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Last Success Problem

Decision Rule and Application

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Abstract

Where is the most likely position for the last success in n events, if each event has the same probability $\Pr(A)$? What is the probability for the last success? This situation assumes returning successes which is different to the stop waiting problem (see [1], [5]) where a single best event is assumed. We set F. Thomas Bruss's theory into a simple probability framework and develop a decision rule for the last success. The results are applied to the Apple share prices.

1 When does the Last Success Come Up?

This question often appears in situations where repeating successes come up and a decision should be moved to the last success within a fixed period, for example for selling assets with a high price to finance an investment.

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Example 1.1

A 6 is defined as a success on a die. All positions between the second last and the last success must be counted. For $w = 1$ the last success is on $k = 8$, but between the first and eighth position no success occurs, so the positions from 1 to 8 are possible to obtain the last success. For $w = 10$ no last success appears.

	w= 1	w= 2	w= 3	w= 4	w= 5	w= 6	w= 7	w= 8	w= 9	w= 10
k= 1	3	3	2	2	6	1	6	4	4	1
k= 2	2	1	3	4	2	1	6	3	5	5
k= 3	5	3	4	6	3	1	5	5	2	2
k= 4	4	4	2	2	6	3	4	5	5	4
k= 5	2	1	4	3	3	6	1	2	4	4
k= 6	5	6	6	3	2	2	1	4	2	2
k= 7	2	1	1	6	4	5	2	3	3	4
k= 8	6	2	4	6	2	3	1	6	2	3
k= 9	5	2	4	3	6	1	6	5	5	2
k= 10	4	4	5	5	1	6	2	6	2	3

k= 1	k= 2	k= 3	k= 4	k= 5	k= 6	k= 7	k= 8	k= 9	k= 10
0.3	0.3	0.4	0.4	0.5	0.6	0.4	0.5	0.4	0.2

Counting out on which position the last success most likely occurs, we see in this small simulation that this happens on position $k = 6$.

2 The Probability for $\Pr(A_k)$

The event A denotes a success and the event A_k denotes the last success on position k . We develop the probability for $\Pr(A_k)$ in 2 steps and assume that $\Pr(A)$ is constant.

1. First we look on A_{jk} which describes the last success in k , if after the j th event the last success appears with $j < k$. We fix k and let j go reversely from $k - 1$ to i (see fig. 1).

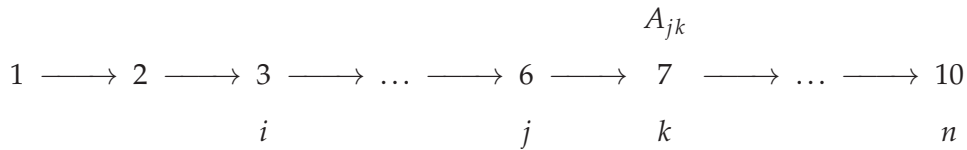


Figure 1: Last success $\Pr(A_{jk})$

Example 2.1

We use a die and define $A = 6$, so that the $\Pr(A) = \frac{1}{6}$ and set $n = 10$. If we set $k = 7$ and $j = 6$ (see fig. 1) the probability is $\Pr(A_{67}) = \frac{1}{6}$, because the chance that

after the 6th throw on $k = 7$ a 6 appears is $\Pr(A)$. If we set $j = 5$ the situation is the following: A 6 on $k = 7$ and no 6 on $j = 6$ or a 6 on $j = 6$ and no 6 on $k = 7$. Both situations will show up a 6 after $j = 5$. The probability for no 6 is $\Pr(\neg A) = \frac{5}{6}$ and therefore the probability for $\Pr(A_{57}) = \frac{1}{6} \frac{5}{6} + \frac{5}{6} \frac{1}{6} = 0.278$. Let us look on $j = 4$ and $k = 7$. Now a 6 can appears on position 5 or 6. So all possible combinations are: $+-$ or $-+$ or $--$, where $+$ stands for a 6 and $-$ for no 6. The probability is $\Pr(A_{47}) = \frac{1}{6} \left(\frac{5}{6}\right)^2 + \frac{1}{6} \left(\frac{5}{6}\right)^2 + \frac{1}{6} \left(\frac{5}{6}\right)^2 = 0.347$. In total we have

	k= 1	k= 2	k= 3	k= 4	k= 5	k= 6	k= 7	k= 8	k= 9	k= 10
j= 0	0.167	0.278	0.347	0.386	0.402	0.402	0.391	0.372	0.349	0.323
j= 1	NA	0.167	0.278	0.347	0.386	0.402	0.402	0.391	0.372	0.349
j= 2	NA	NA	0.167	0.278	0.347	0.386	0.402	0.402	0.391	0.372
j= 3	NA	NA	NA	0.167	0.278	0.347	0.386	0.402	0.402	0.391
j= 4	NA	NA	NA	NA	0.167	0.278	0.347	0.386	0.402	0.402
j= 5	NA	NA	NA	NA	NA	0.167	0.278	0.347	0.386	0.402
j= 6	NA	NA	NA	NA	NA	NA	0.167	0.278	0.347	0.386
j= 7	NA	NA	NA	NA	NA	NA	NA	0.167	0.278	0.347
j= 8	NA	NA	NA	NA	NA	NA	NA	NA	0.167	0.278
j= 9	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.167

