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Risk-Sharing and Contagion in Networks

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Risk-Sharing and Contagion in Networks

Abstract

We investigate the trade-off between the risk-sharing gains enjoyed by more interconnected firms and the costs resulting from an increased risk exposure. We find that when the shock distribution displays "fat" tails, extreme segmentation into small components is optimal, while minimal segmentation and high density of connections are optimal when the distribution exhibits "thin" tails. For less regular distributions, intermediate degrees of segmentation and sparser connections are optimal. Also, if firms are heterogeneous, optimality requires perfect assortativity in a component. In general, however, a conflict arises between efficiency and pairwise stability, due to a "size externality" not internalized by firms.

JEL-Code: D850, C720, G210.

Keywords: firm networks, contagion, risk sharing.

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1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation, abstracting from their linkages, gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. In the words of Acharya *et al.* (2010) "current financial regulations, such as Basel I and Basel II, are designed to limit each institution's risk seen in isolation; they are not sufficiently focused on systemic risk even though systemic risk is often the rationale provided for such regulation." The aim of this paper is precisely to investigate how the capacity of the system to absorb shocks depends on the pattern of interconnections established among firms, say banks.

More specifically, we intend to study the extent to which the risk-sharing benefits to firms of becoming more highly interconnected (which provides some insurance against relatively small shocks, absorbable within the whole system) may be offset by the large-scale costs resulting from an increased risk exposure (which, for large shocks, could entail a large wave of induced bankruptcies). That is, we want to analyze the trade-off between risk-sharing and contagion. Clearly, this trade-off must be at the center of any regulatory efforts of the financial world that takes a truly systemic view of the problem. This paper highlights some of the considerations that should play a key role in this endeavor. In particular, by formulating the problem in a stylized and analytically tractable framework, it examines how the segmentation of the system into separate components, the density of the connections within each component as well as asymmetries in the pattern of connections should be tailored to the underlying shock structure. It also sheds light on the key issue of whether the normative prescriptions on the optimal pattern of linkages are consistent with the individual incentives to form or remove links.

We analyze a model in which there is a network consisting of N nodes, each of them interpreted as a firm. For simplicity, in most of the paper we shall consider the case where all firms are *ex ante* identical and are endowed with the same level of assets and liabilities. But *ex post* they will be different since we assume that, with some probability, a shock hits a randomly selected firm. The first *direct* effect of such a shock is to decrease the income generated by the firm's assets, thus possibly leading to the default of the firm if its resulting income falls short of its liabilities. But if this firm has links to other firms, the latter will also be affected. To be specific, let us think of the presence of a link between two firms as reflecting an exchange of the assets they are endowed with. The exchanges of assets allows firms to diversify risks, but also exposes them to the shocks that arrive to counter-parties. For example, think of the securitization of mortgage loans, which are then sold to investors different from the originator. Those investors can, in turn, sell claims to their own portfolios and thus generate further diversification for themselves, but this also makes the buyers of such claims exposed, in an indirect way, to the returns of the original mortgage loans. In the end, the overall network of connections generates patterns of mutual exposure between any pair of directly or indirectly connected firms. Consequently, when a shock hits a firm,

any other firm connected with it becomes affected in proportion to its exposure to that firm and has to default when its share of the shock exceeds the value of the firm's assets, net of its liabilities.

In this way, the exposure to losses in commonly held assets induces a sort of contagion, the process of asset exchange in our model being analogous to the transmission of pathogens in disease contagion. An asset that turns "bad" (think of a loan to a developer that is unpaid) exposes all common owners of that asset to a shock. So when the shock hits, it is transmitted from the originator bank to others who purchased its mortgage-backed securities, and further down the line to the companies which purchased the securities issued by those firms. To render the contagion analogy precise, we must conceive the "disease" in our case as one that can be transmitted well before symptoms are present (AIDS is a good example of such a disease). It must also have the feature that not all infected firms must die, for in our context what causes the default of one firm need not lead to the default of another. In the end, it is the *overall* 'network' structure that determines how any given shock affects different firms and what is its overall aggregate impact on the whole system.

Clearly, the maximum extent of risk sharing obtains when all firms belong to a single and fully connected network. This configuration, however, exhibits the widest exposure of firms in the system to shocks and a large shock would affect all firms in the system and could lead to extensive default. There are two alternative (and in some cases complementary) ways of reducing such exposure. One is by 'segmentation, which isolates the firms in each component from the shocks that hit any other component.[1] The second route to lowering exposure is by making firms directly connected only to a subset of the other firms in their component, and by having exposure decline in the network distance between firms within the component. To obtain a sharp understanding of the effect of network distance on contagion, our analysis will contrast two cases: (i) completely connected components, where there is a direct link between any pair of firms in each component, so that the mutual exposure between any pair of firms is exactly the same; (ii) ring components, where the reciprocal exposure between firms is heterogeneous, falling with network distance.

The key objective of the paper will be to identify the architecture of the system that best tackles the trade-off between risk sharing and contagion, hence minimizing the expected number of defaults in the system. Our model captures the essence of the problem and allows the study of such a trade-off under a fairly general structure of the random shocks. As this shock structure changes, our analysis will deliver a precise determination of how the *optimal degree of segmentation* as well as *optimal link density* correspondingly adapt.

A first set of our results can be summarized as follows. We find that when the probability distribution of the shocks exhibits "fat tails" (i.e. attributes a high mass to large shocks), the optimal configuration involves a maximum degree of segmentation – that is, components should be of the minimum possible size. This reflects a situation where the priority is to

[1] A component of the network is defined as a maximal set of firms that are directly or indirectly connected.

minimize contagion. Instead, in the opposite case where the probability distribution places high enough mass on relatively small shocks ("thin tails), the best configuration has all firms arranged in a single component. The main aim in this latter case is to achieve the highest level of risk sharing. These two polar cases, however, do not exhaust all possibilities. For we also find that for other, more complex specifications of the shock structure (e.g. mixtures of fat and thin tails) intermediate arrangements are optimal, i.e. the optimal degree of segmentation involves medium-sized components.

It is interesting to note that all of the previous conclusions apply irrespectively of whether one considers structures involving completely connected components or, alternatively, minimally connected ones. As explained, the potential advantage of minimally connected structures is that the exposure between firms in a component is not uniform but decays with network distance, which limits the number of defaults induced when the shocks are large. A natural implication of this feature is that, among minimally connected structures, the optimal degree of segmentation is lower (or, equivalently, the component size is larger) than for completely connected ones. This conclusion notwithstanding, we find that when segmentation in complete components is implemented optimally, it always dominates segmentation in minimally connected structures if the shock distributions are as described in the previous paragraph. It is however important to emphasize that for other, less regular shock distributions a minimally connected structure is optimal (i.e. better than any segmentation in complete components). This happens, for instance, when the shock distribution assigns a high probability mass on a small range of large shocks as well as on a wide range of small and medium size shocks. In those cases, we find that low density is preferred to more segmentation as the mechanism for limiting contagion when the shocks are large.

The paper also addresses the issue of whether the requirements for the optimality of the network structure of the system are compatible with the incentives of individual firms to establish links. Formally, we analyze this issue by examining the *Coalition-Proof Equilibria* (CPE) of a network formation game, where any group (i.e. coalition) of firms can jointly deviate, to avoid the multiplicity issues associated to the coordination problems present in these games. Our main conclusion is that there is typically a conflict between efficiency and individual incentives. This conflict derives from the fact that CPE typically exhibit heterogeneities in component size and these are inconsistent with efficiency. There are, therefore, positive externalities associated to displaying a uniform size for all components that are not internalized at CPE – in general, the equilibria have some firms that lie in a component that is inefficiently small.

The results summarized so far refer to environments where all firms are *ex ante* identical and thus operate at the same scale and face the same shock distribution. The paper, however, also studies the case where firms may be different, either with regard to their size or to their shock distribution. Under these circumstances, our main conclusion is particularly sharp: if we focus on completely connected components, optimality requires perfect assortativity. That is, any optimal configuration must have all firms arranged in

homogeneous components (where within each of them firms are identical both in size and in shock distributions). We also consider the implications of allowing for asymmetric network structures, where the network position of firms within a component matters. In particular, we contrast two situations: the symmetric configuration where components are completely connected (and hence all firms are in a symmetric position), and the case where firms are arranged in star networks (thus there is a large central firm connected to the smaller peripheral ones). We find that the symmetric structure is optimal when the shocks are not too large (because this maximizes risk-sharing possibilities) while the star structure is optimal for bigger shocks. The latter reflects the fact that star networks limit contagion when shocks are large, by restricting overall connectivity and having the central larger nodes act as buffers.

Our model, although stylized, allows for the guidance of concrete policy advice, particularly in light of the robustness of many of our implications found by Loepfe, Cabrales and Sanchez (2013).[2] For example, as already anticipated, we find that one cannot generally expect that social and individual incentives be aligned. But there is more than that, in particular concerning how actual policy should deal with the issue of systemic risk. In view of the increasing trend of financial markets integration that has substantially increased firms' interconnections (see Diebold and Yilmaz 2009), it is now widely judged to be insufficient to rely primarily on the usual approach to the problem, i.e. the regulation of capital requirements.[3]. But even when network characteristics have entered the policy discussion, they have done so by focusing attention on institutions that are "too big to fail"– or in the more enlightened versions, "too central to fail," as in Battiston et al. (2012). Our paper, however, makes it clear that the derivation of an optimal structure goes far beyond understanding which are the most systemic institutions, because even in a model with symmetric firms, the structure of the network plays a very important role for the stability of the system. In particular, the optimal structure crucially depends on the nature of the shocks affecting the firms in the system. A clear recommendation that emerges from our paper is that as the distribution puts more weight on larger shocks, the degree of interconnection should become smaller.

Another important aspect we have highlighted is that when different institutions are hit by shocks coming from different distributions, they should belong to separate components. This result could possibly add a novel justification for the separation of investment and commercial banking which was enforced by the Glass-Steagall act from 1933 to 1999. The traditional view was that the separation is necessary to protect consumers from conflict of interest between these institutions, but Kroszner and Rajan (1994), for example, find little evidence for this argument. In our model, the separation may prove beneficial as a way to

[2]We postpone a summary of the robustness exercise conducted in Lopefe *et al.* (2013) to the concluding Section 6, once the analysis of the present paper is complete.

[3]The Basel III agreements require higher overall equity requirements than previous rounds plus and additional capital conservation buffer of 2.5% during expansions to combat the pro-cyclicality of credit.

cope with different types of shocks hitting alternative types of institutions. It is conceivable that, for this reason, commercial banks should have a much larger component size than investment banks.

We must nevertheless be cautious in interpreting our policy implications. The financial network structures we observe in reality are rather different from the ones we identify as optimal in this paper. They are often polarized, with some firms being far more central and well-connected than others. It would be premature to argue that this is not optimal, since we ignore forces beyond risk-sharing and limiting contagion (intermediation, for instance) that also contribute to determining the form of these networks . We argue, however, that the elements highlighted in this paper are quite relevant and important and should be taken into account by policy makers in shaping their decisions.

1.1 Related literature

We end this introduction with a brief review of the related literature. The research on financial contagion and systemic risk is quite diverse and also fast-growing. Hence we shall provide here only a brief summary of some of the more closely related papers.[4]

Allen and Gale (2000) pioneered the study of the stability of interconnected financial systems. They analyze a model in the Diamond and Dybvig (1983) tradition, where a single, completely connected component is *always* the efficient network structure, i.e. the one that minimizes the extent of default. Our model, in contrast, shows that a richer shock structure can generate a genuine trade-off between risk-sharing and contagion and that in some circumstances a certain degree of segmentation and/or low density may turn out to be efficient.

Freixas *et al.* (2000) consider an environment where a *lower density* of interaction, even though it limits risk sharing, has the positive consequence of reducing the incentives for deposit withdrawal. Concerning *segmentation*, on the other hand, a positive role for it is obtained by Leitner (2005) in a model where, unlike ours, no role is played by the interaction among agents in a component, since risk is always shared completely within each component.

