \text{ for each } h = 1, \dots, H.$$

The segregated society assumption is about the density of social ties between the different groups of the society. Condition (i) stipulates that the number of social ties each consumer has to consumers from his own group exceeds the number of social ties he has to consumers from other groups. Condition (ii) is merely a network regularity requirement. It stipulates that the number of social ties a consumer in a given group has to consumers in any group is independent of the choice of the consumer.

**Theorem 2.** *Assume the society is segregated. Then  $\mathbf{b}^{TI}(\mathbf{G}, -a) \leq \frac{\kappa}{2\sqrt{n}}$ .*

**Proof.** Condition (ii) of segregated society implies that the partition  $\pi = \{C_1, \dots, C_H\}$  of consumers defines an equitable partition of the set of consumers (see Powers and

<sup>7</sup>Notice that since, for ease of exposition, we assume that the network of social interactions  $\mathbf{g}$  is connected, we rule out the full segregation case, which corresponds to  $\cup_{i \in C_h} \mathcal{N}_i \subset C_h$  for a group  $C_h$ .

Sulaiman (1982) for a basic reference). An equitable partition gives rise to a quotient graph  $\mathbf{g}/\pi$  characterized by the adjacency matrix  $\mathbf{G}/\pi = [d_{lz}]_{1 \leq l, z \leq H}$ , where  $d_{lz}$  denotes the number of links from a consumer in group  $C_l$  to consumers in group  $C_z$ .<sup>8</sup> Notice that the adjacency matrix,  $\mathbf{G}/\pi$ , is not necessarily symmetric, since in general  $d_{lz} \neq d_{zl}$ . The quotient graph plays an important role in the study of the main part of spectrum  $\mathcal{M}$  since it holds that (see, for example, Cvetković, Rowlinson, and Simić (1997), Theorems 2.4.3 and 2.4.5)

$$\mathcal{M} \subset \text{spec}(\mathbf{G}/\pi) \subset \text{spec}(\mathbf{G}). \quad (3)$$

Observe that all the eigenvalues of  $\mathbf{G}$  are real and so the eigenvalues of  $\mathbf{G}/\pi$  are also real. Moreover, condition (i) of segregated society implies that  $\mathbf{G}/\pi$  is a diagonally dominant matrix. From the Geršgorin Circle Theorem (see Varga (2004), Theorem 1.1), it follows that all eigenvalues of  $\mathbf{G}/\pi$  are positive. From (3), one obtains  $\mu_i > 0$  for each  $\mu_i \in \mathcal{M}$ . This implies that  $0 < \frac{1}{1+a\mu_i} < 1$  for each  $\mu_i \in \mathcal{M}$ . Since the Jensen's gap is less than  $\frac{1}{4}$  ( $= \max_{x \in ]0,1[} \{x - x^2\}$ ), the maximum possible Jensen's gap, it follows that

$$\mathbf{b}^{TI}(\mathbf{G}, -a) = \kappa \sqrt{\frac{1}{n} \left( \left( \sum_{i=1}^s \frac{\beta_i^2}{(1+a\mu_i)^2} \right) - \left( \sum_{i=1}^s \frac{\beta_i^2}{1+a\mu_i} \right)^2 \right)} \leq \frac{\kappa}{2\sqrt{n}}. \square$$

**Corollary 3.** *Assume the society is segregated. Then  $\lim_{n \rightarrow +\infty} \mathbf{b}^{TI}(\mathbf{G}, -a) = 0$ .*

**Proof.** An immediate consequence of Theorem 2.  $\square$

Theorem 2 provides an upper bound for the Bonacich transfer index in a segregated society. Corollary 3 shows that the Bonacich transfer index vanishes in large segregated societies, which suggests an asymptotic neutrality of income redistribution. Albeit quite different in interpretation, this result mirrors the neutrality result for pure public goods since it shows that social interactions in large segregated societies may limit the impact of redistributive policies and, by the same token, a wide range of other closely-related policies.