The table shows that it is not necessary to use j , because on all diagonals the same probabilities appears. So it is sufficient to calculate the probability only for each $k = n, \dots, 1$.

2. $\Pr(A_{k=1})$ is now the probability of the former $\Pr(A_{0,10})$, which denotes that in the first throw a 6 appears and 9 following throws without a success: $\Pr(A_1) = 10 \times \Pr(A) (1 - \Pr(A))^9 = 0.323$. $\Pr(A_{10})$ is hence the situation where the last success occurs one throw later: $(j = 0, k = 1), (j = 1, k = 2), \dots, (j = 9, k = 10)$ with $\Pr(A_{10}) = 1 \times \Pr(A) (1 - \Pr(A))^0 = 0.167$.

In general we receive with a constant $\Pr(A)$ the formula

$$\Pr(A_k) = \sum_{j=k}^n \Pr(A) (1 - \Pr(A))^{n-k} \quad \text{for } k = 1, \dots, n$$

The sum is not dependent on index j , so

$$\Pr(A_k) = (n - k + 1) \Pr(A) (1 - \Pr(A))^{n-k} \quad (1)$$

Example 2.2 (cont.)

The most likely position for the last success in this example is $k = 6$ (looking from n to 1).

$k= 1$	$k= 2$	$k= 3$	$k= 4$	$k= 5$	$k= 6$	$k= 7$	$k= 8$	$k= 9$	$k= 10$
0.323	0.349	0.372	0.391	0.402	0.402	0.386	0.347	0.278	0.167

Remark: If we assume that the probability $\Pr(A)$ is not constant the probability $\Pr(A_k)$ is dependent on j : $\Pr(A_{jk})$. The above table (see page 3) has then different probabilities on each diagonal.

$$\Pr(A_{jk}) = \sum_{i=j+1}^k \left[\Pr(A_i) \prod_{\substack{h=j+1 \\ h \neq i}}^k (1 - \Pr(A_h)) \right] \quad \text{for } j < k = 1, \dots, n$$

3 Decision Rule

We set $\Pr(A)$ constant. A heuristic approach is that the chance for a success must be greater than for k fails. This leads to the condition where $\Pr(A_k)$ is the last time greater than $(1 - \Pr(A))^{n-k}$. The largest k which fulfils the following inequality leads to $\sup \Pr(A_k)$ (see fig.2 orange line).

$$\sup_{k=1, \dots, n} \left\{ k : \Pr(A_k) \geq (1 - \Pr(A))^{n-k+1} \right\} \Leftrightarrow \sup_{k=1, \dots, n} \left\{ k : (n - k + 1) \Pr(A) \geq (1 - \Pr(A)) \right\}$$

A more mathematically approach is to find the roots of the derivation of $\Pr(A_k)$.

$$\begin{aligned} \frac{\partial \Pr(A_k)}{\partial k} &= -\Pr(A) (1 - \Pr(A))^{n-k} - (n - k + 1) \Pr(A) (1 - \Pr(A))^{n-k} \dots \\ &\quad \times \ln(1 - \Pr(A)) \stackrel{!}{=} 0 \end{aligned} \quad (2)$$

Solving eq. (2) for k leads to k_{\sup} .

$$k_{\sup} = n + 1 + \frac{1}{\ln(1 - \Pr(A))} \quad (3)$$

Because we have a discrete function we round k_{\sup} to the nearest integer: $\lfloor k_{\sup} + 0.5 \rfloor$. Rewriting the necessary condition (2) shows the above heuristic condition.

$$-\frac{\Pr(A)}{1 - \Pr(A)} (1 - \Pr(A))^{n-k+1} = \Pr(A_k) \ln(1 - \Pr(A))$$

For small $\Pr(A)$ (less than 0.2) $-\frac{\Pr(A)}{1 - \Pr(A)} \approx \ln(1 - \Pr(A))$, so for k_{\sup}

$$(1 - \Pr(A))^{n-k+1} \approx \Pr(A_k)$$

holds. $(1 - \Pr(A))^{n-k+1}$ is for $k = 0 \dots, n$ a monotonic increasing function and for $k \leq k_{\sup}$ less than $\Pr(A_k)$ so the maximum for $\Pr(A_k)$ is found for that k where $(1 - \Pr(A))^{n-k+1}$ is the last time smaller or equal $\Pr(A_k)$ (see fig. 2).

