

Pérez-Castrillo, David; Wettstein, David

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## Innovation Contests

David Pérez-Castrillo  
David Wettstein

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# Innovation Contests

## Abstract

We study innovation contests with asymmetric information and identical contestants, where contestants' efforts and innate abilities generate inventions of varying qualities. The designer offers a reward to the contestant achieving the highest quality and receives the revenue generated by the innovation. We characterize the equilibrium behavior, outcomes and payoffs for both nondiscriminatory and discriminatory (where the reward is contestant-dependent) contests. We derive conditions under which the designer obtains a larger payoff when using a discriminatory contest and describe settings where these conditions are satisfied.

JEL-Code: O310, D440, J710.

Keywords: contests, auctions, innovation, discrimination.

*David Pérez-Castrillo*  
*Department of Economics & CODE*  
*Universitat Autònoma de Barcelona &*  
*Barcelona GSE*  
*Spain – 08193 Bellaterra (Barcelona)*  
*david.perez@uab.es*

*David Wettstein\**  
*Department of Economics*  
*Ben-Gurion University of the Negev*  
*Israel*  
*wettstn@bgu.ac.il*

\*corresponding author

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# 1 Introduction

Innovative activity has traditionally been rewarded mainly through the patent system. An alternate approach to generating and rewarding innovations is to design contests that solicit proposals to solve targeted objectives (see, for instance, Scotchmer, 2004). A sponsor interested in technological improvement can launch a contest where agents compete by submitting prototypes, the best of which will be adopted by the sponsor. Such contests have been held to obtain innovations in various fields including mathematics, food preservation and maritime navigation.<sup>1</sup>

Today, contests are frequently used by organizations such as the Institute for Advanced Architecture of Catalonia and HP “to promote discussion and research . . . that can help us to envisage how the city and the habitat of the 21st century will turn out.” Similarly, the U.S.-based Knight Foundation has set innovation contests to elicit digital news experiments that inform and engage communities. Contests have also been suggested as a future means to address a variety of issues. For example, Newell and Wilson (2005) and Newell (2008) proposed that the U.S. Department of Energy should hold contests to resolve specific technical and scientific challenges related to greenhouse gases mitigation.

In this paper, we introduce a new model of contests with asymmetric information to study innovation contests. The environment described is general enough to encompass a range of economic activities and provides a tractable analytical setting to consider, among others, procurement decisions, government contracts, research budgets and promotions. For expositional purposes, we choose the particular setting of innovation contests to describe the model.

Our framework is the following. A designer wishes to obtain an innovation that can be produced by  $n \geq 2$  agents. The quality of the innovation achieved by an agent depends on his ability and the effort devoted to the task. The agents’ abilities are independently drawn from the same distribution function and once an agent observes his ability he decides on his effort. Both ability and effort are an agent’s private information. Furthermore, the quality of an innovation is not verifiable. However, an independent party can determine the best-quality innovation. The designer sets up a contest whereby the highest-quality

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<sup>1</sup>Taylor (1995) analyzed the optimal contest in an environment with symmetric information where the quality of the innovation obtained by a firm is a random variable. More recently, the design of optimal contests in an R&D environment has been studied by Che and Gale (2003).

innovation receives a prize.<sup>2</sup> A contest is called nondiscriminatory if the prize does not depend on the winner's identity, otherwise it is called discriminatory.

We characterize the equilibrium behavior in a nondiscriminatory contest with  $n$  agents for arbitrary distributions of abilities and determine the regions where the equilibrium involves positive effort levels for all agents and those where the agents exert zero effort. The boundaries of the regions depend on the distribution function of the ability parameter and the prize.

We then consider discriminatory contests where the reward depends on the identity of the winner. We restrict attention in this part of the analysis to the case where  $n = 2$ .<sup>3</sup> The introduction of discrimination leads to qualitatively different equilibrium behavior, as compared to agents' behavior in nondiscriminatory contests. The two agents' equilibrium behavior in discriminatory contests again involves regions with zero and positive effort levels. The main new feature is the discontinuity in the behavior of the agent assigned the larger reward, who moves discretely from zero to a positive effort level. We provide a full characterization of the equilibrium for large classes of distribution functions, in particular for either convex or concave distribution functions.

We then study the designer's payoff in both discriminatory and nondiscriminatory contests with two agents and find conditions under which the use of a discriminatory contest increases the payoff. That is, in our environment, where the only feasible mechanisms are contests, the optimal mechanism may lead to discrimination. Discrimination dominates nondiscrimination in some set ups because it leads to an increase in the expected efforts of the agents, thereby resulting in larger expected revenues. In particular, we show a class of simple examples where discrimination increases the designer's payoff. We also prove that a discriminatory contest is better if the distribution of the ability parameter is a convex function, with very low density at zero, and the designer has a high enough valuation of the quality of the innovation.

Contests in symmetric information environments have been extensively analyzed. Baye

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<sup>2</sup>Contests like those that we analyze, due to their simplicity and popularity, are also prevalent in many situations where quality is verifiable.

<sup>3</sup>The case of  $n > 2$  agents presents several technical difficulties because the conditions characterizing the equilibrium behavior do not lead to closed form solutions for general distribution functions. The case where  $n = 2$  allows to highlight the effect discrimination might have and, in particular, the possibility that it might increase the designer's profits.

et al. (1996) studied the contestants' equilibrium behavior in standard all-pay auctions with symmetric information where agents bid for an object, all bids are paid and the highest bidder receives the object. Kaplan et al. (2003) investigated all-pay auctions where the size of the reward depends on the effort. In particular, they applied this framework to an analysis of R&D races. Che and Gale (2003) derived the equilibrium behavior and characterized the optimal research contest in an environment where each contestant submits a quality-price pair. The cost of producing the quality is sunk. The contestant offering the largest surplus, defined as the difference between quality and price, is paid the price. Casas-Arce and Martinez-Jerez (2011) and Siegel (forthcoming) analyzed all-pay auctions with heterogeneous agents which differ in their ability. The former studied the relationship between ability and bids, and used their results to derive equilibrium behavior in two-stage all-pay auctions. The latter studies both single-prize and multiple-prize contests and allows for a wide range of asymmetries among any number of players. Jönsson and Schmutzler (2013) considered all-pay auctions and analyzed the relationship between the prize structure (a specification of effort-dependent rewards) and the efforts expended by the contestants. Siegel (2009) and (2010) introduced a general framework encompassing a very large class of all-pay auctions and provided a general method of solving and calculating equilibrium payoffs. Konrad (2009) provided an excellent survey of equilibrium and optimal design in contests. In more specialized settings, Konrad (2002) and (2004), studies two-player contests where the contestants take decisions (investment to increase the output or to obtain and release information) before engaging in the actual contest.

In scenarios where agents' valuations are private information, Amann and Leininger (1996) characterized the equilibrium bids for all-pay auctions where valuations are independently drawn from a common distribution function. Kirkegaard (2012a) considered a similar setup where bids are converted into scores, with several agents that differ in their cost of bidding, and analyzed the effect of handicaps, where the bid of an agent is translated into a lesser score. Seel (2012) studied a two-player all-pay auction where one bidder has an initial advantage whose size is private information. Krishna and Morgan (1997) analyzed the case in which the bidders' information regarding valuations is affiliated. Moldovanu and Sela (2001), on the other hand, considered the case where the agents' cost of bidding is private information. The uncertainty was introduced via

a constant multiplying a basic cost function common to all agents. In their model, the designer’s goal is to maximize the sum of agents’ efforts, by deciding upon the allocation of a fixed budget among one or more prizes. Finally, Siegel (2012) provides a constructive characterization of the unique equilibrium of asymmetric two-player all-pay auctions when there are a finite number of signals for each player, an asymmetric signal distribution and interdependent valuations, and a non-restricted reserve prize.

Our model differs with respect to the literature discussed above in several ways. For one thing, the designer chooses the size of the reward and may discriminate among the contestants. Moreover, the private information, regarding the agents’ cost, is introduced in a novel and tractable way with ability and effort being perfect substitutes. In this sense, the model is suitable for the analysis of common day scenarios of competition where both innate ability and effort generate the final outcome.

Singh and Wittman (2001) also analyzed a model where quality is a function of both agents’ effort and type. They provided properties of the agents’ equilibrium behavior and, applying the revelation principle, characterized the structure of an optimal nondiscriminatory reward structure. While the environment considered by Singh and Wittman (2001) and their results are quite general, their assumptions exclude the simpler model studied in our paper. Furthermore, their mechanism design analysis is appropriate for environments where quality is verifiable, as the quality variable features prominently in the rules of the mechanism. In contrast, quality is not verifiable in our environment, except for the requirement that an independent party can determine the best-quality innovation. This informational structure leaves little scope for the application of mechanism design techniques as the designer can only specify a reward for the winner, independent of the quality offered. Our simple model allows the derivation of explicit solutions for the agents’ behavior. These solutions are then used to study discriminatory contests as well and analyze the desirability of discrimination from the designer’s point of view.

Measures to increase or decrease the discrepancy between contestants have indeed attracted considerable attention in the literature. Several studies have shown that it is, in general, beneficial to handicap a stronger contestant so as to level the field when starting from an asymmetric contest (Baye et al., 1993, Clark and Riis, 2000, Che and Gale, 2003). Our finding that discriminating, by offering contestant specific prizes, can increase the designer’s payoff runs contrary to those studies and is in line with some of the

findings in Fu et al. (2012) for symmetric information settings and Kirkegaard (2012b) for asymmetric information environments.

Fu et al. (2012) considered R&D races and analyzed the optimal allocation of a fixed budget between a prize to the winning firm and subsidies to the competing firms with the goal of maximizing the (expected) quality of the winning innovation. In their setting, similar to a Tullock contest, they have addressed both issues of which contestant (strong or weak) should be favored and whether it pays to offer contestant specific prizes rather than a single uniform prize. They have shown that favoring the “weaker” contestant is optimal when R&D effort is quite effective, whereas it pays to favor the “stronger” contestant when the R&D effort is quite ineffective. As regards the prize schedule, they have shown it is actually optimal to offer a uniform prize.

Kirkegaard (2012b), allowing both handicaps and head starts, analyzed their use in environments with private valuations where the designer is interested in maximizing total expenditures. While demonstrating several settings where it pays to strengthen the weaker contestant, he also showed that there are cases where the weaker contestant should be handicapped and instances where one of two symmetric contestants should be given a small head start.

Discriminating among identical agents has also been shown to be optimal in models very different from ours. In particular, Levitt (1995) and Winter (2004) found that discriminating is optimal in environments where a principal wants to provide several identical agents with incentives to carry out a task.<sup>4</sup>

The rest of the paper proceeds as follows. In Section 2, we introduce the model. Section 3 analyzes agents’ equilibrium behavior in nondiscriminatory contests and agents’ behavior in discriminatory contests is analyzed in Section 4. Section 5 calculates the designer’s payoff as a function of the rewards and shows conditions under which it is optimal to discriminate. Section 6 concludes and proposes directions for further research.

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<sup>4</sup>Kräkel (2012) considers contests to elicit effort where the contestants have different, known abilities. Effort and ability are perfect substitutes, prizes are given and the designer’s decision is whether to hold “fair” contests among identical or “unfair” among non-identical contestants. He shows that unfair contests may be optimal when a weak agent has a higher perceived winner prize when beating a stronger opponent.



## 2 The model

We consider the problem facing an organization that wishes to procure an innovation. The benefits derived from this innovation depend on its quality  $q$  and are given by  $I(q)$ , with  $I'(q) > 0, I''(q) < 0$ .

There is a set  $N$  of  $n \geq 2$  identical risk-neutral agents who can realize the desired innovation. The quality of the innovation produced by an agent depends on his type and his choice of nonnegative effort. The agents' types represent their proficiency to develop the particular innovation. Both the types and choices of effort are private information. Denoting the type of agent  $i$  by  $\theta_i$  and his effort by  $e_i$ , the quality of the innovation realized by agent  $i$  is given by  $q_i(\theta_i, e_i) = \theta_i + e_i$ .

Agents' types are independently distributed according to the same differentiable and atomless distribution function  $F(\cdot)$  on  $[0, 1]$ , with  $F'(\cdot) > 0$  for all  $\theta \in [0, 1]$ . Types are revealed to the agents prior to their choice of effort. The quality of the innovation, while observed by the designer, cannot be verified. However, an independent authority can verify which innovation is best.

To procure the innovation, the organization holds a *contest* among the agents. Henceforth, we refer to the organization as the *designer* of the contest. The winner of the contest is the agent who offers the innovation of the highest quality, with ties broken by having each agent submitting the highest quality win with equal probability. In a *nondiscriminatory contest*, the designer offers a prize  $R$  to the winning agent. The contest is discriminatory when the designer offers different prizes depending on the identity of the winner. In a discriminatory contest, the prize offered to agent  $i \in N$ , were he to win the contest, is  $R_i$  and  $R_j \neq R_{j'}$  for some  $j, j' \in N$ .

Given a contest, agents simultaneously choose their effort. In a nondiscriminatory contest, the payoff of any agent when he chooses effort  $e$  is  $R - e$  in case he wins the contest and  $-e$  otherwise. The payoff to the designer is  $I(q_W) - R$ , where  $W$  is the identity of the agent winning the contest. In a discriminatory contest, the payoffs to the winning agent and to the designer are  $R_W - e$  and  $I(q_W) - R_W$ , respectively.

The agents' strategies are denoted by the functions  $(q_i(\theta))_{i \in N}$ , with  $q_i(\theta) \geq \theta$ , where  $q_i(\theta)$  indicates the choice of quality by agent  $i$  when his type is  $\theta$ . We consider *regular* strategies. A strategy is *regular* if it is strictly increasing for all  $\theta$  and differentiable whenever  $q(\theta) > \theta$ . Given the agents' strategies, the efforts exerted by agent  $i$  of type  $\theta_i$

are  $e_i(\theta_i) = q_i(\theta_i) - \theta_i$ .

### 3 Agents' equilibrium strategies in nondiscriminatory contests

In this section, we provide the agents' strategies in any symmetric equilibrium  $q^*(\theta)$  when they compete in a nondiscriminatory contest with reward  $R$ .

To formulate agent  $i$ 's maximization problem, for  $i \in N$ , we let agent  $j$ 's strategy be  $q^*(\theta)$  for every  $j \in N \setminus i$ . Then, agent  $i$ 's expected profits when he is of type  $\theta$  and offers quality  $q \geq \theta$  are given by his probability of winning times the prize minus the effort; that is,<sup>5</sup>

$$\Pr_{(\theta_j)_{j \in N \setminus i}} \left( q \geq \max_{j \in N \setminus i} q^*(\theta_j) \right) R - (q - \theta).$$

When  $q \in [q^*(0), q^*(1)]$ , and  $q \geq \theta$ , agent  $i$ 's expected profits can be written as

$$(F(q^{*-1}(q)))^{n-1} R - (q - \theta).$$

If the level of quality  $q$  that maximizes agent  $A$ 's expected profits is interior, that is,  $q > \theta$ , the following first-order condition (FOC) must hold:

$$q^{*'}(q^{*-1}(q)) = (n-1)F'(q^{*-1}(q)) (F(q^{*-1}(q)))^{n-2} R, \quad (1)$$

that is, the optimal  $q(\theta)$  satisfies

$$q^{*'}(q^{*-1}(q(\theta))) = (n-1)F'(q^{*-1}(q(\theta))) (F(q^{*-1}(q(\theta))))^{n-2} R.$$

In the symmetric equilibrium,  $q(\theta) = q^*(\theta)$ . Therefore, the FOC yields the following differential equation that  $q^*(\theta)$  must satisfy in an interior solution:

$$q^{*'}(\theta) = (n-1)F'(\theta) (F(\theta))^{n-2} R. \quad (2)$$

The solution of this equation is given by:

$$q^*(\theta) = F(\theta)^{n-1} R + \eta \quad (3)$$

for some  $\eta \in \mathbb{R}$ .

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<sup>5</sup>Since  $q^*(\theta)$  is strictly increasing, we ignore the possibility of ties as they occur with probability zero.

Equation (3) describes the equilibrium strategies over the range of  $\theta$ 's that lead to an interior solution, where efforts chosen by agents are strictly positive. However, there may be regions of parameters where agents choose corner solutions and provide zero effort,  $q^*(\theta) = \theta$ . Theorem 1 states the explicit agents' equilibrium strategies for any differentiable and atomless distribution function  $F(\cdot)$ . Prior to presenting the theorem, we outline below an intuitive explanation regarding the nature of the symmetric equilibrium strategies, which are necessarily continuous.

There are two types of regions for the parameter  $\theta$ . We let Region  $I$  be the region where agents choose positive levels of effort, thus the equilibrium is characterized by equation (3).<sup>6</sup> We let Region  $C$  be the one where agents choose zero effort,  $q^*(\theta) = \theta$ .

