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FUZZY REJECTIVE CORE OF SATIATED ECONOMIES WITH UNBOUNDED CONSUMPTION SETS

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ABSTRACT. For an exchange economy, under assumptions which did not bring about the existence of equilibrium with dividends as yet, we prove the non-emptiness of the fuzzy rejective core. Then, via Konovalov (1998, 2005)'s equivalence result, we solve the equilibrium with dividends existence problem. Adding to the same assumptions a weak non-satiation condition which differs from the weak non-satiation assumption introduced by Allouch–Le Van (2009), we show in a last section the existence of a Walrasian quasiequilibrium. This result, which fits with exchange economies whose consumers' utility functions are not assumed to be upper semicontinuous, complements the one obtained by Martins-da-Rocha and Monteiro (2009).

JEL classification: D 50.

Keywords: Exchange economy; Satiation; Equilibrium with dividends; Rejective core; Fuzzy rejective core; Core equivalence.

1. INTRODUCTION

In this note, we consider an exchange economy $\mathcal{E} = ((X_i, u_i, e_i)_{i \in I})$ defined on the commodity space \mathbb{R}^ℓ . Each of a finite set I of consumers has a consumption set $X_i \subset \mathbb{R}^\ell$, an initial endowment $e_i \in \mathbb{R}^\ell$ and transitive and complete preferences on X_i represented by a utility function $u_i: X_i \rightarrow \mathbb{R}$. We normalize utility functions by requiring for each i , $u_i(e_i) = 0$. As usual,

$$\mathcal{A}(\mathcal{E}) := \left\{ x = (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} x_i = \sum_{i \in I} e_i \right\}$$

denotes the set of all attainable allocations of the economy \mathcal{E} . We let

$$\mathcal{A}_{IR}(\mathcal{E}) = \{x \in \mathcal{A}(\mathcal{E}), \text{ s.t. } 0 \leq u_i(x_i) \forall i \in I\}$$

$$\mathcal{U} = \{v = (v_i)_{i \in I} \in \mathbb{R}^I : \exists x \in \mathcal{A}(\mathcal{E}), \text{ s.t. } 0 \leq v_i \leq u_i(x_i) \forall i \in I\}$$

be respectively the set of all individually rational attainable allocations and the *individually rational utility set*, sometimes called *utility set*. We will occasionally denote by \mathcal{A}_i the projection on X_i of $\mathcal{A}_{IR}(\mathcal{E})$.

Definition 1.1. A couple (p, x) where $0 \neq p \in \mathbb{R}^\ell$ and $x = (x_i)_{i \in I} \in \mathcal{A}(\mathcal{E})$ is a (**Walrasian**) **quasiequilibrium** of \mathcal{E} if for each $i \in I$,

$$p \cdot x_i = p \cdot e_i \text{ and } u_i(x'_i) > u_i(x_i) \implies p \cdot x'_i \geq p \cdot e_i.$$

It is an **equilibrium** if $u_i(x'_i) > u_i(x_i)$ actually implies $p \cdot x'_i > p \cdot e_i$

The possibility that the current assumptions do not imply local non-satiation of preferences at each consumption component of equilibrium has motivated the following definition, going back to different authors in different contexts, formalized with a different name by Mas-Colell (1992).

Definition 1.2. A couple (p, x) of a price $p \in \mathbb{R}^\ell$ and of an attainable allocation $x = (x_i)_{i \in I} \in \mathcal{A}(\mathcal{E})$ is a **quasiequilibrium with dividends (slacks)** if for each $i \in I$, there exists $m_i \in \mathbb{R}_+$ such that

$$p \cdot x_i \leq p \cdot e_i + m_i \text{ and } u_i(x'_i) > u_i(x_i) \implies p \cdot x'_i \geq p \cdot e_i + m_i.$$

It is an **equilibrium** (with dividends $(m_i)_{i \in I}$) if $u_i(x'_i) > u_i(x_i)$ actually implies $p \cdot x'_i > p \cdot e_i + m_i$.

It is worth noticing that, in the previous definition, consumers are not anymore required to bind their budget constraint at quasiequilibrium. If all dividends are not equal to 0, the fact that x is an attainable allocation does not imply such an equality between quasiequilibrium expenditure and revenue that only local non-satiation at quasiequilibrium could explain. Moreover, a null equilibrium price is compatible with the previous definition. It simply implies that all consumers are satiated at equilibrium. Obviously, a Walrasian equilibrium (resp. quasiequilibrium) is nothing else but an equilibrium (resp. a quasiequilibrium) with a non-null price and dividends all equal to 0.

On the other hand, let us now introduce the core notions which will allow to deduce the existence of equilibria with dividends from the core equivalence theorem proved by Kononov (2005).

Definition 1.3. A coalition S **rejects** an attainable allocation $x \in \mathcal{A}(\mathcal{E})$ if there exists a partition $S = S^1 \cup S^2$ and consumption bundles $y_i \in X_i$, $i \in S$, such that

- (a) $\sum_{i \in S} y_i = \sum_{i \in S^1} e_i + \sum_{i \in S^2} x_i$,
- (b) $u_i(y_i) > u_i(x_i) \forall i \in S$.

Following Aubin (1979), a non-null element $t = (t_i)_{i \in I}$ of the set $[0, 1]^I$ is called a **fuzzy coalition**. Its i th component is interpreted as the rate of participation of agent i in the fuzzy coalition. Obviously, a coalition $S \subset I$ can be seen as a fuzzy coalition with rates of participation equal to 0 or 1.

Definition 1.4. A fuzzy coalition $t = (t_i)_{i \in I}$ **rejects** an attainable allocation $x \in \mathcal{A}(\mathcal{E})$ if there exists t^1, t^2 in $[0, 1]^I$ and $y \in \prod_{i \in I} X_i$ such that $t = t^1 + t^2$ and

- (a) $\sum_{i \in I} t_i y_i = \sum_{i \in I} t_i^1 e_i + \sum_{i \in I} t_i^2 x_i$,
- (b) $u_i(y_i) > u_i(x_i) \forall i: t_i > 0$.