The relative position of k_{\sup} – with a constant $\Pr(A)$ – is always $1 + \frac{1}{\ln(1-\Pr(A))}$ below n .

Example 3.1 (cont.)

For $\Pr(A) = \frac{1}{6}$ the number of $k_{\sup} - n$ is $\left\lfloor 1 + \frac{1}{\ln(1-\frac{1}{6})} + 0.5 \right\rfloor \approx \lfloor -4.48 + 0.5 \rfloor = -4$.

Rounding to the next integer the supremum of $\Pr(A_k)$ is on the fifth last position of n . For $n = 10$ the position of $\sup \Pr(A_k) = 0.402$ is $k_{\sup} = 6$. If n increases to 100, k_{\sup} moves to 96. The position will stay on the fifth last position of n as long as $\Pr(A)$ is unchanged.

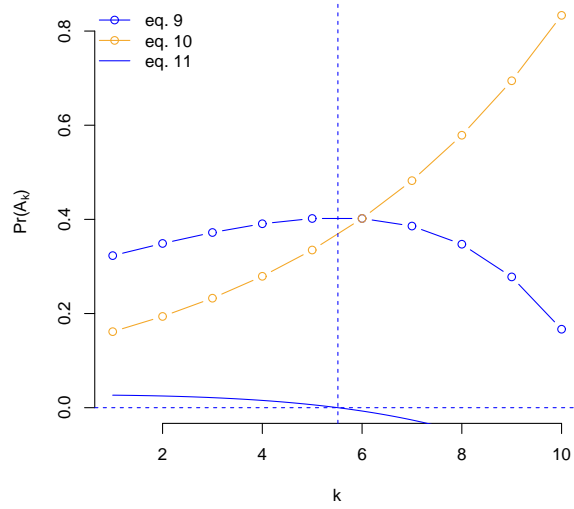


Figure 2: Decision Rule for $\sup \Pr(A_k)$

4 Simulation and Convergence of $\sup \Pr(A_k)$

In our simulation we use again 1000 repeats each with $n = 100$ values. One simulation is with $\Pr(A) = \frac{1}{6}$ which is a die. The other simulation assumes a probability for a success of $\Pr(A) = \frac{1}{60}$. Both simulations confirm the theoretical calculation (see fig. 3, left).

With decreasing probabilities for $\Pr(A)$ the probabilities of the last success $\Pr(A_k)$ converges to $\frac{1}{e}$ (see fig. 3, right). But if only n increases with a constant $\Pr(A)$ the probability

of $\Pr(A_k)$ remains unchanged and only the position of k change.

With $k_{\sup} = n + 1 + \frac{1}{\ln(1-\Pr(A))}$ the maximum probability of $\Pr(A_k)$ is

$$\sup \Pr(A_k) = -\frac{1}{\ln(1-\Pr(A))} \Pr(A) (1-\Pr(A))^{-\left(1+\frac{1}{\ln(1-\Pr(A))}\right)}$$

We let $\Pr(A) \rightarrow 0$. For small $\Pr(A)$ the expression $\ln(1-\Pr(A)) \approx -\frac{\Pr(A)}{1-\Pr(A)}$. So we can write for $\sup \Pr(A_k)$

$$\begin{aligned} \lim_{\Pr(A) \rightarrow 0} \sup \Pr(A_k) &= \lim_{\Pr(A) \rightarrow 0} \frac{1-\Pr(A)}{\Pr(A)} \Pr(A) (1-\Pr(A))^{\frac{1-\Pr(A)}{\Pr(A)}-1} \\ &= \lim_{\Pr(A) \rightarrow 0} (1-\Pr(A))^{\frac{1-\Pr(A)}{\Pr(A)}} = \lim_{\Pr(A) \rightarrow 0} \frac{(1-\Pr(A))^{\frac{1}{\Pr(A)}}}{1-\Pr(A)} = \frac{1}{e} \end{aligned} \quad (4)$$