To derive the conditions that an equilibrium must satisfy, we consider the case where agents' strategies lie in Region  $I$  for  $\theta \in [\theta_1, \theta_2)$  and in Region  $C$  for  $\theta \in [\theta_2, \theta_3]$ . First, the continuity of the equilibrium strategies implies that  $F(\theta_2)^{n-1}R + \eta = \theta_2$ ; hence,  $\eta = \theta_2 - F(\theta_2)^{n-1}R$ . Second, for any  $\theta \in [\theta_2, \theta_3]$ , for  $q^*(\theta) = \theta$  to be the optimal choice, an agent's profits for this choice cannot be lower than his profits for any  $q \in (\theta, \theta_3]$ , i.e.,  $F(\theta)^{n-1}R \geq F(q)^{n-1}R - (q - \theta)$ , or

$$F(q)^{n-1}R - q \leq F(\theta)^{n-1}R - \theta \text{ for any } \theta \in [\theta_2, \theta_3], q \in (\theta, \theta_3]. \quad (4)$$

Finally, the non-negativity of effort implies that  $F(\theta)^{n-1}R + \eta \geq \theta$  for any  $\theta$  in  $[\theta_1, \theta_2]$ , or

$$F(\theta_2)^{n-1}R - \theta_2 \leq F(\theta)^{n-1}R - \theta \text{ for any } \theta \in [\theta_1, \theta_2]. \quad (5)$$

Equation (4) must be satisfied for any interval of parameters where the equilibrium lies in Region  $C$ , thereby implying that the function  $F(\theta)^{n-1}R - \theta$  is nonincreasing in such an interval. Equation (5) must be satisfied for any interval of parameters where the equilibrium lies in Region  $I$ . This equation requires that the value of the function  $F(\theta)^{n-1}R - \theta$  at the upper bound of the interval cannot be higher than the value at any other  $\theta$  in the interval. These conditions are shown to describe an equilibrium in Theorem 1, where the boundaries of the regions are characterized as well.<sup>7</sup> The theorem is stated for the case where the function  $F(\theta)^{n-1}R - \theta$  is decreasing at  $\theta = 0$ , that is,  $(n-1)F(0)^{n-2}F'(0)R - 1 < 0$ , which always holds when  $n > 2$ . After the theorem,

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<sup>6</sup>Region  $I$  can be the union of several intervals. If this is the case, the parameter  $\eta$  changes from one interval to another.

<sup>7</sup>All proofs are relegated to the appendix.

we indicate the small necessary changes to state the results for all other cases with two agents.

**Theorem 1** *Consider a nondiscriminatory contest with a reward  $R$ . Assume that  $F(\theta)^{n-1}R - \theta$  has a finite set of local minima.<sup>8</sup> Also, assume that  $F(\theta)^{n-1}R - \theta$  is decreasing at  $\theta = 0$ . We define a finite sequence of parameters  $\beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \dots$  that starts with  $\beta_0 = 0$ . The other parameters of the sequence are defined recursively as follows:*

(i) *given that  $\beta_{m-1} < 1$  has been defined,  $\alpha_m$ , for  $m \geq 1$ , is equal to 1 if  $F(\theta)^{n-1}R - \theta$  is decreasing in  $[\beta_{m-1}, 1]$ ; otherwise,  $\alpha_m$  is the first parameter  $\theta$  larger than  $\beta_{m-1}$  that is a strict local minimum of  $F(\theta)^{n-1}R - \theta$ .*

(ii) *given that  $\alpha_m < 1$  has been defined,  $\beta_m$ , for  $m \geq 1$ , is equal to 1 if  $\alpha_m$  is a global minimum; otherwise,  $\beta_m$  is the first parameter  $\theta$  larger than  $\alpha_m$  for which  $F(\beta_m)^{n-1}R - \beta_m = F(\alpha_m)^{n-1}R - \alpha_m$  that is not a local minimum of the function  $F(\theta)^{n-1}R - \theta$ .*

*Then, the unique symmetric equilibrium is:*

$$\begin{aligned} q^*(\theta) &= F(\theta)^{n-1}R + \alpha_m - F(\alpha_m)^{n-1}R \text{ for } \theta \in [\alpha_m, \beta_m], m \geq 1 \text{ (Region I)} \\ q^*(\theta) &= \theta \text{ for } \theta \in [\beta_{m-1}, \alpha_m], m \geq 1 \text{ (Region C).} \end{aligned}$$

Figure 1 represents for the case  $n = 2$  the equilibrium effort levels  $e(\theta)$  and the function  $F(\theta)R - \theta$  if  $F'(0)R - 1 < 0$ .<sup>9 10</sup>

When  $F(\theta)^{n-1}R - \theta$  is increasing, Theorem 1 still holds, and the recursive definition of the sequences  $(\alpha_m)_m$  and  $(\beta_m)_m$  is similar, except that we start with  $\alpha_1 = 0$  and  $\beta_0$  does not exist. Similarly, if the derivative of  $F(\theta)^{n-1}R - \theta$  at  $\theta = 0$  is zero, the sequences will be the same as in Theorem 1 unless  $F(\theta)^{n-1}R - \theta > 0$  for some interval  $(0, \hat{\theta})$  with  $\hat{\theta} > 0$ , in which case the definition of the sequences starts with  $\alpha_1 = 0$ .

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<sup>8</sup>This is a very mild requirement that is satisfied by any polynomial and any commonly used distribution function.

<sup>9</sup>We note that, for  $n = 2$ , the proposed strategies constitute the unique equilibrium in monotonic strategies if the set  $\{\theta \in [0, 1] / F'(\theta) = 1/R\}$  has zero measure (see Pérez-Castrillo and Wettstein, 2012).

<sup>10</sup>The equilibrium construction bears some similarities to the construction in Seel (2012), where the informed player's bid can also be zero for an interval of types (in his case, the type corresponds to the size of the player's initial advantage).

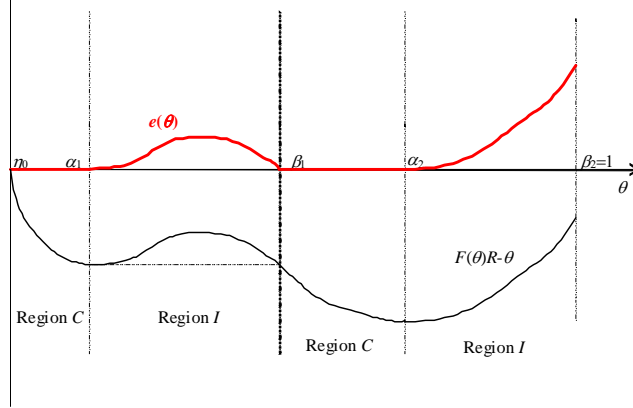


Figure 1: Equilibrium effort in a nondiscriminatory contest for two agents.

## 4 Agents' equilibrium strategies in discriminatory contests

We now analyze agents' strategies in discriminatory contests when  $n = 2$ . We denote  $N = \{A, B\}$  and assume that  $R_A > R_B$ . Formulating the agents' maximization problems in the same manner as in the nondiscriminatory case, we obtain the following FOCs for agents  $A$  and  $B$ , when they exert positive efforts:

$$q'_B(q_B^{-1}(q)) = F'(q_B^{-1}(q)) R_A. \quad (6)$$

$$q'_A(q_A^{-1}(q)) = F'(q_A^{-1}(q)) R_B. \quad (7)$$

The solution of the system of differential equations is given by

$$q_A(\theta) = F(\theta)R_B + \eta_A, \quad (8)$$

$$q_B(\theta) = F(\theta)R_A + \eta_B, \quad (9)$$

for some  $\eta_A, \eta_B \in \mathbb{R}$ <sup>11</sup>.

As we found in our analysis of the agents' equilibrium strategies in nondiscriminatory contests, there are also regions where one or both agents put in zero effort. Furthermore, in discriminatory contests, discontinuities in an agent's strategy cannot be ruled out.

<sup>11</sup>When there are  $n > 2$  agents there exist several possibilities for discrimination, each leading to a different set of conditions and equations. Furthermore, the solution of the resulting systems of differential equations cannot be obtained in a closed form.

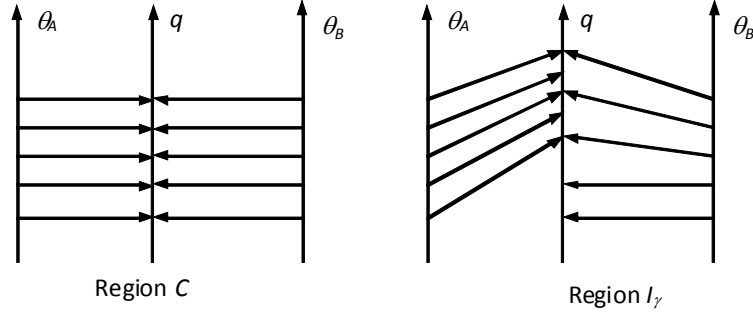


Figure 2: Choice of quality by the two agents in an interval of Region  $C$  and in an interval of Region  $I_\gamma$ .

Therefore, there may exist a quality interval that is never reached by an agent even though he offers qualities below and above that interval.

As we show in the Appendix (Lemmas 1 and 2), only three strategy configurations can emerge in equilibrium. We call Region  $I$  the set of quality levels  $q$  that are reached when both agents play according to the interior solution; that is,  $q \in I$  if  $q_A^{-1}(q) < q$  and  $q_B^{-1}(q) < q$ . Also, we call Region  $C$  the set of qualities reached when both agents put in zero effort; that is,  $q \in C$  if  $q_A^{-1}(q) = q_B^{-1}(q) = q$ . The remaining region, which does not exist in nondiscriminatory contests, corresponds to quality levels reached by  $B$  through zero effort, but never offered by agent  $A$ . More precisely, Region  $J$  is given by the set of qualities  $q$  such that  $q_A^{-1}(q)$  does not exist and  $q_B^{-1}(q) = q$ . Moreover, Region  $J$  always appears before Region  $I$ . Therefore, the range of qualities in equilibrium can be split into two types of intervals, each of which belong to either Region  $C$  or Region  $I_\gamma$ , where Region  $I_\gamma$  is given by an interval of (lower) qualities that is never reached by agent  $A$  and an interval of (higher) qualities that is reached by both agents contributing a positive effort. We note that in discriminatory contests, Region  $I_\gamma$  plays a similar role to Region  $I$  in nondiscriminatory contests. Figure 2 depicts the choice of quality (represented by the middle vertical axis) for both agents as a function of the parameters  $\theta_A$  and  $\theta_B$  (represented by the vertical axes on both sides) in two quality intervals, one belonging to region  $C$  and another to region  $I_\gamma$ .<sup>12</sup>

<sup>12</sup>The interval in region  $I_\gamma$  depicted in Figure 2 corresponds to an interval whose upper bounds (in terms of  $\theta$ ) are  $\theta_A = \theta_B = 1$ . If this interval were followed by an interval in region  $C$ , the transition to the  $C$  interval would be continuous, that is the top arrows would be horizontal.

At equilibrium, for any  $\theta$ , agent  $A$ 's effort is greater than or equal to agent  $B$ 's effort, that is,  $e_A(\theta) - e_B(\theta) \geq 0$ . Efforts are the same in Region  $C$  and the inequality is strict for the values of  $\theta$  in the lower part of Region  $I_\gamma$  (we called that Region  $J$  previously): agent  $B$  provides zero effort whereas agent  $A$  chooses a positive effort level. In the upper part of Region  $I_\gamma$ ,  $e_A(\theta) - e_B(\theta) = q_A(\theta) - q_B(\theta) = \eta_A - \eta_B - F(\theta)(R_A - R_B)$ . If we denote by  $\bar{\theta}$  the upper bound (in terms of  $\theta$ ) of the interval which is in Region  $I_\gamma$ , then at equilibrium  $q_A(\bar{\theta}) = q_B(\bar{\theta})$ , which implies  $\eta_A - \eta_B = F(\bar{\theta})(R_A - R_B)$ . Therefore, in this part of Region  $I_\gamma$ ,  $e_A(\theta) - e_B(\theta) = (F(\bar{\theta}) - F(\theta))(R_A - R_B) > 0$  for all  $\theta < \bar{\theta}$ . The distance in effort decreases as  $\theta$  increases and it vanishes at  $\theta = \bar{\theta}$ .

So far we have highlighted several properties of equilibrium strategies in discriminatory contests for general distribution functions. We proceed to analyze particular classes of distribution functions for which equilibrium strategies can be fully described in a simple manner.

First, we discuss the possibility of a corner solution. It is intuitive that such a solution would emerge if the prizes allocated in the contest are very small. Proposition 1 goes a step forward and provides a necessary and sufficient condition for Region  $C$  to constitute the only equilibrium in the contest. In such an equilibrium, both agents choose zero effort for every  $\theta \in [0, 1]$ , a strategy profile that we denote by  $(q_A^C, q_B^C)$ :

$$q_A^C(\theta) = q_B^C(\theta) = 0 \text{ for all } \theta \in [0, 1].$$

**Proposition 1** *The strategy profile  $(q_A^C, q_B^C)$  constitutes an equilibrium of the contest  $(R_A, R_B)$  if and only if the function  $F(\theta)R_A - \theta$  is non-increasing in  $\theta$  for all  $\theta \in [0, 1]$ . Moreover, if the set  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_A\}$  has zero measure, then no other equilibrium exists.*

Second, if the function  $F(\theta)R_A - \theta$  is nondecreasing for all  $\theta \in [0, 1]$  (which implies  $R_A$  is high enough), then, in equilibrium, the qualities offered always lie in Region  $I_\gamma$ . That

is, agents follow the strategy profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ ,<sup>13</sup> defined by

$$q_A^{I_\gamma}(\theta) = F(\theta)R_B + \gamma \quad \text{for all } \theta \in [0, 1]$$

$$q_B^{I_\gamma}(\theta) = \begin{cases} \theta & \text{for all } \theta \in [0, \gamma) \\ F(\theta)R_A + \gamma - F(\gamma)R_A & \text{for all } \theta \in [\gamma, 1] \end{cases}$$

where  $\gamma$  solves  $R_B = [1 - F(\gamma)]R_A$ ; hence,  $\gamma = F^{-1}\left(1 - \frac{R_B}{R_A}\right)$ .

The whole class of distribution functions and rewards for which the strategy profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is an equilibrium is provided in Proposition 2.

**Proposition 2** *The strategy profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  constitutes an equilibrium of the contest  $(R_A, R_B)$  if and only if the following two conditions hold:*

$$F(\theta)R_A - \theta \leq F(\gamma)R_A - \gamma \text{ for all } \theta \leq \gamma \quad (10)$$

$$F(\theta)R_A - \theta \geq F(\gamma)R_A - \gamma \text{ for all } \theta \geq \gamma. \quad (11)$$

In particular,  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is an equilibrium if  $F(\theta)R_A - \theta$  is nondecreasing in  $\theta$  for all  $\theta \in [0, 1]$ . Furthermore, the equilibrium is unique if inequalities (10) and (11) are strict.

When the function  $F(\theta)R_A - \theta$  is increasing in some intervals of  $\theta$ 's and decreasing in others, then the equilibrium will include intervals of quality that lie in Region  $I_\gamma$  and, often, others that lie in Region  $C$ . While a full characterization of the equilibrium strategies is cumbersome for general distribution functions, there are large families of functions that allow for quite simple characterizations. Their analysis will also provide robust intuitions on the agents' equilibrium behavior. Next, we consider two such families, namely strictly convex or concave distribution functions.

The following theorem characterizes the structure of equilibrium strategies when the function  $F(\cdot)$  is convex. We note that for a convex  $F(\cdot)$ , it is always the case that  $F'(0) < 1 < F'(1)$ . To simplify the presentation of the theorem, we define the strategy

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<sup>13</sup>The qualities chosen by the agents as well as the cut-off  $\gamma$  in the profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  depend on the particular function  $F(\cdot)$  and the rewards  $R_A$  and  $R_B$ . We do not express this dependence on  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  to keep the notation simple.



profile  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ :

$$q_A^{CI_\gamma}(\theta) = \begin{cases} \theta & \text{for all } \theta \in [0, \alpha) \\ F(\theta)R_B + \gamma - F(\alpha)R_B & \text{for all } \theta \in [\alpha, 1] \end{cases}$$

$$q_B^{CI_\gamma}(\theta) = \begin{cases} \theta & \text{for all } \theta \in [0, \gamma) \\ F(\theta)R_A + \gamma - F(\gamma)R_A & \text{for all } \theta \in [\gamma, 1] \end{cases}$$

where  $\alpha$  and  $\gamma$  solve

$$[1 - F(\alpha)] R_B = [1 - F(\gamma)] R_A \quad (12)$$

$$F(\alpha) R_A - \alpha = F(\gamma) R_A - \gamma. \quad (13)$$

Under the strategy profile  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ , both agents exert zero effort up to a threshold value of  $\theta$  and choose positive effort levels for higher values. Agent  $A$ 's threshold is lower than agent  $B$ 's, ( $\alpha < \gamma$ ) due to the higher reward he obtains if he wins the contest. Finally, while agent  $B$ 's strategy is continuous in the parameter  $\theta$ , agent  $A$ 's strategy entails a discrete jump from  $\alpha$  to  $\gamma$  at his threshold.