<sup>8</sup>Equitable partitions are referred to as colorations while the quotient graph  $\mathbf{g}/\pi$  is also known as divisor.

## 6. AN EXAMPLE: INTEGRATED VERSUS SEGREGATED SOCIETY

We now provide an example to understand the welfare impact of income redistribution in an integrated versus segregated society. Consider a society comprising two groups of consumers  $C_1$  and  $C_2$  of sizes, respectively,  $n_1$  and  $n_2$  such that  $n_1 = 4n_2$ . The society has a particular network of social interactions  $\mathbf{g}$  defined as follows. For each consumer in  $C_1$ , the number of social ties to consumers from  $C_1$  is  $d$  and the number of social ties to consumers from  $C_2$  is  $r$ . For each consumer in  $C_2$ , the number of social ties to consumers from  $C_2$  is  $d$  and the number of social ties to consumers from  $C_1$  is (obviously)  $4r$ . We assume that the network  $\mathbf{g}$  is connected, which implies  $r > 0$ . Let us consider the adjacency matrix of the quotient graph  $\mathbf{g}/\pi$  corresponding to the partition  $\pi = \{C_1, C_2\}$ :

$$\mathbf{G}/\pi = \begin{pmatrix} d & r \\ 4r & d \end{pmatrix}.$$

Then, from (3) and the fact that the network  $\mathbf{g}$  is not regular, it follows that  $\mathbf{G}$  has exactly two main eigenvalues, which are

$$\lambda_{\max}(\mathbf{G}) = \mu_1 = d + 2r \quad \text{and} \quad \mu_2 = d - 2r.$$

**6.1. Integrated society:  $d < r$ .** For each consumer, the number of social ties to consumers from his own group is smaller than the number of social ties to consumers from the other group. Then it holds that

$$\mu_1 = d + 2r > 0 \quad \text{and} \quad \mu_2 = d - 2r < 0.$$

Hence, for  $a \in ]0, \frac{1}{d+2r}[$ , it follows that

$$0 < \frac{1}{1 + a\mu_1} < 1 < \frac{1}{1 + a\mu_2}.$$

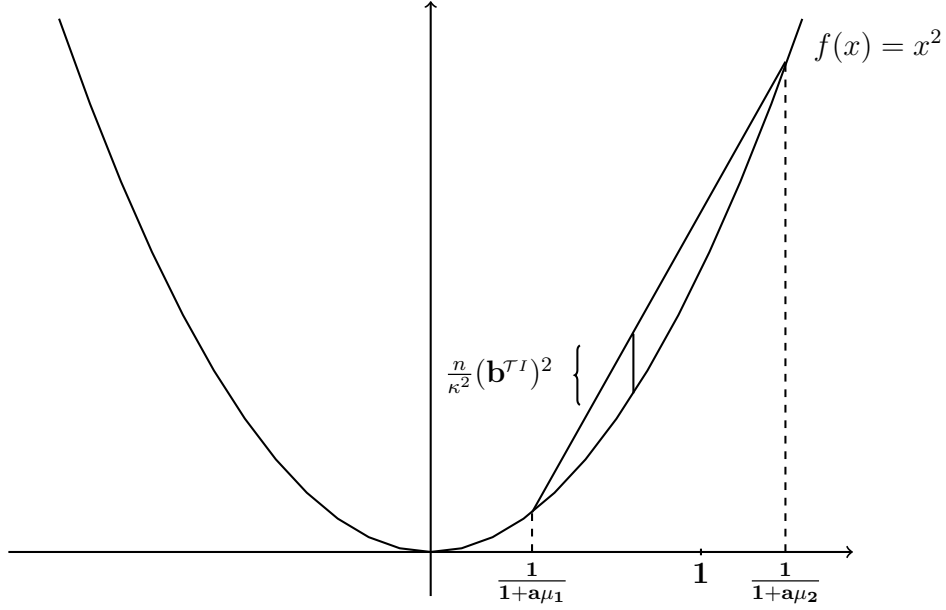


Figure 3: The Bonacich transfer index of an integrated society

Observe that if  $d = 0$  then we have a “fully” integrated society. Moreover, since

$$\mu_2 = -\mu_1 = -2r = \lambda_{\min}(\mathbf{G}),$$

it follows from Corollary 2 that  $\lim_{a \uparrow -\frac{1}{\mu_2}} \mathbf{b}^{TI}(\mathbf{G}, -a) = +\infty$ .