Definition 1.5. An attainable allocation $x \in \mathcal{A}(\mathcal{E})$ is an element of the **rejective core** $\mathcal{C}_r(\mathcal{E})$ (resp. is an element of the **fuzzy rejective core** $\mathcal{C}_r^f(\mathcal{E})$) if it cannot be rejected by a coalition (resp. a fuzzy coalition). It is an element of the **Edgeworth rejective core** $\mathcal{C}_r^e(\mathcal{E})$ if it cannot be rejected by a fuzzy coalition with rational rates t_i, t_i^1, t_i^2 of participation.

If, in definitions 1.3 and 1.4, $S^2 = \emptyset$ or if $t^2 = 0$, the previous definitions coincide with the standard definitions of blocking, so that $\mathcal{C}_r(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$, $\mathcal{C}_r^f(\mathcal{E}) \subset \mathcal{C}^f(\mathcal{E})$, $\mathcal{C}_r^e(\mathcal{E}) \subset \mathcal{C}^e(\mathcal{E})$. It easily follows from the definitions that an attainable allocation of the rejective core (resp. fuzzy rejective core, Edgeworth rejective core) is individually rational, that is, belongs to $\mathcal{A}_{IR}(\mathcal{E})$. Moreover, if satiation at all components of an attainable allocation is not ruled out, such an attainable allocation is a trivial element of $\mathcal{C}_r(\mathcal{E})$ (resp. $\mathcal{C}_r^e(\mathcal{E})$, $\mathcal{C}_r^f(\mathcal{E})$). It is also a trivial equilibrium with dividends allocation with any null or non-null equilibrium price.

The main purpose of this note is to give sufficient assumptions on the economy \mathcal{E} to guarantee the non-emptiness of its fuzzy rejective core. This will be done in Section 2. A direct proof of this result is by itself of interest. At the end of the section, using Konovalov(1998, 2005)'s core-equivalence result, we will deduce from this result sufficient assumptions for the existence of equilibrium with dividends in \mathcal{E} . These assumptions have been proved to be strictly weaker than comparable assumptions found in the literature. Under the same assumptions, we will show in Section 3 the existence of a Walrasian equilibrium under a weak non-satiation condition which differs from the weak non-satiation assumption introduced by Allouch–Le Van (2009) and used by Martins-da-Rocha–Monteiro (2009).

2. THE NON-EMPTINESS OF THE FUZZY REJECTIVE CORE AND ITS CONSEQUENCE FOR THE EXISTENCE OF EQUILIBRIUM WITH DIVIDENDS

In this section, we will maintain on \mathcal{E} the following assumptions.

A.1: For each i , X_i is convex, containing e_i ;

A.2: For each i , u_i is strictly quasi-concave;

A.3: For each i , the set $P_i(x_i) := \{y_i \in X_i : u_i(y_i) > u_i(x_i)\}$ is open in X_i at every attainable and individually rational consumption vector;¹

A.4: For each i , $e_i \in \text{int } X_i$.

In addition, when consumption sets are not bounded below, a compactness assumption should limit the arbitrage possibilities for consumers. We borrow this assumption from Martins-da-Rocha and Monteiro (2009). Let us denote for each $i \in I$, $S_i = \text{argmax}\{u_i(x_i) : x_i \in X_i\}$ the set of satiation points of u_i on X_i , and for every attainable allocation $x \in \mathcal{A}(\mathcal{E})$, $S(x) = \{i \in I : x_i \in S_i\}$, $N(x) = I \setminus S(x)$.

Definition 2.1. *The individually rational utility set \mathcal{U} is **strongly compact** if for every sequence x^ν in $\mathcal{A}_{IR}(\mathcal{E})$ there exists an attainable allocation $y \in \mathcal{A}(\mathcal{E})$ and a subsequence x^{ν_k} satisfying*

- (a) $\forall i \in I$, $u_i(y_i) \geq \lim_{k \rightarrow \infty} u_i(x_i^{\nu_k})$,
(b) $\forall i \in I$, $\lim_{k \rightarrow \infty} \frac{\mathbf{1}_{S_i}(x_i^{\nu_k})}{1 + \|x_i^{\nu_k}\|^2} (y_i - x_i^{\nu_k}) = 0$.²

As explained by Martins-da-Rocha and Monteiro, condition *b.* means that for the sequence $x_i^{\nu_k}$, only three cases can occur:

- (1) the subsequence $x_i^{\nu_k}$ is unbounded,
- (2) the subsequence $x_i^{\nu_k}$ converges to y_i ,
- (3) for k large enough, $x_i^{\nu_k}$ is not a satiation point.

The two following implications are proved in Martins-da-Rocha–Monteiro (2009) where are also provided examples showing that the reverse implications are not true.

- If the utility functions u_i are upper semicontinuous,³ compactness of $\mathcal{A}_{IR}(\mathcal{E})$ implies the strong compactness of \mathcal{U} ,
- strong compactness of \mathcal{U} implies its compactness.

¹As noticed several times in the literature, this assumption is not necessary if in **A.2**, the functions u_i are assumed to be concave.

²If A is a subset of \mathbb{R}^ℓ , for $z \in \mathbb{R}^\ell$, $\mathbf{1}_A(z)$ is defined by $\mathbf{1}_A(z) = z$ if $z \in A$, $\mathbf{1}_A(z) = 0$ otherwise.

³It is obvious that compactness properties of \mathcal{U} subsume some upper semicontinuity properties of utility function. When the compactness or the strong compactness of \mathcal{U} is directly assumed, according to the results to be got, it may be not necessary to assume in addition that the utility functions are upper semicontinuous.