More in line with our approach, Allen et al. (2011) consider a six-firm environment where each firm needs funds for its investment. Since these investments are risky, firms may gain from risk diversification, which is achieved by exchanging shares with other firms. This gives rise to a financial network for which two possibilities are considered: a segmented and an unsegmented structure. The paper then analyzes the different effects in these two structures of the arrival of a signal indicating that *some* firm in the system will have to

[4]The reader is referred to Allen and Babus (2009) for a recent survey on how risk sharing in financial contexts can lead, through contagion, to large systemic effects. There is also a large body of literature that studies the general problem of risk sharing in non financial contexts, largely motivated by its application to consumption sharing in poor economies that lack formal insurance mechanisms. Paradigmatic examples are the papers by Bramoullé and Kranton (2007), Bloch *et al.* (2008), and Ambrus *et al.* (2011).

default. Depending on parameters, either of these structures can dominate the other.

Two more recent, related papers are Elliott et al. (2012) and Acemoglu et al. (2013). In the first one financial contagion is modeled in a similar way to ours, as the consequence of cross-holdings among firms. Such cross-holdings, however, embody cross ownership of equity shares among firms, rather than being the result of the securitization and exchange of assets as here. This, together with the assumption that default brings about some fixed bankruptcy costs, imply that the default of any given firm induces a discontinuous drop in the revenue of all the firms owning its equity and may thus trigger a default chain. No such discontinuities occur in our setup, where contagion arises only from the correlation in returns that is induced by the firms' exchanges of assets (i.e. by their joint *ownership of assets*). Another difference with our paper concerns the main focus of the analysis. While the aim of the aforementioned paper is to characterize conditions on the structure of the network under which default cascades occur, the primary objective of ours is to characterize the optimal financial structures in diverse scenarios and their consistency with individual incentives.

The second paper, Acemoglu *et al* (2013), shares with the present one its concern with the optimality and incentive-compatibility of financial networks but displays important modeling differences in some key respects. In their set-up firms are required to engage in credit-borrowing relationships, and hence contagion is channeled throughout the system in a manner that is different from our model. One of the main results of their paper is that, depending on whether the magnitude of shocks is small or large, the optimal network is either complete or has components almost isolated from the rest. In contrast we characterize the optimal network structure for a non trivial distribution function of the shocks, finding a richer pattern of optimal structures, which may exhibit intermediate degrees of segmentation or different levels of link density. Finally, both their approach and ours find that a strategically stable configuration is not optimal, as agents do *not* internalize the global network-based externalities imposed on others. The reasons, however, are different in the two cases. While in Acemoglu *et al* (2013) the indicated externalities pertain to the enhanced spreading of risk brought about by individual connections, in our case they are induced by the need to meet an overall feasibility constraint.

There is also a complementary line of literature that, in contrast with the papers just mentioned, studies the issue of contagion and systemic risk in the context of large networks (usually, randomly generated). In many of these papers, the approach is numerical, based on large-scale simulations (see e.g. Nier *et al.* (2007)). A notable exception is the recent paper by Blume *et al.* (2011), which integrates the mathematical theory of random networks with the strategic analysis of network formation. Its primary aim is to address the question of what is the socially optimal connectivity of the system (i.e. the average degree of the network) and whether such optimal connectivity is consistent with agents' incentives to create or destroy links. They find that social optimality is attained around the threshold where a large component emerges, but individuals will generally want to connect beyond

this point. As in our paper, this induces a conflict between social and individual optimality, which in their case is due to the fact that agents do not internalize the effect on others of adding new channels (i.e. links) that facilitate the spread of the effects of the shocks.

Finally, we should refer to the large empirical and policy-oriented literature whose main objective is to devise summary measures of the network of inter-firm (mostly banks) relationships that is able to anticipate systemic failures. For example, Battiston *et al.* (2012) propose a measure of centrality (*Debtrank*) that is inspired on the measure of *Pagerank* centrality that Google uses to rank web-pages, while Denbee *et al.* (2011) propose a measure of centrality á la Katz-Bonacich (following Ballester, Calvó-Armengol and Zenou (2006)) and apply it to English data. Of particular interest in this respect is Elsinger, Lehar and Summer (2011) who, using Austrian data, show that correlation in banks' asset portfolios is the main source of systemic risk. A related emphasis on cross-ownership and investment has been pursued as well by the literature on "balance sheet effects" to understand the Asian financial turmoil in the late 90's (Krugman 1999) as well as the current crisis (Ahrend and Goujard 2011). These latter approaches are very much in line with our model, which precisely highlights portfolio correlation as the key driver of default risk.

The rest of the paper is organized as follows. Section 2 presents the model, including the convenient continuum approximation that simplifies the analysis. Section 3 characterizes optimal financial structures for various properties of the shock distribution. Section 4 addresses the network formation problem and studies the tension between strategic stability and optimality. Section 5 considers different extensions to environments with asymmetries, in particular concerning firm size, the distribution of shocks, or the underlying network architecture. Finally, Section 6 concludes with a summary and an outline of pending research issues. For convenience, all formal proofs of our results are relegated to an Appendix.

2 The Model

2.1 The Environment

As an initial benchmark scenario, we consider an environment with N ex ante identical, risk-neutral financial firms and a continuum of small investors. At any given point in time, each firm has an investment opportunity - a project - which requires an initial payment $I > 0$ and yields a random gross return $\widetilde{R}$ at the end of the period. The resources needed to undertake the project are obtained by issuing liabilities (e.g. deposits or bonds) on which a deterministic rate of return must be paid.

The gross return of the project is random, as with some probability q the firm is hit by a negative shock. If no shock hits, the return equals some normal level R. The shock can in turn be small (which we label as s) or large (labeled b), with respective conditional probabilities π_s and π_b. When a small shock hits, the firm experiences a loss of some fixed size L_s, so its gross return is $R - L_s$. When the shock is large, the loss $\widetilde{L}_b$ is a random

variable, with support $[L_s, \infty)$ and cumulative distribution function $\Phi(L_b)$. Summarizing, the gross return on a firm's project is:

$$\widetilde{R} = \begin{cases} R \text{ with prob. } 1-q \\ R - L_s \text{ with prob. } q\,\pi_s \\ R - \widetilde{L}_b \text{ with prob. } q\,\pi_b \end{cases} \tag{1}$$

Since the return on a firm's investment is subject to shocks, while the return promised to its creditors is deterministic, when the firm is hit by a shock it may be unable to meet the required payments on its liabilities, in which case it must *default*. Default entails two types of costs. First, there are the liquidations costs – for simplicity, we assume that these costs leave no resources available to make any payment to creditors at the time of default. In addition, there are additional costs deriving from the fact that a defaulting firm stops operating and hence loses any future earnings possibility. These costs are assumed to be substantial, so that the value of a firm is maximized when its probability of default at any point in time is minimized.

There is a large set of investors, who are the source of the supply of funds to firms. Investors are risk neutral and require an expected gross rate of return equal to r in order to lend their funds in any given period.[5]

Since firms may default, in which case creditors receive a payment equal to zero, the nominal gross rate of return M on the deposits to the firms must be greater or equal than r. Specifically, if we denote by φ_i the probability that any given firm i defaults (an endogenous variable), we must have:

$$M_i = \frac{r}{1-\varphi_i}. \tag{2}$$

For the risk of default to be an issue, we assume:

A1. (i) $R\,(1-q) > r$
(ii) $R - L_s < r$.

The first inequality ensures that a firm's project is viable, that is, its expected return exceeds what must be paid to lenders. On the other hand, the second inequality implies (since $r \leq M$ and $\widetilde{L}_b \geq L_s$) that if a firm can only draw on the revenue generated by its project, it is surely unable to pay depositors (and hence must default) whenever a shock (whether s or b) hits its return.

Since, as stated above, default entails a significant cost for a firm, a firm may benefit from entering *risk sharing arrangements* with other firms and hence diversify risks. Here we consider the case where these arrangements take the form of swaps of assets between firms, that is, exchanges of claims to the yields of the firms' investments, prior to the realization of the uncertainty (similarly to Allen, Babus and Carletti (2011)).

[5] We allow the support of $\widetilde{L}_b$ to go to ∞ purely for technical convenience. The analysis could have been equivalently carried out by truncating the distribution at some upper bound $\bar{L}_b$ smaller than R so as not to violate limited liability, but with R, $\bar{L}_b$, r big enough.

More precisely, let us posit that each firm exchanges a fraction $1-\theta$ of its standing shares, giving rights to the return on its investments, for shares held by other firms. Such an exchange entitles the firm to a fraction of the returns on the other firms' investments. The specific pattern of exchanges among firms is formalized by a network, where a direct linkage between two firms reflects the fact that they undertake a *direct* exchange of their assets. We allow for these asset swaps to occur repeatedly. *Indirect* connections are then also formed, whereby a firm ends up having claims on the returns of projects of firms who swapped assets with the firms it exchanges assets with, and so on.

As a result of this process of asset exchanges, the return on a firm's assets becomes a weighted average of the yield of its own project and the yields of the projects of the firms it exchanged assets with, directly or indirectly. A convenient way of representing a network structure is through a matrix A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{3}$$

where, for each i, j, $(i \neq j)$, $a_{ij} \geq 0$ denotes the fraction of shares in the investment project run by firm j that is owned by firm i. By construction, the following adding-up constraints must then hold:

$$\sum_{i=1}^{N} a_{ij} = 1 \qquad\qquad j = 1, 2...., N \tag{4}$$

In addition, given that, for the moment, all firms are assumed ex-ante symmetric it is natural to focus our attention on the case where the pattern of asset exchanges within each component is also symmetric across firms. This exchange takes place on a one for one basis, as it involves assets of equal expected return. This implies that A is symmetric, i.e.

$$a_{ij} = a_{ji} \quad \text{for all} \quad i, j = 1, 2, ..., N.$$

One of the primary objectives of this paper is to compare different network configurations in terms of social welfare. Our assumptions that default costs are very significant and that all firms are risk neutral and *ex ante* identical imply that social welfare is maximized when the expected number of defaults in the system is minimized.[6] Since investors are risk-neutral and they always earn an expected rate of return r, their welfare is always the same across the different configurations. It can be easily checked that this criterion is equivalent to that

[6] Beale et al. (2011) propose a similar criterion to evaluate and compare different financial systems, based on the minimization of a "systemic cost function" defined as the expectation of a convex function of the number of defaults in the system. Thus, in this case, not only the expected number of defaults matters, but also its variability.

of minimizing the individual probability that any single firm defaults.[7] Thus, from an *ex ante* viewpoint, we have that social and individual objectives are aligned.

We shall assume that all network structures involve the same "externalization of risk" by each firm. That is, each firm, after all rounds of asset exchanges have been completed, holds a claim to the same fraction α of the yield of its own project, and the residual fraction $1-\alpha$ of claims to other firms' projects. Formally, this amount to making $a_{ii} = \alpha$ for each $i = 1, .., N$ and *every* network configuration. Alternative structures, therefore, differ only in terms of how the fraction $1-\alpha$ of the risk that is externalized is distributed among the rest of the firms. This allows a meaningful comparison across different network structures.

The value of α, a parameter of the model, is taken to be at least as large as $1/2$. Such a lower bound on α may be motivated by considerations of moral hazard.[8] Indeed, if in order to operate one's project some costly effort needs to be exerted, a standard way to induce a high enough effort level is to make sure that the firm retains some minimum share of the project's returns.[9] On the other hand, an upper bound on the fraction of claims retained to its own project is explained by the benefits of risk sharing and the fact that, as we will see below, a situation with all firms in isolation is never optimal.

We assume that shocks are rare and thus each period at most one firm is hit by a shock. Given (1), this can be motivated by postulating that, even if shocks hit firms in a stochastically independent manner, the probability q that a shock hits any given firm is so low that the probability that two or more shocks arrive in a single period is of an order of magnitude that can be ignored.[10]

When a shock of size L hits the return on the project run by some firm i, the exposure to it by each one of the firms in the system is given by the vector Ae_iL, where e_i is the i-th unit vector $[0, .., 1, ..0]^\top$. Hence firm i defaults in response to such a shock when

$$\alpha(R-L) + \sum_{j \neq i} a_{ij} R < M, \quad \text{that is,} \quad \alpha L > R - M,$$

[7] To see this, note that the *ex ante* probability of default of an individual firm φ is equal to $\sum_m \rho(m)\varphi(m)$, where $\rho(m)$ stands for the probability that m firms default and $\varphi(m)$ for the conditional probability that any particular firm defaults when there are a total of m defaults in the system. Then, since

$$\varphi(m) = \frac{1}{N} + (1 - \frac{1}{N})\frac{1}{N-1} + \cdots + (1 - \frac{1}{N} - \cdots - \frac{1}{N})\frac{1}{N-m+1} = \frac{m}{N}$$

we obtain that the expected number of defaults in the system is $\sum_m \rho(m)\, m = \varphi N$, i.e. is proportional to the individual default probability.