**6.2. Segregated society:  $d > 4r$ .** For each consumer, the number of social ties to consumers from his own group is bigger than the number of social ties to consumers from the other group. Then it holds that

$$\mu_1 = d + 2r > 0 \quad \text{and} \quad \mu_2 = d - 2r > 0.$$

Hence, for  $a \in ]0, \frac{1}{d+2r}[$ , it holds that

$$0 < \frac{1}{1 + a\mu_1} < \frac{1}{1 + a\mu_2} < 1.$$

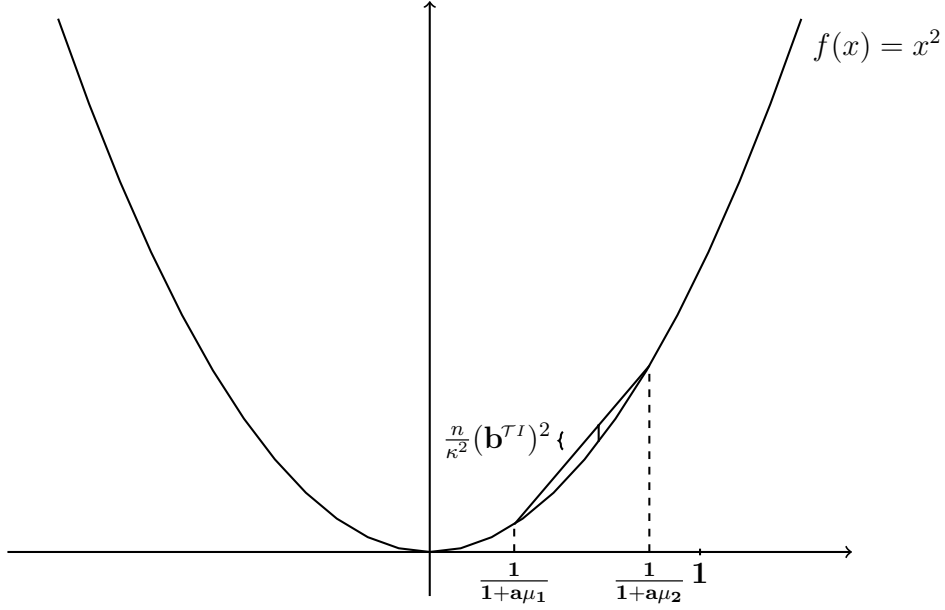


Figure 4: The Bonacich transfer index of a segregated society

The economic intuition for asymptotic neutrality in this simple example of a segregated society may be explained as follows. First, due to their low social cost to produce public goods, there are welfare gains from transferring income to consumers in  $C_2$ . However, these welfare gains will be offset for consumers in  $C_1$  with both an income reduction and, due to segregation, a limited benefit from the increased public good provision by consumers in  $C_2$ . On balance, these different effects will gradually, as society grows large, exhaust the potential per capita welfare gains and, equivalently, cause the Bonacich transfer index to shrink to zero.

Finally, observe that if  $r = 0$  (and  $d > 0$ ) then, contrary to our assumption, the network of social interactions is no longer connected. In this case, we have a fully segregated society with two unconnected groups. We may deduce from Corollary 1 that the Bonacich transfer index is zero for each subsociety defined by a group.