**Theorem 2** *Let  $F(\cdot)$  be convex.*

- (a) *If  $R_A \leq \frac{1}{F'(1)}$ , then an equilibrium is given by  $(q_A^C, q_B^C)$ .*
- (b) *If either  $R_A \in (\frac{1}{F'(1)}, 1)$  or both  $R_A \in [1, \frac{1}{F'(0)})$  and  $F(R_A - R_B) R_A - (R_A - R_B) < 0$ , then an equilibrium is given by  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ .*
- (c) *If either  $R_A \geq \frac{1}{F'(0)}$  or both  $R_A \in [1, \frac{1}{F'(0)})$  and  $F(R_A - R_B) R_A - (R_A - R_B) \geq 0$ , then an equilibrium is given by  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ .*

*Furthermore, these are the unique equilibrium configurations.*

We point out that when  $R_A \in [1, \frac{1}{F'(0)})$ , the condition  $F(R_A - R_B) R_A - (R_A - R_B) \geq 0$  is equivalent to  $R_A - R_B \geq p(R_A)$ , where we denote by  $p(R_A)$  the unique strictly positive  $p$  that satisfies  $F(p)R_A - p = 0$ . Thus, when  $R_A \in [1, \frac{1}{F'(0)})$ , we are in case (c) if  $R_B$  is not “close” to  $R_A$ .

Figure 3 depicts the equilibrium configuration as a function of the prizes  $R_A$  and  $R_B$  for the class of convex distribution functions. When  $R_A$  is small enough, the whole equilibrium is in Region  $C$ , and when  $R_A$  is large enough, the equilibrium is in Region  $I_\gamma$  where agent  $A$  puts in a positive effort at  $\theta = 0$  and agent  $B$  starts to exert a positive effort only for

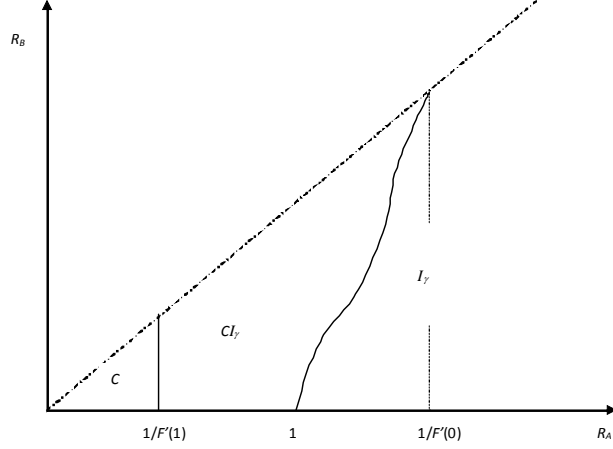


Figure 3: Equilibrium configuration when  $F(\cdot)$  is convex.

large enough  $\theta$ 's. The same occurs when  $R_A$  is intermediate but substantially larger than  $R_B$ . The intuition for these two cases is similar to that provided after Propositions 1 and 2, namely agents do not have incentives to exert any effort if the reward is low whereas competition in efforts arises if the reward is high.

When  $R_A$  is intermediate and, depending on  $R_A$ , not too much higher than  $R_B$ , the equilibrium entails a corner solution for low values of  $\theta$  and an interior solution for high values (the Region  $CI_\gamma$ ). The reason for a positive effort being exerted at higher values of  $\theta$  can be traced back to the larger density at higher values of  $\theta$  due to the convexity of the distribution function  $F(\theta)$ . If the density is high, increasing the effort implies a large increase in the probability of winning the contest hence, a larger payoff.

Figure 4 illustrates the quality choices by the agents in an example where  $F(\theta) = \theta^3$ ,  $R_A = 1$  and  $R_B = 0.666$ , which implies that the solution lies in region (b) in Theorem 2. Both agents choose corner solution as long as  $\theta < \alpha = 0.416$  at which point agent  $A$  jumps to  $q_A = \gamma = 0.725$  and chooses a positive effort hereafter. On the other hand, agent  $B$  continues with zero effort up to  $\theta = 0.725$  and selects a positive effort for higher  $\theta$ 's. As this figure suggests,  $q_A(\theta)$  (or  $e_A(\theta)$ ) is always higher than  $q_B(\theta)$  (or  $e_B(\theta)$ ) whenever we are in region  $I_\gamma$ , and it is strictly higher except at the top of the interval.

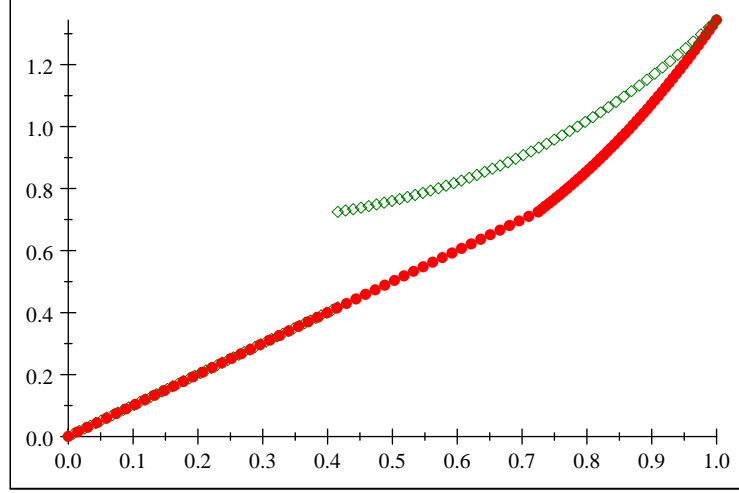


Figure 4: Quality choices  $q_A$  (green diamonds) and  $q_B$  (red dots) if  $F(\theta) = \theta^3$ ,  $R_A = 1$  and  $R_B = 0,666$ .

We now characterize the equilibrium strategies when the function  $F(\cdot)$  is concave, in which case  $F'(1) < 1 < F'(0)$ . We define the strategy profile  $(q_A^{I_\gamma^C}, q_B^{I_\gamma^C})$  as follows:

$$q_A^{I_\gamma^C}(\theta) = \begin{cases} F(\theta)R_B + \gamma & \text{for all } \theta \in [0, \beta) \\ \theta & \text{for all } \theta \in [\beta, 1] \end{cases}$$

$$q_B^{I_\gamma^C}(\theta) = \begin{cases} \theta & \text{for all } \theta \in [0, \gamma) \cup [\beta, 1] \\ F(\theta)R_A + \gamma - F(\gamma)R_A & \text{for all } \theta \in [\gamma, \beta) \end{cases}$$

where  $\gamma$  and  $\beta$  solve

$$\gamma = \beta - F(\beta)R_B \quad (14)$$

$$F(\gamma)R_A - \gamma = F(\beta)R_A - \beta. \quad (15)$$

When agents follow the strategy profile  $(q_A^{I_\gamma^C}, q_B^{I_\gamma^C})$ , they exert zero effort above the same threshold value of  $\theta$ . For low values of  $\theta$ , they follow an interior solution where agent  $A$  has an incentive to exert a strictly positive effort even when  $\theta_A = 0$ , whereas agent  $B$  only exerts a positive effort above a certain threshold of  $\theta_B$ .

**Theorem 3** *Let  $F(\cdot)$  be concave.*

- (a) *If  $R_A \leq \frac{1}{F'(0)}$ , then an equilibrium is given by  $(q_A^C, q_B^C)$ .*
- (b) *If either  $R_A \in \left(\frac{1}{F'(0)}, 1\right)$  or both  $R_A \in \left[1, \frac{1}{F'(1)}\right)$  and  $F(1-R_B)R_A - (1-R_B) > R_A - 1$ ,*

then an equilibrium is given by  $(q_A^{I_\gamma C}, q_B^{I_\gamma C})$ .

(c) If either  $R_A \geq \frac{1}{F'(1)}$  or both  $R_A \in \left[1, \frac{1}{F'(1)}\right)$  and  $F(1 - R_B)R_A - (1 - R_B) \leq R_A - 1$ , then an equilibrium is given by  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ .

Furthermore, these are the unique equilibrium configurations.

Note that when  $R_A \in \left[1, \frac{1}{F'(1)}\right)$ , the condition  $F(1 - R_B)R_A - (1 - R_B) > R_A - 1$  is equivalent to  $z(R_A) < 1 - R_B$ , where we denote by  $z(R_A)$  the unique strictly positive  $z$  that satisfies  $F(z)R_A - z = R_A - 1$ . Thus, when  $R_A \in \left[1, \frac{1}{F'(0)}\right)$ , we are in case (b) if  $R_B$  is “small enough” or not too close to  $R_A$ .

When rewards are quite low or quite high, the agents’ behavior is similar to that for the convex case. For intermediate values of reward, they play according to the interior strategy profile for low levels of  $\theta$  and exert zero effort for high values of  $\theta$ . Notice that the corner strategy profile emerges now for high values of  $\theta$  because of the low density of  $\theta$  due to the concavity of the distribution function.

We remark that the propositions and theorems derived in the current section also apply to nondiscriminatory contests. However, if  $R_A = R_B$  the systems (12)-(13) and (14)-(15) do not have a unique solution. The equilibrium behavior in a nondiscriminatory contest when the distribution function is convex is the solution of (12)-(13) that satisfies  $\gamma = \alpha$  and  $F'(\alpha)R = 1$  in the strategy profile  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ . When the distribution function is concave, the equilibrium is the solution of (14)-(15) that satisfies  $\gamma = 0$  and  $\beta$  given by the unique positive value for which  $F(\beta)R - \beta = 0$  in the strategy profile  $(q_A^{I_\gamma C}, q_B^{I_\gamma C})$ . Naturally, these two equilibria coincide with the contestants’ equilibrium behavior in the nondiscriminatory contest identified in Theorem 1.

## 5 Designer’s payoff and discrimination

In this section, we address the desirability of discrimination when  $n = 2$ . The designer’s benefit  $I(q)$  depends on the best quality obtained in the contest. We can determine the expected designer’s payoff as a function of the agents’ strategies, which allows us to discuss the change in the designer’s payoff due to discrimination separately for the four possible equilibrium strategy profiles that we identified in Section 4.

The discussion of one of the profiles is trivial. In any contest for which the agents’ equilibrium strategy profile is  $(q_A^C, q_B^C)$ , the designer’s revenue is the same, whereas the

cost increases with the rewards. Therefore, any contest with positive rewards is dominated by the nondiscriminatory contest with  $R = 0$  in the sense that the designer's profits are higher if  $R = 0$ . This implies that discrimination can only decrease the designer's payoff. Proposition 3 addresses the other three cases.

**Proposition 3** *Consider a nondiscriminatory contest  $R_A = R_B = R$  for which an equilibrium is given by the strategy profile  $(q_A, q_B)$ . Consider a marginal change in  $(R_A, R_B)$  that increases  $R_A$  by  $\varepsilon > 0$  and decreases  $R_B$  by  $\varepsilon$ .*

- (a) If  $(q_A, q_B) = (q_A^I, q_B^I)$  and the equilibrium following the change is given by  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ , then the contest is dominated by a discriminatory contest if  $R < \frac{2}{F'(0)}$ .*
- (b) If  $(q_A, q_B) = (q_A^{CI}, q_B^{CI})$  and the equilibrium following the change is given by  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ , then the contest is dominated by a discriminatory contest.*
- (c) If  $(q_A, q_B) = (q_A^{IC}, q_B^{IC})$  and the equilibrium following the change is given by  $(q_A^{I_\gamma C}, q_B^{I_\gamma C})$ , then the contest is dominated by a discriminatory contest if  $R < \frac{2}{F'(0)}$ .*

Discrimination in the cases addressed in Proposition 3 is desirable because it elicits higher efforts on the part of the agents. A marginal change in the rewards that leads to a discriminatory contest has a second-order effect on the designer's expected cost because the infinitesimal change in the rewards is multiplied by infinitesimal changes in the probability of  $A$  winning (which go up) and  $B$  winning (which go down). However, the marginal change in the rewards has a first-order effect (which, in principle, can be positive or negative) on the expected quality of the innovation, hence, on the designer's expected revenue.

Consider a shift from a nondiscriminatory contest  $R$  to a discriminatory contest ( $R_A = R + \varepsilon, R_B = R - \varepsilon$ ) when the initial equilibrium is  $(q_A^I, q_B^I)$  (case (a)), which requires that  $F'(0)R \geq 1$ . The increase in  $R_A$  (and the decrease in  $R_B$ ) makes player  $A$  more aggressive: it gives him incentives to provide a strictly positive level of quality ( $\gamma$ ) even if his type were the lowest. Player  $B$  becomes less aggressive and prefers not to provide any effort if his type is low (lower than  $\gamma$ ). While the initial increase in the effort by player  $A$  is  $\gamma$  for all the low types, the initial loss of effort by player  $B$  depends on the effort that he was exerting in the equilibrium  $(q_A^I, q_B^I)$ , which follows the function  $F(\theta)R - \theta$ . If the initial effort was very low, i.e., if  $F'(0)R$  is close to 1, then the increase in  $q_A$  is larger than the decrease in  $q_B$ . Moreover, this change in qualities offered by low levels of  $\theta$  moves

up the equilibrium qualities for the whole interval  $[0, 1]$ . This leads to the condition that discrimination is optimal if  $F'(0)R < 2$  whereas it is not otherwise. Moreover, the same intuition explains the condition in case (c), where the equilibrium in the nondiscriminatory contest is  $(q_A^{IC}, q_B^{IC})$ , because the region for low values of types under discrimination is also an  $I_\gamma$  Region.

If we look at the changes induced by discrimination when agents' equilibrium behavior under a nondiscriminatory contest is  $(q_A^{CI}, q_B^{CI})$  (case (b)), the positive effect that we have identified in the other two cases also exists. Moreover, a new effect appears: the stronger incentives for agent  $A$  to provide a higher effort translate to a decrease in the cut-off value  $\alpha$  from which on he provides a positive effort. Therefore, the interior region  $I$  (which becomes  $I_\gamma$ ) expands. This additional positive effect makes discrimination always beneficial.

While Proposition 3 shows when it is beneficial to discriminate for a given strategy profile, it fails to provide actual conditions under which a discriminatory contest dominates a nondiscriminatory one. To show that such an equilibrium profile arises in an optimal nondiscriminatory contest, we present first an illustrative example and then provide sufficient conditions in the case of a convex distribution function.

**Example 1.** Consider the case where the designer wants to achieve a minimum level of quality  $\underline{q}$ . That is, any lower quality innovation yields zero revenues for the designer whereas innovations of quality  $\underline{q}$  or higher yield a revenue of  $I > 0$ :  $I(q) = 0$  for  $q \in [0, \underline{q}]$  and  $I(q) = I$  for  $q \geq \underline{q}$ .<sup>14</sup> We analyze the case where  $\underline{q} \leq 1$ . Furthermore, we assume that an agent's type is uniformly distributed, that is,  $F(\theta) = \theta$ .

In a nondiscriminatory contest  $R$ , the equilibrium is  $(q_A^C, q_B^C)$  if  $R \leq 1$  and  $(q_A^I, q_B^I)$  if  $R > 1$ . Therefore, the designer's payoff increases under discrimination if the optimal nondiscriminatory contest implies a reward  $R^* \in (1, 2)$ .

For  $R > 1$ , easy calculations show that the designer's payoff is  $U(R) = I - \frac{q^2}{R^2}I - R$ . The optimal  $R^*$  (if it is higher than 1) is  $R^* = 2^{\frac{1}{3}} (\underline{q}^2 I)^{\frac{1}{3}}$ . Therefore, discrimination dominates nondiscrimination if two conditions are met:  $U(2^{\frac{1}{3}} (\underline{q}^2 I)^{\frac{1}{3}}) > U(0) = (1 - \underline{q}^2)I$  (which implies that  $R^* > 1$ ) and  $2^{\frac{1}{3}} (\underline{q}^2 I)^{\frac{1}{3}} < 2$ , that is,  $\underline{q}^2 I \in (\frac{3^{\frac{3}{2}}}{2}, 4)$ , which is satisfied, for example, if  $\underline{q} = 1$  and  $I = 3$ .

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<sup>14</sup>We note that in this simple, illustrative example, the designer's benefit function does not satisfy the assumptions  $I'(q) > 0$  and  $I''(q) < 0$ .

For illustration, we can compute the optimal discriminatory contest in this numerical example, which is  $R_A = 1.8393$  and  $R_B = 1.6639$ . The discriminatory contest yields a payoff of  $U(1.8393, 1.6639) = 0.28228$ , improving the payoff obtained in the optimal nondiscriminatory contest, which is  $U\left(6^{\frac{1}{3}}\right) = 0.27432$ .

A more general class of environments where discrimination increases the designer's payoff is given by the following proposition.

**Proposition 4** *If  $F(\theta)$  is a convex function with  $F'(0) = 0$ , and  $I(q) = vi(q)$ , with  $v > 0$  and  $i(q)$  a bounded and strictly increasing function of  $q$ , then discrimination increases the designer's payoff if  $v$  is large enough.*

Proposition 4 shows that when the distribution of types puts more weight on the higher types, then the designer has an incentive to discriminate if the innovation is very profitable.