[8] This requirement is analogous, for example, to a well-known provision in the recent Dodd-Frank act passed in the USA to strengthen the regulation of the financial system. By virtue of this new legislation, under certain circumstances "a securitizer is required to retain not less than 5 percent of the credit risk..." (see http://www.sec.gov/about/laws/wallstreetreform-cpa.pdf).

[9] In particular, the lower bound of $1/2$ ensures that a firm always retains a larger share of claims on its project than any other firm.

[10] Or, as an extreme formalization of this idea, we could model time continuously and assume that the arrivals of small and big shocks to each firm are governed by independent Poisson processes with fixed rates π_s and π_b, respectively.

while any firm $k \neq i$ defaults whenever

$$\left(\alpha + \sum_{j \neq i,k} a_{kj}\right) R + a_{ki}(R - L) < M.$$

We readily see from the above expressions that when a firm exchanges its assets with other firms, the firm reduces its exposure to the shocks hitting its own project but at the same time becomes exposed to the shocks affecting the projects of those firms with which the firm in question is directly or indirectly connected.

On the nature of those shocks, we make the following key assumption:

A2. (i) $\pi_s > N\pi_b$.
(ii) $\alpha(R - L_s) + (1 - \alpha)R \geq \frac{r}{1 - Nq\pi_b}$.

Part (i) of the above assumption says that s (small) shocks are significantly more likely than b (big) ones. In particular, it is more likely that a given firm is hit directly by a s shock than a b shock hits any of the firms. Part (ii) then ensures that, by exchanging a fraction $1 - \alpha$ of its shares with shares of other firms, no firm defaults when a s shock hits any firm in the environment. The term on the right-hand side of A2(ii) constitutes an upper bound on the gross rate of return M on deposits when no default occurs with an s shock, since the probability of default φ of a firm is no larger than $Nq\pi_b$, which is the probability that a b shock hits anywhere in the system. On the other hand, the term on the left-hand side of the inequality in A2(ii) constitutes the gross return of a firm that swapped a fraction $1 - \alpha$ of its shares in the event where the firm is hit by a s shock. Since $\alpha \geq 1/2$ this term is also a lower bound on the firm's revenue when any other firm in the system is hit by a s shock.

The inequality in A2(ii), therefore, implies that the asset exchange – i.e. the establishment of links – allows firms to fully insure against the s shocks. But, on the other hand, the asset exchange also exposes a firm to the risk of contagion when a b shock hits any of the firms directly or indirectly linked to the firm. Recall, however, that Assumptions A1(ii) and A2(i) imply that a firm in isolation always defaults when hit by a s shock and that s shocks are much more likely than b shocks. This implies that the probability of default of a firm is always lower when it exchanges assets with other firms than when it is in autarky, and therefore guarantees that every firm will always want to display *some* links to others.

While *any* pattern of asset exchanges consistent with our assumptions allows firms to attain full insurance against s shocks, the default performance in the event of b shocks will typically not be the same across different financial network structures. In general, that is, these structures will be markedly different in terms of the extent of contagion they induce when big shocks hit the system. Indeed, to understand which configurations minimize those detrimental side effects of risk-sharing is one of the primary aims of this paper.

Remark 1 *We should point out that in the environment considered here the mutual exposure between firms comes from the cross-ownership of their shares. Hence the default of one firm has*

no direct implication for the solvency of other firms, and the possibility of contagion of a large shock hitting a firm only comes from the correlation of their portfolio returns. An alternative mechanism to generate exposure between firms is via chains of lending relationships in the form of credit lines that are accessed when a shock hits a firm (subject to appropriate constraints requiring firms to enter simultaneous borrowing and lending relationships). This is the modeling approach followed by Acemoglu et al (2013).

2.2 Financial Structures

We shall consider financial structures that differ along two dimensions: segmentation and network density. For the moment, we shall also restrict our analysis to symmetric configurations – which means, in particular, that all components of the network are of identical size and have the same network form. This, however, implies no loss of generality for our normative analysis since, as we will show, the welfare maximizing configurations are symmetric. In contrast, in Section 4, where the implications of strategic stability are examined, we will allow for asymmetries in component sizes.

By *segmentation* we mean the partition of the N firms into disjoint components. Each component is formed by firms that are either directly or indirectly linked by the exchange of assets (and hence the cross-ownership of shares), while there is no exchange across components. The measure of the segmentation is given by the number C of equal-sized components in which the set of firms is divided, or equivalently by the number $K \equiv \frac{N}{C} - 1$ of other firms to whom every firm is linked.[11] In terms of the matrix A, this amounts to having a block diagonal structure with C blocks along the diagonal. The larger the segmentation, the lower the number K of firms indirectly affected by a given shock but, *ceteris paribus*, the larger their mutual exposure and hence the probability of default if a b shock hits them. At the two extremes of segmentation, we have the case $K = N - 1$, where all firms are connected (directly or indirectly), and $K = 1$, where each firm only engages in trade with a single other firm. We shall allow for all possible values of K between 1 and $N - 1$ and explore their welfare implications.

By *network density* we refer to the presence or absence of indirect linkages within each component. As we said in the previous section, a firm is directly linked to another one if the two are involved in a direct exchange of assets. Instead, two firms are only indirectly linked if they are connected by a multiple-link path in the financial network. In this case, they will end up holding claims to each other's projects once there has been a repeated exchange of assets (repeated rounds of securitization). For simplicity our analysis will just compare the case where all linkages are direct with the case of a ring in which firms connect directly with a few firm.

[11] We ignore for now all possible integer problems regarding these magnitudes.

Complete Structures This corresponds to the situation where all firms in each component of size $K+1$ are directly linked among them – so, in the language of networks, all components are completely connected. In this case, the exchange of assets among firms within the component is captured by a square matrix A_K of the form:

$$A_K = \begin{pmatrix} \alpha & (1-\alpha)/K & \cdots & (1-\alpha)/K \\ (1-\alpha)/K & \alpha & \cdots & (1-\alpha)/K \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)/K & (1-\alpha)/K & \cdots & \alpha \end{pmatrix}, \tag{5}$$

where, as before, α is the fraction of its own assets a firm keeps for itself. In these structures each firm is equally exposed to the shocks hitting any other firm in the component.

Ring Structures In this case the pattern of exposure between firms in a component is not constant. Rather, it falls with the network distance among any two firms in the ring. More specifically, the entries of A_K satisfy:

$$a_{ij} > a_{lq} \; when \; |i-j| < |l-q| \,. \tag{6}$$

Naturally, it must also be the case that the exposure $a_{ij} > 0$ for any pair of distinct nodes i and j in the same component (while being zero if they lie in different components).

To illustrate the fact that this pattern of exposure can be generated by iterating the exchange of assets among each firm and the few others to which it is directly connected, we now present the following stylized example:

Example 1 *[Risk-sharing through securitization: the ring component]*
Each firm in a ring component of size $K+1$ exchanges first a fraction $1-\theta$ of the claims to its project's returns with each of its two neighboring firms in the ring. Next, the exchange is repeated, but now it concerns shares in the composite portfolio acquired with the previous trades. We keep iterating these trades m times. We can interpret this as a process of repeated securitization. Suppose, for simplicity, that K is even. Then, if $m \geq K/2$, the final outcome would have all firms holding some shares on the project run by every firm in its component. And, clearly, the actual share held by a firm i on some other firm j must decrease with the ring distance between i and j.

Formally, each round of asset exchange in the mechanism described can be represented by the composition of the matrix

$$B_K = \begin{pmatrix} \theta & (1-\theta)/2 & 0 & \cdots & 0 & (1-\theta)/2 \\ (1-\theta)/2 & \theta & (1-\theta)/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\theta)/2 & 0 & 0 & \cdots & (1-\theta)/2 & \theta \end{pmatrix}, \tag{7}$$

with the outcome of the previous round, so that iterating m times the process, the final pattern of exposure is:

$$A_K = (B_K)^m. \tag{8}$$

It is immediate to see that the matrix A_K satisfies the decay condition (6) and all its entries are positive if we suppose that $s \geq K/2$. Finally, note that θ can always be chosen so that all the main-diagonal elements $\tilde{a}_{ii}(K, m)$ of A_K are equal to α.

The design problem In what follows we will compare the performance of the alternative financial structures described above in different environments, characterized by different distributions of the shocks. As explained, the socially optimal structure is the one for which the expected number of defaults is minimal. We will then determine the optimal degree of segmentation (that is, the optimal size of the components) as well as the optimal network density (that is, whether it is preferable to have a ring or a completely connected structure), depending on the characteristics of the shock distribution.

For the choice of the financial structure to be non trivial, we assume that the number of firms in the system is not too small, that is:

A3. $N > 4$.

Let $\varphi(A; \Phi(\cdot))$ denote the probability of bankruptcy of an arbitrary firm i when the financial network structure is described by the matrix A and the distribution of the b shocks is given by the cumulative distribution function $\Phi(\cdot)$ defined on $\mathbb{R}$. We have so:

$$\varphi(A; \Phi(\cdot)) = q\pi_b \Pr\{\alpha(R - L_b) + (1-\alpha)R) < M\} + q\pi_b \sum_{j \neq i}^{N} \Pr\{R - M < a_{ij} L_b\} \tag{9}$$

where M satisfies (2). The first term on the right hand side is the probability of default when a b shock hits the firm under consideration, the second term is the probability of default of the same firm when a b shock hits any of the other $N-1$ firms. By the symmetry of A the overall probability of default is the same and given by (9) for each firm i.

As argued before,[12] the expected number of defaults in the system is $\varphi(A; \Phi(\cdot))N$. Hence the optimal financial structure is given by the value of K and the network density for which $\varphi(A; \Phi(\cdot))$ is minimal. It is worth highlighting that the first term on the right hand side of (9) is independent of the financial structure, as it does not depend on A. That is, the probability of default of a firm whose project is hit by a b shock is the same across all structures considered. These only differ in their ability to limit contagion – that is, in how effective they are in preventing large b shocks from inducing the default of firms connected to the one directly hit by the shock.

In the setup we consider there is no cost of forming linkages, still the problem of identifying the optimal structure is a non trivial one and this one may differ from the single

[12] See footnote 7.

completely connected component. There is in fact always a positive probability that a sufficiently large $\tilde{L}_b$ shock hits so that all firms in a single completely connected component would default. This is a key feature of our analysis that stands in marked contrast with the conclusions derived from the models studied in earlier work as, for instance, Allen and Gale (2001) or Allen, Babus, Carletti (2011).

The Continuum Approximation

The pattern of risk exposure induced by either the complete or the ring structures can be graphically depicted through a function that, for each firm i, specifies the fraction of the claims to the yield of the project of firm i that is held by any other firm j in the component. This function depends on the network distance of firms j to i and is captured by the terms on the corresponding row of the matrix A_K. Given the assumption that $\alpha \geq 1/2$ the "exposure function" always reaches the highest value when $i = j$. In the particular case of the complete structure it takes a constant value for all other $j \neq i$, while for the ring it is a step function that is monotonically decreasing.

Because of the discreteness of the domain of this function – in particular, in the case of the ring structure – and the integer problems faced in the determination of the optimal degree of segmentation a formal analysis of the firms' probability of default for different network configurations becomes quite involved and tedious to carry out. To render the analysis more tractable, we study in what follows a continuum approximation of our model that abstracts from these considerations but still allows to capture the essential features of the problem. It involves approximating the discrete function $\varphi(A; \Phi(\cdot))$ by a continuous real function that embodies the same features as the original one.[13]

In the continuum approximation of the model, N is taken to be the measure of firms in the system, while $K + 1$ is the generic notation used to represent the measure of firms belonging to a certain component. The returns on a firm's project are the same as in (1). To keep a formal parallelism with the discrete formulation, when a shock occurs it is taken to hit directly a unit mass of firms in one component. This mass of firms, therefore, plays the role of the single firm directly hit by a shock in the discrete context.