## 7. CONCLUSION

Enhancing private provision of public goods has long been an important policy objective and our paper shows that understanding social networks is a key way to achieve this. To this effect, the Bonacich transfer index, introduced in this paper, may be thought of as an instrument to capture the welfare multiplier effect of income redistribution based on the complex network of social interactions. The computation of the Bonacich transfer index shows that redistributive policies may have a normative significance in integrated societies and may be welfare inconsequential in segregated societies. Surprisingly, the result is obtained only from the network structure of social interactions since consumers care about their neighbors only insofar as they affect public good provision and not their group identities. Nevertheless, while undergoing the process of formation, the underlying network structure of social interactions of a society may have been largely affected by the linking preferences of consumers. Hence, a straightforward implication of our result suggests that the optimal policy to increase the welfare multiplier effect of income redistribution within a society, when building/reshuffling the network structure of social interactions, stipulates fostering social bridges and outward social ties between groups while discouraging homogenous and inward social ties within groups.

## 8. APPENDIX

**Proof of Proposition 2.** First, it follows from (2) that for each consumer  $i$

$$\begin{aligned} q_i^t - q_i^* &= ((1-a)(w_i + t_i + Q_{-i}^t) - Q_{-i}^t) - ((1-a)(w_i + Q_{-i}^*) - Q_{-i}^*) \\ &= (1-a)t_i - a(Q_{-i}^t - Q_{-i}^*). \end{aligned}$$

Rearranging terms, it follows that  $(\mathbf{I} + a\mathbf{G})(\mathbf{q}^t - \mathbf{q}^*) = (1-a)\mathbf{t}$ , and therefore

$$\mathbf{q}^t - \mathbf{q}^* = (1-a)(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{t}.$$

Hence, it holds that

$$Q^t - Q^* = \mathbf{1} \cdot (\mathbf{q}^t - \mathbf{q}^*) = (1-a)\mathbf{b}(\mathbf{G}, -a) \cdot \mathbf{t}. \square$$

**Proof of Proposition 3.** When preferences of consumers are of the Gorman polar form, the indirect utility function for each consumer  $i$ , at given price  $\mathbf{p}$  and income  $w_i$ , can be written as

$$v_i(\mathbf{p}, w_i) = \alpha(\mathbf{p})w_i + \beta_i(\mathbf{p}),$$

where  $\alpha(\mathbf{p}) > 0$  is common to all consumers. From the utility maximization in (1), it follows that at the unique Nash equilibrium each consumer maximizes his utility with respect to the price  $\mathbf{p} = (1, 1)$  and the social income  $w_i + Q_{-i}^*$ . Therefore it holds that

$$u_i(x_i^*, q_i^* + Q_{-i}^*) = v_i(\mathbf{p}, w_i + Q_{-i}^*) = \alpha(\mathbf{p})(w_i + Q_{-i}^*) + \beta_i(\mathbf{p}).$$

Hence, it follows from Proposition 2 that

$$\begin{aligned} \mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w}) &= \sum_{i=1}^n [u_i(x_i^{\mathbf{t}}, q_i^{\mathbf{t}} + Q_{-i}^{\mathbf{t}}) - u_i(x_i^*, q_i^* + Q_{-i}^*)] \\ &= \sum_{i=1}^n [\alpha(\mathbf{p})(w_i + t_i + Q_{-i}^{\mathbf{t}}) - \alpha(\mathbf{p})(w_i + Q_{-i}^*)] \\ &= \alpha(\mathbf{p}) \left[ \sum_{i=1}^n (Q_{-i}^{\mathbf{t}} - Q_{-i}^*) + \sum_{i=1}^n t_i \right] = \alpha(\mathbf{p}) \sum_{i=1}^n (Q_{-i}^{\mathbf{t}} - Q_{-i}^*) \\ &= \alpha(\mathbf{p}) \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} (q_j^{\mathbf{t}} - q_j^*) = \alpha(\mathbf{p}) \sum_{i=1}^n [\mathbf{G}(\mathbf{q}^{\mathbf{t}} - \mathbf{q}^*)]_i \\ &= \alpha(\mathbf{p}) \mathbf{1}^T \mathbf{G}(\mathbf{q}^{\mathbf{t}} - \mathbf{q}^*) = \alpha(\mathbf{p})(1-a) \mathbf{1}^T \mathbf{G}(\mathbf{I} + a\mathbf{G})^{-1} \mathbf{t} \\ &= \alpha(\mathbf{p})(1-a) \mathbf{1}^T \left( -\frac{1}{a} \mathbf{I} + \frac{1}{a} (\mathbf{I} + a\mathbf{G}) \right) (\mathbf{I} + a\mathbf{G})^{-1} \mathbf{t} \\ &= \frac{\alpha(\mathbf{p})(1-a)}{a} \mathbf{1}^T (-\mathbf{I} + a\mathbf{G})^{-1} \mathbf{t} \\ &= -\frac{\alpha(\mathbf{p})(1-a)}{a} \mathbf{1}^T (\mathbf{I} + a\mathbf{G})^{-1} \mathbf{t}. \end{aligned}$$