Finally, let us note that the advantages of discrimination do not rely on the hypothesis that the designer only cares about the best quality offered in the contest. Consider an organization that benefits from the inputs provided by all the contestants. For example, in a promotion contest, the efforts of all the contestants benefit the firm; in a sport or a talent contest, the designer (the spectators) benefit from the performance of all the participants; in a sales contest, the firm benefits from the outcomes realized by all the sales people. The revenues of such an organization (again restricting attention to  $n = 2$ ) are captured by a function  $I(q_A, q_B)$ , increasing in both qualities  $q_A$  and  $q_B$  offered by agents  $A$  and  $B$ . Then, Propositions 3 and 4 (considering  $I(q_A, q_B) = vi(q_A, q_B)$ ) easily extend to this environment.<sup>15</sup>

## 6 Conclusion

We provided a new setting of contests with asymmetric information where innate abilities and effort combine to generate innovations of various qualities. Both the ability and effort of an agent are his private information. The designer, whose revenue depends on

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<sup>15</sup>The results also extend to environments where entering a contest entails a fixed cost independent of the effort exerted in the contest itself. The formal proofs of these results are available from the authors upon request.

the quality of the bid, specifies a contest where the innovation of the highest quality is rewarded. We first analyzed strategic behavior in a nondiscriminatory contest, where the reward does not depend on the identity of the winner. We allowed for arbitrary distribution functions and determined the structure of equilibrium strategies and outcomes. The equilibrium agents' strategies consisted of two types of quality intervals: regions where all agents put in a positive effort and those where all agents put in zero effort.

We then analyzed strategic behavior in discriminatory contests where rewards depend on the identity of the winner. Here equilibrium strategies were more complex. We provided a qualitative analysis of the structure of the equilibrium strategies for general distribution functions and a full characterization of equilibrium behavior for several classes of functions, in particular if the distribution function is either convex or concave.

We then used the equilibrium analysis to evaluate the designer's payoff. This generated conditions under which the designer prefers a discriminatory to a nondiscriminatory contest. To show that the conditions are not vacuous, we provided parameterized classes of environments where discrimination is beneficial. Our result that discrimination is desirable in a symmetric setting goes against the intuition that when agents are asymmetric, some restrictions imposed on the stronger contestant may increase the designer's payoff.

Our model can handle many familiar scenarios in addition to innovation. It can be used, for example, to analyze, lobbying activity, procurement settings, promotion competitions and even the design of sporting events. It can also be used to study contest design in the presence of asymmetric contestants and shed further light on the imposition of handicaps or favoritism.

The model can be extended in several dimensions. A dynamic version would consider two-stage contests where the winners of the first round are paired against each other in the second round. The designer's objective function may also be expanded to include explicit dependence on the agent's innate ability. Moreover, since the analysis was carried out only from the point of view of a single designer, a challenging task for further research would be to consider environments with several competing designers.



## 7 Appendix

**Proof of Theorem 1.**  $q^*(\theta)$  is differentiable whenever  $q^*(\theta) > \theta$  by construction. We now show through a series of claims that  $q^*(\theta)$  is indeed a symmetric equilibrium strategy.

*Claim 1.* The equilibrium quality is well defined, that is,  $q^*(\theta) \geq \theta$  for any  $\theta \in [\alpha_m, \beta_m]$  for any  $m$ .

By construction,  $\alpha_m$  is a local minimum of the function  $F(\theta)^{n-1}R - \theta$ .  $\beta_m$  is the first instance for which  $F(\theta)^{n-1}R - \theta$  also reaches this minimum for  $\theta > \alpha_m$  and, if this minimum is never reached again,  $\beta_m = 1$ . Hence,  $q^*(\theta) = F(\theta)^{n-1}R - (F(\alpha_m)^{n-1}R - \alpha_m) \geq \theta$  for any  $\theta \in [\alpha_m, \beta_m]$ .

*Claim 2.* The equilibrium quality  $q^*(\theta)$  is continuous. It is also strictly increasing in  $\theta$  and hence  $q^*(\theta)$  is a regular strategy.

By definition,  $q^*(\theta) = \theta$  over all intervals in  $C$ , and  $q^*(\theta) = F(\theta)^{n-1}R + \eta$  over all intervals in  $I$  (where  $\eta$  is different for different intervals). Hence,  $q^*(\theta)$  is continuous over any interval. Moreover, for any interval  $[\alpha_m, \beta_m]$  in  $I$ ,  $q^*(\alpha_m) = F(\alpha_m)^{n-1}R + \alpha_m - F(\alpha_m)^{n-1}R = \alpha_m$  and  $q^*(\beta_m) = F(\beta_m)^{n-1}R + \alpha_m - F(\alpha_m)^{n-1}R = \beta_m$ . Therefore,  $q^*(\theta)$  is continuous everywhere. Finally, given that  $F(\theta)$  is strictly increasing,  $q^*(\theta)$  is also strictly increasing over any interval. Hence,  $q^*(\theta)$  is strictly increasing.

*Claim 3.* The profits of any agent of type  $\theta$  as a function of his choice of  $q$  are constant in any interval in Region  $I$ ; that is, for any  $q \in [\alpha_m, \beta_m]$  with  $q \geq \theta$ .

For any  $q \in [\alpha_m, \beta_m]$  with  $q \geq \theta$ , the agent's profits are

$$\left( F \left( F^{-1} \left( \frac{1}{R^{\frac{1}{n-1}}} (q - \alpha_m + F(\alpha_m)^{n-1}R)^{\frac{1}{n-1}} \right) \right) \right)^{n-1} R - (q - \theta) = F(\alpha_m)^{n-1}R - (\alpha_m - \theta).$$

These profits are independent of  $q$ .

*Claim 4.* The profits of any agent of type  $\theta$ , as a function of his choice of  $q$  are decreasing in any interval in Region  $C$ , that is; for any  $q \in [\beta_{m-1}, \alpha_m]$  with  $q \geq \theta$ .

For  $q \in [\beta_{m-1}, \alpha_m]$  with  $q \geq \theta$ , the agent's profits are  $F(q)^{n-1}R - (q - \theta)$ . They are decreasing in  $q$  since  $F(q)^{n-1}R - q$  is decreasing in  $q$  by construction of  $\beta_{m-1}$  and  $\alpha_m$ .

*Claim 5.* The profits of any agent of type  $\theta$  are non-increasing in his choice of  $q$ .

This follows from continuity and claims 3 and 4.

*Claim 6.*  $q^*(\theta)$  is a best response for an agent of type  $\theta$ .

If  $q^*(\theta) = \theta$ , then the only possible change in the strategy is to increase  $q$  which results, according to Claim 5, in a lower payoff. If  $q^*(\theta) > \theta$ , increasing  $q$  is also non-profitable.

Moreover, any decrease in  $q$  for which the effort is still non-negative implies that the agent stays within the same region (recall that effort is zero in the left boundary of the region). Hence, according to Claim 3, profits remain the same.

Claims 1 to 6 imply that  $q^*(\theta)$  is an equilibrium of the nondiscriminatory contest.

We now prove that  $q^*(\theta)$  is the unique symmetric equilibrium. We again proceed through a series of claims. Consider any symmetric equilibrium  $q(\theta)$ . By arguments similar to those used in auction theory,  $q(\theta)$  is necessarily continuous.

*Claim 7.* If  $q(\theta^o) = \theta^o$  and  $q(\theta^{oo}) = \theta^{oo}$  for  $\theta^{oo} \geq \theta^o$  then  $F(\theta^o)^{n-1}R - \theta^o \geq F(\theta^{oo})^{n-1}R - \theta^{oo}$ .

The expected payoff of an agent of type  $\theta^o$  when he chooses  $q(\theta^o) = \theta^o$  is equal to  $F(\theta^o)^{n-1}R$ , which must not be less than  $F(\theta^{oo})^{n-1}R - (\theta^{oo} - \theta^o)$ , his expected payoff if he offers quality  $\theta^{oo}$ . Therefore, the claim holds.

*Claim 8.* Consider a maximal interval  $[\theta^o, \theta^{oo}]$  where  $q(\theta)$  is an interior solution. Then,  $q(\theta) = F(\theta)^{n-1}R + \eta$  for all  $\theta \in [\theta^o, \theta^{oo}]$  with  $\eta = \theta^o - F(\theta^o)^{n-1}R$ . Moreover, either  $\theta^{oo} = 1$  or  $\theta^{oo}$  is the first parameter larger than  $\theta^o$  for which  $F(\theta^{oo})^{n-1}R - \theta^{oo} = F(\theta^o)^{n-1}R - \theta^o$  that is not a local minimum of the function  $F(\theta)^{n-1}R - \theta$ .

The property that  $q(\theta) = F(\theta)^{n-1}R + \eta$  follows from the FOCs characterizing an interior equilibrium. To show that  $\eta = \theta^o - F(\theta^o)^{n-1}R$  we distinguish between two cases.

(1) If  $\theta^o > 0$ , then there exists an interval in Region  $C$  just to the left of  $\theta^o$ . By continuity of  $q(\theta)$ , it must be the case that  $q(\theta^0) = \theta^0$  which implies that  $\eta = \theta^o - F(\theta^o)^{n-1}R$ .

(2) If  $\theta^o = 0$ , the probability of winning is zero because the quality offered is strictly increasing in  $\theta$ . Hence, it cannot be that in equilibrium both agents choose  $q(0) > 0$ , since it would lead to a negative payoff. Therefore,  $F(0)^{n-1}R + \eta = 0$ , i.e.,  $\eta = 0$  and  $\eta = \theta^o - F(\theta^o)^{n-1}R$  holds in this case as well.

Also, by continuity, if  $\theta^{oo} < 1$ , it must be the case that  $F(\theta^{oo})^{n-1}R + \eta = \theta^{oo}$ ; that is,  $\theta^o - F(\theta^o)^{n-1}R = \theta^{oo} - F(\theta^{oo})^{n-1}R$ . Finally, suppose by way of contradiction, that  $\theta^{oo} > \hat{\theta}$ , where  $\hat{\theta}$  is the first parameter which is not a local minimum of the function  $F(\theta)^{n-1}R - \theta$  that satisfies  $\theta^o - F(\theta^o)^{n-1}R = \hat{\theta} - F(\hat{\theta})^{n-1}R$ . Then,  $q(\theta)$  is an interior solution in an interval  $[\hat{\theta}, \hat{\theta}]$  where  $F(\theta)^{n-1}R - \theta$  is a decreasing function. However, this is not possible because for  $\theta \in (\hat{\theta}, \hat{\theta}]$ ,  $q(\theta) = F(\theta)^{n-1}R + \eta = F(\theta)^{n-1}R + \theta^o - F(\theta^o)^{n-1}R =$

$F(\theta)^{n-1}R + \hat{\theta} - F(\hat{\theta})^{n-1}R < \theta$ , since  $F(\theta)^{n-1}R - \theta$  is decreasing in this interval.

*Claim 9.* In a maximal interval  $[\theta^o, \theta^{oo}]$  where  $q(\theta) = \theta$ , either  $\theta^{oo} = 1$  or  $\theta^{oo}$  is the first parameter such that the function  $F(\theta)^{n-1}R - \theta$  is increasing on an interval  $(\theta^{oo}, \hat{\theta}]$ .

Suppose, by way of contradiction, that  $\theta^{oo}$  is such that the function  $F(\theta)^{n-1}R - \theta$  is not increasing in an interval  $(\theta^{oo}, \hat{\theta}]$  for some  $\hat{\theta} > \theta^{oo}$ . Recall that by maximality of the interval  $[\theta^o, \theta^{oo}]$  it must be the case that  $q(\theta) > \theta$  for  $\theta \in (\theta^{oo}, \tilde{\theta}]$  with  $\tilde{\theta} < \hat{\theta}$ . Hence,  $q(\theta) = F(\theta)^{n-1}R + \eta = F(\theta)^{n-1}R + \theta^{oo} - F(\theta^{oo})^{n-1}R > \theta$  for  $\theta \in (\theta^{oo}, \tilde{\theta}]$ . But this cannot happen if  $F(\theta)^{n-1}R - \theta$  is not increasing. Furthermore, by Claim 7,  $\theta^{oo}$  must be the first parameter where this happens after  $\theta^o$ .

Therefore,  $q^*(\theta)$  is the unique symmetric equilibrium, given that it is the only candidate compatible with claims 8 to 9. Indeed, given that  $F(\theta)^{n-1}R - \theta$  is decreasing for  $\theta = 0$ , Claim 8 implies that the first interval (at equilibrium) can not be in Region *I* because otherwise  $q(\theta) < \theta$  for  $\theta$  close to 0. Once we have identified the first interval, the sequential application of claims 9 and 8 characterized the upper bounds of the following equilibrium intervals in regions *C* and *I*. Finally, the sequence of intervals is finite because the function  $F(\theta)^{n-1}R - \theta$  has a finite set of local minima. ■

Before we proceed to the proof of Propositions 1 and 2, and Theorems 2 and 3, we state and prove two lemmas. Lemma 1 rules out many possible strategy configurations in equilibrium. Lemma 2 states the three strategic equilibrium configurations and the order in which they can appear.

**Lemma 1** *Assume the sets  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_A\}$  and  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_B\}$  have zero measure. Equilibrium strategies in a contest  $(R_A, R_B)$  cannot give rise to a nonempty interval of qualities  $(q_1, q_2) \subseteq [\min\{q_A(0), q_B(0)\}, q_A(1)]$  such that one of the following holds:*

- (a)  $q_A^{-1}(q) < q$  and  $q_B^{-1}(q)$  does not exist, for all  $q \in (q_1, q_2)$ ,
- (b)  $q_A^{-1}(q)$  does not exist and  $q_B^{-1}(q) < q$ , for all  $q \in (q_1, q_2)$ ,
- (c) both  $q_A^{-1}(q)$  and  $q_B^{-1}(q)$  do not exist, for all  $q \in (q_1, q_2)$ ,
- (d)  $q_A^{-1}(q) < q$  and  $q_B^{-1}(q) = q$ , for all  $q \in (q_1, q_2)$ ,
- (e)  $q_A^{-1}(q) = q$  and  $q_B^{-1}(q) < q$ , for all  $q \in (q_1, q_2)$ ,
- (f)  $q_A^{-1}(q) = q$  and  $q_B^{-1}(q)$  does not exist, for all  $q \in (q_1, q_2)$ .

**Proof of Lemma 1.** We prove the six properties by way of contradiction.

(a) If such an interval  $(q_1, q_2)$  exists, then agent  $A$  of type  $\theta \in (q_A^{-1}(q_1), q_A^{-1}(q_2))$  can increase his payoff by lowering the quality offered to another  $q' < q(\theta)$  such  $q' \geq \max\{q_1, \theta\}$ . This change reduces the cost and does not affect his probability of winning the contest.

(b) The proof is similar to the proof of (a).

(c) If such an interval  $(q_1, q_2)$  exists, let  $q_3 = \{\inf q \mid q > q_2 \text{ and } q = q_i(\theta) \text{ for some } i = A, B \text{ and } \theta \in [0, 1]\}$ . If it is the case that  $q_3$  is offered, that is,  $q_A(\theta) = q_3$  for some  $\theta \in [0, 1]$  (we take agent  $A$  to be the one offering  $q_3$  without loss of generality), then  $\theta < q_3$  (it is certainly true if  $q_2 > 1$ , and if  $q_2 \leq 1$  it is true since the equilibrium strategies are monotonic) and agent  $A$  of type  $\theta$  can increase his payoff by lowering the quality offered to another  $q' > \theta$  in the interval  $(q_1, q_2)$  because this change does not affect his probability of winning the contest. By continuity, a similar argument goes through if  $q_3$  is not reached.

(d) Suppose, by contradiction, that such an interval exists. For any type  $\theta \in (q_A^{-1}(q_1), q_A^{-1}(q_2))$ ,  $q(\theta)$  maximizes firm  $A$ 's profits  $F(q)R_A - (q - \theta)$ . Therefore, the following FOC is necessarily satisfied:  $F'(q_A(\theta))R_A - 1 = 0$ . However, this is not possible for an interval  $(q_A^{-1}(q_1), q_A^{-1}(q_2))$  provided the set  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_A\}$  has zero measure.

(e) The proof is similar to the proof of (d).

(f) Consider the maximal last interval  $(q_1, q_2)$  of this type.<sup>16</sup> Since  $q_B(\theta) \geq q_2$  for every  $\theta \in (q_1, q_2)$  and there cannot be a mass point, it must be the case that  $q_2 < 1$ . Moreover, we claim that agent  $B$  must be offering quality levels arbitrarily close to  $q_2$ . Indeed, if this were not the case, the maximality of  $(q_1, q_2)$  implies that agent  $A$  is either reaching qualities just above  $q_2$  by putting in positive effort or he is not reaching these qualities. The first possibility is ruled out by part (a) while the second is ruled out by part (c) of the lemma. Note that these qualities arbitrarily close to  $q_2$  must be offered through positive effort levels by agent  $B$  since they are offered by types  $\theta < q_2$ .