Segmentation is modeled as in the previous section, except that the size of a component, $K + 1$, need not be an integer and can now be any real positive number lying between 1 and N. Differences in network density, on the other hand, are modeled as follows.

If the component is complete, the pattern of exposure to the returns on other firms' projects is as in the discrete model: the indirect exposure, to a shock that hits any other firm in the component, is constant and equal to $(1-\alpha)/K$. On the other hand, if the component has a ring structure, the pattern of risk exposure is now described by a continuous function

[13] This continuum formulation can be seen as representing a limit description of a context consisting of a large number of firms, each of them of small (relative) size. This interpretation, however, is not essential, so we prefer to view it simply as a continuous approximation of a context where the number of firms is not necessarily large.

$f(d;K)$, where $d \in [0, K/2]$ is the ring distance to the set of firms directly hit by the shock. This function has the following form:

$$\begin{aligned} f(d;K) &= \alpha - \tfrac{\alpha - H}{H} d && \text{for } 0 \leq d \leq H \\ &= \tfrac{HK}{K-2H} - \tfrac{2H}{K-2H} d && \text{for } H < d < K/2 \\ &= 0 && \text{for } d = K/2. \end{aligned} \tag{10}$$

for some suitably computed H. The function $f(\cdot)$ is defined for all $K \geq 1$, with the value of H being determined by the following adding-up constraint:

$$2 \int_0^{K/2} f(x;K) \mathrm{d}x = (\alpha - H) H + 2H^2 + H (K/2 - H) = 1 - \alpha, \tag{11}$$

which reflects the requirement that firms externalize the fraction $(1-\alpha)$ of their risk. This condition leads to

$$H = \frac{2(1-\alpha)}{K + 2\alpha}. \tag{12}$$

Recalling that $\alpha \geq 1/2$ and $K \geq 1$, it is immediate to verify that the function $f(d;K)$ exhibits the following properties that match those of the exposure function in the discrete set-up:

$$f(d;K) \text{ is positive and decreasing for all } K/2 > d > 0, \tag{13}$$

$$f(0) = \alpha, \tag{14}$$

$$f(K/2) = 0. \tag{15}$$

First, (13) says that every firm in a component is affected by a shock hitting any other firm in the component, but with an intensity that decreases with the distance to the source of the shock. In addition, (14) states that the level of exposure to a firm at minimal distance is the same as that to a direct shock, while (15) says that the exposure becomes vanishing small when the distance to the source of the shock is maximal in the component. Conditions (13), (14), and (15), together with the adding-up constraint (11) on the total fraction of the risk externalized by firms, capture the essential features displayed by the exposure function in the discrete setup.

Note that $f(d;K)$ is a two-piece linear function with the kink at the bisectrix (i.e. at a distance $d = H$ such that $f(H) = H$). It is concave or convex depending on whether $K > 2(1-\alpha)/\alpha$, and since $\alpha \geq 1/2$, and $K \geq 2$, $f(.;K)$ is always convex. An illustration of how it approximates the original function for the discrete setup is displayed in Figure 1.

Using the above specification of the exposure function for the ring and the complete structures, for any possible level of K, we are now in a position to determine the extent of default induced by any given shock of magnitude L for alternative financial structures. Note first that, as already argued for the discrete context, whether the unit mass of adjacent firms directly hit by a shock default or not is independent of the underlying financial network structure. For those firms will default if, and only if,

$$L > \frac{R - M}{\alpha}.$$

Figure 1: Continuum approximation of the exposure function. The step function describes the exposure in a ring with $K+1=21$ firms, as a function of the distance d to either side of the location occupied by any given firm. The exposure is the one induced by the matrix $A_{20} \equiv (B_{20})^{10}$, with B_{20} being the matrix of asset exchange specified in (7). In contrast, the solid line describes the exposure for the continuum approximation $f(d, 10)$.

If the former inequality is violated and the firms hit by the shock do not default, no other firm in the corresponding component defaults either. This simply follows from the fact that $\alpha \geq (1-\alpha)/K$ since $\alpha \geq 1/2$, $K \geq 1$ and $f(d;K) \leq \alpha$ for any $d \geq 0$. But if those firms directly affected by the shock do default, what happens to all the others in the component naturally depends on the size $K+1$ of the component and on the pattern of the connections within it.

In the case where firms are connected through a *complete interaction pattern*, the uniformity of the exposure has the following immediate implication: *all* firms indirectly affected (i.e. not hit by the shock but lying in the component affected) will default if

$$L > \frac{K}{1-\alpha}(R-M) \tag{16}$$

whereas *none* of those firms will default otherwise. Thus, if we let $g_c(L;K)$ stand for the mass of firms that default when the shock hits some other firm in their component, that magnitude is given by the following step function:

$$g_c(L;K;M) = \begin{cases} 0 & \text{if } L \leq \frac{K}{1-\alpha}(R-M) \\ K & \text{if } L > \frac{K}{1-\alpha}(R-M) \end{cases} \tag{17}$$

In contrast, when the firms in the component are connected through a *ring interaction pattern* (as captured by the exposure pattern $f(\cdot)$), the conclusion is, in general, not so dichotomic. For, in this case, whether any particular firm in the component defaults or not

depends on its ring distance d to those firms that have been directly affected. Such a firm defaults if, and only if,

$$L > \frac{1}{f(d;K)}(R-M),$$

which is to be contrasted with (16). Hence the threshold that marks the relevant "default range" is given by the distance $\hat{d}$ such that

$$f(\hat{d};K) = \frac{R-M}{L}, \tag{18}$$

so that a firm defaults if, and only if, its distance d from the set of firms directly hit by the shock is such that $d < \hat{d}$. Under a ring structure, therefore, the effect of different levels of the shock on the mass of firms defaulting is not discontinuous as in the complete structure. Rather, as the magnitude L of the shock increases, the mass of firms defaulting among those indirectly affected by it grows gradually (see Figure 2 for an illustration), as determined by the function $g_r(L;K,M) \equiv 2f^{-1}((R-M)/L;K)$ given by:

$$g_r(L;K,M) = \begin{cases} 0 & \text{for } L \leq \frac{R-M}{\alpha} \\ \frac{2\alpha H}{\alpha - H} - \frac{2H}{\alpha - H}\frac{R-M}{L} & \text{for } \frac{R-M}{\alpha} \leq L \leq \frac{R-M}{H} \\ K - \frac{K-2H}{H}\frac{R-M}{L} & \text{for } L \geq \frac{R-M}{H} \end{cases} \tag{19}$$

Figure 2: The function $g_r(L;20,M)$, specifying the mass of firms that default when indirectly hit in a component of size $K+1 = 21$ for any given magnitude L of the b shock, plotted for $\alpha = 1/2$ and the normalization $R - M = 1$, as posited below.

3 Optimal Financial Structures

In this section we identify the optimal network segmentation and network density. That is, we determine the financial structure that minimizes the expected mass of firms that default, in the continuum approximation of the model. To this end, we rely on the functions given in (19), and (17) that specify, respectively, the mass of firms defaulting in a component with a complete or ring structure. The choice variables are not only the number C of different components in the system, but also the size K_i+1 of each component i, $i = 1,..,C$. We allow, therefore, for the possibility of asymmetric structures, with components of possibly different

size. As explained in the previous section, the minimization of the expected mass of defaults is equivalent to the minimization of the probability that any given firm defaults. Hence, focusing on the latter, our optimization problem can be formulated as the minimization of the following magnitudes:

$$\varphi_\nu = qN\pi_b \left(\frac{1}{N}\Pi + \sum_{i=1}^{C} \frac{K_i+1}{N} \frac{K_i}{N} \frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M)}{K_i} \right) \tag{20}$$

where the subindex $\nu = c$, r signifies that either a complete or a ring structure is considered, and Π is the probability that a firm defaults when directly hit by a b shock (which is the same for all structures since α is also the same across structures). The above expression reflects the fact that, when heterogeneous components are allowed, the ex-ante probability that any firm defaults is the average weighted probability that firms in each component default.

Consider first a fixed structure $\nu \in \{c, r\}$. Then, discarding irrelevant terms in (20), the optimal degree of segmentation is obtained as a solution of the following optimization problem:

$$\begin{array}{c} \min_{K_i,C} \sum_{i=1}^{C} \frac{K_i+1}{N} \mathbb{E}g_\nu(\widetilde{L}_b; K_i, M) \\ \text{s.t.} \sum_{i=1}^{C} \frac{K_i+1}{N} = 1 \end{array} \tag{21}$$

We will show that the solution is (essentially) symmetric for all cases under consideration, with $K_i = K$ for all i. This implies that problem (21) can be reduced to minimizing $\mathbb{E}g_\nu(\widetilde{L}_b; K, M)$ with respect to K. Then, by comparing the optimum values of the solution of (21) for $\nu = r$ and $\nu = c$ we obtain the optimal network density.

Note that the value of $\mathbb{E}g_\nu(\widetilde{L}_b; K, M)$ depends not only on the distribution $\Phi(L_b)$ but also on the other parameters of the model, α, R, r, L_s. The specific values of these parameters have however little interesting bearing on the analysis and our primary focus is on the effects of the shock distribution on the optimal network structure. Hence in what follows we set[14] $\alpha = 1/2$, normalize $R - r$ to unity and set also $L_s = 1$. Furthermore, even though M is an endogenous variable (determined by (2)) and varies with the underlying network structure, for the purpose of determining the optimal structure we can make it equal to r (and hence $R - M = 1$) if $qN\pi_b$ is small. More formally, we can show that:

Lemma 1 $\lim_{qN\pi_b \to 0} M = r$.

Therefore, in identifying the optimal structure, we can set $M = M_F(0) = r$ and thus $R - M = 1$.

We organize our formal analysis in the rest of this section in three parts. First, in Subsection 3.1 we identify some clear-cut conditions on the probability distribution of the b shocks under which the optimal segmentation is one of the two polar extremes – maximal

[14] A convenient consequence of this choice of α is that, just as for the discrete version of the model, the pattern of exposure is exactly the same for the complete and the ring structure if the component size is the smallest (i.e. $K = 1$). The difference between the two structures then grows wider the higher is K.

or minimal – and the optimal degree of connectivity is complete. Then, in Subsection 3.2 we extend the analysis to more general specifications of the shock distribution, for which intermediate levels of segmentation are optimal. Finally, in Subsection 3.3 we identify scenarios where the optimal structure exhibits not only intermediate levels of segmentation but also low density of connections, as embodied by the ring structure.

3.1 Polarized segmentation

In order to get a clear understanding of the forces at work, we shall start by examining the case where the probability distribution of the b shocks is of the Pareto family with support $[1,\infty)$ and density $\gamma/L_b^{\gamma+1}$. By modulating the decay parameter γ, this formulation already allows the discussion of many questions of interest such as the contrast between fat and thin tails in the shock distribution (i.e. between scenarios where large shocks are relatively frequent or not). As mentioned above, our analysis will be carried out in two steps. Firstly, we shall characterize how γ affects the optimal degree of segmentation (as described by K) for the ring and for the complete structure. Secondly, we shall compare these two structures.