It is well known that if \mathcal{U} is compact, the economy \mathcal{E} has a (strong) Pareto optimum⁴ which is also individually rational. Strong compactness of \mathcal{U} allows to prove the following sharper result to be used later (see the discussion between Claim 2 and Claim 3 in the proof of Proposition 2.2). For $\bar{x} \in \mathcal{A}_{IR}(\mathcal{E})$, let us define the following subset of \mathcal{U}

$$\mathcal{U}(\bar{x}) = \left\{ v = (v_i)_{i \in I} \in \mathbb{R}^I : \exists x \in \mathcal{A}(\mathcal{E}), \text{ s.t. } \begin{array}{l} 0 \leq v_i \leq u_i(x_i) \quad \forall i \in I, x_i = \bar{x}_i \quad \forall i \in S(\bar{x}), \\ \text{and } u_i(x_i) \geq u_i(\bar{x}_i) \quad \forall i \in N(\bar{x}) \end{array} \right\}.$$

Proposition 2.1. *Assume that \mathcal{U} is nonempty and strongly compact. Then for $\bar{x} \in \mathcal{A}_{IR}(\mathcal{E})$, the set $\mathcal{U}(\bar{x})$ has a maximal element.*

Proof. it suffices to verify that $\mathcal{U}(\bar{x})$ is closed. Let v^ν a sequence of elements of $\mathcal{U}(\bar{x})$ converging to v . There exists a sequence x^ν of elements of $\mathcal{A}(\mathcal{E})$ such that $0 \leq v_i^\nu \leq u_i(x_i^\nu) \quad \forall i \in I$ and $x_i^\nu = \bar{x}_i \quad \forall i : \bar{x}_i \in S_i, u_i(x_i^\nu) \geq u_i(\bar{x}_i) \quad \forall i : \bar{x}_i \notin S_i$. By strong compactness of \mathcal{U} , there exists a subsequence $x_i^{\nu_k}$ and $y \in \mathcal{A}(\mathcal{E})$ satisfying for all $i \in I$, $u_i(y_i) \geq \lim_{k \rightarrow \infty} u_i(x_i^{\nu_k})$ and one of the three properties above. It then follows that $y_i = \bar{x}_i$ if $\bar{x}_i \in S_i$ and $u_i(y_i) \geq u_i(\bar{x}_i)$ if $\bar{x}_i \notin S_i$, thus that $v \in \mathcal{U}(\bar{x})$. ■

We will keep in mind that if $v^* = (u_i(x_i^*))_{i \in I}$ is a maximal element of $\mathcal{U}(\bar{x})$ then $S(\bar{x}) \subset S(x^*)$.

From now on, we add to the assumptions **A.1** – **A.4** on the economy \mathcal{E} the following compactness assumption.

A.5: \mathcal{U} is strongly compact.

The next proposition is the main result of the note.

Proposition 2.2. *Under Assumptions **A.1** – **A.5**, the economy \mathcal{E} has a non-empty fuzzy rejective core.*

Proof. As Le Van–Minh (2007), given some $\mu > 0$ and $\delta = (\delta_i)_{i \in I} \in \mathbb{R}_{++}^I$, we associate to the economy \mathcal{E} , the economy

$$\widehat{\mathcal{E}} = ((\widehat{X}_i, \widehat{u}_i, \widehat{e}_i)_{i \in I})$$

defined as follows. For each $i \in I$, $\widehat{X}_i = X_i \times \mathbb{R}_+$, $\widehat{e}_i = (e_i, \delta_i)$, and, for any $(x_i, d_i) \in \widehat{X}_i$,

$$\widehat{u}_i(x_i, d_i) = \begin{cases} u_i(x_i) & \text{if, in } \mathcal{E}, x_i \notin S_i \\ u_i(x_i) + \mu d_i & \text{if, in } \mathcal{E}, x_i \in S_i \end{cases}$$

Without loss of generality, we can assume that no e_i is satiation point of u_i on X_i , so that $\widehat{u}_i(e_i, \delta_i) = 0$.⁵ For the economy $\widehat{\mathcal{E}}$, the sets $\mathcal{A}(\widehat{\mathcal{E}})$, $\mathcal{A}_{IR}(\widehat{\mathcal{E}})$, \mathcal{U} are defined as in \mathcal{E} . In particular,

$$\widehat{\mathcal{U}} = \{v = (v_i)_{i \in I} : \exists (x, d) = (x_i, d_i)_{i \in I} \in \mathcal{A}(\widehat{\mathcal{E}}), \text{ s.t. } 0 \leq v_i \leq \widehat{u}_i(x_i, d_i) \quad \forall i \in I\}.$$

The proposition will be proved through a sequence of claims. Two first claims will verify the conditions, stated in Proposition 4.1 of Allouch–Florenzano (2004), for existence of Edgeworth equilibria in $\widehat{\mathcal{E}}$. Then, the following claims will show first that to an Edgeworth equilibrium of $\widehat{\mathcal{E}}$ can be associated an element of the Edgeworth rejective core of \mathcal{E} , which is, in view of Assumption **A.3**, an element of the fuzzy rejective core of \mathcal{E} .

⁴That is, $x^* \in \mathcal{A}(\mathcal{E})$ such that there is no $y \in \mathcal{A}(\mathcal{E})$ with $u_i(y_i) \geq u_i(x_i^*)$ for each $i \in I$ and $u_{i_0}(y_{i_0}) > u_{i_0}(x_{i_0}^*)$ for some $i_0 \in I$.

⁵A consumer who is satiated at e_i does not play any role in the rejection process.

Claim 1. For each i , \widehat{X}_i is convex, containing \widehat{e}_i . Moreover, the function \widehat{u}_i is strictly quasiconcave.

The claim is proved in Le Van–Minh (2007).

Claim 2. The set \widehat{U} is compact.