Given parts (b) and (e), it is necessarily the case that if  $B$  puts in positive effort to reach a certain interval  $(q_2, q_3)$ ,  $A$  also puts in positive effort to reach this interval.

Consider now the largest such  $q_3$ , we show that  $\theta_{A3} \equiv q_A^{-1}(q_3) > q_B^{-1}(q_3) \equiv \theta_{B3}$ . Notice first that  $q_2 = q_A^{-1}(q_2) > q_B^{-1}(q_2) \equiv \theta_{B2}$  (we assume for convenience that both  $q_2$  and  $q_3$  are reached, otherwise we can make a limiting argument). Given that the qualities offered

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<sup>16</sup>We say that  $(q_1, q_2)$  is a maximal interval in a region if there does not exist an interval  $(q'_1, q'_2)$  in that region with  $q'_1 \leq q_1$ ,  $q'_2 \geq q_2$  where one of the two inequalities is strict.

in an interior equilibrium are given by  $q_A(\theta) = F(\theta)R_B + \eta_A$  and  $q_B(\theta) = F(\theta)R_A + \eta_B$ ,  $q_3 = F(\theta_{A3})R_B + \eta_A = F(\theta_{B3})R_A + \eta_B$  and  $q_2 = F(q_2)R_B + \eta_A = F(\theta_{B2})R_A + \eta_B$ . Therefore,  $q_3 - q_2 = [F(\theta_{A3}) - F(q_2)]R_B = [F(\theta_{B3}) - F(\theta_{B2})]R_A$ , which implies that  $F(\theta_{A3}) > F(\theta_{B3}) - F(\theta_{B2}) + F(q_2) > F(\theta_{B3})$ , i.e.,  $\theta_{A3} > \theta_{B3}$  as we wanted to show. Therefore, there still exists another interval  $(q_3, q_4)$  above  $(q_2, q_3)$  where agents bid. Given that agent  $B$  is putting in positive effort to reach  $q_3$ , he cannot, since there are no atoms, switch to a region of qualities that are reached by him through zero effort. Therefore, in the new interval it is again the case that  $B$  does not offer any quality in it while  $A$  puts in zero effort. This is the type of region we started with, in contradiction to the assumption that it is the last region of this kind. Hence, such a region cannot exist in equilibrium. ■

**Lemma 2** (a) Consider an equilibrium  $(q_A(\theta), q_B(\theta))$  of the contest  $(R_A, R_B)$  and assume the sets  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_A\}$  and  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_B\}$  have zero measure. Then the range of qualities offered in equilibrium  $[q_A(0), q_A(1)]$  can be split into intervals, each of which belongs to either Region  $I$ ,  $C$ , or  $J$ .

(b) Consider an equilibrium where there exists a (maximal) interval  $(q_1, q_2)$  in Region  $J$ . Then, it must be followed by another interval  $(q_2, q_3)$  in Region  $I$ .

(c) Consider an equilibrium where there exists a (maximal) interval  $(q_1, q_2)$  in Region  $I$ . Then, it must be preceded by another interval  $(q_3, q_1)$  in Region  $J$ .

**Proof of Lemma 2.** (a) It follows from Lemma 1.

(b) Consider the maximal interval  $(q_1, q_2)$  in Region  $J$ . We notice that  $q_A^{-1}(q_2) \leq q_1 < 1$ . Therefore, there are types of agent  $A$  (higher than  $q_1$ ) that offer qualities above Region  $J$ . In this new interval just above  $J$ , agent  $A$  puts in strictly positive effort. Thus, it must be the case (according to Lemma 1) that agent  $B$  also puts in positive effort; that is, this new interval belongs to Region  $I$ .

(c) Consider the maximal interval  $(q_1, q_2)$  in Region  $I$ . We prove this part if we show that the interval can not be preceded by an interval in Region  $C$  and that it can not be the initial interval. Suppose by contradiction that either  $q_1 = 0$  or that  $(q_1, q_2)$  is preceded by a interval in Region  $C$ . In both cases,  $q_A(q_1) = q_B(q_1) = q_1$ . Given the equilibrium strategies in an interior region,  $q_A(\theta) = F(\theta)R_B + q_1 - F(q_1)R_B$  and  $q_B(\theta) = F(\theta)R_A + q_1 - F(q_1)R_A$  for any  $\theta \in (q_1, q_2)$ . Therefore,  $q_A(\theta) < q_B(\theta)$  for any  $\theta \in (q_1, q_2)$ , which implies that  $q_B^{-1}(q_2) < q_A^{-1}(q_2)$  (or that  $\lim_{q \rightarrow q_2} q_B^{-1}(q) < \lim_{q \rightarrow q_2} q_A^{-1}(q)$ ). In particular, there must be

an interval of qualities reached above  $q_2$  and  $q_B^{-1}(q_2) < q_2$ . Therefore, in the interval of qualities just above  $q_2$ , agent  $B$  exerts positive effort, which must be matched by agent  $A$  also offering positive effort, contradicting the maximality of  $(q_1, q_2)$  in Region  $I$ . ■

**Proof of Proposition 1.** We first prove a claim that will be used in the current proof as well as in several proofs in Section 5.

*Claim 10.* Suppose that agent  $i$ , for  $i = A, B$ , chooses  $q_i(\theta) = \theta$  for all  $\theta \in (\theta^o, \theta^{oo})$  and that the function  $F(\theta)R_j - \theta$ , for  $j \neq i$ , is non-increasing in  $\theta$  for  $\theta \in (\theta^o, \theta^{oo})$ . Then, the payoff of agent  $j$  of type  $\theta_j$  is non-increasing in the quality  $q$ , for  $q \in (\theta^o, \theta^{oo})$  with  $q \geq \theta_j$ .

The proof of Claim 10 follows from the fact that  $F(q)R_j - (q - \theta_j) \leq F(q')R_j - (q' - \theta_j)$  when  $q \geq q'$  if the function  $F(\theta)R_j - \theta$  is non-increasing between  $q$  and  $q'$ .

We now prove Proposition 1. If  $F(\theta)R_A - \theta$  is non-increasing in  $\theta$  for all  $\theta \in [0, 1]$ , then the function  $F(\theta)R_B - \theta$  is also non-increasing in  $\theta$  for all  $\theta \in [0, 1]$  because  $R_B < R_A$ . Therefore, if agent  $i$  chooses  $q_i(\theta) = \theta$  for all  $\theta \in [0, 1]$ , then agent  $j \neq i$  maximizes his payoff by choosing  $q_j(\theta) = \theta$  as well, according to Claim 10. It follows that there is an equilibrium where the agents' strategies lie in Region  $C$  for all  $\theta \in [0, 1]$ .

Moreover, if agents' equilibrium strategies lie in Region  $C$  for all  $\theta \in [0, 1]$ , then it is necessarily the case that  $F(\theta)R_A - \theta \geq F(q)R_A - q$  for any  $\theta \in [0, 1]$  and for any  $q \geq \theta$ . Therefore, the function  $F(\theta)R_A - \theta$  is non-increasing in  $\theta$  for all  $\theta \in [0, 1]$ .

Finally, suppose by contradiction that there exists another equilibrium. It must either start with an interval in Region  $I_\gamma$  or with an interval in Region  $C$  followed by an interval in Region  $I_\gamma$ . Therefore, there is a jump, that is, there exist two values  $q_1$  and  $q_2$  (where  $q_1$  is possibly 0) with  $q_1 < q_2$  such that  $F(q_1)R_A - q_1 \leq F(q_2)R_A - q_2$ . If the inequality is strict, then this contradicts the fact that the function  $F(\theta)R_A - \theta$  is non-increasing in  $\theta$ . If this an equality, then  $F(\theta)R_A - \theta$  is constant for all  $\theta \in [q_1, q_2]$ , which contradicts the property that the set  $\{\theta \in [0, 1] \mid F'(\theta) = 1/R_A\}$  has zero measure. ■

**Proof of Proposition 2.** We show first that  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is well defined. (i)  $q_B^{I_\gamma}(\theta) \geq \theta$  for all  $\theta \in [0, 1]$  because  $q_B^{I_\gamma}(\theta) = \theta$  for all  $\theta \in [0, \gamma)$  and  $q_B^{I_\gamma}(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A \geq \theta$  for all  $\theta \geq \gamma$  according to (11). (ii)  $q_A^{I_\gamma}(\theta) \geq \theta$  for all  $\theta \in [0, 1]$  because  $q_A^{I_\gamma}(\theta) \geq q_B^{I_\gamma}(\theta)$  due to the properties that  $q_A^{I_\gamma}(1) = q_B^{I_\gamma}(1)$  and  $q_A^{I_\gamma'}(\theta) = R_B < R_A = q_B^{I_\gamma'}(\theta)$ .

Second, we prove a claim that will be useful at several proofs:

*Claim 11.* Suppose that agent  $i$ , for  $i = A, B$ , chooses  $q_i(\theta) = F(\theta)R_j + \eta$  for all  $\theta \in (\theta^o, \theta^{oo})$ , with  $j \neq i$ . Then, the payoff of agent  $j$  of type  $\theta_j$  is constant and equal to

$\theta_j - \eta$  when he offers any quality  $q \in (F(\theta^o) R_j + \eta, F(\theta^{oo}) R_j + \eta)$  with  $q \geq \theta_j$ .

Given  $q_i(\theta)$ , the payoff of agent  $j$  of type  $\theta_j$  when he offers  $q \in (F(\theta^o) R_j + \eta, F(\theta^{oo}) R_j + \eta)$  with  $q \geq \theta_j$  is

$$R_j \Pr_\theta(F(\theta) R_j + \eta \leq q) - (q - \theta_j) = R_j \left( \frac{q - \eta}{R_j} \right) - q + \theta_j = \theta_j - \eta.$$

Third, from Claim 11 and given agent  $B$ 's strategy, the payoff of agent  $A$  of type  $\theta$  when he offers any quality  $q \in [\gamma, q_B(1)]$  with  $q \geq \theta$  is  $\theta - \gamma + F(\gamma) R_A$ . Similarly, the payoff of agent  $B$  of type  $\theta$  when he offers quality  $q \in [\gamma, q_B(1)]$  with  $q \geq \theta$  is  $\theta - \gamma$ , also independent of  $q$ . In particular, the strategies suggested are best responses one to the other for agents of type  $\theta \in [\gamma, 1]$ .

Agent  $B$ 's payoff when offering quality  $q(\theta) = \theta$  for  $\theta \in [0, \gamma]$  is zero. His payoff would be negative if he were to offer any  $q \in (\theta, \gamma]$  since he still has a probability zero of winning and it would be  $\theta - \gamma < 0$  if he were to offer any  $q \in (\gamma, q_B(1)]$ . Therefore, agent  $B$ 's strategy is a best response for all  $\theta \in [0, \gamma)$  as well.

Agent  $A$ 's payoff when following the strategy suggested for  $\theta \in [0, \gamma]$  is  $\theta - \gamma + F(\gamma) R_A$ . As shown above, his payoff is the same for any  $q \geq \gamma$ . If he offers  $q \in [\theta, \gamma)$ , then his payoff is  $F(q) R_A - (q - \theta)$ . Hence, agent  $A$ 's proposed strategy is his best response if  $F(q) R_A - (q - \theta) \leq \theta - \gamma + F(\gamma) R_A$  for all  $q \leq \gamma$ , that is,  $F(q) R_A - q \leq F(\gamma) R_A - \gamma$ , which is implied by (10).

We also prove that conditions (10) and (11) are necessary for  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  to be an equilibrium. If  $F(\theta) R_A - \theta < F(\gamma) R_A - \gamma$  for some  $\theta > \gamma$ , then  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  cannot be an equilibrium because  $q_B^{I_\gamma}(\theta)$  would not be well defined ( $q_B^{I_\gamma}(\theta) < \theta$ ). Moreover, if  $F(\theta) R_A - \theta > F(\gamma) R_A - \gamma$  for some  $\theta < \gamma$ , then  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  also cannot be an equilibrium because agent  $A$  of type  $\theta$  would strictly prefer  $\theta$  to  $q_A^{I_\gamma}(\theta)$  (because his benefits under  $q_A^{I_\gamma}(\theta)$  are the same as under  $\gamma$ ), contradicting the optimality of  $q_A^{I_\gamma}$ .

Finally, we show that the equilibrium  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is unique if the inequalities are strict. We start by proving two claims:

*Claim 12.* Consider a maximal interval  $[\theta_1, \theta_2]$  in Region  $I_\gamma$  and denote by  $\gamma$  the cut-off corresponding to this interval. Then,  $\gamma$  is increasing in both  $\theta_1$  and  $\theta_2$ .

Indeed, the parameters  $\eta_A$ ,  $\eta_B$  and  $\gamma$  in the interval  $[\theta_1, \theta_2]$  are characterized by  $q_A(\theta_1) = \gamma$ , that is,  $\eta_A = \gamma - F(\theta_1) R_B$ ;  $q_B(\gamma) = \gamma$ , that is,  $\eta_B = \gamma - F(\gamma) R_A$ ; and  $q_A(\theta_2) = q_B(\theta_2)$ , that is,  $F(\theta_2) R_B + \gamma - F(\theta_1) R_B = F(\theta_2) R_A + \gamma - F(\gamma) R_A$ .

Therefore,  $\gamma$  is characterized by  $F(\gamma) R_A = F(\theta_2)(R_A - R_B) + F(\theta_1) R_B$ , which implies that  $\gamma$  is increasing in  $\theta_1$  and  $\theta_2$ .

Claim 13. If  $F(\theta)R_A - \theta$  is increasing in  $\theta$  for all  $\theta \in [\theta_1, \theta_2]$ , then, at equilibrium, the interval  $[\theta_1, \theta_2]$  is in the region  $I_\gamma$ .

Otherwise, there must exist an interval  $[\theta^\circ, \theta^{oo}] \subseteq [\theta_1, \theta_2]$ , with  $\theta^{oo} > \theta^\circ$ , such that  $q_A(\theta) = q_B(\theta) = \theta$  for all  $\theta \in [\theta^\circ, \theta^{oo}]$ . However, we claim that, for example,  $q = \theta^\circ$  is not a best response for player  $A$  if his type is  $\theta^\circ$ . Indeed, this player's expected profit if he chooses  $q \in [\theta^\circ, \theta^{oo}]$  is  $F(q)R_A - q + \theta^\circ$ , which is increasing in  $q$  if  $F(\theta)R_A - \theta$  is increasing in  $\theta$ . Therefore, his expected profit is higher with, say,  $q = \theta^{oo}$  than with  $q = \theta^\circ$ .

To prove that the equilibrium is unique, we denote by  $\gamma^*$  the cut-off in the profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ , that is,  $\gamma^* \equiv F^{-1}\left(1 - \frac{R_B}{R_A}\right)$ . We note that  $\gamma^*$  is in the increasing part of the function  $F(\theta)R_A - \theta$ , according to the equations (10) and (11). Assume by way of contradiction there exists another equilibrium denoted by  $E$ , different from  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ . According to Claim 13, in this equilibrium  $\gamma^*$  must still be in an interval in the Region  $I_\gamma$ , we denote by  $\Omega$  such an interval. It cannot be that  $\Omega = [0, 1]$  since then the two equilibria would coincide, thus, there must be an interval in Region  $C$  either before or after  $\gamma^*$ . Consider first the case where there exists an interval  $[\theta^\circ, \theta^{oo}]$  in Region  $C$  before  $\gamma^*$ . If there are several such intervals, we take the one adjacent to  $\Omega$  on the left, that is,  $\theta^{oo}$  is the lower bound of  $\Omega$ . There is then a  $\gamma$  (not necessarily  $\gamma^*$ ) in  $\Omega$  such that player  $B$  chooses  $q_B(\theta) = \theta$  for  $\theta \in [\theta^{oo}, \gamma]$  and player  $A$ 's profits, if of type  $\theta^{oo}$ , are  $F(\theta^{oo})R_A = F(\gamma)R_A - \gamma + \theta^{oo}$ . Since  $\theta^{oo} < \gamma^*$ , inequality (10) implies that  $F(\gamma)R_A - \gamma < F(\gamma^*)R_A - \gamma^*$  and inequality (11) implies that  $\gamma < \gamma^*$ . ■

If  $\Omega = [\theta^{oo}, 1]$ , then by Claim 12, it must be that  $\gamma > \gamma^*$ , hence,  $\Omega \neq [\theta^{oo}, 1]$  and there must exist another interval in Region  $C$  with values of  $\theta$  higher than  $\gamma$ . This, however, implies that  $E$  cannot be an equilibrium. Take a particular value  $\theta^{ooo}$  in that interval. The profit of player  $A$  of type  $\theta^{oo}$  in equilibrium  $E$  (following  $q_A(\theta^{oo}) = \gamma$ ) is  $F(\gamma)R_A - \gamma + \theta^{oo}$ . If he were to choose  $\theta^{ooo}$  his profit would increase to  $F(\theta^{ooo})R_A - \theta^{ooo} + \theta^{oo}$ , in contradiction to  $E$  being an equilibrium. Profit increases since,  $\gamma < \gamma^* < \theta^{ooo}$  implies (if inequalities (10) and (11) are strict) that  $F(\gamma)R_A - \gamma < F(\theta^{ooo})R_A - \theta^{ooo}$ .