Let $D_r(K,\gamma) = \mathbb{E}_\gamma g_r(\widetilde{L}_b;K)$ denote the expected mass of firms in a ring of size $K+1$ who default when indirectly hit by a b shock with a Pareto distribution with parameter γ (that is, when the shock hits some other firm in the component). We have:[15]:

$$\begin{aligned} D_r(K,\gamma) &= \int_1^\infty g_r(L;K)\ \mathrm{d}\Phi(L;\gamma) \\ &= \int_{\frac{R-M}{H}}^\infty \left(K - \frac{K-2H}{H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L + \int_{\frac{R-M}{\alpha}}^{\frac{R-M}{H}} \left(\frac{2\alpha H}{\alpha-H} - \frac{2H}{\alpha-H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L \\ &= \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right] + \\ &2\gamma\left[-\frac{1}{K-1}\frac{1}{\gamma(K+1)^\gamma} + \frac{2}{K-1}\frac{1}{(\gamma+1)(K+1)^{\gamma+1}} + \frac{1}{K-1}\frac{1}{\gamma\, 2^\gamma} - \frac{2}{K-1}\frac{1}{(\gamma+1)2^{\gamma+1}}\right] \end{aligned} \tag{22}$$

We study next how the above expression behaves as K varies in its admissible range (recall that the minimum[16] admissible value of K is 1 and its maximal value is $N-1$). We establish in the following proposition that whenever $\gamma > 1$ (i.e. the distribution function of $\widetilde{L}_b$ does not have fat tails) $D_r(K,\gamma)$ attains a minimum at the highest value of K, i.e. $N-1$. On the other hand, when $\gamma < 1$ (i.e. the distribution has fat tails) the function attains a minimum at $K=1$.

Lemma 2 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the component size which minimizes the mass of firms defaulting is minimal ($K=1$) if $\gamma > 1$, and maximal ($K = N-1$) if $\gamma < 1$.*

On this basis, we can easily determine the optimal segmentation pattern in the system. To this end we must also take into account the constraint present in (21), $\sum_{i=1}^C (K_i+1)/N =$

[15] Henceforth we shall use the parameter values $\alpha = 1/2$ and $R-M = 1$ specified at the end of the previous section and omit the argument M $(= R-1)$ in the functions throughout. Also, these values imply that $H = 1/(K+1)$.

[16] Strictly speaking, the expression in (22) holds for $K > 1$. For $K = 1$ we have $D_r(1,\gamma) = \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right]$. However, since we show in the proof of Lemma 2 that $D_r(K,\gamma)$ is continuous at $K=1$, in what follows it suffices to work with (22).

1. Given the findings of the above lemma, this constraint only binds when N is odd and $\gamma < 1$, as in such case a structure with all components of the optimal (minimal) size 2 is not feasible. To find the optimal structure in this case we establish the concavity of $D_r(K, \gamma)$:

Lemma 3 *When $\gamma < 1$ the function $D_r(K, \gamma)$ is strictly concave in K, for all $K > 1$.*

From Lemmas 2 and 3 it immediately follows that the optimal ring structure when $\gamma < 1$ and N is odd is "almost" symmetric, with $\frac{N}{2} - 1$ components of minimal size 2 and a residual component of size 3. In all other cases, the optimal ring structure is the one with all components of the same optimal size, determined in Lemma 2. We can then summarize our findings in the following statement:

Proposition 1 *Suppose the shock $\widetilde{L}_b$ has a Pareto distribution. With $\gamma > 1$ (no fat tails) the optimal degree of segmentation for the ring structure is minimal, with a single component including all N firms. Otherwise, with $\gamma < 1$ segmentation is maximal, with all components of the minimal size 2 (except one, of size 3, when N is odd).*

The previous result shows that there is indeed a trade-off between risk sharing and contagion. On the one hand, when the distribution of the shocks exhibits fat tails (hence large shocks are relatively likely), the predominant consideration is to control contagion rather than achieve risk sharing. Hence the expected number of defaults is minimized by breaking the network into disjoint components of minimal size, which limits the extent to which a shock may spread its consequences far into the system. Instead, when the distribution has no fat tails, the most important consideration becomes risk-sharing, which is maximized by placing all firms in a single component.

Next, we turn to studying the analogous question for the case where the components are completely connected. In this case, the expected mass of firms who default in a completely connected component of size $K + 1$ when indirectly hit by a b shock is:

$$D_c(K, \gamma) = \mathbb{E}_\gamma g_c(\widetilde{L}_b; K) = K \ \Pr\left(L \geq 2K\right) = K \ \left(\frac{1}{2K}\right)^\gamma. \tag{23}$$

Hence

$$\frac{\partial D_c}{\partial K} = -\left(\gamma - 1\right)\left(\frac{1}{2K}\right)^\gamma \gtreqless 0 \quad \Longleftrightarrow \quad \gamma \lesseqgtr 1,$$

which readily implies that the optimal component size is again minimal when $\gamma < 1$ (the shock distribution has fat tails), while it is maximal in the case where $\gamma > 1$. Hence the optimal segmentation structure is the same as for the ring, with all components of the optimal size when the constraint $\sum_{i=1}^{C}(K_i + 1)/N = 1$ does not bind. In contrast, when $\gamma < 1$ and N odd, and hence the former constraint binds, the optimal structure is different from the one obtained for the ring in this situation. It is now exactly symmetric, with $\frac{N-1}{2}$ components, all of the same size (slightly larger than 2). This is because, as can be seen from the proof of the proposition 4, $D_c(\cdot)$ is a convex function of K (remember that $D_r(\cdot)$ is a concave function of K). Hence we have the following:

Proposition 2 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the optimal degree of segmentation for the completely connected structure is minimal (one single component) if $\gamma > 1$, and maximal, with $N/2$ ($(N-1)/2$ if N is odd) identical components, if $\gamma < 1$.*

Finally, we need to compare the optimally segmented ring and complete network structures in order to identify which of the two is optimal when not only segmentation but also network density can be chosen. In view of Propositions 1 and 2, it is enough to compare the expected mass of firms defaulting under either maximal or minimum segmentation for the ring and the complete structures when, respectively, γ is lower or higher than one. Note also that, when K is at its minimal admissible value (1), the pattern of exposure is the same for the two structures, $g_c(L_b; 1) = g_r(L_b; 1)$, hence a difference only arises when the optimal component size is greater than 1 for at least one structure. This leads to the following result.

Proposition 3 *If the shock $\widetilde{L}_b$ has a Pareto distribution, for all values of $\gamma < 1$, when N is even the optimal structure is equivalently completely connected or a ring structure, while for all $\gamma > 1$ and for N not too small ($N > 1 + (1+\gamma)^{\frac{1}{\gamma-1}}$) the complete network strictly dominates the ring structure.*

Combining the results obtained in this subsection, we conclude that when the distribution of the shocks has a simple Pareto structure and thus it either has, or does not have, fat tails, the optimal network always displays maximal density and polarized (maximum or minimum) segmentation. In this case, therefore, the whole adjustment to the underlying risk conditions (i.e. to the different values of γ) is obtained only by varying the segmentation pattern. However, as our subsequent analysis will show, neither the optimality of the polarized segmentation pattern nor of the complete connectivity within components are features maintained for other, more complex, shock distributions.

3.2 Intermediate Degrees of Segmentation

We show next that polarized segmentation is no longer optimal when the distribution of the shocks is more complex, as for instance when it is given by the mixture of two Pareto distributions.

Proposition 4 *Suppose that the shock $\widetilde{L}_b$ is distributed as a mixture of a Pareto distribution with parameter $\gamma > 1$ and another Pareto distribution with parameter $\gamma' < 1$, with respective weights p and $1-p$. Then, there exist p_0, p_1, $0 < p_0 < p_1 < 1$, such that, whenever $p \in (p_0, p_1)$, the optimal pattern of segmentation for the completely connected structure is symmetric with components of intermediate size $K^* + 1$, $1 < K^* < N - 1$.*

Abusing slightly previous notation, denote by $D_c(K, \gamma, \gamma', p) = p\mathbb{E}_\gamma g_c(\widetilde{L}_b; K) + (1 - p)\mathbb{E}_{\gamma'} g_c(\widetilde{L}_b; K)$ the expected mass of firms who default in a complete component of size $K+1$ when indirectly hit by a b shock hitting a firm in the component, and the distribution of this shock is as in the statement of the above proposition. We show in the proof of this result that the function $D_c(K, \gamma, \gamma', p)$ attains a minimum at an intermediate value $\hat{K} \in (1, N-1)$. A symmetric structure with all components of size $\hat{K}$ is however generically not feasible now (i.e. violates the condition $\sum_{i=1}^{C}(K_i + 1)/N = 1$). We then show that the optimal structure is still symmetric, with all components of size K^*, smaller or equal to $\hat{K}$.

A numerical analysis of the problem shows that a similar conclusion applies to the case where the components display a ring structure. That is, an intermediate level of segmentation continues to be optimal when the shock distribution involves a mixture of Pareto distributions with both fat and thin tails. However, since an analytic result in this case is hard to obtain, we illustrate matters through the following example.

Example 2 *Set $\gamma = 2$, $\gamma' = 0.5$ and $p = 0.95$. For these values we find that the value of K which minimizes $D_c(K, \gamma, \gamma', p)$ is $\hat{K}^c = 5.65$, and at this value the expected mass of defaults (when a b shock hits some other firms in the component) is 0.13. The value of K that minimizes the corresponding expression for the ring structure, $D_r(K, \gamma, \gamma', p)$, is higher, $\hat{K}^r = 8.02$, and the expected mass of defaults in this case is also higher, equal to 0.145. The fact that $\hat{K}^r > \hat{K}^c$ can be intuitively understood as a reflection of the fact that, when arranged optimally, components with a ring structure compensate for their lower density of connections with a larger size. See Figure 3 for a graphical depiction of these two functions.*

To find the optimal financial structure for the whole system we also need to specify the value of N. Let $N = 10$. In this case it is clear that a symmetric structure with equal components of size $\hat{K}^c + 1$ is not *feasible when the components are completely connected, and the same applies to components of size $\hat{K}^r + 1$ when they are rings. We find that both for the complete and the ring structures the optimal configuration of the system is given by two equal-sized components of size $K^* + 1 = 5$. Moreover, in line with the results stated above for the values of the mass of expected defaults at $\hat{K}^r$ and $\hat{K}^c$, we find that the optimal complete structure still dominates the optimal ring structure: the former yields an expected mass of defaults equal to 0.13 in contrast with 0.16 induced by the latter (see Figure 4). This conclusion is robust to alternative specifications of the parameter values of the environment.*

3.3 Sparse Connections

Let us consider now the case where the probability distribution of the b shocks is not smooth because it has some atoms. More precisely, let $\Phi(L_b)$ be the mixture of a Pareto distribution with $\gamma > 1$ and a Dirac distribution putting all probability mass on a shock of magnitude $\bar{L} > 2(N-1)$. On the one hand, note that the Pareto distribution considered has no fat tails. With such a distribution, as we saw in Section 3.1, minimal segmentation

Figure 3: Expected mass of firms in a component of size K indirectly affected by a b shock who default when this shock hits their component, as given by the function $D_v(K,\cdot)$ of the component's size K, for both complete and ring structures ($v = c, r$). The parameter values are $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

($K = N - 1$) is optimal and, in addition, the completely connected structure dominates the ring structure. On the other hand, the shock $\bar{L}$ selected by the Dirac distribution has the following property. If a shock of that magnitude occurs and firms are arranged in a single component, all firms default when they are completely connected while some survive if arranged in a ring. Combining the previous considerations, we show below that there is an open region of parameter values for which the second effect prevails over the first one and hence the optimal financial structure is a ring. Under such conditions, therefore, sparse connections are optimal.

Proposition 5 *Let $\widetilde{L}_b$ be distributed as a mixture of a Pareto distribution with parameter $2 > \gamma > 1$ and a Dirac distribution with all mass concentrated on $\bar{L} = 2(N-1)+1$, with weights respectively given by p and $1-p$. Then, for all values of N such that*

$$N > 1 + \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}} + \frac{\frac{1}{2^{\gamma-1}} \frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}} \right)^{\frac{1}{2-\gamma}} \tag{24}$$

and provided that

$$\frac{(1-p)}{p} < (\gamma - 1) \left(\frac{1}{2(N-1)} \right)^{\gamma}, \tag{25}$$

there exists an open set of values of p such that the optimal financial structure is a single ring component.