Therefore, if one sets  $\kappa = \frac{\alpha(\mathbf{p})(1-a)}{a} > 0$ , the desired result follows, that is,

$$\mathcal{SW}(\mathbf{w} + \mathbf{t}) - \mathcal{SW}(\mathbf{w}) = -\kappa \mathbf{b}(\mathbf{G}, -a) \cdot \mathbf{t}. \square$$

**Proof of Theorem 1.** Let  $\mathbf{u}_i$  be the unit eigenvector of the main eigenvalue  $\mu_i$  orthogonal to  $\mathcal{E}_{\mathbf{G}}(\mu_i) \cap \mathbf{1}^\perp$ . The eigenvector  $\mathbf{u}_i$  is determined uniquely since we choose  $\beta_i = \mathbf{u}_i \cdot \frac{\mathbf{1}}{\sqrt{n}}$  to be the cosine of the acute angle between  $\mathcal{E}_{\mathbf{G}}(\mu_i)$  and  $\mathbf{1}$ . Let  $\mathbf{V}$  be a matrix whose columns,  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , are eigenvectors of  $\mathbf{G}$  chosen to extend the eigenvectors  $\mathbf{u}_1, \dots, \mathbf{u}_s$  of  $\mathbf{G}$  to an orthonormal basis of  $\mathbb{R}^n$ . Therefore,  $\mathbf{G} = \mathbf{V}\mathbf{D}\mathbf{V}^T$ , where  $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Therefore, it holds that

$$\begin{aligned} \mathbf{b}(\mathbf{G}, -a) &= (\mathbf{I} + a\mathbf{G})^{-1}\mathbf{1} = \mathbf{V}(\mathbf{I} + a\mathbf{D})^{-1}\mathbf{V}^T\mathbf{1} \\ &= \sum_{i=1}^n \frac{\mathbf{1} \cdot \mathbf{v}_i}{1 + a\lambda_i} \mathbf{v}_i = \sum_{i=1}^s \frac{\mathbf{1} \cdot \mathbf{u}_i}{1 + a\mu_i} \mathbf{u}_i \\ &= \sqrt{n} \sum_{i=1}^s \frac{\beta_i}{1 + a\mu_i} \mathbf{u}_i. \end{aligned}$$

From the Pythagorean theorem, it holds that

$$\|\mathbf{b}(\mathbf{G}, -a)\|^2 = \|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|^2 + \|\text{proj}_{\mathbf{1}} \mathbf{b}(\mathbf{G}, -a)\|^2.$$