Second, if there is an interval  $[\theta^\circ, \theta^{oo}]$  in Region  $C$  after  $\gamma^*$  but there does not exist an interval in Region  $C$  before  $\gamma^*$  then, by Claim 12, it must be that  $\gamma < \gamma^*$ . Player  $A$ 's profit in equilibrium  $E$ , if of type  $\theta = 0$ , is  $F(\gamma)R_A - \gamma$  whereas it would be  $F(\theta^\circ)R_A - \theta^\circ$



if it would choose  $q = \theta^o$ . However,  $F(\gamma)R_A - \gamma < F(\theta^o)R_A - \theta^o$  because  $\gamma < \gamma^* < \theta^o$  and strict inequalities (10) and (11), in contradiction to  $q_A(0) = \gamma$  being a best response.

**Proof of Theorem 2.** (a) Given the convexity of  $F(\cdot)$ ,  $F'(1)R_A \leq 1$  is a necessary and sufficient condition for  $F(\theta)R_A - \theta$  to be non-increasing in  $\theta$  for all  $\theta \in [0, 1]$ . Therefore, this part follows from Proposition 1.

(b) First, we show that  $\alpha$  and  $\gamma$  are well defined in this region and that  $\gamma > \alpha$ . Equation (12) defines a function  $\gamma^1(\alpha)$  which is increasing and such that  $\gamma^1(1) = 1$  and  $\gamma^1(\alpha) > \alpha$  for  $\alpha \in [0, 1]$  (because  $R_A > R_B$ ). Equation (13), together with the condition that  $\gamma \geq \alpha$  defines another function  $\gamma^2(\alpha)$ .  $\gamma^2(\alpha)$  is defined only for values of  $\alpha$  where the function  $F(\theta)R_A - \theta$  is non-increasing, but not necessarily for all of them. Note that  $\gamma^2(\alpha)$  is defined for all such values when  $R_A \geq 1$ ; furthermore, it is certainly defined for values of  $\alpha$  close enough to the minimum of the function  $F(\theta)R_A - \theta$ , which we denote  $\theta_{\min}$ . Also note that  $\gamma^2(\alpha)$  always lies in the increasing part of  $F(\theta)R_A - \theta$ . The function  $\gamma^2(\alpha)$  is strictly decreasing and converges to  $\theta_{\min}$  when  $\alpha$  converges to  $\theta_{\min}$ . We distinguish between two cases.

When  $R_A \in \left(\frac{1}{F'(1)}, 1\right)$ , then  $F(1)R_A - 1 < 0$ . Therefore, there is some  $\alpha^o$  for which  $\gamma^2(\alpha^o) = 1$ , from which on the function is strictly decreasing until it reaches  $\theta_{\min}$ , where  $\gamma^2(\theta_{\min}) = \theta_{\min}$ . Given that  $\gamma^1(\alpha)$  is strictly increasing,  $\gamma^1(1) = 1$  and  $\gamma^1(\alpha) > \alpha$  for  $\alpha \in [0, 1)$ , then a solution to the system of equations always exists.

When  $R_A \in \left[1, \frac{1}{F'(0)}\right)$ , then the function  $\gamma^2(\alpha)$  is defined for  $\alpha \in [0, \theta_{\min})$  and it takes values always lower than 1. In this case, given that  $\gamma^2(\alpha)$  is decreasing and  $\gamma^1(\alpha)$  is increasing, a solution exists if and only if  $\gamma^2(0) \geq \gamma^1(0)$ , that is  $\gamma^2(0) \geq F^{-1}\left(\frac{R_A - R_B}{R_A}\right)$ , which we write as,  $F(\gamma^2(0))R_A \geq R_A - R_B$ , or,  $\gamma^2(0) \geq R_A - R_B$ . Given that  $\gamma^2(0)$  is the increasing part of  $F(\theta)R_A - \theta$ , the previous inequality is equivalent to  $F(R_A - R_B)R_A - (R_A - R_B) \leq 0$ , which we assume in Region (b).

Second, we show that agents' strategies are well defined, that is, the functions  $\phi_A(\theta) \equiv q_A^{CI_\gamma}(\theta) - \theta$  and  $\phi_B(\theta) \equiv q_B^{CI_\gamma}(\theta) - \theta$  are non-negative for all  $\theta \in [0, 1]$ . This trivially holds for all regions where players choose zero effort.

For  $\theta \in [\gamma, 1]$  we have  $q_B^{CI_\gamma}(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A$ . Given that  $\gamma$  lies in the increasing part of  $F(\theta)R_A - \theta$ , we have  $\phi_B(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A - \theta \geq 0$  for  $\theta \in [\gamma, 1]$ . For  $\theta \in [\alpha, \gamma)$ , the convexity of  $\phi_A(\theta) = F(\theta)R_B + \gamma - F(\alpha)R_B - \theta$  implies that  $\phi_A(\theta) \geq \phi_A(\alpha) + \phi'_A(\alpha)(\theta - \alpha) = \gamma - \alpha + (F'(\alpha)R_B - 1)(\theta - \alpha) \geq \gamma - \alpha - (\theta - \alpha) \geq 0$ . For

$\theta \in [\gamma, 1]$ , we note that both  $\phi_A(\theta)$  and  $\phi_B(\theta)$  are convex functions. Furthermore,  $\phi_A(\gamma) > \phi_B(\gamma) = 0$ ,  $\phi_A(1) = \phi_B(1)$  (since  $R_A - F(\gamma)R_A = R_B - F(\alpha)R_B$ ) and  $\phi'_A(\theta) < \phi'_B(\theta)$  which implies that  $\phi_A(\theta) \geq \phi_B(\theta)$  for all  $\theta \in [\gamma, 1]$  and thus  $\phi_A(\theta) \geq 0$  for all  $\theta \in [\gamma, 1]$  as well.

Third, we prove that each agent's strategy is best response to each other.

Given agent  $B$ 's strategy, the payoff of agent  $A$  of type  $\theta$  when he offers quality  $q \in [\gamma, q_B(1)]$  with  $q \geq \theta$  is (see Claim 11)  $\theta - \gamma + F(\gamma)R_A$ , which is independent of  $q$ . Similarly, the payoff of agent  $B$  of type  $\theta$  when offering quality  $q \in [\gamma, q_B(1)]$  with  $q \geq \theta$  is  $\theta - \gamma + F(\alpha)R_B$ , also independent of  $q$ . This implies, in particular, that the strategies suggested are best responses one to the other for agents of type  $\theta \in [\gamma, 1]$ .

The payoff of agent  $B$  of type  $\theta \in (\alpha, \gamma)$  is decreasing in  $q$  for  $q \in (\theta, \gamma)$ , because no type of agent  $A$  chooses qualities in  $(\alpha, \gamma)$  and the payoff is constant for  $q \in [\gamma, q_B(1)]$ . Therefore,  $q_B^{CI_\gamma}(\theta) = \theta$  is a best response for all  $\theta \in (\alpha, \gamma)$ . The payoff of agent  $B$  of type  $\theta \in [0, \alpha]$  is decreasing in  $q$  for  $q \in [\theta, \alpha]$  because the interval  $[\theta, \alpha]$  is in the decreasing part of the function  $F(\theta)R_A - \theta$  (see Claim 10). Therefore,  $q_B^{CI_\gamma}(\theta) = \theta$  is a best response because it is first decreasing and then constant for  $q \in [\alpha, q_B(1)]$ .

Agent  $A$  of type  $\theta$  that chooses  $q \in [0, \gamma]$ , with  $q \geq \theta$ , obtains a payoff of  $F(q)R_A - q + \theta$ . The function  $F(q)R_A - q$  is decreasing until  $\alpha$ , then it further decreases, then increases until it recovers the same value  $F(\alpha)R_A - \alpha$  at  $\gamma$  (see (13)). As we saw above,  $A$ 's payoff is constant for  $q \in [\gamma, q_B(1)]$ . Therefore,  $q_A^{CI_\gamma}(\theta) = \theta$  is a best response for all  $\theta \in [0, \alpha]$  and  $q_A^{CI_\gamma}(\theta) = F(\theta)R_B + \gamma - F(\alpha)R_B$  is a best response for all  $\theta \in [\alpha, \gamma]$ .

To show this is the unique equilibrium note that for this range of  $R_A$ , the function  $F(\theta)R_A - \theta$  is first decreasing and then increasing. Also, if there is an interval in Region  $C$  in the decreasing part of  $F(\theta)R_A - \theta$  (such an interval cannot exist in the increasing part, according to Claim 13), then it cannot be preceded by another interval in Region  $I_\gamma$ . Suppose otherwise. Let  $[\theta^\circ, \theta^{\circ\circ}]$  be a maximal interval where the equilibrium lies in Region  $I_\gamma$  and  $[\theta^{\circ\circ}, \theta^{\circ\circ\circ}]$  be an interval where it lies in Region  $C$ , both in the decreasing part of  $F(\theta)R_A - \theta$ . The strategies in  $[\theta^\circ, \theta^{\circ\circ}]$  are  $q_A(\theta) = F(\theta)R_B + \eta_A$  and  $q_B(\theta) = F(\theta)R_A + \eta_B$ , with  $\eta_B = \theta^{\circ\circ} - F(\theta^{\circ\circ})R_A$ . Also, there must exist some  $\gamma \in (\theta^\circ, \theta^{\circ\circ})$  such that  $F(\gamma)R_A + \eta_B = \gamma$ , that is,  $\eta_B = \gamma - F(\gamma)R_A$ . Therefore,  $\theta^{\circ\circ} - F(\theta^{\circ\circ})R_A = \gamma - F(\gamma)R_A$ , which is not possible because  $\gamma < \theta^{\circ\circ}$  and  $F(\theta)R_A - \theta$  is decreasing between the two values.

Therefore, there are only two candidate equilibria:  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  and  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ . By

Proposition 2,  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  can be an equilibrium only if the inequalities of Proposition 2 are satisfied for  $\gamma$  defined by  $F(\gamma) = 1 - \frac{R_B}{R_A}$ . This implies, since  $F(\theta)R_A - \theta$  equals 0 at  $\theta = 0$ , and is first decreasing and then increasing, that  $F(\gamma)R_A - \gamma \geq 0$  or  $R_A - R_B \geq \gamma$  or finally,  $F(R_A - R_B) \geq F(\gamma) = 1 - \frac{R_B}{R_A}$ . Each of the two conditions in this part implies this inequality is violated.  $F(R_A - R_B)R_A - (R_A - R_B) < 0$  implies it directly. Also if  $R_A < 1$ , then  $F(\gamma)R_A - \gamma < 0$  since  $F(\gamma) < \gamma$  by convexity of  $F$ . Thus, the equilibrium must be of the form  $CI_\gamma$ , the parameters of which are uniquely determined.

(c) Given the convexity of  $F(\theta)$ , the function  $F(\theta)R_A - \theta$  is always increasing when  $R_A \geq \frac{1}{F'(0)}$ . Moreover, given the definition of  $\gamma$ , when  $R_A \in \left[1, \frac{1}{F'(0)}\right)$  the condition  $F(R_A - R_B)R_A - (R_A - R_B) \geq 0$  is equivalent to  $F(R_A - R_B)R_A - F(\gamma)R_A \geq 0$ , or  $R_A - R_B \geq \gamma$ , which is equivalent to  $F(\gamma)R_A - \gamma \geq 0$ . Given that, in this region,  $F(\theta)R_A - \theta$  is first decreasing and then increasing, if  $F(\gamma)R_A - \gamma > 0$ , then inequalities (10) and (11) are strictly satisfied. Therefore, by Proposition 2 part (c) holds and the equilibrium is unique. If  $F(\gamma)R_A - \gamma = 0$ , then inequalities (10) and (11) are strictly satisfied for any  $\theta > 0$ . Similar arguments to those in the proof of Proposition 2 imply that  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is the unique equilibrium in this case as well. ■

**Proof of Theorem 3.** (a) This part follows from Proposition 1 because  $F'(0)R_A \leq 1$  and the concavity of  $F$  imply that  $F(\theta)R_A - \theta$  is non-increasing in  $\theta$  for all  $\theta \in [0, 1]$ .

(b) We first show that  $\gamma$  and  $\beta$  are well defined in this region and that  $\beta > \gamma$ . Similar to its behavior in Theorem 2, equation (15) defines a function  $\beta^2(\gamma)$  for those values of  $\gamma$  where  $F(\theta)R_A - \theta$  is non-decreasing, but not necessarily for all of them.  $\beta^2(\gamma)$  is defined for all such values when  $R_A \leq 1$ ; furthermore it is certainly defined for values of  $\gamma$  close enough to the maximum of the function  $F(\theta)R_A - \theta$ , which we denote  $\theta_{\max}$ . Also note that,  $\beta^2(\gamma)$  always lies in the decreasing part of  $F(\theta)R_A - \theta$ . The function  $\beta^2(\gamma)$  is strictly decreasing (in the interval of  $\gamma$  where it is defined) and converges to  $\theta_{\max}$  when  $\gamma$  converges to  $\theta_{\max}$ .

Equation (14) defines a function  $\gamma^1(\beta)$ . The function is increasing at least for  $\beta \geq \theta_{\max}$  because  $F(\theta)R_A - \theta$  is decreasing for  $\theta \geq \theta_{\max}$  and  $R_A > R_B$ . Moreover,  $\gamma^1(1) = 1 - R_B$ . We distinguish between two cases. When  $\beta^2(0)$  is well defined, that is, when  $R_A - 1 \leq 0$  then, since  $R_B < R_A$ ,  $\gamma^1(1) = 1 - R_B$  is positive. Therefore, the functions  $\gamma^1(\beta)$  and  $\beta^2(\gamma)$  intersect and a solution to equations (14) and (15) exists. When  $R_A \in \left[1, \frac{1}{F'(1)}\right]$ , then  $F(1)R_A - 1 \leq 0$ , therefore there is some  $\gamma$  for which  $\beta^2(\gamma) = 1$ . We denote this

value by  $z(R_A)$ . The necessary and sufficient condition for (14) and (15) to intersect is that  $z(R_A) < 1 - R_B$  or, equivalently,  $F(1 - R_B)R_A - (1 - R_B) > R_A - 1$ .

Second, we show that the functions  $\delta_A(\theta) \equiv q_A^{I_\gamma C}(\theta) - \theta$  and  $\delta_B(\theta) \equiv q_B^{I_\gamma C}(\theta) - \theta$  are non-negative for all  $\theta \in [0, 1]$ . This trivially holds if players choose zero effort.

For  $\theta \in [\gamma, \beta]$ , we have  $F(\theta)R_A - \theta \geq F(\gamma)R_A - \gamma$  because  $\gamma$  lies in the increasing part of  $F(\theta)R_A - \theta$  and the function takes the same value for  $\gamma$  and  $\beta$ . Hence,  $\delta_B(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A - \theta \geq 0$  for  $\theta \in [\gamma, \beta]$ . For  $\theta \in [0, \beta]$ ,  $\delta_A(\theta) = F(\theta)R_B + \gamma - \theta > 0$  because it is a concave function of  $\theta$ ,  $\delta(0) = \gamma > 0$  and  $\delta(\beta) = 0$  by equation (14).

Third, we prove that each agent's strategy is best response to each other.

Given agent  $B$ 's strategy, the payoff of agent  $A$  of type  $\theta \in [\beta, 1]$  is decreasing in  $q$  for  $q > \theta$  because the function  $F(q)R_A - q$  is decreasing (see Claim 10); thus  $q_A^{I_\gamma C}(\theta) = \theta$  is agent  $A$ 's best response. For  $\theta \in [0, \beta]$ , the payoff of agent  $A$  is equal to  $\theta + F(\gamma)R_A - \gamma$  for any  $q \in [\gamma, \beta]$  with  $q \geq \theta$  (by Claim 11) and it is decreasing for  $q \in [\beta, 1]$  (by Claim 10). If agent  $A$  offers quality  $q \in [0, \gamma]$  with  $q \geq \theta$  his payoff is  $F(q)R_A - (q - \theta)$ , which is smaller than  $\theta + F(\gamma)R_A - \gamma$  because the function  $R_A F(\theta) - \theta$  is increasing in that interval. Therefore,  $q_A^{I_\gamma C}(\theta)$  is an agent  $A$ 's best response.

Given agent  $A$ 's strategy, the payoff of agent  $B$  of type  $\theta \in [\beta, 1]$  when offering  $q_B^{I_\gamma C}(\theta) = \theta$  is  $F(\theta)R_B$ , which is higher than his payoff for any  $q > \theta$  because we are in the decreasing part of  $F(\theta)R_A - \theta$ . The payoff of agent  $B$  of type  $\theta \in [\gamma, \beta]$  when offering  $q_B^{I_\gamma C}(\theta)$  is  $\theta - \gamma$ , which is higher than his payoff if he offers quality  $q \in [\beta, 1]$  because  $F(\theta)R_B - \theta$  is decreasing in  $\theta$  for  $\theta \geq \beta$ . Finally, the payoff of agent  $B$  of type  $\theta \in [0, \gamma]$  when offering quality  $\theta$  is zero. It would be negative for any  $q \in [0, \gamma]$  with  $q > \theta$  and, as shown above, the payoff would be first constant and then decreasing as  $q$  is higher. Therefore,  $q_B^{I_\gamma C}(\theta)$  constitutes a best response to agent  $A$ 's strategy.