Let $(v^\nu) = ((v_i^\nu)_{i \in I})$ be a sequence of elements of \widehat{U} . By definition, there exists a sequence $((x_i^\nu, d_i^\nu)_{i \in I})$ of elements of $\mathcal{A}_{IR}(\widehat{E})$ such that

$$0 \leq v_i^\nu \leq \begin{cases} u_i(x_i^\nu) + \mu d_i^\nu & \text{if } x_i^\nu \text{ is a satiation point} \\ u_i(x_i^\nu) & \text{if } x_i^\nu \text{ is not a satiation point} \end{cases}$$

Using the strong compactness of \mathcal{U} , there exists $y = (y_i)_{i \in I} \in \mathcal{A}(\mathcal{E})$ and, for each $i \in I$, subsequences $(x_i^{\nu_k})$ satisfying the conditions of Definition 2.1, together with subsequences $(v_i^{\nu_k})$ converging to v_i , $(d_i^{\nu_k})$ converging to $d_i \in \mathbb{R}_+$ such that $(y_i, d_i)_{i \in I} \in \mathcal{A}(\widehat{E})$.

Passing to limit in the above relation, distinguishing whether y_i is or not a satiation point, and observing that if y_i is not a satiation point then for k large enough, $x_i^{\nu_k}$ is not a satiation point, we get for each $i \in I$

$$0 \leq v_i \leq \begin{cases} u_i(y_i) + \mu d_i & \text{if } y_i \text{ is a satiation point} \\ u_i(y_i) & \text{if } y_i \text{ is not a satiation point} \end{cases}$$

so that $0 \leq v_i \leq \widehat{u}_i(y_i, d_i)$, which proves that $v = (v_i)_{i \in I} \in \widehat{U}$.

In view of the previous claims, the economy $\widehat{\mathcal{E}}$ satisfies Assumptions **A.1–A.3** of Theorem 3.1 in Allouch–Florenzano (2004), thus has an Edgeworth equilibrium (\bar{x}, \bar{d}) .

At this stage, it is worthwhile noticing that in $\widehat{\mathcal{E}}$, the set

$$\widehat{P}_i(\bar{x}_i, \bar{d}_i) = \begin{cases} P_i(\bar{x}_i) \times \mathbb{R}_+ & \text{if } \bar{x}_i \notin S_i \\ S_i \times \{d_i > \bar{d}_i\} & \text{otherwise} \end{cases}$$

is not necessarily open in \widehat{X}_i . We thus cannot infer from the previous conclusion that (\bar{x}, \bar{d}) belongs to the fuzzy core of $\widehat{\mathcal{E}}$. We now indicate how to circumvent this difficulty.

If, in view of Proposition 2.1, $v^* = (u_i(x_i^*))_{i \in I}$ is a maximal element of $\mathcal{U}(\bar{x})$, the reader will verify that, in the economy $\widehat{\mathcal{E}}$, $\widehat{u}_i(x_i^*, \bar{d}_i) \geq \widehat{u}_i(\bar{x}_i, \bar{d}_i) \forall i \in I$, and thus that (x^*, \bar{d}) is likewise an element of $\mathcal{C}^e(\widehat{\mathcal{E}})$. In the following claim, we intend to show that $x^* = (x_i^*)_{i \in I} \in \mathcal{C}_r^e(\mathcal{E})$, and assume from now on, by way of contradiction, that there exists t^1, t^2 in $([0, 1] \cap \mathbb{Q})^I$ and $y \in \prod_{i \in I} X_i$ such that if $t = t^1 + t^2 \neq 0$ then

- (a) $\sum_{i \in I} t_i y_i = \sum_{i \in I} t_i^1 e_i + \sum_{i \in I} t_i^2 x_i^*$,
- (b) $u_i(y_i) > u_i(x_i^*) \forall i: t_i > 0$.

Taking in account Proposition 1 in the appendix, there is no loss of generality to assume that $t_i = 0$ if and only if $i \in S(x^*)$.

Claim 3. $x^* \in \mathcal{C}_r^e(\mathcal{E})$.

Setting, as Konovalov (2005), $t_{i \max}^2 = \max_i t_i^2$, we will distinguish two cases.

Case 1: $t_{i \max}^2 = 0$. In this case, $\sum_{i \in I} t_i y_i = \sum_{i \in I} t_i e_i$ and $\sum_{i \in I} t_i (y_i, \delta_i) = \sum_{i \in I} t_i (e_i, \delta_i)$. As $\widehat{u}_i(y_i, \delta_i) > \widehat{u}_i(x_i^*, \bar{d}_i) \forall i: t_i > 0$, the coalition t blocks the allocation (x^*, \bar{d}) , which contradicts the fact that (x^*, \bar{d}) is an Edgeworth equilibrium of $\widehat{\mathcal{E}}$.

Case 2: $t_{i \max}^2 > 0$. We first remark that, in this case, there exists i_0 such that $t_{i_0}^1 > 0$. Indeed, if not, from $\sum_{i \in I} t_i^2 y_i = \sum_{i \in I} t_i^2 x_i^*$ with $u_i(y_i) > u_i(x_i^*) \forall i: t_i^2 > 0$, we easily deduce: $\sum_{i \in I} t_{i \max}^2 \left(\frac{t_i^2}{t_{i \max}^2} y_i + \frac{t_{i \max}^2 - t_i^2}{t_{i \max}^2} x_i^* \right) = \sum_{i \in I} t_{i \max}^2 e_i$ with if $z_i = \frac{t_i^2}{t_{i \max}^2} y_i + \frac{t_{i \max}^2 - t_i^2}{t_{i \max}^2} x_i^*$, $u_i(z_i) > u_i(x_i^*)$ if $t_i^2 > 0$, $z_i = x_i^*$ if $t_i^2 = 0$. As $t_{i \max}^2 > 0 \implies z = (z_i)_{i \in I} \in \mathcal{A}(\mathcal{E})$, recalling that $S(\bar{x}) \subset S(x^*)$, it follows that $(u_i(z_i))_{i \in I} \in \mathcal{U}(\bar{x})$, which contradicts the definition of x^* .