Figure 4: Expected mass of firms in a system with N firms indirectly hit by a b shock who default when this shock hits a component in the system, as given by the function $W_v(K,\cdot) \equiv \frac{K+1}{N} D_v(K,\cdot) + \frac{N-K-1}{N} D_v(N-K-2,\cdot)$, for the complete and ring structures ($v = c, r$). The system consists of $N = 10$ firms divided into two components of size K and $N - K$. The parameter values are as in Figure 3, i.e. $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

The role of inequality (25) is to guarantee that the weight p on the Pareto distribution is sufficiently high that the optimal segmentation structure is determined by it and hence a *single* component is optimal for the completely connected structures. But then, given that there is also a non negligible probability that a large shock arrives that cannot be fully absorbed, some attempt at "controlling the induced damage" may be in order. And this is indeed what the ring achieves – a suitable compromise between the extent of *risk sharing* allowed by extensive *indirect* connectivity (i.e. minimal segmentation) and the limits to wide *risk contagion* resulting from sparse *direct* connections.

4 Stability and optimality

We now examine the relationship between the optimal pattern of linkages derived in the previous section and the individual incentives to form those linkages. We explore, in other words, whether social welfare is aligned with the maximization of individual payoffs. To model the strategic considerations involved in the creation of networks structures, we will model a network-formation game. Since we have only explored the payoffs accruing to structures formed by complete components and rings, we need to restrict our analysis to such kinds of structures. For simplicity we concentrate on completely connected component structures in what follows. The network-formation game is then assumed to be conducted as follows:

- Firms independently submit their proposals concerning the components to which they

want to belong. Formally, a strategy of each firm i is a subset $S_i \subset N$, with the property that $i \in S_i$.

- A particular proposal S_i is accepted only when all the firms $j \in S_i$ make the same proposal, i.e. $S_j = S_i$. Formally, given a profile of strategies $\mathbf{S} \equiv (S_i)_{i \in N}$ a component S is established if and only if for all the firms $i \in S$, $S_i = S$. If the proposal S_i of firm i is rejected, that is if for some $j \in S_i$, $S_j \neq S_i$, firm i remains in isolation.

Given the above network-formation rules, a specific network $\Gamma(\mathbf{S})$ is induced by each strategy profile $\mathbf{S}$. As maintained throughout, the payoff of each firm i is decreasing in $\varphi_i(\Gamma(\mathbf{S}))$, its own default probability resulting from the network induced by $\mathbf{S}$. Formally, we can w.l.o.g. identify its payoff with $-\varphi_i(\Gamma(\mathbf{S}))$, the opposite of that probability.

In such a network-formation game, an undesirable feature of the standard concept of Nash Equilibrium is that it leads to a vast multiplicity of equilibrium networks, a consequence of the fact that the formation of any link induces a coordination problem between the two agents involved. (As an extreme illustration, note that the empty network can always be supported by a Nash equilibrium where every agent proposes nobody to link with.) To address this issue, it is common in the literature to consider a strengthening of the Nash equilibrium notion that reduces miscoordination by allowing sets of agents to deviate jointly (see e.g. Goyal and Vega Redondo (2007) or Calvó-Armengol and Ikiliç (2009)). In the context of our model, we shall capture this idea by means of the concept we label *Coalition-Proof Equilibrium* (CPE), where any group (i.e. coalition) of agents can coordinate their deviations:

Definition 1 *A strategy profile* $\mathbf{S} \equiv (S_i)_{i \in N}$ *of the network-formation game defines a* Coalition-Proof equilibrium *(CPE) if there is* no subset *of firms* W *and a strategy profile for these firms,* $(S'_j)_{j \in W}$*, such that*[17]

$$\forall i \in W, \quad \varphi_i \left[\Gamma \left((S'_j)_{j \in W}, (S_k)_{k \in N \setminus W} \right) \right] < \varphi_i[\Gamma(\mathbf{S})].$$

The CPE configuration is immune not only to individual but also to joint profitable deviations by any arbitrary set of firms, so it is obviously a refinement of the standard notion of Nash Equilibrium. A requirement typically imposed on coalition-based notions of equilibrium is that the coalitional deviations considered should be robust, in the sense of being themselves immune to a subcoalition profitably deviating from it. This requirement has no bite for our analysis since we can show that all the profitable deviations that need to be allowed are robust in the aforementioned sense.[18]

[17] In line with what was postulated in Section 2, we shall continue to assume that, after any change in connections has been implemented, the firms involved in the change continue to distribute the fraction $1-\alpha$ of their own assets among all their neighbors (old and new) in a uniform manner

[18] See Remark 2 in the Appendix.

Consider first the case where the optimal completely connected structure consists of identical components of the size that minimizes the mass of defaults in a component – thus, in this case, the feasibility constraint $\sum_{i=1}^{C}(K_i+1)/N = 1$ does not bind (as for instance in Proposition 2). Then it is immediate to verify that the optimal structure is also a CPE. In contrast, we show below (Proposition 6) that this is no longer necessarily true when the feasibility constraint binds.

To make the point, we analyze the CPE for the class of distributions considered in Section 3.2 and show that, in that context, there is typically a conflict between social optimality and individual incentives. To understand the conflict, first recall from Proposition 4 that there is an open set of the parameter space for which, under the aforementioned class of shock distributions, the optimal completely connected structure involves a symmetric segmentation of the whole population into several components of equal size, K^*+1. Moreover, K^*+1 is typically smaller than the individually optimal component size $\hat{K}+1$ that minimizes default *within* a given component.

This gap produces a conflict between individual and social incentives, which is reflected by the inefficiency of CPE configurations. In a CPE the outcome is asymmetric: individual incentives (supported by coalitional deviations) give rise to all but one component being of the individually optimal size $\hat{K}+1$ and one other component of a size smaller than K^*+1. But this, to repeat, cannot be socially optimal because standard convexity considerations favor a more balanced configuration where the sizes of all components are equal.

Proposition 6 *Consider the same environment as in Proposition 4. Then, for $p \in (p_0, p_1)$, the socially optimal completely connected structure cannot be supported at a CPE of the network formation game. Among the completely connected structures, the only CPE configuration is asymmetric with all but one component displaying the size $\hat{K}$ that minimizes $D_c(K, \gamma, \gamma', p)$ and one component of a lower size.*

The basis for the above result is two-fold. First, it is clear that no component prevailing at a CPE can be larger than $\hat{K}+1$, the size that is individually optimal. For, if it were, an obvious coalitional deviation would be available. Moreover, if there is any component of a size lower than $\hat{K}+1$, there can be no more than one. For otherwise a profitable deviation towards a larger component would be admissible for a suitable coalition. In this context, the source of the conflict between social and individual optimality lies in the externality that is imposed by the fact that the feasibility constraint must hold in the aggregate across all components. Therefore, as firms in any component strive for the size that is individually (or component-wise) optimal, this will generally force other firms to remain in a component that is too small to share risk efficiently.

As a simple illustration of the problem, refer back to Example 2, where the efficient configuration involves two completely connected components of common size equal to $K^*+1 = 5$, while the individually optimal size for each component is $\hat{K}^c+1 = 6.65$, the value that minimizes $D_c(K, 2, 0.5, 0.95)$. In contrast the only CPE configuration is asymmetric

with two components of sizes 6.65 and 3.35, the latter being a "residual" group.[19]

The existence of a conflict between efficiency and strategic stability is of course hardly novel nor surprising in the field of social networks (see e.g. Jackson and Wolinsky (1996) for an early instance of it). For, typically, the creation or destruction of any link between two agents impose externalities on others that are not internalized by the two agents involved in the linking decision. In the context of risk-sharing, this tension has been studied in a recent paper by Bramoullé and Kranton (2007) – hereafter labeled BK – and it is interesting to understand its differences with our approach. We close, therefore, this section with a brief comparison of the two models.

BK consider an environment consisting of a finite number of agents affected by i.i.d. income shocks. Linkages generate risk sharing in a way similar to that of our model, except in two important respects: (a) risk-sharing is complete (i.e. uniform) across all members in a component; (b) there is no risk of contagion, so the size of optimal components is just limited by the fact that links are assumed to be costly. Focusing on the notion of strategic stability (which is weaker than ours),[20] BK are also interested in comparing efficient and equilibrium configurations. They find that whenever equilibrium structures exist (not always), there are at most two asymmetric components, with sizes smaller than the optimal one.

The contrast between BK's conclusions and ours derives from the nature of the externality in the two cases. In BK, given that the cost of any new link is borne only by the two agents involved, the equilibrium induces an underinvestment in link formation (which is a "public good"). Instead, in our case there are no linking costs, so the nature of the externality that is not internalized is quite different. It has to do with the fact that, when firms deviate to reach a larger component of individually optimal size, they do not internalize that the firms that are left behind will be forced to components that are inefficiently small. Thus, in the end, it is the need to meet an overall feasibility constraint imposed by the finite measure of the population that typically generates inefficiencies.

5 Asymmetric environments

So far we have concentrated the discussion on a situation where all firms in the system are identical. Although this allows us to obtain analytical results and gather intuition, it is important to extend our analysis to situations where firms are significantly different in size or in their shock distribution, as often happens in the real world. To this end, we shall simplify the analysis and focus our attention in the next subsection on the case where structures are completely connected. In the following subsection we also allow for

[19] The example also makes clear that this is not the outcome of an *integer problem* in the sense that it does not disappear when the size of the economy grows large. One can easily renormalize the parameters to construct examples where $\hat{K}^c + 1 = 665$ when $N = 1000$.

[20] Strategic stability allows only for coalitions of at most two players and rules out as well the simultaneous creation and destruction of links. See Jackson and Wolinsky (1996) for details.

some asymmetry in network structures. Under these circumstances, there is no real gain in using the continuum approximation of the patterns of firms' exposure and thus we revert to the original specification, where firms are conceived as discrete entities (which can be of different types and sizes, as we explain below).

5.1 Shock and size asymmetry

To begin with, consider the case where all firms have equal resources R in the absence of shocks but they differ in the distribution of the shocks that may hit their project's returns. More precisely, suppose that the N firms are partitioned into subsets $N_1, ...N_n$, so that $N = \cup_{l=1}^{n} N_l$, and for every firm $j \in N_l$ the big shock follows a distribution F_{N_l} with common expected value for all $l \in \{1, ..., n\}$.

For each $l = 1, .., n$ we denote by $D_c(K, \Phi_{N_l})$ the expected number of firms who default in a (complete) component of size $K + 1$ when indirectly hit by a b shock hitting a firm j belonging to subset N_l . Note that, since all firms are of equal size, each firm has the same capacity of absorbing shocks. This is what implies that the function $D_c(\cdot)$ depend only on the size of the component (as determined by K) and the distribution Φ_{N_l} of the shock hitting a firm j, and not also of the types of other firms in the component. In particular, each firm retains a fraction α of its own project and acquires the same exposure $(1 - \alpha)/K$ to shocks hitting any other firm in the component[21]. We thus have:

$$D_c(K, \Phi_{N_l}) = K \left[1 - \Phi_{N_l} \left(\frac{K}{1 - \alpha} \right) \right]. \tag{26}$$

Let $K^*_{N_l}$ be the minimizer in K of this function.

We are interested in identifying the optimal segmentation structure, which now requires determining not only the number of firms that should lie in each component but also the composition of each component in terms of the different types of firms included in it. One important concern will be to understand whether firm matching within components should be assortative or dissortative – that is, whether components should be more or less homogeneous. In this context, therefore, a structure must specify a set of components $\mathcal{C} \equiv \{C_i\}_{i=1,...,m}$ and, for each component i, the corresponding type distribution $\{N_l^i\}_{i=1,...,n}$, where $C_i = \cup_{l=1}^{n} N_l^i$ and each N_l^i stands for the subset of firms in N_l that belong to component i.

Proposition 7 *Suppose that, for every l, the cardinality $|N_l|$ of the set of firms of type l is a multiple of $K^*_{N_l} + 1$. Then, the optimal (completely connected) structure has all firms belonging to homogeneous components, in each of which firms are all of the same type, and $K^*_{N_l} + 1$ is the common size of every component consisting of firms of any given type l.*

[21] This is a consequence of the fact that, even though the risks are different, the exchange of assets is still one for one for every pair of firms.