Hence, it follows that

$$\begin{aligned} \mathbf{b}^{T\mathbf{I}}(\mathbf{G}, -a)^2 &= \frac{\kappa^2}{n^2} \|\text{proj}_{\mathbf{1}^\perp} \mathbf{b}(\mathbf{G}, -a)\|^2 = \frac{\kappa^2}{n^2} (\|\mathbf{b}(\mathbf{G}, -a)\|^2 - \|\text{proj}_{\mathbf{1}} \mathbf{b}(\mathbf{G}, -a)\|^2) \\ &= \frac{\kappa^2}{n^2} (n \|\sum_{i=1}^s \frac{\beta_i}{1 + a\mu_i} \mathbf{u}_i\|^2 - n \|(\sum_{i=1}^s \frac{\beta_i}{1 + a\mu_i} \mathbf{u}_i) \cdot \frac{\mathbf{1}}{\sqrt{n}}\|^2) \\ &= \frac{\kappa^2}{n} ((\sum_{i=1}^s \frac{\beta_i}{1 + a\mu_i} \mathbf{u}_i) \cdot (\sum_{i=1}^s \frac{\beta_i}{1 + a\mu_i} \mathbf{u}_i) - (\sum_{i=1}^s \frac{\beta_i^2}{1 + a\mu_i})^2 \|\frac{\mathbf{1}}{\sqrt{n}}\|^2) \\ &= \frac{\kappa^2}{n} ((\sum_{i=1}^s \frac{\beta_i^2}{(1 + a\mu_i)^2}) - (\sum_{i=1}^s \frac{\beta_i^2}{1 + a\mu_i})^2). \end{aligned}$$

Therefore, it holds that

$$\mathbf{b}^{T\mathbf{I}}(\mathbf{G}, -a) = \kappa \sqrt{\frac{1}{n} (\sum_{i=1}^s \frac{\beta_i^2}{(1 + a\mu_i)^2}) - (\sum_{i=1}^s \frac{\beta_i^2}{1 + a\mu_i})^2}. \square$$

**Proof of Corollary 1.** From the Jensen's gap interpretation of the Bonacich transfer index, it follows that  $\mathbf{b}^{T\mathbf{I}}(\mathbf{G}, -a) = 0$  if and only if  $s = 1$ , which is equivalent



to  $\beta_1^2 = 1$ . From the definition of main angles, it holds that  $\beta_1^2 = 1$  is equivalent to  $\frac{1}{\sqrt{n}}$  is an eigenvalue of  $\mathbf{G}$ , which is also equivalent to  $\mathbf{g}$  is a regular network.  $\square$

**Proof of Corollary 2.** First, observe that if  $\lambda_{\min}(\mathbf{G}) = \mu_s$  then it follows that  $\mu_s < 0$ , which implies  $s \geq 2$ . Moreover,

$$\begin{aligned} \mathbf{b}^{TI}(\mathbf{G}, -a) &= \kappa \sqrt{\frac{1}{n} \left( \left( \sum_{i=1}^s \frac{\beta_i^2}{(1+a\mu_i)^2} \right) - \left( \sum_{i=1}^s \frac{\beta_i^2}{1+a\mu_i} \right)^2 \right)} \\ &= \frac{\kappa}{1+a\mu_s} \sqrt{\frac{1}{n} \left( \sum_{i=1}^{s-1} \beta_i^2 \left( \frac{1+a\mu_s}{1+a\mu_i} \right)^2 + \beta_s^2 - \left( \sum_{i=1}^{s-1} \beta_i^2 \frac{1+a\mu_s}{1+a\mu_i} + \beta_s^2 \right)^2 \right)}. \end{aligned}$$

Obviously,  $s \geq 2$  implies that  $\beta_s < 1$ . Therefore, it holds that

$$\lim_{a \uparrow -\frac{1}{\mu_s}} \sqrt{\frac{1}{n} \left( \sum_{i=1}^{s-1} \beta_i^2 \left( \frac{1+a\mu_s}{1+a\mu_i} \right)^2 + \beta_s^2 - \left( \sum_{i=1}^{s-1} \beta_i^2 \frac{1+a\mu_s}{1+a\mu_i} + \beta_s^2 \right)^2 \right)} = \sqrt{\frac{\beta_s^2 - \beta_s^4}{n}} > 0.$$

Hence it follows that

$$\lim_{a \uparrow -\frac{1}{\lambda_{\min}(\mathbf{G})}} \mathbf{b}^{TI}(\mathbf{G}, -a) = \lim_{a \uparrow -\frac{1}{\mu_s}} \frac{\kappa}{1+a\mu_s} \sqrt{\frac{\beta_s^2 - \beta_s^4}{n}} = +\infty. \square$$

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