In view of this remark, using

$$\sum_{i \in I} t_i y_i = \sum_{i \in I} t_i^1 e_i + \sum_{i \in I} t_i^2 x_i^* \quad (2.1)$$

and

$$0 < \sum_{i \in I} t_i^1 \delta_i + \sum_{i \in I} t_i^2 \bar{d}_i \quad (2.2)$$

we can choose for $i \in S(x^*)$ $d_i > \bar{d}_i$ and rational τ_i^2 such that

$$\sum_{i \in S(x^*)} \tau_i^2 (d_i - \bar{d}_i) = \sum_{i \in I} t_i^1 \delta_i + \sum_{i \in I} t_i^2 \bar{d}_i \quad (2.3)$$

with $0 < \tau_i^2 < t_{i \max}^2$.

Let in $\widehat{\mathcal{E}}$, the allocation $z = (z_i)_{i \in I}$ with $z_i = \begin{cases} (y_i, 0) & \text{if } i \notin S(x^*) \\ (x_i^*, d_i) & \text{if } i \in S(x^*) \end{cases}$.

Setting $\tau_i^2 = 0$ for $i \in S(x^*)$, $\tau_i^1 = t_i^1$ and $\tau_i^2 = t_i^2$ for $i \notin S(x^*)$, it is easily verified that

$$\sum_{i \in I} (\tau_i^1 + \tau_i^2) z_i = \sum_{i \in I} t_i^1 (e_i, \delta_i) + \sum_{i \in I} \tau_i^2 (x_i^*, \bar{d}_i) \quad (2.4)$$

and that the fuzzy coalition $\tau = \tau^1 + \tau^2$ rejects with z the allocation (x^*, \bar{d}) . Notice, in addition, that $\tau_i > 0 \forall i \in I$.

From (2.4), using the attainability of the allocation $(x_i^*, \bar{d}_i)_{i \in I}$, we successively deduce:

$$\sum_{i \in I} t_i^1 z_i + \sum_{i \in I} \tau_i^2 z_i + \sum_{i \in I} (t_{i \max}^2 - \tau_i^2) (x_i^*, \bar{d}_i) = \sum_{i \in I} t_i^1 (e_i, \delta_i) + \sum_{i \in I} t_{i \max}^2 (x_i^*, \bar{d}_i) \quad (2.5)$$

$$\sum_{i \in I} t_i^1 (y_i, 0) + \sum_{i \in I} t_{i \max}^2 \left(\frac{\tau_i^2}{t_{i \max}^2} z_i + \left(1 - \frac{\tau_i^2}{t_{i \max}^2}\right) (x_i^*, \bar{d}_i) \right) = \sum_{i \in I} (t_i^1 + t_{i \max}^2) (e_i, \delta_i) \quad (2.6)$$

Set

$$z'_i = \frac{\tau_i^2}{t_{i \max}^2} z_i + \left(1 - \frac{\tau_i^2}{t_{i \max}^2}\right) (x_i^*, \bar{d}_i),$$

$$z''_i = \frac{t_i^1}{t_i^1 + t_{i \max}^2} (y_i, 0) + \frac{t_{i \max}^2}{t_i^1 + t_{i \max}^2} z'_i.$$

For every $i \in I$, let us first assume that $\tau_i^2 > 0$. Then, either $i \in S(x^*)$, $z_i = (x_i^*, d_i)$, and $\widehat{u}_i(z'_i) > \widehat{u}_i(x_i^*, \bar{d}_i)$ follows from the strict monotonicity of \widehat{u}_i relative to its second argument when $i \in S(x^*)$ and its strict quasi-concavity; or $i \notin S(x^*)$, $z_i = (y_i, 0)$, $\widehat{u}_i(z_i) > \widehat{u}_i(x_i^*, \bar{d}_i)$

and $\widehat{u}_i(z'_i) > \widehat{u}_i(x_i^*, \bar{d}_i)$ by strict quasi-concavity of \widehat{u}_i . In both cases, whether $t_i^1 > 0$ or not, $\widehat{u}_i(z''_i) > u_i(x_i^*, \bar{d}_i)$.

Let us now assume that $\tau_i^2 = 0$. Then, $i \notin S(x^*)$, $\tau_i^1 = t_i^1 > 0$, $z'_i = (x_i^*, \bar{d}_i)$, $\widehat{u}_i(y_i, 0) > \widehat{u}_i(x_i^*, \bar{d}_i)$, and $\widehat{u}_i(z''_i) > u_i(x_i^*, \bar{d}_i)$ follows from the strict quasi-concavity of \widehat{u}_i .

Summarizing,

$$\sum_{i \in I} \frac{t_i^1 + t_{i \max}^2}{\sum_{j \in J} (t_j^1 + t_{j \max}^2)} z''_i = \sum_{i \in I} \frac{t_i^1 + t_{i \max}^2}{\sum_{j \in J} (t_j^1 + t_{j \max}^2)} (e_i, \delta_i)$$

with $\widehat{u}_i(z''_i) > \widehat{u}_i(x_i^*, \bar{d}_i) \forall i \in I$, which contradicts $(x^*, \bar{d}) \in \mathcal{C}^e(\widehat{\mathcal{E}})$. The proof of Claim 3 is now complete.

Claim 4. *In view of A.3, one actually has $x^* \in \mathcal{C}_r^f(\mathcal{E})$.*

We omit the proof which is very similar to the one used for proving that, under such a continuity assumption, an Edgeworth equilibrium of an economy is actually an element of the fuzzy core. Claim 4 completes the proof of the proposition. ■

Decentralizing with prices the elements of the fuzzy rejective core, we get as a consequence of Proposition 2.2 the following proposition which does not follow from any of the existence results proposed so far in the literature.