The intuition for this result is as follows. For each l, conditional on the b shock hitting a firm of type l, the optimal arrangement is to have all components where firms l are present with $K^*_{N_l} + 1$ firms in them, as $K^*_{N_l}$ is the value that minimizes the function $D_c(K, \Phi_{N_l})$ given in (26). If we had some heterogeneous component, with firms of type j, in addition to l in it, its size could not be set to equal *simultaneously* the (generally different) optimal sizes for the cases of shocks hitting type j and type l firms. Hence the expected number of defaults could be reduced by rearranging firms within homogenous components (if this is feasible), as this would allow to attain the minimum of $D_c(K, \Phi_{N_l})$ for each l. Of course, a key feature required for this conclusion to hold is that, as already noticed, the expected number of indirect defaults in a component only depends on the type l of the firm directly hit by the shock and the size of the component. That is, the indirect effect of the shock does *not* depend on the distribution of types among the firms in the component not directly hit by the shock.

Next, we turn our attention to the case where firms are heterogeneous in size. In light of the previous result, which establishes the optimality of extreme assortativity by distributional types, we now concentrate on environments where all firms are of the same type as far as the distribution of the shock hitting them is concerned. For simplicity, let us consider a situation with just two possible sizes. On the one hand, there are firms of unit size, identical to the ones we have been considering so far. On the other hand, there are firms of size $\beta > 1$, such larger size having the following two implications. First, the return on the projects of these larger firms when no shock affects them is βR, so it is scaled up by β compared to those of the smaller firms. (Naturally, we also assume that the same factor β applies to the value of their liabilities.) Second, the larger firms are supposed to face a probability of being directly hit by a shock that is β times larger (that is, equal to βq) Hence, in this case, we make an assumption that is polar to the case considered before (and thus leads to a complementary analysis): size affects the probability of arrival of the shocks, *not* their distribution. In a sense, therefore, one can view a large firm as a full merge of β small firms of the sort we have been considering so far.

Of course, in order to maintain equal values in trade, a firm of size β must exchange assets with a firm of size 1 in a proportion $1/\beta$ to 1. Let us denote by N_1 the subset of firms of size 1, and by N_β the subset of firms of size β, while N_1^i and N_β^i stand for the corresponding subsets of small and big firms within a component C_i. The size of this component is then given by $K_i + 1 = \left|N_1^i\right| + \beta \left|N_\beta^i\right|$ where $|\cdot|$ stands for the cardinality of the set in question. Then the expected number of defaults in the component if the firm hit is small (i.e. of size 1) is:[22]

[22] To derive the expression below note that, as a result of the fact that a small firm exchanges equity at a rate $1/\beta$ with the big firm, each small firm ends up having a share $(1-\alpha)/(K_i + 1 - \beta)$ of the big firm, while the big firm has a share $(1-\alpha)\beta/K_i$ of each small firm.

$$
\begin{aligned}
D_c(K_i,1) &= \left(\left|N_1^i\right|-1\right)\Pr\left\{\frac{(1-\alpha)L}{K_i}>1\right\}+\beta\left|N_\beta^i\right|\Pr\left\{\frac{\beta(1-\alpha)L}{K_i}>\beta\right\} \\
&= K_i\Pr\left\{\frac{(1-\alpha)L}{K_i}>1\right\},
\end{aligned}
\tag{27}
$$

while if the firm hit is large (of size β) the expected number of defaults is

$$
\begin{aligned}
D_c(K_i,\beta) &= \left|N_1^i\right|\Pr\left\{\tfrac{(1-\alpha)L}{K_i+1-\beta}>1\right\}+\beta\left(\left|N_\beta^i\right|-1\right)\Pr\left\{\tfrac{\beta(1-\alpha)L}{K_i+1-\beta}>\beta\right\} \\
&= (K_i+1-\beta)\Pr\left\{\tfrac{(1-\alpha)L}{K_i+1-\beta}>1\right\}.
\end{aligned}
\tag{28}
$$

For each $s\in\{1,\beta\}$, define K_s^* as the minimizer in K of $D_c(K,s)$. The total expected number of defaults in a system divided into m components when a b shock hits a firm is then:

$$
\begin{aligned}
& \sum_{i=1}^{m}\frac{K_i+1}{N}\left(\frac{\left|N_1^i\right|}{K_i+1}D_c(K_i,1)+\frac{\left|N_\beta^i\right|}{K_i+1}D_c(K_i,\beta)\right). \\
= & \sum_{i=1}^{m}\left(\frac{\left|N_1^i\right|}{N}D_c(K_i,1)+\frac{\left|N_\beta^i\right|}{N}D_c(K_i,\beta)\right) \\
\geq & \sum_{i=1}^{m}\left(\frac{\left|N_1^i\right|}{N}D_c(K_1^*,1)+\frac{\left|N_\beta^i\right|}{N}D_c\left(K_\beta^*,\beta\right)\right) \\
= & D_c(K_1^*,1)\frac{|N_1|}{N}+D_c\left(K_\beta^*,\beta\right)\frac{|N_\beta|}{N}
\end{aligned}
$$

This establishes the following result:

Proposition 8 *Suppose that $|N_1|$ and $|N_\beta|$ are a multiple, respectively, of K_1^*+1 and K_β^*+1. Then the optimal (completely connected) structure is such that all firms are in components with all firms of the same size (either small or large) and the common size of every component consisting of small (resp. large) firms is K_1^*+1 (K_β^*+1).*

It is worth noting that the same result would hold if firm size, 1 or β, instead of scaling up the probability were to scale the size of the b shock hitting firms – that is, if the probability that a firm of size 1 is hit by a shock of magnitude L were the same as the probability that a firm of size β is hit by a shock of size βL. It can be easily checked that, even though the counterpart of expressions (27) and (28) would be different, they would still retain the property of being independent of the composition between small and large firms in the component. And, as explained before, this is the key feature that yields the above conclusion, that the optimal configuration involves the segmentation of the population into homogeneous components, with all firms of equal size. The intuition is similar to the one discussed when motivating Proposition 7: allowing for heterogeneous components

would mean failing to take advantage of the generally one-to-one correspondence between component size and type – be it associated to size or shock distribution – of the firm hit by the shock, that assures optimality.

5.2 Asymmetry in structure

Once we allow for different sizes of firms, we may also examine new types of structures in which large and small firms may play asymmetric roles. The key question we now study is whether it is optimal that firms of different size have a different position in the network, and hence a different role in the pattern of risk sharing trades. To this end, our analysis will focus on comparing two structures: (a) one *symmetric*, where components are completely connected and each component involves only firms of identical size; (b) another one asymmetric, where components have a *"star" structure*, consisting of a large firm that acts as a central hub and is directly connected to various small firms.

More precisely, we consider for simplicity the case where there are only two large firms and 2β small ones. Hence in structure (a) we have two completely connected components of the same total size: one with the two large firms; the other with the 2β small firms. In contrast, in structure (b) we have two identical star components, each consisting of β small firms that are solely connected to a large firm (that is, the large firm acts as a hub and there are β spokes of unit size). Hence in this case each component is heterogeneous and not fully connected.

The first step is to specify the pattern of risk exposure in each of the two cases. In structure (a), the situation is analogous to that of the completely connected structures considered in Subsection 2.2. The exposure pattern can then be described by a matrix A_K of the form specified in (5) for a component of size $K+1$, with $K=1$ and $K=2\beta-1$ for the complete components consisting of large and small firms, respectively.

In a star structure (b) we face the new problem that not only firms are heterogenous within each component, but also the structure of the component is asymmetric. The pattern of mutual exposure obtained in each component after the direct exchanges of assets is:[23]

$$\widetilde{B} = \begin{pmatrix} \theta & (1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ (1-\theta) & \theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\theta) & 0 & 0 & \cdots & \theta \end{pmatrix}$$

The entries of the matrix $\widetilde{B}$ reflect the fact the exchange of assets among firms of different size is no longer one for one: the large firm (indexed by $j=1$) must offer only a share $(1-\theta)/\beta$ of its assets for a larger share $(1-\theta)$ in the assets of small firms (indexed by $j=2,3,...,\beta+1$). Iterating once these trades we obtain the following pattern of exposure,

[23] We dispense here for notational simplicity with any reference to the component size ($K=\beta+1$).

$\widetilde{A} = \widetilde{B}^2$, where each firm has a non-zero exposure to any other firm in the component

$$\widetilde{A} = \begin{pmatrix} \theta^2 + (1-\theta)^2 & 2\theta(1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ 2\theta(1-\theta) & \theta^2 + (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & (1-\theta)^2/\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\theta(1-\theta) & (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & \theta^2 + (1-\theta)^2/\beta \end{pmatrix}.$$

From the above expression for $\widetilde{A}$ we see that, for each value of θ, the fraction of claims held by each small firm on its own project:

$$\alpha = \theta^2 + (1-\theta)^2/\beta, \tag{29}$$

is smaller than the analogous fraction held by the large firm:

$$\alpha' = \theta^2 + (1-\theta)^2. \tag{30}$$

Naturally, the difference $\alpha' - \alpha > 0$ grows larger with β. This is simply a reflection of the fact that the higher is the size asymmetry between large and small firms the lower the share of its own project that the large firm needs to trade for any given share on the project of a small firm. As in Subsection 2.2, the value of θ will be set so that each firm maximizes the extent of risk sharing consistent with *every* firm in the system (large or small) retaining at least a share of 1/2 on the yields of its own project. From (29) and (30), it is clear that this happens when $\alpha = 1/2$ and $\alpha' = 1/2 + (1-\theta)\frac{\beta-1}{\beta}$.

As in Section 2.1, any firm i defaults when a shock L hits the project of firm j in the same component (possibly $i = j$) and the firm's exposure to the shock, $\tilde{a}_{ij}L$, exceeds the firm's net returns (here $\tilde{a}_{ij}$ denotes the ij entry of $\widetilde{A}$). This now amounts to a different condition if the firm under consideration is big or small, since each firm has a different "return buffer": if i is small, default occurs when $\tilde{a}_{ij}L > 1$; instead, if i is large the condition is $\tilde{a}_{ij}L > \beta$.

The next result compares the performance of structures (a) and (b), for different values L of the magnitude of the b shock hitting a randomly selected firm:

Proposition 9 *Assume $\beta > 2$. Then, if*

$$\frac{1}{1-\alpha'} < L \leq 2\beta, \tag{31}$$

expected defaults are lower in structure (a) with homogeneous components than in structure (b) with star components, while if

$$2(2\beta - 1) < L \leq \max\left\{\frac{\beta-1}{\alpha' - 1/2}, \frac{\beta^2}{1-\alpha'}\right\} \tag{32}$$

the opposite conclusion holds.

First, the superiority of the symmetric (completely connected) structure in the range given by (31) is easy to explain. In the star structure a large firm (the hub) engages in risk

sharing trades with a set of smaller firms (the spokes). This ends up limiting the possibilities of risk sharing when a shock hits a large firm because, in order to match the lower value of the assets of the smaller firms, the large firm is forced to hold more of its own assets than in the symmetric structure. As a consequence, when a shock of an intermediate size as in (31) hits a large firm, it triggers the default of the small firms in the star structure as well as, possibly, that of the large firm. Instead such a shock hitting the big firm would be absorbed with no defaults in a symmetric structure, due to the enhanced risk sharing possibilities it would enjoy when connected to a firm of the same size.

On the other hand, it is precisely this limitation to the risk sharing possibilities in a star structure that protects the hub as well as the other spokes against the shocks hitting a small firm. As a consequence, the star structure performs better than the symmetric structure for shocks of larger size, as in (32). In a symmetric component, these shocks are so large that they trigger the default of all the firms linked to a firm directly hit by the shock. In contrast, in a star component none of these firms defaults when the shock hits one of the spoke firms – only the small firm hit defaults. In this case, therefore, the large firm acts a sort of buffer, preventing a sizable fraction of the shock to spread and cause further defaults in the component.

6 Conclusion

We have proposed a stylized model to study the problem that arises when firms need to share resources to weather shocks that can threaten their survival, but by so doing they become exposed to the risk coming from those same connections that help them in the time of need. Depending on the characteristics of the shock distribution, a wide variety of different configurations turn out to be optimal. For example, maximal segmentation in small groups is optimal if big shocks are likely, while very large groups are optimal when most shocks are of moderate magnitude. There are also conditions, however, when an intermediate group size is optimal or when groups should be large but display some internal "detachment" (i.e. sparse connectivity).