To show that this is a unique equilibrium note that, for this range of  $R_A$ , the function  $F(\theta)R_A - \theta$  is first increasing and then decreasing. Hence, an equilibrium must start with an interval in  $I_\gamma$ , according to Claim 13. Moreover, if there is an interval in Region  $C$  in the decreasing part of  $F(\theta)R_A - \theta$ , then it cannot be followed by another interval in Region  $I_\gamma$ . Suppose otherwise. Let  $[\theta^\circ, \theta^{\circ\circ}]$  be a maximal interval where the equilibrium lies in Region  $C$  and  $[\theta^{\circ\circ}, \theta^{\circ\circ\circ}]$  be an interval where it lies in Region  $I_\gamma$ . Player  $A$ 's profits must be the same at  $\theta^{\circ\circ}$  and at some  $\gamma > \theta^{\circ\circ}$ , that is (taking into account that  $B$  plays  $q_B(\theta) = \theta$  for all  $\theta \in [\theta^\circ, \gamma]$ ),  $F(\theta^{\circ\circ})R_A - \theta^{\circ\circ} = F(\gamma)R_A - \gamma$ , which is not possible because

$F(\theta)R_A - \theta$  is decreasing between the two values.

Therefore, there are only two candidate equilibria:  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  and  $(q_A^{I_\gamma C}, q_B^{I_\gamma C})$ .  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  can be an equilibrium only if the inequalities of Proposition 2 are satisfied for  $\gamma$  such that:  $F(\gamma) = 1 - \frac{R_B}{R_A}$ , and. This is equivalent, since  $F(\theta)R_A - \theta$  equals 0 at  $\theta = 0$ , and is first increasing and then decreasing, to  $F(\gamma)R_A - \gamma \leq R_A - 1$  or  $1 - R_B \leq \gamma$  or finally,  $F(1 - R_B) \leq F(\gamma) = 1 - \frac{R_B}{R_A}$ . Each of the two conditions in this part implies this inequality is violated.  $F(1 - R_B)R_A - (1 - R_B) > R_A - 1$  implies it directly. Also if  $R_A < 1$ , then since  $R_A F(\gamma) - \gamma > 0$ , the inequality is violated as well. Thus, the equilibrium must be of the form  $I_\gamma C$ , the parameters of which are uniquely determined.

(c) We use Proposition 2, which we can apply directly if  $R_A \geq \frac{1}{F'(1)}$  because the function  $F(\theta)R_A - \theta$  is always increasing. If  $R_A \in \left[1, \frac{1}{F'(1)}\right)$ , then conditions (10) and (11) hold if and only if  $F(\gamma)R_A - \gamma \leq F(1)R_A - 1$ , that is,  $F(1 - R_B)R_A - (1 - R_B) \leq R_A - 1$ , which is the condition appearing in (c). If the inequality is strict,  $(q_A^{I_\gamma}, q_B^{I_\gamma})$  is the unique equilibrium according to Proposition 2. Arguments similar to those in the proof of Proposition 2 imply that the equilibrium is unique also when  $F(1 - R_B)R_A - (1 - R_B) = R_A - 1$ , in which case (10) and (11) are strict except for  $\theta = 1$ . ■

Before we proceed to the proof of Proposition 3 we state and prove a lemma describing the designer's payoff as a function of the agents' strategies.

**Lemma 3** *The designer's payoff  $U(R_A, R_B)$ , for  $R_A \geq R_B$ , as a function of the agents' strategies, is the following:*

(a) *If agents follow the strategy profile  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ , then*

$$U(R_A, R_B) = \frac{1}{R_A R_B} \int_{\gamma}^{R_B + \gamma} I(q) [2(q - \gamma) + R_A - R_B] dq - R_A + \frac{1}{2} R_B \left(1 - \frac{R_B}{R_A}\right).$$

(b) *If agents follow the strategy profile  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ , then*

$$\begin{aligned} U(R_A, R_B) = & 2 \int_0^\alpha I(q) F(q) F'(q) dq + \int_\alpha^\gamma I(q) F(\alpha) F'(q) dq + \\ & \frac{1}{R_A R_B} \int_\gamma^{[1 - F(\gamma)]R_A + \gamma} I(q) [2(q - \gamma + F(\gamma)R_A) - (R_A - R_B)] dq - \\ & \frac{1}{2} \left[ R_A + R_B + (1 - F(\alpha))^2 (R_A - R_B) \left(1 - \frac{R_B}{R_A}\right) \right]. \end{aligned}$$

(c) If agents follow the strategy profile  $(q_A^{I_\gamma C}, q_B^{I_\gamma C})$ , then

$$U(R_A, R_B) = \frac{1}{R_A R_B} \int_\gamma^\beta I(q) [2(q - \gamma) + F(\gamma) R_A] dq + \\ 2 \int_\beta^1 I(q) F(q) F'(q) dq - \frac{1}{2} [R_A + R_B + F(\gamma) F(\beta) (R_A - R_B)].$$

**Proof of Lemma 3.** (a) If agents follow  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ , the interval of qualities  $q$  that may be offered is  $[\gamma, R_B + \gamma]$ , according to the distribution function

$$F^*(q) = \frac{1}{R_B} (q - \gamma) \frac{1}{R_A} [(q - \gamma) + R_A - R_B], \\ dF^*(q) = \frac{1}{R_A R_B} [2(q - \gamma) + R_A - R_B] dq.$$

Therefore, the designer's expected income is the first part of the expression  $U(R_A, R_B)$ . The expected cost depends on the probability that either agent wins the contest. An agent  $A$  of type  $\theta$  wins the contest with probability

$$\Pr_{\theta_B}(F(\theta_B) R_A + \gamma - (R_A - R_B) \leq F(\theta) R_B + \gamma) = \frac{1}{R_A} [F(\theta) R_B + R_A - R_B].$$

It follows that the probability that agent  $A$  wins the contest is

$$\int_0^1 \frac{1}{R_A} [F(\theta) R_B + R_A - R_B] F'(\theta) d\theta = \frac{R_B}{R_A} \frac{1}{2} [F(\theta)^2]_0^1 + \frac{(R_A - R_B)}{R_A} [F(\theta)]_0^1 = 1 - \frac{1}{2} \frac{R_B}{R_A}$$

while the probability that  $B$  wins the contest is  $\frac{1}{2} \frac{R_B}{R_A}$ . Therefore, the designer's expected cost is  $R_A \left(1 - \frac{1}{2} \frac{R_B}{R_A}\right) + R_B \frac{1}{2} \frac{R_B}{R_A}$ , from which the second part of the expression  $U(R_A, R_B)$  is obtained.

(b) If agents follow  $(q_A^{CI_\gamma}, q_B^{CI_\gamma})$ , the set of qualities that is reached is  $[0, (1 - F(\gamma)) R_A + \gamma]$ . For  $q \in [0, \alpha)$ ,  $dF^*(q) = 2F(q)F'(q)dq$ . For  $q \in [\alpha, \gamma)$ ,  $F^*(q) = F(\alpha)F(q)$  and

$$dF^*(q) = F(\alpha)F'(q)dq.$$

Finally, for  $q \in [\gamma, (1 - F(\gamma)) R_A + \gamma]$ ,

$$F^*(q) = \frac{1}{R_B} (q - \gamma + F(\alpha) R_B) \frac{1}{R_A} [q - \gamma + F(\gamma) R_A],$$

$$dF^*(q) = \frac{1}{R_A R_B} [2(q - \gamma) + F(\alpha) R_B + F(\gamma) R_A] dq = \\ \frac{1}{R_A R_B} [2(q - \gamma + F(\gamma) R_A) - (R_A - R_B)] dq$$

and the expression for the designer's income follows. Concerning the probability that either agent wins the contest, agent  $A$  of type  $\theta \in [0, \alpha)$  wins with probability  $F(\theta)$  whereas, if his type is  $\theta \in [\alpha, 1]$ , he wins with probability

$$\Pr_{\theta_B}(F(\theta_B)R_A + \gamma - F(\gamma)R_A \leq F(\theta)R_B + \gamma - F(\alpha)R_B) = \frac{1}{R_A} [F(\theta)R_B - F(\alpha)R_B + F(\gamma)R_A] = \frac{1}{R_A} [F(\theta)R_B + R_A - R_B].$$

Therefore, the probability that  $A$  wins the contest is

$$\begin{aligned} \int_0^\alpha F(\theta)F'(\theta)d\theta + \int_\alpha^1 \frac{1}{R_A} [F(\theta)R_B + R_A - R_B] F'(\theta)d\theta = \\ \frac{1}{2} [F(\theta)^2]_0^\alpha + \frac{R_B}{R_A} \frac{1}{2} [F(\theta)^2]_\alpha^1 + \frac{(R_A - R_B)}{R_A} [F(\theta)]_\alpha^1 = \\ 1 - \frac{1}{2} \frac{R_B}{R_A} + \frac{1}{2} \left(1 - \frac{R_B}{R_A}\right) F(\alpha)^2 - \left(1 - \frac{R_B}{R_A}\right) F(\alpha) = \frac{1}{2} + \frac{1}{2} [1 - F(\alpha)]^2 \left(1 - \frac{R_B}{R_A}\right) \end{aligned}$$

and the designer's expected cost is

$$\begin{aligned} R_A \left[ \frac{1}{2} + \frac{1}{2} \left(1 - \frac{R_B}{R_A}\right) (1 - F(\alpha))^2 \right] + R_B \left[ \frac{1}{2} - \frac{1}{2} \left(1 - \frac{R_B}{R_A}\right) (1 - F(\alpha))^2 \right] = \\ \frac{1}{2} \left[ R_A + R_B + (1 - F(\alpha))^2 (R_A - R_B) \left(1 - \frac{R_B}{R_A}\right) \right], \end{aligned}$$

which corresponds to the last term of  $U(R_A, R_B)$  in part (b) of the lemma.

(c) If agents follow the strategy profile  $(q_A^{I_\gamma^C}, q_B^{I_\gamma^C})$ , the space of qualities that is reached is  $[\gamma, 1]$ . For  $q \in [\gamma, \beta)$ ,

$$F^*(q) = \frac{1}{R_B} (q - \gamma) \frac{1}{R_A} [q - \gamma + F(\gamma)R_A],$$

$$dF^*(q) = \frac{1}{R_A R_B} [2(q - \gamma) + F(\gamma)R_A] dq.$$

For  $q \in [\beta, 1]$ ,  $dF^*(q) = 2F(q)F'(q)dq$ . Therefore, the expression for the designer's income follows. We compute now the probability that agent  $A$  wins the contest. If his type is  $\theta \in [0, \beta)$ , he wins with probability

$$\Pr_{\theta_B}(F(\theta_B)R_A + \gamma - F(\gamma)R_A \leq F(\theta)R_B + \gamma) = \frac{1}{R_A} [F(\theta)R_B + F(\gamma)R_A].$$

Moreover, agent  $A$  with type  $\theta \in [\beta, 1]$  wins with probability  $F(\theta)$ . Therefore, the proba-

bility that  $A$  wins the contest is

$$\begin{aligned} \int_0^\beta \frac{1}{R_A} [F(\theta) R_B + F(\gamma) R_A] F'(\theta) d\theta + \int_\beta^1 F(\theta) F'(\theta) d\theta = \\ \frac{R_B}{R_A} \frac{1}{2} [F(\theta)^2]_0^\beta + F(\gamma) [F(\theta)]_0^\beta + \frac{1}{2} [F(\theta)^2]_\beta^1 = \frac{R_B}{R_A} \frac{1}{2} F(\beta)^2 + F(\gamma) F(\beta) + \frac{1}{2} - \frac{1}{2} F(\beta)^2 = \\ \frac{1}{2} + \frac{1}{2R_A} [F(\beta) R_B + 2F(\gamma) R_A - F(\beta) R_A] F(\beta) = \frac{1}{2} + \frac{1}{2} F(\gamma) F(\beta) \end{aligned}$$

(where the last equality is derived from the two equations that define  $\gamma$  and  $\beta$ ) and the designer's expected costs are

$$\frac{1}{2} [R_A + R_B + F(\gamma) F(\beta) (R_A - R_B)],$$

which corresponds to the expression for the cost in part (c) of the lemma. ■

**Proof of Proposition 3.** (a) Consider a marginal change from a nondiscriminatory contest where agents play  $(q_A^I, q_B^I)$  in equilibrium to a discriminatory contest where the new equilibrium is  $(q_A^{I_\gamma}, q_B^{I_\gamma})$ . To evaluate the optimality of such a change, we take the partial derivatives of the designer's payoff function  $U(R_A, R_B)$  obtained in part (a) of Lemma 3 with respect to  $R_A$  and  $R_B$ .

$$\begin{aligned} \frac{\partial U}{\partial R_A}(R_A, R_B) = & -\frac{1}{R_A^2 R_B} \int_\gamma^{R_B+\gamma} I(q) [2(q - \gamma) + R_A - R_B] dq + \\ & \frac{1}{R_A R_B} \int_\gamma^{R_B+\gamma} I(q) \left[ -2 \frac{\partial \gamma}{\partial R_A} + 1 \right] dq + \frac{1}{R_A R_B} I(R_B + \gamma) [R_A + R_B] \frac{\partial \gamma}{\partial R_A} - \\ & \frac{1}{R_A R_B} I(\gamma) [R_A - R_B] \frac{\partial \gamma}{\partial R_A} - 1 + \frac{1}{2} \frac{R_B^2}{R_A^2}. \end{aligned}$$

In particular, when  $R_A = R_B = R$ , then  $\gamma = 0$  and

$$\frac{\partial U}{\partial R_A}(R_A = R, R_B = R) = -\frac{2}{R^3} \int_0^R I(q) q dq + \frac{1}{R^2} \left[ 1 - 2 \frac{\partial \gamma}{\partial R_A} \right] \int_0^R I(q) dq + \frac{2}{R^2} I(R) R \frac{\partial \gamma}{\partial R_A} - \frac{1}{2}.$$

Similarly,

$$\begin{aligned} \frac{\partial U}{\partial R_B}(R_A, R_B) = & -\frac{1}{R_A R_B^2} \int_\gamma^{R_B+\gamma} I(q) [2(q - \gamma) + R_A - R_B] dq + \\ & \frac{1}{R_A R_B} \int_\gamma^{R_B+\gamma} I(q) \left[ -2 \frac{\partial \gamma}{\partial R_B} - 1 \right] dq + \frac{1}{R_A R_B} I(R_B + \gamma) [R_A + R_B] \left[ 1 + \frac{\partial \gamma}{\partial R_B} \right] - \\ & \frac{1}{R_A R_B} I(\gamma) [R_A - R_B] \frac{\partial \gamma}{\partial R_B} + \frac{1}{2} - \frac{R_B}{R_A}. \end{aligned}$$



Therefore,

$$\begin{aligned} \frac{\partial U}{\partial R_B}(R_A = R, R_B = R) = & -\frac{2}{R^3} \int_0^R I(q) q dq + \frac{1}{R^2} \left[ -2 \frac{\partial \gamma}{\partial R_B} - 1 \right] \int_0^R I(q) dq + \\ & \frac{2}{R^2} I(R) R \left[ 1 + \frac{\partial \gamma}{\partial R_B} \right] - \frac{1}{2}. \end{aligned}$$

Consider now a nondiscriminatory contest  $R$ . If we marginally increase  $R_A$  and simultaneously marginally decrease  $R_B$ , then the total effect is

$$\begin{aligned} \left[ \frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B} \right] (R_A = R, R_B = R) = & \frac{2}{R^2} \left[ 1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} \right] \int_0^R I(q) dq - \\ & \frac{2}{R^2} I(R) R \left[ 1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} \right] = \frac{2}{R^2} \left[ 1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} \right] \left[ \int_0^R I(q) dq - I(R) R \right]. \end{aligned}$$

The integral  $\int_0^R I(q) dq - I(R) R < 0$  because  $I(q)$  is an increasing function. Therefore,  $\left[ \frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B} \right] (R_A = R, R_B = R) > 0$ , that is, discriminating marginally increases the designer's payoff if and only if  $1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} < 0$ . From  $\gamma = F^{-1} \left( 1 - \frac{R_B}{R_A} \right)$  we have

$$\frac{\partial \gamma}{\partial R_A}(R_A, R_B) = \frac{R_B}{R_A^2} \frac{1}{F'(\gamma)} \text{ and } \frac{\partial \gamma}{\partial R_B}(R_A, R_B) = -\frac{1}{R_A} \frac{1}{F'(\gamma)}.$$

When we evaluate these derivatives at  $R_A = R_B = R$ , we obtain

$$1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} = 1 - \frac{1}{R} \frac{1}{F'(0)} - \frac{1}{R} \frac{1}{F'(0)} = 1 - \frac{2}{R F'(0)}$$

and the result follows.