Proposition 2.3. *Under Assumptions A.1–A.5, the economy \mathcal{E} has an equilibrium with dividends.*

Proof. In view of Proposition 2.2, let $\bar{x} \in \mathcal{C}_r^f(\mathcal{E})$. As in Konovalov (2005), it can be proved that if $N(\bar{x}) \neq \emptyset$, then there exists a nonnull $\bar{p} \in \mathbb{R}^\ell$ such that (\bar{p}, \bar{x}) is a quasiequilibrium with dividends of \mathcal{E} . Given Assumption A.4, this quasiequilibrium is actually an equilibrium with dividends. If $N(\bar{x}) = \emptyset$, \bar{x} is trivially the allocation of an equilibrium with dividends. ■

Remark 2.2. In several respects, Proposition 2.3 goes less far than results which, following Mas-Colell (1992), do not assume transitive preferences of consumers. In our framework, that is, in the case where consumer's preferences are transitive and represented by utility functions, the reference paper is Le Van–Minh (2007). Recall that the compactness assumption used by Le Van and Minh (2007) (compactness of the set of individually rational attainable allocations together with the upper semicontinuity of utility functions) was proved by Martins-da-Rocha and Monteiro (2009)⁶ to be strictly stronger than the assumption of strong compactness of the individually rational utility set (A.5) that we use in Propositions 2.2 and 2.3

Remark 2.3. Before closing Section 2, it is worth noticing the role of Assumption A.4 in the previous proofs. As well-known, such an assumption is not necessary for the non-emptiness of the fuzzy core of an economy. Its role in the proof of the non-emptiness of the fuzzy rejective core (Proposition 2.2) is stressed just before the statement of Claim 3, with a reference to Proposition 1 in the appendix. In the proof of Proposition 2.3, together with A.3, Assumption A.4 is classically used for proving that a quasiequilibrium (with dividends) is actually an equilibrium with dividends.

⁶See Example 7.4 in their paper.

3. EXISTENCE OF (WALRASIAN) EQUILIBRIUM UNDER A WEAK NON-SATIATION ASSUMPTION

According to Definition 1.1, a (Walrasian) quasiequilibrium (resp. equilibrium) is a quasiequilibrium (resp. equilibrium) with dividends (x^*, p^*) such that $p^* \neq 0$ and all dividends m_i^* are equal to 0. If for each $i \in I$, $x_i^* \notin S_i$, in view of the strict quasi-concavity of utility functions, both conditions are obviously satisfied at equilibrium. This explains why the classical assumption of (local) non-satiation at every component of an attainable allocation has been for a long time the only one non-satiation assumption considered for proving Walrasian quasiequilibrium (resp. equilibrium) existence. Assuming, in addition to **A.1** and **A.2**, that $\mathcal{A}_{IR}(\mathcal{E})$ is compact and that the utility functions u_i are upper semicontinuous, Allouch and Le Van (2009) have stated a weaker non-satiation condition sufficient for the existence of a (Walrasian) quasiequilibrium. An interesting result immediately extended by Martins-da-Rocha and Monteiro (2009) to their weaker compactness assumption (Assumption **A.5**) on the economy. In this section, we use **A.5** and formulate a different weak non-satiation assumption (Assumption **A.6** below) especially adapted to the exchange economies considered in this paper, that is, to economies where consumers' utility functions are not assumed to be upper semicontinuous on their consumption set. This assumption will be proved to be sufficient for quasiequilibrium existence under **A.1–A.3** and **A.5**.

We consider on \mathcal{E} the assumptions **A.1 – A.3**, **A.5**, still assume without loss of generality that for each $i \in I$, $e_i \notin S_i$, and add the following weak non-satiation assumption:

A.6: For each consumer $i \in I$, if $S_i \cap \mathcal{A}_i \neq \emptyset$ then for all $x_i \in S_i$, $\lambda_i(x_i) := \inf\{\lambda > 0 : e_i + \lambda(x_i - e_i) \in S_i\} < 1$.

Note that for each i , the set of satiated points S_i is convex and would be closed if the utility function u_i was supposed to be upper semicontinuous. Assumption **A.6** is thus inconsistent with any assumption of upper semicontinuity of consumers' utility functions.

Let us define on X_i , the following utility functions:

$$\widehat{u}_i(x_i) = \begin{cases} u_i(x_i) & \text{if } x_i \notin S_i \\ u_i(x_i) + e^{-d(e_i, x_i)} & \text{if } x_i \in S_i \end{cases}$$

and consider the economy $\widehat{\mathcal{E}}$, obtained from \mathcal{E} replacing by \widehat{u}_i the consumers' utility functions u_i .

For proving equilibrium existence in \mathcal{E} , the strategy is different from the one used in Section 2. We first establish the non-emptiness of the fuzzy core of $\widehat{\mathcal{E}}$ and thus the existence of a (Walrasian) quasi-equilibrium for $\widehat{\mathcal{E}}$. The equilibrium existence for \mathcal{E} will then follow from the equilibrium existence for $\widehat{\mathcal{E}}$.

Proposition 3.1. *Under the assumptions **A.1 – A.3**, **A.5** and **A.6**, $\mathcal{C}^f(\widehat{\mathcal{E}}) \neq \emptyset$.*

Proof. It is easily verified that the functions $\widehat{u}_i: X_i \rightarrow \mathbb{R}$ are quasi-concave. We now prove that the set

$$\widehat{\mathcal{U}} = \{v = (v_i)_{i \in I} : \exists x = (x_i)_{i \in I} \in \mathcal{A}(\mathcal{E}), \text{ s.t. } 0 \leq v_i \leq \widehat{u}_i(x_i) \forall i \in I\}$$

is compact. Indeed let v^ν and $x^\nu \in \mathcal{A}(\mathcal{E})$ be such that for each $i \in I$, $\widehat{u}_i(e_i) = u_i(e_i) \leq v_i \leq \widehat{u}_i(x_i^\nu)$. Since $\widehat{\mathcal{U}}$ is strongly compact, there exist a subsequence x^{ν_k} of x^ν and an allocation $y \in \mathcal{A}(\mathcal{E})$ such that

$$\forall i \in I, \quad u_i(y_i) \geq \lim_{k \rightarrow \infty} u_i(x_i^{\nu_k}) \quad (3.1)$$

and one of the three cases stated in Definition 2.1 occur. From (3.1) and the definition of \widehat{u}_i , we first deduce that for each $i \in I$, the sequence $v_i^{\nu_k}$ converges to $v_i \geq 0$.