The former consideration pertains to social optimality, i.e. to the minimization of the default rate within the whole system. We have also explored whether such social optimum is aligned with individual optimality. And we have seen that, in general, there is a conflict between strategic incentives and social welfare. This tension arises from the fact that firms have always an incentive to form connected components of the size that minimizes the default probability of their members, thus ignoring the negative externality this behavior imposes on other firms. Finally, we have found that when heterogeneity among firms is allowed but network components are taken to be fully and symmetrically connected, optimality is achieved under perfect assortativity. But if the network structure can display some asymmetries (e.g. a "large central" agent can act as a hub), it may be optimal to take advantage of this possibility as a *firebreak* under certain conditions.

The present analysis represents a first step in studying the welfare implications of alternative risk-sharing network structures, as well as their (in)consistency with individual incentives. By restricting attention to suitable families of network structures, we have arrived at clear-cut conclusions as to what kind of segmentation, sparseness, or asymmetries in connections attain optimality. The basic insights obtained from this analysis could inform the regulation of financial systems, which are indeed subject to some of the key mechanisms contemplated by our theoretical framework. Should one separate commercial and investment banking? Should the financial systems of different regions be insulated from each other, or should overall integration be pursued? Is it advisable to allow for some large central institutions to have a buffer role in mitigating the shocks affecting smaller ones? And, in the end, how effectively can we trust any prescription along these lines to be indeed implemented by the economic agents themselves (i.e. to be compatible with their own individual incentives)? These are some of the questions that will naturally and crucially arise in the design and regulation of financial systems in the real world.

Even though our stylized model cannot possibly provide *direct* policy advice on the previous questions, it does highlight some of the key considerations that should underlie a systematic analysis of them. As mentioned in the Introduction, the companion paper by Loepfe, Cabrales and Sánchez (2013) extends our analysis through numerical methods to environments with more realistic features and verifies the robustness of our main results. More precisely, it finds that the optimality of forming smaller/less dense networks when shocks come from distributions that put more weight on large shocks extends even when one allows for the possibility of multiple simultaneous shocks, for other kinds of distributions and for networks that present heterogeneity in the firm sizes and the distribution of links among the different agents. In a different vein, they also find similar effects on the real-life network of corporate control studied by Battiston *et al.* (2012), where the consequences of link removal on the vulnerability of the system (increasing or decreasing it) happens to depend on the nature of the shock distribution as intuitively suggested by the theory.

But, of course, a proper discussion of the risk-sharing and contagion phenomena in the real world must also account for many important dimensions that our model abstracts from. One of them is the consideration of problems of moral hazard and, in general, asymmetries of information that have been repeatedly singled out as a key factor underlying the recent financial crisis. Any improvement in the regulatory framework should tackle these serious problems carefully. Finally, another point that is worthy of further research is the integration of the risk-management decisions studied here with the considerations of inter-agent cooperation and exploitation of synergies that also underlie economic connections in the real world. A proper study of all these crucial issues will demand a richer theoretical framework and a more powerful methodology, the development of which can hopefully build in a fruitful manner upon the present effort.

Appendix

Proof of Lemma 1: Let $M_F(qN\pi_b)$ denote the function specifying the value of M that satisfies the required arbitrage condition (2) under financial structure F, given by a pattern of interaction $\nu \in \{c, r\}$ and component sizes $K_1, ..., K_n$ all displaying the pattern ν, when the probability of the arrival of a b shock is $qN\pi_b$. Substituting equation (20) into (2), $M_F(qN\pi_b)$ is defined implicitly by the following equation:

$$M_F(qN\pi_b)\left[1 - qN\pi_b\left(\frac{1}{N}\Pi + \sum_{i=1}^{C}\frac{K_i+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_F(qN\pi_b))}{N}\right)\right] = r.$$

Since the function $\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_F(qN\pi_b))$ is bounded for each $\nu \in \{c, r\}$, as we see from (19), (17), $M_F(qN\pi_b)$ is continuous in $qN\pi_b$ at $qN\pi_b = 0$. Hence, if structure F_1 yields a higher expected mass of defaults when indirectly hit by a b shock than structure F_2, i.e.

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_1}(0))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_2}(0))}{N},$$

then we also have, for $qN\pi_b$ sufficiently small,

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_1}(qN\pi_b))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_2}(qN\pi_b))}{N},$$

■

Proof of Proposition 6: Let $\mathbf{S} \equiv (S_i)_{i\in N}$ be a CPE of the network-formation game with completely connected components. Denote by C^S the number and by $\{K_j+1\}_{j=1}^{C^S}$ the sizes of the different components of the CPE network $\Gamma(\mathbf{S})$. Recall that $\hat{K}$ denotes the unique value that minimizes $D_c(K, \gamma, \gamma', p)$ (c.f. the proof of Proposition 4).

The proof has two main steps. We show first that for each component $j = 1, .., C^S$ we have $K_j \leq \hat{K}$. That is, all components in the CPE have size smaller or equal than the individually optimal one. Suppose not: $K_j > \hat{K}$ for some j. Choose then a subset $\mathcal{D}$ of the firms belonging to component j whose measure satisfies $K_j - \hat{K} \geq |\mathcal{D}| > 0$. Since the function $D_c(K, \gamma, \gamma', p)$ is increasing in K for $K > \hat{K}$, $D_c(K_q - |\mathcal{D}|, \gamma, \gamma', p) < D_c(K_q, \gamma, \gamma', p)$. This implies that all the firms who are in component j except those in $\mathcal{D}$ could enjoy a lower default probability by proposing a component with all the firms in j except those with the firms in $\mathcal{D}$. This establishes the desired conclusion.

Next, we show that at a CPE we have $K_{\widetilde{h}} < \hat{K}$ for some $\widetilde{h} \in \{1, 2, ..., C^S\}$ and $K_j = \hat{K}$ for all $j \neq \widetilde{h}$. We proceed again by contradiction. Suppose that there are two components of $\Gamma(\mathbf{S})$, j and q, such that $0 < K_j < K_q < \hat{K}$. Pick then a subset of firms belonging to the first component: $\mathcal{D}_j \subset K_j$, of measure $|\mathcal{D}_j| \leq \hat{K} - K_q$. Consider the following joint deviation for the firms in $\mathcal{D}_j$ as well as for all those in component q from their strategies in

S. Each of the firms in $\mathcal{D}_j$ together with the firms in component q propose a component consisting of all of them (and only them). Then, we have:

$$D_c(K_q + |\mathcal{D}_j|, \gamma, \gamma', p) < D_c(K_q, \gamma, \gamma', p) < D_c(K_j, \gamma, \gamma', p)$$

Hence all firms involved benefit from the deviation, which contradicts the fact that at a CPE we have $0 < K_j < K_q < \hat{K}$. This contradiction establishes the above claim and completes so the proof of the Proposition. ■

Remark 2 *Note that the deviations contemplated in the proof of Proposition 6 are "internally consistent" in the following sense. Given any set of firms that find optimal to deviate, this deviation is itself in the interest of any subset of this set that might reconsider the situation. This requirement (commonly demanded in the game-theoretic literature for coalition-based notions of equilibrium) is clearly satisfied in our case since, by refusing to follow suit with the deviation, any firm in this subset can only lose, because of the monotonicity of the function $D_c(.)$.*

Proof of Proposition 7: If a component i is hit a by a b shock, the expected number of defaults among firms not directly hit by it is:

$$\sum_{l=1}^{n} \frac{\left|N_l^i\right|}{K_i+1} D_c\left(K_i, \Phi_{N_l}\right),$$

where $K_i + 1$ is as usual the size of component i, in this case given by $\sum_{l=1}^{n} \left|N_l^i\right|$. Then the total expected number of defaults in the system when a b shock hits a firm is

$$\begin{aligned} & \sum_{i=1}^{I} \frac{K_i+1}{N} \sum_{l=1}^{n} \frac{\left|N_l^i\right|}{K_i+1} D_c\left(K_i, \Phi_{N_l}\right). \\ = & \sum_{i=1}^{I} \sum_{l=1}^{n} \frac{\left|N_l^i\right|}{N} D_c\left(K_i, \Phi_{N_l}\right). \end{aligned}$$

Exchanging the summation indices, we get

$$\begin{aligned} \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{\left|N_l^i\right|}{N} D_c\left(K_i, \Phi_{N_l}\right) & \geq \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{\left|N_l^i\right|}{N} D_c\left(K_{N_l}^*, \Phi_{N_l}\right) \\ & = \sum_{l=1}^{n} D_c\left(K_{N_l}^*, \Phi_{N_l}\right) \sum_{G_i=1}^{I} \frac{\left|N_l^i\right|}{N} = \sum_{l=1}^{n} \frac{|N_l|}{N} D_c\left(K_{N_l}^*, \Phi_{N_l}\right). \end{aligned}$$

Since $|N_l|$ is a multiple of $K_{N_l}^* + 1$ the lower bound

$$\sum_{l=1}^{n} \frac{|N_l|}{N} D_c\left(K_{N_l}^*, \Phi_{N_l}\right)$$

is also the expected number of defaults if every firm is part of a group of optimal size $K^*_{N_l}+1$ with all firms of the same type. ■

Proof of Proposition 9: To prove the result, we compute the expected number of bankruptcies associated to the two structures (star and symmetric) for all possible levels of the b shock. Denote by $G_{star}(L)$ and $G_{sym}(L)$ the functions that specify the total size of all[24] the firms who default, resp. for the star and the symmetric structure, as a function of the magnitude L of the b shock.

Let us begin by determining the total size of firms defaulting if the shock hits a *small firm* and the structure of the component to which it belongs is a star. For any given value of L, it is given by the following function[25]:

$$G^s_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} = 2 \\ 1 & \text{for } 2 < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ 1+\beta \text{ otherwise} & \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}$$

Instead, if the structure is still a star but the shock hits a *large firm* (i.e. the hub), the expected number of defaults is given by the following function[26]:

$$G^\ell_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{1-\alpha'} \\ \beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ 2\beta & \text{for } L > \frac{1}{1-\alpha'} \end{cases},$$

Ex ante, a shock hitting a component of the network has the same probability of striking a large firm or a small firm. Hence, $G_{star}(L) = \left(G^s_{star}(L) + G^\ell_{star}(L)\right)/2$.

Consider next the case where the structure is symmetric, with two completely connected components, the first one with the two large firms, the second one with the 2β small firms. In this case, since there is no asymmetry within each component, every firm retains a fraction exactly equal to $\alpha = 1/2$ of claims to the returns of its own project. The off diagonal terms of the exposure matrix are then equal to $1/2$ for the component with the two large firms and $1/[2(2\beta-1)]$ for the second one, with 2β firms. Since the shock reaches with equal probability each of the two components, the expected number of defaults is given by the following function:

[24] Note that, in contrast to the functions $g_\nu(L; K, M)$, $\nu = c, r$, introduced in Section 2.2, the functions $G_{star}(\cdot)$, $G_{sym}(\cdot)$ describe the number of *all firms* who default, that is including the firm that is directly hit by the shock L. Since the degree of risk externalization, as we noticed, may now differ across different structures, so does the probability that a firm directly hit by a shock defaults.

[25] Recall that a small firm i defaults when a shock hits a firm j if and only if $\tilde{a}_{ij}L > 1$, while a large firm k defaults if and only if $\tilde{a}_{kj}L > \beta$.

[26] The upper and lower bounds on α' established in Lemma 3 in Appendix B imply that $\beta/\alpha' > 1/(1-\alpha')$.

$$G_{sym}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq 2\beta \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } 2\beta < L \leq 2(2\beta - 1) \\ 2\beta & \text{for } L > 2(2\beta - 1) \end{cases}$$

where it can be easily verified that $2 < 2\beta < 2(2\beta - 1)$.

A straightforward comparison of the functions $G_{star}(L)$ and $G_{sym}(L)$, noting that $\frac{\beta}{\alpha'} < 2\beta$ and $\frac{\beta}{\alpha'} < 2(2\beta - 1) < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}$, yields then the claim in the proposition. ■

The remaining proofs of Lemmas 2, 3 and Propositions 3, 4 and 5, as well as some further details of the proof of Proposition 9 can be found in Appendix B[27].

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