(b) We proceed as in part (a).

$$\begin{aligned} \frac{\partial U}{\partial R_A}(R_A, R_B) = & 2I(\alpha)F(\alpha)F'(\alpha) \frac{\partial \alpha}{\partial R_A} + \int_{\alpha}^{\gamma} I(q)F'(\alpha)F'(q) \frac{\partial \alpha}{\partial R_A} dq + \\ & I(\gamma)F(\alpha)F'(\gamma) \frac{\partial \gamma}{\partial R_A} - I(\alpha)F(\alpha)F'(\alpha) \frac{\partial \alpha}{\partial R_A} - \\ & \frac{1}{R_A^2 R_B} \int_{\gamma}^{[1-F(\gamma)]R_A + \gamma} I(q) [2(q - \gamma + F(\gamma) R_A) - (R_A - R_B)] dq + \\ & \frac{1}{R_A R_B} \int_{\gamma}^{[1-F(\gamma)]R_A + \gamma} I(q) \left[ 2(-1 + F'(\gamma) R_A) \frac{\partial \gamma}{\partial R_A} + [2F(\gamma) - 1] \right] dq + \\ & \frac{1}{R_A R_B} I([1 - F(\gamma)] R_A + \gamma) (R_A + R_B) \left[ 1 - F(\gamma) + (1 - F'(\gamma) R_A) \frac{\partial \gamma}{\partial R_A} \right] - \\ & \frac{1}{R_A R_B} I(\gamma) (2F(\gamma) R_A - (R_A - R_B)) \frac{\partial \gamma}{\partial R_A} - \\ & \frac{1}{2} \left[ 1 + \left( 1 - \frac{R_B^2}{R_A^2} \right) (1 - F(\alpha))^2 - 2 \left( R_A - 2R_B + \frac{R_B^2}{R_A} \right) (1 - F(\alpha)) F'(\alpha) \frac{\partial \alpha}{\partial R_A} \right]. \end{aligned}$$

When  $R_A = R_B = R$ , then  $\gamma = \alpha$  and  $\alpha$  satisfies  $F'(\alpha)R = 1$ . Therefore,

$$\begin{aligned} \frac{\partial U}{\partial R_A}(R_A = R, R_B = R) &= \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \alpha}{\partial R_A} - \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \gamma}{\partial R_A} - \\ &\quad \frac{2}{R^3} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q) [q - \alpha + F(\alpha)R] dq + \frac{1}{R^2} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q) [2F(\alpha) - 1] dq + \\ &\quad \frac{2}{R}I\left(\left[1 - F(\widehat{\theta})\right]R + \alpha\right) [1 - F(\alpha)] - \frac{1}{2}. \end{aligned}$$

The derivative of the designer's payoff with respect to  $R_B$  is

$$\begin{aligned} \frac{\partial U}{\partial R_B}(R_A, R_B) &= 2I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_B} + \int_{\alpha}^{\gamma} I(q)F'(\alpha)F'(q)\frac{\partial \alpha}{\partial R_B} dq + \\ &\quad I(\gamma)F(\alpha)F'(\gamma)\frac{\partial \gamma}{\partial R_B} - I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_B} - \\ &\quad \frac{1}{R_A R_B^2} \int_{\gamma}^{[1-F(\gamma)]R_A+\gamma} I(q) [2(q - \gamma + F(\gamma)R_A) - (R_A - R_B)] dq + \\ &\quad \frac{1}{R_A R_B} \int_{\gamma}^{[1-F(\gamma)]R_A+\gamma} I(q) \left[ 2(-1 + F'(\gamma)R_A)\frac{\partial \gamma}{\partial R_B} + 1 \right] dq + \\ &\quad \frac{1}{R_A R_B} (1 - F'(\gamma)R_A) \frac{\partial \gamma}{\partial R_B} I([1 - F(\gamma)]R_A + \gamma)(R_A + R_B) - \\ &\quad \frac{1}{R_A R_B} \frac{\partial \gamma}{\partial R_B} I(\gamma)(2F(\gamma)R_A - (R_A - R_B)) - \\ &\quad \frac{1}{2} \left[ 1 + \left( -2 + 2\frac{R_B}{R_A} \right) (1 - F(\alpha))^2 - 2 \left( R_A - 2R_B + \frac{R_B^2}{R_A} \right) (1 - F(\alpha)) F'(\alpha) \frac{\partial \alpha}{\partial R_B} \right], \end{aligned}$$

which implies

$$\begin{aligned} \frac{\partial U}{\partial R_B}(R_A = R, R_B = R) &= \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \alpha}{\partial R_B} - \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \gamma}{\partial R_B} - \\ &\quad \frac{2}{R^3} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q) (q - \alpha + F(\alpha)R) dq + \frac{1}{R^2} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q) dq - \frac{1}{2}. \end{aligned}$$

A marginal increase in  $R_A$  and a simultaneous marginal decrease in  $R_B$  from a nondiscriminatory contest  $R$  lead to

$$\begin{aligned} \left[ \frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B} \right](R_A = R, R_B = R) &= \frac{1}{R}I(\alpha)F(\alpha) \left[ \frac{\partial \alpha}{\partial R_A} - \frac{\partial \gamma}{\partial R_A} - \frac{\partial \alpha}{\partial R_B} + \frac{\partial \gamma}{\partial R_B} \right] - \\ &\quad \frac{2}{R^2} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q) [1 - F(\alpha)] dq + \frac{2}{R}I([1 - F(\alpha)]R + \alpha) [1 - F(\alpha)]. \end{aligned}$$

To compute the partial derivatives of  $\alpha$  and  $\gamma$ , we use equations (12) and (13) that implicitly define these variables. Then,

$$\begin{pmatrix} -F'(\alpha) R_B & F'(\gamma) R_A \\ F'(\alpha) R_A - 1 & -F'(\gamma) R_A + 1 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\gamma \end{pmatrix} = \begin{pmatrix} 1 - F(\gamma) & -1 + F(\alpha) \\ F(\gamma) - F(\alpha) & 0 \end{pmatrix} \begin{pmatrix} dR_A \\ dR_B \end{pmatrix}$$

from which,

$$\begin{pmatrix} d\alpha \\ d\gamma \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} -F'(\gamma) R_A + 1 & -F'(\gamma) R_A \\ -F'(\alpha) R_A + 1 & -F'(\alpha) R_B \end{pmatrix} \begin{pmatrix} 1 - F(\gamma) & -1 + F(\alpha) \\ F(\gamma) - F(\alpha) & 0 \end{pmatrix} \begin{pmatrix} dR_A \\ dR_B \end{pmatrix}$$

where

$$\Delta = F'(\alpha) R_B [F'(\gamma) R_A - 1] + F'(\gamma) R_A [1 - F'(\alpha) R_A].$$

We notice that  $\Delta > 0$  because  $\alpha$  is in the decreasing part, while  $\gamma$  is in the increasing part, of  $F(\theta) R_A - \theta$ , that is,  $F'(\alpha) R_A - 1 < 0$  and  $F'(\gamma) R_A - 1 > 0$ . Therefore,  $\frac{\partial \alpha}{\partial R_A} - \frac{\partial \gamma}{\partial R_A} - \frac{\partial \alpha}{\partial R_B} + \frac{\partial \gamma}{\partial R_B} = \frac{\Omega}{\Delta}$ , where

$$\begin{aligned} \Omega &= [-F'(\gamma) R_A + 1] [1 - F(\gamma)] - F'(\gamma) R_A [F(\gamma) - F(\widehat{\theta}_A)] - \\ &\quad [-F'(\alpha) R_A + 1] [1 - F(\gamma)] + F'(\alpha) R_B [F(\gamma) - F(\alpha)] - \\ &\quad [-F'(\gamma) R_A + 1] [-1 + F(\alpha)] + [-F'(\alpha) R_A + 1] [-1 + F(\alpha)] = \\ &\quad -F'(\gamma) R_A + F'(\gamma) F(\alpha) R_A + F'(\alpha) [1 - F(\gamma)] R_A + \\ &\quad F'(\alpha) [F(\gamma) - F(\alpha)] R_B + F'(\gamma) [-1 + F(\alpha)] R_A - F'(\alpha) [-1 + F(\alpha)] R_A = \\ &\quad -2F'(\gamma) [1 - F(\alpha)] R_A + F'(\alpha) [2R_A - F(\gamma) R_A + F(\gamma) R_B - F(\alpha) R_B - F(\alpha) R_A]. \end{aligned}$$

Both  $\Delta$  and  $\Omega$  depend on  $(R_A, R_B)$  and we need to compute  $\frac{\Omega}{\Delta}$  at  $(R_A = R, R_B = R)$ . We note that  $\Omega(R_A = R, R_B = R) = 0$  and  $\Delta(R_A = R, R_B = R) = 0$ . We use that  $\lim_{R_B \rightarrow R_A} \frac{\Omega}{\Delta}(R_A, R_B) = \frac{\lim_{R_B \rightarrow R_A} \frac{\partial \Omega}{\partial R_B}}{\lim_{R_B \rightarrow R_A} \frac{\partial \Delta}{\partial R_B}}(R_A, R_B)$ .

$$\begin{aligned} \frac{\partial \Omega}{\partial R_B} &= -2F''(\gamma) [1 - F(\alpha)] R_A \frac{\partial \gamma}{\partial R_B} + 2F'(\gamma) F'(\alpha) R_A \frac{\partial \alpha}{\partial R_B} + \\ &\quad F''(\alpha) [2R_A - F(\gamma) R_A + F(\gamma) R_B - F(\alpha) R_B - F(\alpha) R_A] \frac{\partial \alpha}{\partial R_B} + \\ &\quad F'(\alpha) \left[ F(\gamma) - F'(\gamma) (R_A - R_B) \frac{\partial \gamma}{\partial R_B} - F(\alpha) - F'(\alpha) (R_B + R_A) \frac{\partial \alpha}{\partial R_B} \right], \end{aligned}$$

which, taking into account that  $\gamma = \alpha$  and  $F'(\alpha) = \frac{1}{R}$  when  $R_A = R_B = R$ , implies

$$\frac{\partial \Omega}{\partial R_B}(R_A = R, R_B = R) = 2F''(\alpha)[1 - F(\alpha)]R \left[ \frac{\partial \alpha}{\partial R_B} - \frac{\partial \gamma}{\partial R_B} \right].$$

Similarly,

$$\begin{aligned} \frac{\partial \Delta}{\partial R_B} &= F'(\alpha)[F'(\gamma)R_A - 1] + F''(\alpha)R_B[F'(\gamma)R_A - 1]\frac{\partial \alpha}{\partial R_B} + \\ &F'(\alpha)R_B F''(\gamma)R_A \frac{\partial \gamma}{\partial R_B} + F''(\gamma)R_A[1 - F'(\alpha)R_A]\frac{\partial \gamma}{\partial R_B} - F'(\gamma)R_A F''(\alpha)R_A \frac{\partial \alpha}{\partial R_B}, \end{aligned}$$

hence,

$$\frac{\partial \Delta}{\partial R_B}(R_A = R, R_B = R) = F''(\alpha)R \left[ \frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B} \right].$$

We notice that  $\gamma > \alpha$  as soon as  $R_A > R_B$ , which implies that  $\frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B} > 0$  at  $R_A = R_B = R$ . Therefore,

$$\frac{\Omega}{\Delta} = \frac{2F''(\alpha)R[1 - F(\alpha)] \left[ \frac{\partial \alpha}{\partial R_B} - \frac{\partial \gamma}{\partial R_B} \right]}{F''(\alpha)R \left[ \frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B} \right]} = -2[1 - F(\alpha)].$$

We substitute  $\frac{\partial \alpha}{\partial R_A} - \frac{\partial \gamma}{\partial R_A} - \frac{\partial \alpha}{\partial R_B} + \frac{\partial \gamma}{\partial R_B}$  in the derivative  $\left[ \frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B} \right](R_A = R, R_B = R)$  to obtain

$$\begin{aligned} \left[ \frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B} \right](R_A = R, R_B = R) &= -2\frac{1}{R}I(\alpha)F(\alpha)[1 - F(\alpha)] - \\ &\frac{2}{R^2} \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q)[1 - F(\alpha)]dq + \frac{2}{R}[1 - F(\alpha)]I([1 - F(\alpha)]R + \alpha) = \\ &\frac{2}{R^2}[1 - F(\alpha)] \left[ I([1 - F(\alpha)]R + \alpha)R - I(\alpha)F(\alpha)R - \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q)dq \right] > \\ &\frac{2}{R^2}[1 - F(\alpha)] \left[ (I([1 - F(\alpha)]R + \alpha) - I(\alpha))R - \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q)dq \right] > \\ &\frac{2}{R^2}[1 - F(\alpha)] \left[ (I([1 - F(\alpha)]R + \alpha) - I(\alpha))[1 - F(\alpha)]R - \int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q)dq \right] > 0 \end{aligned}$$

given that  $I(q)$  is increasing. Therefore, discriminating marginally always increases the designer's payoff.

(c) As in the previous cases, the marginal change in costs due to marginal discrimination is zero. Therefore, it is enough to examine the change in revenues. Rather than

proceeding directly through the designer's revenue function, we examine the effect on qualities of a marginal shift from  $(q_A^{IC}, q_B^{IC})$  to  $(q_A^{I\gamma C}, q_B^{I\gamma C})$ , which leads at  $\varepsilon = 0$  to the following derivatives:

$$\begin{aligned}\frac{dq_A^{I\gamma C}(\theta)}{d\varepsilon} &= -F(\theta) + \frac{\partial\gamma}{\partial\varepsilon} \text{ for all } \theta \in [0, \beta] \\ \frac{dq_B^{I\gamma C}(\theta)}{d\varepsilon} &= F(\theta) + \frac{\partial\gamma}{\partial\varepsilon} - F'(\gamma)R \frac{\partial\gamma}{\partial\varepsilon} \text{ for all } \theta \in [0, \beta].\end{aligned}$$

Differentiating equations (14) and (15) with respect to  $\varepsilon$ , at  $\varepsilon = 0$ , yields

$$\begin{pmatrix} 1 & -1 + F'(\beta)R \\ 1 - RF'(\beta) & -1 + F'(\beta)R \end{pmatrix} \begin{pmatrix} d\gamma \\ d\beta \end{pmatrix} = \begin{pmatrix} F(\beta) \\ 0 \end{pmatrix} d\varepsilon$$

hence,

$$\frac{d\gamma}{d\varepsilon} = \frac{F(\beta)}{F'(\beta)R} > 0.$$

Summing up the effect of  $\varepsilon$  on both qualities, we obtain

$$\frac{dq_A^{I\gamma C}(\theta)}{d\varepsilon} + \frac{dq_B^{I\gamma C}(\theta)}{d\varepsilon} = [2 - F'(\beta)R] \frac{\partial\gamma}{\partial\varepsilon} \text{ for all } \theta \in [0, \beta]$$

whereas the effect is null for  $\theta \in [\beta, 1]$ , and the result follows. ■

**Proof of Proposition 4.** By Proposition 3 (b), discriminating is optimal if the equilibrium strategy profile in the nondiscriminatory contest is  $(q_A^{CI}, q_B^{CI})$  and if marginal changes in  $(R_A, R_B)$  lead to  $(q_A^{CI\gamma}, q_B^{CI\gamma})$ . Given that  $F(\theta)$  is convex and  $F'(0) = 0$ , the equilibrium profile is  $(q_A^{CI}, q_B^{CI})$  if the optimal  $R$  satisfies  $R > \frac{1}{F'(1)}$ . To show that this is the case if  $v$  is large enough, we compare the profits that the designer obtains by choosing an  $R > \frac{1}{F'(1)}$  with those obtained for  $R = 0$  ( $R = 0$  is the optimal choice among all the rewards that lead to the equilibrium profile of  $(q_A^C, q_B^C)$ ). This difference in profits is equal to  $vh(R) - R$ , where  $h(R)$  is the difference in the designer's expected income when she derives benefits  $i(q)$  from an innovation, between the case where the contestants adopt the strategy profile  $(q_A^{CI\gamma}, q_B^{CI\gamma})$  (for  $R > \frac{1}{F'(1)}$ ) and the case where the contestants adopt the strategy  $(q_A^C, q_B^C)$ . It is immediate that  $h(R) > 0$  if  $i(q)$  is increasing in  $q$ . Therefore,  $vh(R) - R > 0$  if  $v$  is large enough and the optimal contest necessarily implies  $R > \frac{1}{F'(1)}$ .

Finally, the proposition is proved if we show that marginal changes from a nondiscriminatory contest  $R_A = R_B = R > \frac{1}{F'(1)}$  lead to  $(q_A^{CI\gamma}, q_B^{CI\gamma})$ . According to Theorem 2, this property certainly holds if  $R_A F(R_A - R_B) - (R_A - R_B) < 0$ . Taking  $R_A = R + \varepsilon$  and  $R_B = R - \varepsilon$ , the inequality is equivalent to  $(R + \varepsilon) F(2\varepsilon) - 2\varepsilon < 0$ . The inequality holds for  $\varepsilon$  small enough because  $F'(0) = 0$ . ■

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