If $y_i \notin S_i$, then for k large enough, $x_i^{\nu_k} \notin S_i$. Then, from (3.1) and from $v_i^{\nu_k} \leq \widehat{u}_i(x_i^{\nu_k}) = u_i(x_i^{\nu_k})$, we deduce: $v_i \leq u_i(y_i) = \widehat{u}_i(y_i)$,

If $y_i \in S_i$, the third case may still occur. Then from (3.1), and from $v_i^{\nu_k} \leq \widehat{u}_i(x_i^{\nu_k}) = u_i(x_i^{\nu_k})$, we deduce: $v_i \leq u_i(y_i) \leq \widehat{u}_i(y_i)$. In the first case, we can assume without loss of generality $\lim_{k \rightarrow \infty} \|x_i^{\nu_k}\| = \infty$, so that from (3.1) and from $v_i^{\nu_k} \leq \widehat{u}_i(x_i^{\nu_k}) \leq u_i(x_i^{\nu_k}) + e^{-d(e_i, x_i^{\nu_k})}$, we deduce $v_i \leq u_i(y_i) \leq \widehat{u}_i(y_i)$. In the second case, the third one being excluded, we can assume without loss of generality $\widehat{u}_i(x_i^{\nu_k}) = u_i(x_i^{\nu_k}) + e^{-d(e_i, x_i^{\nu_k})}$, so that by using (3.1) and passing to limit in the relations $v_i^{\nu_k} \leq \widehat{u}_i(x_i^{\nu_k})$, we get $v_i \leq u_i(y_i) + d(e_i, y_i) = \widehat{u}_i(y_i)$. The proof of compactness of $\widehat{\mathcal{U}}$ is thus complete.

In view of Theorem 3.1 in Allouch–Florenzano (2004), the economy $\widehat{\mathcal{E}}$ has an Edgeworth equilibrium $\bar{x} = (\bar{x}_i)_{i \in I}$. We now prove that, under the assumption **A.6**, \bar{x} is an element of the fuzzy core of $\widehat{\mathcal{E}}$. To see that, assume by contraposition that there exist $t = (t_i)_{i \in I} \in [0, 1] \setminus \{0\}$ and $x \in \prod_{i \in I} X_i$ such that $\sum_{i \in I} t_i x_i = \sum_{i \in I} t_i e_i$ and $\widehat{u}_i(x_i) > \widehat{u}_i(\bar{x}_i) \quad \forall i \in \text{supp } t$.

We first claim that there exists $\varepsilon > 0$ such that for each $i \in I$, $1 - \varepsilon < \lambda < 1 \implies \widehat{u}_i(e_i + \lambda(x_i - e_i)) > \widehat{u}_i(\bar{x}_i)$.

For each $i \in I$, several cases must be considered:

- If $\bar{x}_i \in S_i$, from which it follows that $x_i \in S_i$ and $d(e_i, x_i) < d(e_i, \bar{x}_i)$, it suffices to take $1 - \varepsilon > \lambda_i(x_i)$.
- If $\bar{x}_i \notin S_i$ and $x_i \in S_i$, from which it follows that $u_i(x_i) + e^{-d(e_i, x_i)} > u_i(\bar{x}_i)$, it suffices to take $1 - \varepsilon > \lambda_i(x_i)$ to get for $\lambda > 1 - \varepsilon$, $\widehat{u}_i(e_i + \lambda(x_i - e_i)) > u_i(x_i) + e^{-d(e_i, x_i)} = \widehat{u}_i(x_i)$.
- If $\bar{x}_i \notin S_i$ and $x_i \notin S_i$, from which it follows that $u_i(x_i) > u_i(\bar{x}_i)$, using **A.3**, it suffices to take $1 - \varepsilon$ such that for $\lambda > 1 - \varepsilon$, $\widehat{u}_i(e_i + \lambda(x_i - e_i)) \geq u_i(e_i + \lambda(x_i - e_i)) > u_i(\bar{x}_i) = \widehat{u}_i(\bar{x}_i)$.

We now classically define for each $i \in I$, $s_i \in Q$ such that $t_i = 0 \implies s_i = 0$ and $t_i > 0 \implies 1 - \varepsilon < \frac{t_i}{s_i} < 1$ and $x_i^s = e_i + \frac{t_i}{s_i}(x_i - e_i)$. If $s = (s_i)_{i \in I}$, it is easily verified that $\sum_{i \in I} s_i x_i^s = \sum_{i \in I} s_i e_i$ and $\widehat{u}_i(x_i^s) > \widehat{u}_i(\bar{x}_i) \quad \forall i \in \text{supp } s$, which contradicts the assumption that \bar{x} is an Edgeworth equilibrium of $\widehat{\mathcal{E}}$. ■

Proposition 3.2. *Under the assumptions **A.1** – **A.3**, **A.5** and **A.6**, the economy $\widehat{\mathcal{E}}$ has a quasiequilibrium.*

Proof. Let \bar{x} be the element of $\mathcal{C}^f(\widehat{\mathcal{E}})$ obtained in the previous proposition. As well known, if for all $i \in I$, $\bar{x}_i \in \text{cl } \widehat{P}_i(\bar{x}_i)$, then there exists $\bar{p} \in \mathbb{R}^\ell$, $\bar{p} \neq 0$ such that (\bar{x}, \bar{p}) is a quasiequilibrium of $\widehat{\mathcal{E}}$. Recall that $\widehat{P}_i(\bar{x}_i) = \{x_i \in X_i : \widehat{u}_i(x_i) > \widehat{u}_i(\bar{x}_i)\}$. If $\bar{x}_i \in S_i$, the above condition follows from the definition of \widehat{u}_i . If $\bar{x}_i \notin S_i$, the above condition follows from Assumption **A.2**. ■

Corollary 3.1. *Under the assumptions **A.1** – **A.3**, **A.5** and **A.6**, the economy \mathcal{E} has a quasiequilibrium. If we assume in addition **A.4**, this quasiequilibrium is actually an equilibrium.*

Proof. A quasiequilibrium of $\widehat{\mathcal{E}}$ is obviously a quasiequilibrium of \mathcal{E} . Proving the second part of the statement of the corollary is standard. ■

APPENDIX

Proposition 1. *Assume Assumptions **A.1** – **A.4** on the economy \mathcal{E} . If $x^* \in \mathcal{A}_{IR}(\mathcal{E}) \setminus \mathcal{C}_r^e(\mathcal{E})$ then there exists τ^1, τ^2 in $([0, 1] \cap \mathbb{Q})^I$ and $y \in \prod_{i \in I} X_i$ such that $\tau = \tau^1 + \tau^2 \neq 0$ and*

- (a) $\sum_{i \in I} \tau_i y_i = \sum_{i \in I} \tau_i^1 e_i + \sum_{i \in I} \tau_i^2 x_i^*$,
(b) $u_i(y_i) > u_i(x_i^*) \forall i: \tau_i > 0$,

with $\tau_i > 0$ if and only if $i \notin S(x^*)$.