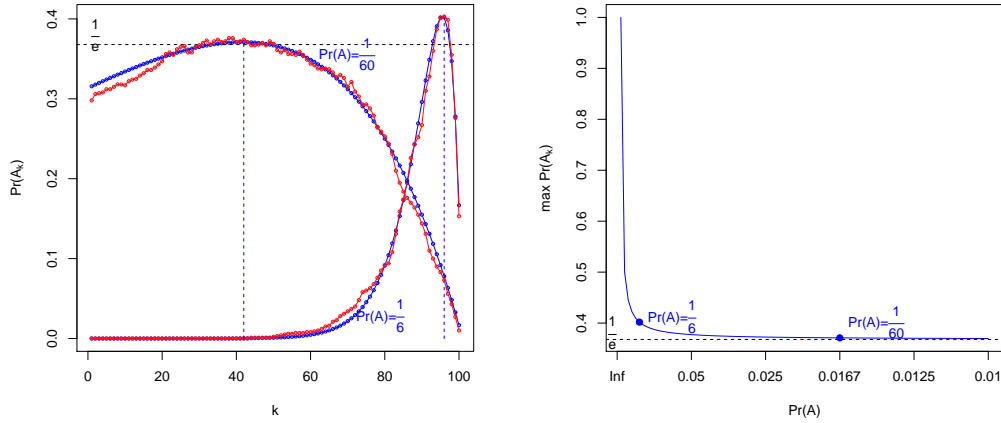


Figure 3: Probabilities of $\Pr(A_k)$ and Distribution of $\sup \Pr(A_k)$

5 Application

For our application to Apple share prices we have to define what a success and what the probability for a success is. The probability for a success is computed from 1000 simulated share prices. For the simulation of share prices all log returns $\leq \pm 0.011$ are removed to avoid a strong drift. The resampled log returns are used to simulate the share prices (see fig. 4, right). The estimation for $\Pr(A)$ is 0.136. A success for selling

is defined, if the price is above $\bar{x} + s = 344.245$. Average and standard deviation are computed from the simulated prices.

For the historical Apple data ($n = 755$) the last success is indicated at $k = 749$ (see eq. 3). In fig. 4 (left) we see that this point is connected with a success (the share price is above $\bar{x} + s$). The probability for the last success is computed with $\Pr(A_k) = 0.396$ (see eq. 1).

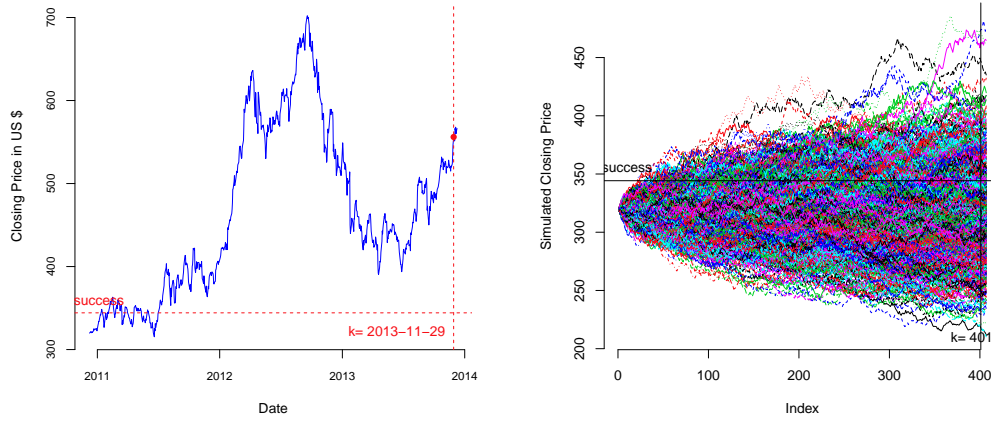


Figure 4: Last Success of Apple Share Prices

To see how the theory of last success performs we use the 1000 simulated share prices ($n = 407$) and use eq. (3) to determine k_{sup} and eq. (1) to compute $\Pr(A_k)$. $\Pr(A)$ is used from the above simulation. The result is at $k = 401$ the last success should appear with $\Pr(A_k) = 0.396 (= \sup \Pr(A_k))$. $\Pr(A_k)$ has the same value, because $\Pr(A)$ is unchanged and therefore the difference $n - k$ is equal.

We compare this result with counting out the frequency at which k the last result appears. The table below shows, that for $k = n$ the last success is most likely ($\Pr(A_k) = 0.228$) and at $k = 401$ the probability is really low. Why does the theory not hold? One possible explanation is, that the drift of the share prices is very much influencing the outcome. Experiments show that for a pure random process the simulations are in line with the theoretical results.

k= 398	k= 399	k= 400	k= 401	k= 402	k= 403	k= 404	k= 405	k= 406	k= 407
0.009	0.011	0.008	0.008	0.006	0.007	0.011	0.005	0.010	0.228

6 Conclusion

The probability theory for the last success has a simple decision rule to find in a series of random events the position with the most likely last success. The probability for the last success with a constant success probability $\Pr(A)$ is as well easy to compute and converges to $\frac{1}{e}$. The application of the theory shows that the above results can be very much influenced by real effects like drifts.

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