Proof. Indeed, let us assume that $x^* \in \mathcal{A}_{IR}(\mathcal{E}) \setminus \mathcal{C}_r^e(\mathcal{E})$ thus that

- (a) $\sum_{i \in I} t_i z_i = \sum_{i \in I} t_i^1 e_i + \sum_{i \in I} t_i^2 x_i^*$,
(b) $u_i(z_i) > u_i(x_i^*) \forall i: t_i > 0$,

for some t^1, t^2 in $([0, 1] \cap \mathbb{Q})^I$, $t = t^1 + t^2 \neq 0$ and $z \in \prod_{i \in I} X_i$.

We first claim that there exist $z' \in \prod_{i \in I} \text{int } X_i$ and t'^1, t'^2 in $([0, 1] \cap \mathbb{Q})^I$, $t' = t'^1 + t'^2 \neq 0$ such that

- (a) $\sum_{i \in I} t'_i z'_i = \sum_{i \in I} t'^1_i e_i + \sum_{i \in I} t'^2_i x_i^*$,
(b) $u_i(z'_i) > u_i(x_i^*) \forall i: t'_i > 0$.

Indeed, it suffices to choose $\lambda \in \mathbb{Q} \cap (0, 1)$ such that if $z'_i = e_i + \lambda(z_i - e_i)$ then $u_i(z'_i) > u_i(x_i^*) \forall i: t_i > 0$. Then $\sum_{i \in I} \frac{t_i}{\lambda} z'_i = \sum_{i \in I} (t_i^1 + t_i(\frac{1}{\lambda})) e_i + \sum_{i \in I} t_i^2 x_i^*$. As it follows from the convexity of X_i (**A.1**) and **A.4** that $z'_i \in \text{int } X_i$, then $z' = (z'_i)_{i \in I}$ and, up to an obvious normalization, $t' = (\frac{t_i}{\lambda})_{i \in I}$ prove our claim.

To prove Proposition 1, starting now from the relations obtained in our first claim, we assume that for some i^0 , $t'_{i^0} = 0$ and $i^0 \in N(x^*)$. Let y_{i^0} be such that $u_{i^0}(y_{i^0}) > u_{i^0}(x_{i^0}^*)$. It follows from **A.2** that $u_{i^0}(x_{i^0}^* + \mu(y_{i^0} - x_{i^0}^*)) > u_{i^0}(x_{i^0}^*)$ for every $\mu \in (0, 1]$. Letting $z'_{i^0} = x_{i^0}^* + \mu(y_{i^0} - x_{i^0}^*)$, we now have: $\sum_{i \in I} t'_i z'_i + z'_{i^0} - \mu(y_{i^0} - x_{i^0}^*) = \sum_{i \in I} t'^1_i e_i + \sum_{i \in I} t'^2_i x_i^* + x_{i^0}^*$. Since for $t'_i > 0$, $z'_i \in \text{int } X_i$, in view of **A.3**, it is possible to choose i^1 such that $t'_{i^1} > 0$ and $\mu \in (0, 1]$ such that, if $z''_{i^1} = z'_{i^1} - \frac{\mu}{t'_{i^1}}(y_{i^0} - x_{i^0}^*)$ then $u_{i^1}(z''_{i^1}) > u_{i^1}(x_{i^1}^*)$ and $\sum_{i \neq i^1} t'_i z'_i + t'_{i^1} z''_{i^1} + z'_{i^0} = \sum_{i \in I} t'^1_i e_i + \sum_{i \in I} t'^2_i x_i^* + x_{i^0}^*$. The same procedure can be repeated until there is no i such that $t'_i = 0$ and $i \in N(x^*)$. ■

REFERENCES

- Allouch, N., Florenzano, M., 2004. Edgeworth and Walras equilibria of an arbitrage-free exchange economy. *Economic Theory* 23, 353–370.
- Allouch, N., Le Van, C., 2009. Erratum to "Walras and dividends equilibrium with possibly satiated consumers". *Journal of Mathematical Economics* 45, 320–328.
- Aubin, J.P., 1979. *Mathematical methods of game and economic theory*. Amsterdam, North-Holland.
- Florenzano, M., 1990. Edgeworth equilibria, Fuzzy core, and equilibria of a production economy without ordered preferences. *Journal of Mathematical Analysis and Applications* 153 (1), 18–36.
- Kononov, A., 1998. Core equivalence in economies with satiation. Center Discussion Paper 9828, Tilburg University.
- Kononov, A., 2005. The core of an economy with satiation. *Economic theory* 25, 711–719.
- Le Van, C., Minh, N.B., 2007. No-arbitrage condition and existence of equilibrium with dividends. *Journal of Mathematical Economics* 43, 135–152.
- Martins-da-Rocha, V.F., Monteiro, P.K., 2009. Unbounded exchange economies with satiation: How far can we go?. *Journal of Mathematical Economics* 45, 465–478.
- Mas-Colell, A., 1992. Equilibrium theory with possibly satiated preferences. In: Majumdar, M. (Ed.), *Equilibrium and dynamics: Proceedings of the Essays in Honour of David Gale*, Macmillan, London, pp. 201–213.

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